Phase fluctuations and first-order correlation functions of dissipative Bose-Einstein condensates

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We investigate the finite-lifetime effects on first-order correlation functions of dissipative Bose-Einstein condensates. By taking into account the phase fluctuations up to all orders, we show that the finite-lifetime effects are negligible for the spatial first-order correlation functions, but have an important effect on the temporal correlations. As an application, we calculate the one-particle density matrix of a quasicondensate of photons. Finally, we also consider the photons in the normal state and we demonstrate that the finite-lifetime effects decrease both the spatial and temporal first-order correlation functions.

I. INTRODUCTION

Bose-Einstein condensation (BEC) was first directly observed by cooling atomic vapors to temperatures in the nK regime [1–3]. These vapors were confined in three-dimensional traps, of which the shape could be altered by changing certain experimental parameters, e.g., the magnetic field. For sufficiently tight confinement in one or two directions, the dynamics in these dimensions can be frozen out and the atoms behave as a quasi one-dimensional or two-dimensional gas. This manipulation opened up the possibility of exploring Bose-Einstein condensates in lower dimensions [4–7].

From a theoretical point of view these low-dimensional Bose-Einstein condensates are particulary interesting because their physics is fundamentally different from three-dimensional condensates. Namely, in two dimensions a homogeneous Bose gas can only undergo BEC at zero temperature, and in one dimension BEC in a homogeneous Bose gas cannot take place at all [8,9]. In the presence of an external potential the situation drastically changes. In particular, harmonically trapped bosons can undergo BEC at nonzero temperatures in both one and two dimensions [10–12].

For homogeneous two-dimensional Bose gases, theoretical studies show that even though the Bose-Einstein condensate does not exist at nonzero temperatures, there still exists a critical temperature in the system. Below the so-called Kosterlitz-Thouless temperature the gas is superfluid, and above this temperature the bosons lose their superfluid property [13]. This is known as the Kosterlitz-Thouless transition, and it implies that superfluidity only requires the presence of a quasicondensate [14] with phase coherence over a distance much less than the system size. This quasicondensate can be roughly interpreted as a system consisting of several patches, each with a fixed phase, whereas the phases of these different patches are uncorrelated.

In addition to atomic gases, there are presently also other low-dimensional systems in which BEC is observed, such as systems consisting of exciton-polaritons [15,16] or photons [17]. Together with BEC of magnons [18], these systems form

a class of condensates that is different from the atomic Bose-Einstein condensates. In particular, the bosonic quasiparticles have a small effective mass resulting in BEC at temperatures in the range of 10–300 K instead of in the nK regime relevant to the atomic Bose-Einstein condensates. Furthermore, these condensates are not in true thermal equilibrium, and the steady state is a dynamical balance between particle losses and external pumping. Therefore, the particles have a finite lifetime, which can be characterized by a single dimensionless damping parameter.

In the context of exciton-polariton condensates the first-order correlation functions are extensively studied theoretically. In Refs. [19,20] the temporal and spatial first-order correlation functions are calculated by introducing a cutoff to handle the ultraviolet divergence at zero temperature. Furthermore, a more general discussion on spatial correlation functions of nonequilibrium condensates in reduced dimension is given in Ref. [21]. In that work, the main results are obtained for a frequency-independent damping, and a first attempt is made to incorporate frequency-dependent damping. Moreover, in Refs. [22,23] the spatial correlations of one-dimensional driven-dissipative nonequilibrium condensates are investigated by studying a stochastic equation for the phase fluctuations.

In this article we study the effect of the appropriate frequency-dependent damping parameter on the first-order correlation functions of low-dimensional Bose-Einstein condensates. In particular, we focus on a Bose-Einstein condensate of photons for which this damping parameter is explicitly calculated in Ref. [24]. First, we derive a general expression for the first-order correlation function for a homogeneous Bose gas in the condensed phase in Sec. II. Thereafter, we use this general expression for the first-order correlation functions to determine the effect of the finite lifetime on the spatial and temporal correlations in Sec. III. In Sec. IV we consider BEC of photons, taking their interaction with the dye molecules into account, and determine the off-diagonal long-range behavior of the one-particle density matrix in the Bose-Einstein condensed phase. We show that for the relevant parameters used in the experiment the photons form a true condensate and Kosterlitz-Thouless physics is not observable. Subsequently, we determine the first-order correlation functions of photons in the normal state, and we end with conclusions and an outlook in Sec. V.

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II. PHASE FLUCTUATIONS

In this section we derive a general expression for the first-order correlation functions for a homogeneous Bose gas consisting of N bosons in a box of volume V. We start from the Euclidean action

$$
\begin{aligned}
&S[\phi^*,\phi] \\
&= \int_0^{\hbar\beta} d\tau\, d\tau' \int d\mathbf{x}\, d\mathbf{x}'\, \phi^*(\mathbf{x},\tau) \\
&\quad \times \left\{ \left[\hbar \frac{\partial}{\partial \tau} - \frac{\hbar^2}{2m}\nabla^2 - \mu + \frac{1}{2}T^{2B}|\phi(\mathbf{x},\tau)|^2 \right] \delta(\tau-\tau') \right. \\
&\quad \left. \times \delta(\mathbf{x}-\mathbf{x}') + \hbar\Sigma(\mathbf{x}-\mathbf{x}',\tau-\tau') \right\} \phi(\mathbf{x}',\tau'),
\end{aligned} \tag{1}
$$

where τ and τ' denote imaginary times, $\beta = 1/k_B T$ with T the temperature, μ is the chemical potential, and T^{2B} is the strength of the self-interaction. Furthermore, we included a self-energy $\hbar\Sigma(\mathbf{x}-\mathbf{x}',\tau-\tau')$ describing additional interaction effects, e.g., in the photon experiment of Klaers *et al.* the interaction of the photons with the dye molecules [17].

To obtain an expression for the first-order correlation functions in the superfluid phase, we split the density and phase fluctuations and substitute $\phi(\mathbf{x},\tau) = \sqrt{n + \delta n(\mathbf{x},\tau)}\, e^{i\theta(\mathbf{x},\tau)}$. Here n is the average density of the gas, $\delta n(\mathbf{x},\tau)$ denotes the density fluctuations, and $\theta(\mathbf{x},\tau)$ represents the phase. We expand up to second order in θ and δn, and define

$$
\begin{aligned}
\theta(\mathbf{x},\tau) &= \frac{1}{\sqrt{\hbar\beta V}} \sum_{\mathbf{k},m} \theta_{\mathbf{k},m} e^{i(\mathbf{k}\cdot\mathbf{x}-\omega_m\tau)}, \\
\delta n(\mathbf{x},\tau) &= \frac{1}{\sqrt{\hbar\beta V}} \sum_{\mathbf{k},m} \delta n_{\mathbf{k},m} e^{i(\mathbf{k}\cdot\mathbf{x}-\omega_m\tau)},
\end{aligned} \tag{2}
$$

to obtain

$$
\begin{aligned}
&S[\delta n,\theta] \\
&= \frac{n}{2} \sum_{\mathbf{k},m} \{\hbar\Sigma^s(\mathbf{k},i\omega_m) + 2\epsilon(\mathbf{k})\}\theta_{\mathbf{k},m}\theta_{-\mathbf{k},-m} \\
&\quad + \frac{1}{8n} \sum_{\mathbf{k},m} \{\hbar\Sigma^s(\mathbf{k},i\omega_m) + 4n\chi^{-1}(\mathbf{k})\}\delta n_{\mathbf{k},m}\delta n_{-\mathbf{k},-m} \\
&\quad + \sum_{\mathbf{k},m} \left\{ \hbar\omega_m + \frac{i}{2}\hbar\Sigma^a(\mathbf{k},i\omega_m) \right\} \delta n_{-\mathbf{k},-m}\theta_{\mathbf{k},m},
\end{aligned} \tag{3}
$$

with the inverse of the static density-density correlation function $\chi^{-1}(\mathbf{k}) = \epsilon(\mathbf{k})/2n + T^{2B}$, the single-particle dispersion $\epsilon(\mathbf{k}) = \hbar^2\mathbf{k}^2/2m$, and the antisymmetric and symmetric parts of the self-energy obeying

$$
\begin{aligned}
\hbar\Sigma^a(\mathbf{k},i\omega_m) &= \hbar\Sigma(\mathbf{k},i\omega_m) - \hbar\Sigma(-\mathbf{k},-i\omega_m), \\
\hbar\Sigma^s(\mathbf{k},i\omega_m) &= \hbar\Sigma(\mathbf{k},i\omega_m) + \hbar\Sigma(-\mathbf{k},-i\omega_m).
\end{aligned} \tag{4}
$$

Here $\hbar\Sigma(\mathbf{k},i\omega_m)$ is defined in a similar way as the Fourier transform of the phase and density fluctuations in Eq. (2) except for the normalization factor, which is $1/\hbar\beta V$ in this case. Furthermore, in Fourier space we take without

loss of generality $\hbar\Sigma(\mathbf{0},0) = 0$. For bosons this assumption is automatically satisfied for the imaginary part, and the constant real part results in a energy shift of the poles of the Green's function and can be absorbed in the chemical potential.

Note that in Eq. (3) we substituted the zero-loop result for the chemical potential $\mu = nT^{2B}$.

By using the classical equations of motion, we can now eliminate the phase $\theta_{\mathbf{k},m}$ and find an action for the density fluctuations $\delta n_{\mathbf{k},m}$ alone. From this action we obtain

$$
\begin{aligned}
&\langle \delta n(\mathbf{x},\tau)\delta n(\mathbf{x}',\tau') \rangle \\
&= \frac{n}{\beta V} \sum_{\mathbf{k},m} \frac{\hbar\Sigma^s(\mathbf{k},i\omega_m) + 2\epsilon(\mathbf{k})}{\text{Det}[\hbar\mathbf{G}_B^{-1}(\mathbf{k},i\omega_m)]} e^{i[\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')-\omega_m(\tau-\tau')]},
\end{aligned} \tag{5}
$$

where the Green's function of the density and phase fluctuations has the matrix structure

$$
\begin{aligned}
&\hbar\mathbf{G}_B^{-1}(\mathbf{k},i\omega_m) \\
&= \begin{bmatrix} \hbar G^{-1}(\mathbf{k},i\omega_m) - nT^{2B} & -nT^{2B} \\ -nT^{2B} & \hbar G^{-1}(-\mathbf{k},-i\omega_m) - nT^{2B} \end{bmatrix},
\end{aligned} \tag{6}
$$

in terms of the single-particle Green's function

$$
\hbar G^{-1}(\mathbf{k},i\omega_m) = i\hbar\omega_m - \epsilon(\mathbf{k}) - \hbar\Sigma(\mathbf{k},i\omega_m). \tag{7}
$$

Similarly we use the equation of motion for $\delta n_{\mathbf{k},m}$ to eliminate the density fluctuations, and we find

$$
\begin{aligned}
&\langle \theta(\mathbf{x},\tau)\theta(\mathbf{x}',\tau') \rangle \\
&= \frac{1}{4n\beta V} \sum_{\mathbf{k},m} \frac{\hbar\Sigma^s(\mathbf{k},i\omega_m) + 2\epsilon(\mathbf{k}) + 4nT^{2B}}{\text{Det}[\hbar\mathbf{G}_B^{-1}(\mathbf{k},i\omega_m)]} \\
&\quad \times e^{i[\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')-\omega_m(\tau-\tau')]}.
\end{aligned} \tag{8}
$$

However, from Ref. [25] we know that the contribution of the phase fluctuations is proportional to the density. The first two terms are not proportional to n and they are an artifact of making an expansion up to second order in $\theta(\mathbf{x},\tau)$ and of neglecting the interaction terms between the density and phase fluctuations. A more accurate approach that takes into account higher-order terms in $\theta(\mathbf{x},\tau)$ would not contain these high-momentum contributions. Therefore, the correct expression for the phase fluctuations is given by

$$
\begin{aligned}
&\langle \theta(\mathbf{x},\tau)\theta(\mathbf{x}',\tau') \rangle \\
&= \frac{1}{n\beta V} \sum_{\mathbf{k},m} \frac{nT^{2B}}{\text{Det}[\hbar\mathbf{G}_B^{-1}(\mathbf{k},i\omega_m)]} e^{i[\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')-\omega_m(\tau-\tau')]}.
\end{aligned} \tag{9}
$$

In certain cases the self-energy is only known for real frequencies. Therefore, we define

$$
\begin{aligned}
\rho_\theta(\mathbf{k},\omega) &= -\frac{nT^{2B}}{2\hbar}\rho_B(\mathbf{k},\omega) \\
&:= -\frac{nT^{2B}}{\pi\hbar}\text{Im}\left[\frac{1}{\text{Det}[\hbar\mathbf{G}_B^{-1}(\mathbf{k},\omega^+)]} \right],
\end{aligned} \tag{10}
$$

and we write for the phase correlation function

$$\langle\theta(\mathbf{x},\tau)\theta(\mathbf{x}',\tau')\rangle$$

$$= \frac{\hbar}{n\hbar\beta V}\sum_{\mathbf{k},m}\int_{-\infty}^{\infty}d(\hbar\omega)\frac{\rho_\theta(\mathbf{k},\omega)}{i\omega_m-\omega}e^{-i\omega_m(\tau-\tau')}e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')}$$

$$= -\frac{\hbar}{nV}\sum_{\mathbf{k}}\int_{-\infty}^{\infty}d(\hbar\omega)\,\rho_\theta(\mathbf{k},\omega)e^{-\omega(\tau-\tau')}e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')}$$

$$\times\{\Theta(\tau'-\tau)N_{\mathrm{BE}}(\hbar\omega)+\Theta(\tau-\tau')[N_{\mathrm{BE}}(\hbar\omega)+1]\},$$

$$(11)$$

where

$$N_{\mathrm{BE}}(\hbar\omega)=\frac{1}{e^{\beta\hbar\omega}-1} \tag{12}$$

is the Bose-Einstein distribution function. For simplicity we take $\tau'>\tau$, and the case $\tau'<\tau$ is treated analogously. In principle we have to consider both the density and phase fluctuations in order to calculate the first-order correlation functions. However, here we consider relatively high condensate fractions for which the density fluctuations are strongly suppressed and the phase fluctuations are most important, especially for the description of the long-range order which is of most interest to us here [26–29]. Hence, we have

$$\langle\phi^*(\mathbf{x},t)\phi(\mathbf{x}',t')\rangle\simeq n_0\langle e^{-i[\theta(\mathbf{x},t)-\theta(\mathbf{x}',t')]}\rangle$$

$$= n_0 e^{-\frac{1}{2}\langle[\theta(\mathbf{x},t)-\theta(\mathbf{x}',t')]^2\rangle}, \tag{13}$$

with n_0 the quasicondensate density. Note that for the Gaussian approach used here, the second line of Eq. (13) is exact. However, in principle there are corrections to this result which can be incorporated by treating the phase of the condensate as a stochastic variable. As is shown in Ref. [23], these corrections are rather small and therefore we neglect these corrections throughout the remainder of this work.

By using Eq. (11) and performing the analytical continuation to real time $\tau=it$, we obtain for the exponent of this expression

$$\langle[\theta(\mathbf{x},t)-\theta(\mathbf{x}',t')]^2\rangle$$

$$= -\frac{T^{2\mathrm{B}}}{V}\sum_{\mathbf{k}}\int_{-\infty}^{\infty}d(\hbar\omega)\rho_{\mathrm{B}}(\mathbf{k},\omega)N_{\mathrm{BE}}(\hbar\omega)$$

$$\times\{1-\cos[\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')]\cos[\omega(t-t')]\}, \tag{14}$$

where we used $\hbar\Sigma(\mathbf{k},\omega)=\hbar\Sigma(-\mathbf{k},\omega)$ for an isotropic system.

To make further progress, we perform a long-wavelength approximation to the self-energy. As mentioned before, the real part of the self-energy can effectively be absorbed in the chemical potential and therefore we neglect this part. The imaginary part is zero for $\mathbf{k}=\mathbf{0}$ and $\omega=0$, and therefore for small frequencies the imaginary part is linear in ω. Since the result of Eq. (14) is dominated by the contributions for small frequencies, the large frequency behavior is not visible in the final result. Therefore we can safely assume that the self-energy obeys $\hbar\Sigma(\mathbf{k},\omega)=\hbar\Sigma^*(\mathbf{k},-\omega)$ for the imaginary part of both the retarded and advanced self-energy. This allows

us to rewrite

$$\langle[\theta(\mathbf{x},t)-\theta(\mathbf{x}',t')]^2\rangle$$

$$= -\frac{T^{2\mathrm{B}}}{V}\sum_{\mathbf{k}}\int_0^{\infty}d(\hbar\omega)\rho_{\mathrm{B}}(\mathbf{k},\omega)\{1+2N_{\mathrm{BE}}(\hbar\omega)\}$$

$$\times\{1-\cos[\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')]\cos[\omega(t-t')]\}. \tag{15}$$

This expression for the phase fluctuations contains an ultraviolet divergence. This divergence is a consequence of not taking into account the proper energy dependence of the self-interaction of the bosons. For atoms this problem was already encountered in Ref. [25], and in that case the divergence was handled by appropriate renormalization of the interactions. In our case the form of the divergence is the same, since the self-energy must vanish for large momenta. However, we do not know the exact energy dependence of $T^{2\mathrm{B}}$. Therefore, the cancellation of the ultraviolet divergence requires us to introduce another energy scale $\gamma nT^{2\mathrm{B}}$ that models the correct energy dependence of the self-interaction of the bosons. We come back to the precise determination of γ in the next section. We thus write

$$\langle[\theta(\mathbf{x},t)-\theta(\mathbf{x}',t')]^2\rangle$$

$$= -\frac{T^{2\mathrm{B}}}{V}\sum_{\mathbf{k}}\int_0^{\infty}d(\hbar\omega)\rho_{\mathrm{B}}(\mathbf{k},\omega)$$

$$\times\left\{1+2N_{\mathrm{BE}}(\hbar\omega)-\frac{\hbar\omega_{\mathrm{B}}(\mathbf{k})}{\epsilon(\mathbf{k})+\gamma nT^{2\mathrm{B}}}\right\}$$

$$\times\{1-\cos[\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')]\cos[\omega(t-t')]\}, \tag{16}$$

with $\hbar\omega_{\mathrm{B}}(\mathbf{k})=\sqrt{\epsilon(\mathbf{k})[\epsilon(\mathbf{k})+2n_0T^{2\mathrm{B}}]}$ the Bogoliubov dispersion. The integrant in Eq. (16) must be positive for all $\mathbf{k}$ and all temperatures, because this term corresponds to the expectation value of $|\theta_{\mathbf{k},n}|^2$. Therefore, we have the restriction that $\gamma\geqslant 1$. Finally, note that this result is consistent with the expressions found in Ref. [25].

III. CORRELATION FUNCTIONS IN THE CONDENSED PHASE

In the previous section we found an expression for the first-order correlation function by taking into account the phase fluctuations up to all orders. As mentioned before, the phase fluctuations are dominated by the small-frequency contributions, and for bosons the imaginary part of the self-energy is linear in ω for small frequencies. Therefore, in this section we take the retarded self-energy equal to $\hbar\Sigma^+(\mathbf{k},\omega)=-i\alpha\hbar\omega$ and we investigate the effect of α on the first-order correlation functions.

A. Spatial correlations

From Eq. (16) we obtain that the phase fluctuations contain a zero-temperature part, and a contribution that is temperature dependent. For the equal-time phase fluctuations at zero

temperature, we have

$$\langle[\theta(\mathbf{x},0) - \theta(\mathbf{0},0)]^2\rangle$$

$$= -\frac{T^{2B}}{V}\sum_{\mathbf{k}}\int_0^\infty d(\hbar\omega)\rho_B(\mathbf{k},\omega)$$

$$\times\left\{1 - \frac{\hbar\omega_B(\mathbf{k})}{\epsilon(\mathbf{k}) + \gamma nT^{2B}}\right\}\{1 - \cos(\mathbf{k}\cdot\mathbf{x})\}. \quad (17)$$

Without loss of generality we have set $\mathbf{x}' = 0$ and we have put t equal to zero. Before we consider the effect of the self-energy on the spatial correlations, we investigate the effect of γ. We first consider the case without a self-energy. By writing the sum over $\mathbf{k}$ as an integral, we find in two dimensions

$$\langle[\theta(\mathbf{x},0) - \theta(\mathbf{0},0)]^2\rangle$$

$$= \int_0^\infty dk \frac{1 - J_0(kx)}{4\pi n\xi^2}\left\{\frac{1}{\sqrt{k^2 + 1}} - \frac{2k}{2k^2 + \gamma}\right\}, \quad (18)$$

where $J_0(kx)$ is the Bessel function of the first kind, $\xi = \hbar/[4mn_0T^{2B}]^{1/2}$ is the correlation length and $x = |\mathbf{x}|$. In the limit $x \to \infty$, the Bessel function vanishes and we obtain

$$\langle[\theta(\mathbf{x},0) - \theta(\mathbf{0},0)]^2\rangle \to \frac{\ln(2\gamma)}{2\pi}\frac{mT^{2B}}{\hbar^2}. \quad (19)$$

Thus, the condensate density is given by the right-hand side of the following equation

$$\langle\phi^*(\mathbf{x},0)\phi(\mathbf{0},0)\rangle \to n_0\exp\left\{-\frac{\ln(2\gamma)}{4\pi}\frac{mT^{2B}}{\hbar^2}\right\}, \quad (20)$$

where we again considered the limit $\mathbf{x} \to \infty$. Therefore, by increasing γ we effectively increase the interaction strength and thereby decrease the condensate density of the gas. This dependence can in principle be used to determine the value of γ from experiment. To determine the effect of the self-energy we here just fix γ and set it equal to 1. In Fig. 1 we show the result for the zero-temperature part of the phase fluctuations for different values of α. If α increases, the contribution of the phase fluctuations decreases. Therefore, for increasing α we obtain that the quantum depletion of the condensate decreases.

For systems at low temperatures this is the dominating contribution. However, here we are interested in BEC at higher temperatures such as the BEC of photons. For these condensates the temperature-dependent part is the most relevant contribution. The temperature-dependent part of the phase

fluctuations is free of ultraviolet divergences and given by

$$\langle[\theta(\mathbf{x},0) - \theta(\mathbf{0},0)]^2\rangle = -\frac{2T^{2B}}{\pi}\int_0^\infty dk\, k\int_0^\infty d(\hbar\omega)\rho_B(k,\omega)$$

$$\times N_{BE}(\hbar\omega)\{1 - J_0(kx)\}. \quad (21)$$

We evaluate this quantity for $\beta n_0 T^{2B} \simeq 1.3 \times 10^{-2}$. This corresponds to a typical value for BEC of photons in the regime where the density fluctuations are suppressed and we can focus on the phase fluctuations [30]. By looking at Fig. 1, we observe that the temperature-dependent contribution is indeed several orders of magnitude larger than the zero-temperature part. Furthermore, it turns out that the α dependence of the nonzero-temperature part is negligible.

To understand this feature, we distinguish between two different frequency regimes. Namely, $\beta\hbar\omega < 1$ and $\beta\hbar\omega > 1$. Since we are at room temperature, the latter corresponds to relatively high values of the momentum k. In the Bose-Einstein condensed phase the contributions for small momenta are dominant. Therefore, the contributions coming from $\beta\hbar\omega > 1$ are suppressed, and we can focus on the first regime.

To make analytical progress, we use $2N_{BE}(\hbar\omega) \simeq 2/\beta\hbar\omega - 1$ for $\beta\hbar\omega < 1$. Furthermore, we can neglect the -1 since this is a contribution of the same order as the zero-temperature part and is therefore negligible compared to the temperature-dependent part. Hence, we obtain for the nonzero-temperature part of the phase fluctuations

$$\langle[\theta(\mathbf{x},0) - \theta(\mathbf{0},0)]^2\rangle = -\frac{2T^{2B}}{\pi\beta}\int_0^\infty dk\, k\int_0^\infty d(\hbar\omega)$$

$$\times \frac{\rho_B(k,\omega)}{\hbar\omega}\{1 - J_0(kx)\}. \quad (22)$$

We can perform the integral over ω analytically, and we obtain

$$\langle[\theta(\mathbf{x},0) - \theta(\mathbf{0},0)]^2\rangle = \frac{T^{2B}}{\pi\beta}\int_0^\infty dk\, k\frac{1 - J_0(kx)}{[\hbar\omega_B(k)]^2}. \quad (23)$$

This expression is indeed independent of α, and this explains why the α dependence of the spatial phase fluctuations is negligible. Note that this argument is independent of the number of dimensions, and therefore also in one dimension the α dependence of the spatial correlations is negligible. For a frequency-independent damping, this independence of the spatial correlations is also encountered in exciton-polariton condensates [31].
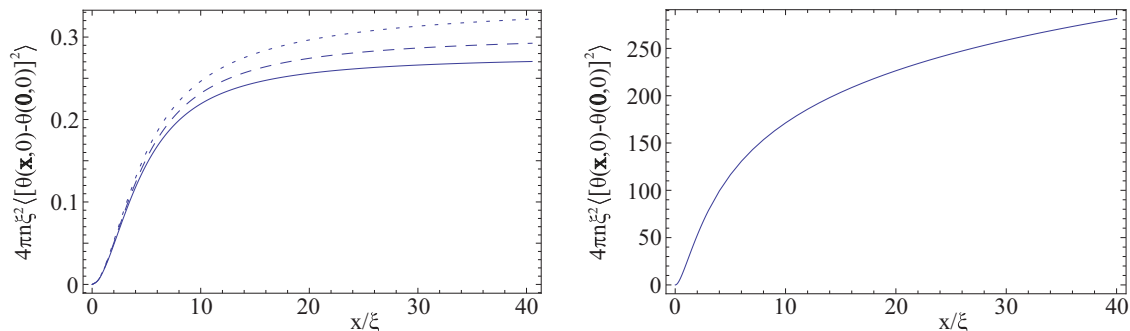

FIG. 1. (Color online) Zero temperature part (left) and the nonzero temperature part (right) of the phase fluctuations in two dimensions for a condensate density $n \simeq 5 \times 10^{13} \mathrm{m}^{-2}$ and correlation length $\xi \simeq 2.8 \times 10^{-6}$ m. In the nonzero temperature part we take $\beta n_0 T^{2B} \simeq 1.3 \times 10^{-2}$. The dotted, dashed, and solid curves correspond to, respectively, $\alpha = 0$, $\alpha = 5 \times 10^{-2}$, and $\alpha = 10^{-1}$.
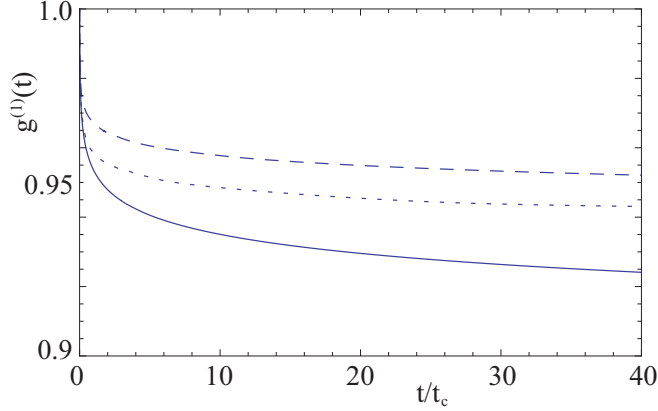
FIG. 2. (Color online) Normalized first-order correlation function $g^{(1)}(t)$ for a two-dimensional Bose-Einstein condensate as a function of t/t_c, with $t_c = \hbar(n_0 T^{2B}\sqrt{\beta})^{-2} \simeq 1.5 \times 10^{-10}$ s. Here $\beta n_0 T^{2B} \simeq 1.3 \times 10^{-2}$ and the dashed, dotted, and solid curves are for, respectively, $\alpha = 10^{-1}$, $\alpha = 10^{-2}$, and $\alpha = 0$.

B. Temporal correlations

To study the temporal correlation function, we define the first-order correlation function $g^{(1)}(\mathbf{x},t)$ as

$$g^{(1)}(\mathbf{x},t) := \frac{\langle \phi^*(\mathbf{x},t)\phi(\mathbf{0},0)\rangle}{\langle |\phi(\mathbf{0},0)|^2\rangle}, \qquad (24)$$

with the temporal correlations defined as $g^{(1)}(t) = g^{(1)}(\mathbf{0},t)$. Similar to the spatial correlations, we consider the regime in which the phase fluctuations are dominant. Therefore, we can directly calculate $g^{(1)}(t)$ by using Eqs. (13) and (16), where we again consider $\beta n_0 T^{2B} \simeq 1.3 \times 10^{-2}$. In Fig. 2 we show $g^{(1)}(t)$ in two dimensions for several values of α. As can be seen from the figure, $g^{(1)}(t)$ increases for increasing α. Furthermore, we find the same qualitative behavior in one dimension. Thus as opposed to the spatial correlations, the finite-lifetime effects are important for the temporal correlations.

IV. PHOTONS

In the previous section we gave a general discussion on finite-lifetime effects on correlation functions of bosons in the Bose-Einstein condensed phase. In this section we will focus on a specific example of such a system, namely, the BEC of photons [17]. Since this system is two-dimensional and of finite size due to the presence of a trap for the photons, we first investigate whether the photons form a quasicondensate or a true condensate.

A. One-particle density matrix

To determine whether the photons form a quasicondensate or a true condensate, we need to calculate the off-diagonal long-range behavior of the one-particle density matrix, and compare the size of the condensate with the distance over which the one-particle density matrix falls off. In particular, if the size of the condensate is smaller than the distance over which the one-particle density matrix reduces to, say, half the maximum value, we have a true condensate. Otherwise, we are in the quasicondensate regime. Furthermore, we consider

the regime where the density fluctuations are suppressed and therefore we are have large condensate fractions. Therefore we are allowed to use the Thomas-Fermi approximation, and we obtain for the number of condensed photons

$$N_0 = \frac{2\pi}{T^{2B}} \int_0^{R_{TF}} dr \, r \left(\mu - \frac{1}{2}m\Omega^2 r^2 \right), \qquad (25)$$

where the Thomas-Fermi radius $R_{TF} = \sqrt{2\mu/m\Omega^2}$ is the size of the condensate and Ω is the frequency of the isotropic harmonic trap. Note that the constant energy mc^2, with c the speed of the photons in the medium, is absorbed in the definition of the chemical potential μ. Furthermore, N_0 can be related to the total number of photons N according to [32]

$$N_0 = N - \frac{\pi^2}{3}\left(\frac{k_B T}{\hbar\Omega}\right)^2. \qquad (26)$$

By performing the integral in Eq. (25), we can relate the chemical potential to the total number of photons in our system. This then implies

$$R_{TF} = \left(\frac{4T^{2B}N_0}{\pi m\Omega^2}\right)^{1/4}. \qquad (27)$$

Furthermore, for the density of condensed photons n_0 we take the density in the center of the trap. Hence,

$$n_0 = \sqrt{\frac{m\Omega^2 N_0}{\pi T^{2B}}}. \qquad (28)$$

Experimentally, the relevant parameter is the condensate fraction N_0/N. Therefore given a condensate fraction, we use Eq. (26) to determine N_0 and with this value we obtain the size and density of the condensate via Eqs. (27) and (28). Furthermore, we use Ref. [17] to obtain numerical values for the parameters m, Ω, and T^{2B}.

In Fig. 3 we show a plot of the normalized first-order spatial correlation function $g^{(1)}(x/R_{TF}) = g^{(1)}(x/R_{TF},0)$ as defined
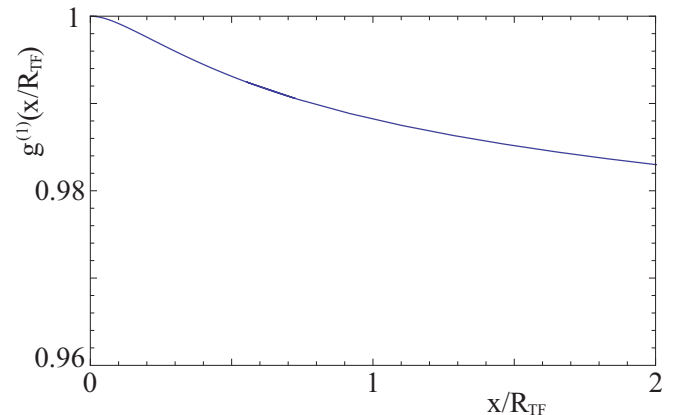

FIG. 3. (Color online) Normalized first-order correlation function $g^{(1)}(x/R_{TF})$ for the Bose-Einstein condensate of photons for a condensate fraction $N_0/N = 0.2$.
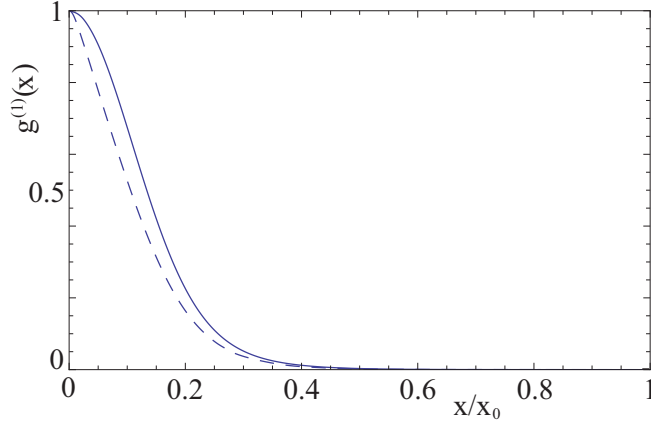
FIG. 4. (Color online) Normalized first-order correlation function $g^{(1)}(x)$ of the photons in the normal state for $\mu = 0.99\mu_c$ as a function of x/x_0, where $x_0 = \hbar\beta c \simeq 6 \times 10^{-6}$ m. The solid curve is the result without taking into account the molecules, and the dashed curve corresponds to a molecular density $n_m = 4.5 \times 10^{24}$ m^{-3}.

in Eq. (24) for a condensate fraction of 20%. From this plot it is clear that the phase correlation function $g^{(1)}(\mathbf{x})$ hardly drops over the condensate size, and that the photons form a true condensate. Note that this conclusion is also valid for a homogeneous system with the same central density and confined to a square with size R_{TF}. Furthermore, in Ref. [33] it is shown that at large length scales, in the regime of Kosterlitz-Thouless physics, isotropic systems undergo a Kardar-Parisi-Zhang (KPZ) crossover. Since these length scales are even larger than the length scales for Kosterlitz-Thouless physics, this KPZ regime is not accessible with the present setup.

In principle we should also include the harmonic trap in the calculation for $g^{(1)}(x/R_{TF})$. As shown in Ref. [25], in this case the phase fluctuations correlation function can also be found by solving the Bogoliubov–de Gennes equations. However, the thermal energy $k_B T$ is roughly two orders of magnitude larger than the energy splitting $\hbar\Omega$ of the harmonic potential $V^{ex}(\mathbf{x})$. Therefore, incorporation of the harmonic potential is only a small correction and since for the homogeneous calculation we are already far in the true condensate regime, this correction will not influence our conclusion.
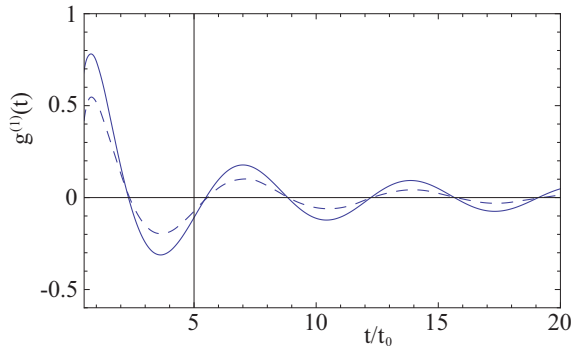
B. Normal state

Apart from the Bose-Einstein condensed phase, the photons can also be in the normal state. Since in this case not only the small frequency behavior of the self-energy is important, the details of the system of interest should also be included and we need the exact expression for the self-energy. Therefore, we use the explicit expression for the self-energy as given in Ref. [24]. In the normal state we write for the first-order correlation function

$$\langle \phi^*(\mathbf{x},\tau)\phi(\mathbf{0},0)\rangle = \frac{1}{\hbar\beta V}\sum_{\mathbf{k},n} G(\mathbf{k},i\omega_n)e^{i(\omega_n\tau - \mathbf{k}\cdot\mathbf{x})}, \quad (29)$$

where

$$G(\mathbf{k},i\omega_n) = \frac{-\hbar}{-i\hbar\omega_n + \epsilon(\mathbf{k}) - \mu + \hbar\Sigma(\mathbf{k},i\omega_n)}. \quad (30)$$

Note that in this calculation we neglect the self-interaction of the photons, since we are primarily interested in the effect of the imaginary part of the self-energy. Furthermore, for the same reasons as mentioned before, we neglect the harmonic potential $V^{ex}(\mathbf{x})$ which in the local-density approximation can be incorporated by the replacement $\mu(\mathbf{x}) = \mu - V^{ex}(\mathbf{x})$. By defining

$$\rho(\mathbf{k},\omega) := \frac{1}{\pi\hbar}\text{Im}[G(\mathbf{k},\omega^+)], \quad (31)$$

we write

$$\langle \phi^*(\mathbf{x},t)\phi(\mathbf{0},0)\rangle \quad (32)$$

$$= \frac{1}{2\pi}\int dk \int d(\hbar\omega)\, k\, \rho(\mathbf{k},\omega)N_{BE}(\hbar\omega)J_0(kx)e^{i\omega t},$$

where $J_0(kx)$ is the Bessel function of the first kind, $N_{BE}(\hbar\omega)$ is the Bose-Einstein distribution function as defined in Eq. (12), and $x = |\mathbf{x}|$.

We study the spatial and temporal correlation functions $g^{(1)}(t)$ and $g^{(1)}(x)$ separately. In general we are interested in the regime where we are close to condensation, and therefore we take $\mu \simeq 0.99\mu_c$. Furthermore, we take the parameters as in the experiment of Ref. [17]. It turns out that for the densities used in these experiments the effect of the molecules is small. To demonstrate the effect of the dye molecules we take $n_m = 4.5 \times 10^{24}$ m^{-3}. In general high molecular densities can
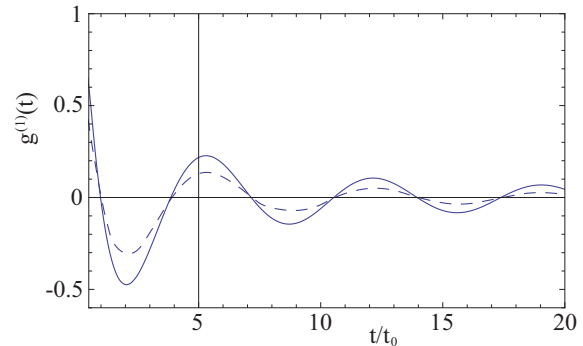

FIG. 5. (Color online) Real (left) and the imaginary (right) parts of the normalized first-order temporal correlation function $g^{(1)}(t)$ for the photons in the normal state for $\mu = 0.99\mu_c$ as a function of t/t_0 with $t_0 = \beta\hbar \simeq 2.5 \times 10^{-14}$ s. For the solid curve we omitted the effect of the molecules and the dashed curve corresponds to the molecular density $n_m = 4.5 \times 10^{24}$ m^{-3}.

spoil the thermalization of the photons, but this value should be within the regime in which the photons can still thermalize.

As can be seen in Fig. 4, the normalized spatial correlation function $g^{(1)}(x)$ is lowered by the effect of the molecules. The normalized first-order temporal correlation $g^{(1)}(t)$ consists of an imaginary and a real part, which we show separately in Fig. 5. For both parts the amplitude of the oscillations are decreased by the interaction with the molecules. Here, we used correlation functions in terms of creation and annihilation operators. In experiments one measures the correlation between the electric field at different times and positions, and therefore experimentally only the real part is relevant.

V. CONCLUSION AND OUTLOOK

In this paper we investigated energy-dependent finite-lifetime effects, characterized by the dimensionless parameter α, on the first-order correlation functions. By taking into account the phase fluctuations up to all orders, we derived an explicit expression for the first-order correlation functions in the Bose-Einstein condensed phase for high condensate fractions. We showed that the value of α does not influence the spatial correlations, but it enhances the temporal correlation function.

Subsequently, we focused on the BEC of photons under the relevant experimental conditions and we showed that the phase of the condensate is coherent over length scales larger than the size of the condensate. Therefore, the photons form a true condensate. Finally, we calculated the normalized first-order correlation functions of the photons in the normal state and we showed that the spatial and temporal correlations are both suppressed by the interaction with the dye molecules.

For future research it would be interesting to investigate the regime with smaller condensate fractions. Here the density fluctuations are important and they also have to be incorporated in the formalism. For the BEC of photons this regime is also accessible experimentally [30], and in this case the effect of the interaction with the dye molecules can be different from the case with high condensate fractions.

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