Correspondence analysis for strong three-valued logic

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abstract. I apply Kooi and Tamminga's (2012) idea of correspondence analysis for many-valued logics to strong three-valued logic (K_3). First, I characterize each possible single entry in the truth-table of a unary or a binary truth-functional operator that could be added to K_3 by a basic inference scheme. Second, I define a class of natural deduction systems on the basis of these characterizing basic inference schemes and a natural deduction system for K_3. Third, I show that each of the resulting natural deduction systems is sound and complete with respect to its particular semantics. Among other things, I thus obtain a new proof system for Łukasiewicz's three-valued logic.

Keywords: three-valued logic, correspondence analysis, proof theory, natural deduction systems

1 Introduction

Strong three-valued logic (K_3) [1] and Łukasiewicz's three-valued logic $Ł_3$ [2] have much in common: their truth-tables for negation, disjunction, and conjunction coincide, and they have the same concept of validity. The two logics differ, however, in their treatment of implication: whereas Kleene's implication is definable in terms of negation, disjunction, and conjunction, this does not hold true for Łukasiewicz's implication ($Ł_3$ is therefore a truth-functional extension of K_3). This fact seriously complicates the construction of proof systems for $Ł_3$.

In this paper, I present a general method for finding natural deduction systems for truth-functional extensions of K_3. To do so, I use the correspondence analysis for many-valued logics that was presented recently by [3]. In their study of the logic of paradox (LP) [4], they characterize every possible single entry in the truth-table of a unary or a binary truth-functional operator by a basic inference scheme. As a consequence, each unary and each binary truth-functional operator is characterized by a set of basic inference schemes. Kooi and Tamminga show that if we add the inference schemes that characterize an operator to a natural deduction system for LP, we immediately obtain a natural deduction system that is sound and complete with respect to the logic that contains, next to LP's negation, disjunction, and conjunction, the additional operator. In this paper, I show that the same thing can be done for K_3.

The structure of my paper is as follows. First, I briefly present K_3. Second, I give a list of basic inference schemes that characterize every possible single entry in the truth-table of a unary or a binary truth-functional operator. Third, I define a class of natural deduction systems on the basis of these characterizing inference schemes and a natural deduction system for K_3. I show that each of the resulting natural deduction systems is sound and complete with respect to its particular semantics.

2 Strong three-valued logic (K_3)

Strong three-valued logic (K_3) provides an alternative way to evaluate formulas from a propositional language $\mathcal{L}$ built from a set $\mathcal{P} = \{p, p', \ldots\}$ of atomic formulas using negation ($\neg$), disjunction ($\vee$), and conjunction ($\wedge$). K_3 adds a third truth-value 'none' to the classical pair 'false' and 'true'. In K_3, a valuation is a function v from the set $\mathcal{P}$ of atomic formulas to the set $\{0, i, 1\}$ of truth-values 'false', 'none', and 'true'. A valuation v on $\mathcal{P}$ is extended recursively to a valuation on $\mathcal{L}$ by the following truth-tables for $\neg$, $\vee$, and $\wedge$:

$f_\neg$	
0	1
i	i
1	0

$f_\vee$	0	i	1
0	0	i	1
i	i	i	1
1	1	1	1

$f_\wedge$	0	i	1
0	0	0	0
i	0	i	i
1	0	i	1

An argument from a set Π of premises to a conclusion ϕ is *valid* (notation: $\Pi \models \phi$) if and only if for each valuation v it holds that if $v(\psi) = 1$ for all ψ in Π, then $v(\phi) = 1$.

3 Correspondence Analysis for K_3

Let $\mathcal{L}_{(\sim)_m(\circ)_n}$ be the language built from the set $\mathcal{P} = \{p, p', \ldots\}$ of atomic formulas using negation ($\neg$), disjunction ($\vee$), conjunction ($\wedge$), m unary operators $\sim_1, \ldots, \sim_m$, and n binary operators $\circ_1, \ldots, \circ_n$. It is obvious that $\mathcal{L}_{(\sim)_m(\circ)_n}$ is an extension of $\mathcal{L}$. To interpret this extended language, I use K_3's concept of validity, the truth-tables $f_\neg$, $f_\vee$, and $f_\wedge$, but also the truth-tables $f_{\sim_1}, \ldots, f_{\sim_m}$ and the truth-tables $f_{\circ_1}, \ldots, f_{\circ_n}$. I refer to the resulting logic as $K_3(\sim)_m(\circ)_n$.

To construct a proof system for $K_3(\sim)_m(\circ)_n$, I follow [3]. I first characterize each possible single entry in the truth-table of a unary or a binary operator by a basic inference scheme. To do so, I need the following notion of single entry correspondence [3, p. 722]:

Definition 1 (Single Entry Correspondence). Let $\Pi \subseteq \mathcal{L}_{(\sim)_m(\circ)_n}$ and let $\phi \in \mathcal{L}_{(\sim)_m(\circ)_n}$. Let $x, y, z \in \{0, i, 1\}$. Let E be a truth-table entry of the type $f_\sim(x) = y$ or $f_\circ(x, y) = z$. Then the truth-table entry E is characterized by an inference scheme Π/ϕ, if

$$E \quad \text{if and only if} \quad \Pi \models \phi.$$

Accordingly, each of the nine possible single entries in a truth-table $f_\sim$ for a unary operator $\sim$ and each of the twenty-seven possible entries in a truth-table $f_\circ$ for binary operator $\circ$ is characterized by an inference scheme (I do the binary operator case first):

Theorem 1. *Let $\phi, \psi, \chi \in \mathcal{L}_{(\sim)_m(\circ)_n}$. Then*

$$f_\circ(0,0) = \begin{cases} 0 & \textit{iff} \quad \neg\phi \wedge \neg\psi \models \neg(\phi \circ \psi) \\ i & \textit{iff} \quad \neg\phi \wedge \neg\psi, (\phi \circ \psi) \vee \neg(\phi \circ \psi) \models \chi \\ 1 & \textit{iff} \quad \neg\phi \wedge \neg\psi \models \phi \circ \psi \end{cases}$$

$$f_\circ(0,i) = \begin{cases} 0 & \textit{iff} \quad \neg\phi \models (\psi \vee \neg\psi) \vee \neg(\phi \circ \psi) \\ i & \textit{iff} \quad \neg\phi, (\phi \circ \psi) \vee \neg(\phi \circ \psi) \models \psi \vee \neg\psi \\ 1 & \textit{iff} \quad \neg\phi \models (\psi \vee \neg\psi) \vee (\phi \circ \psi) \end{cases}$$

$$f_\circ(0,1) = \begin{cases} 0 & \textit{iff} \quad \neg\phi \wedge \psi \models \neg(\phi \circ \psi) \\ i & \textit{iff} \quad \neg\phi \wedge \psi, (\phi \circ \psi) \vee \neg(\phi \circ \psi) \models \chi \\ 1 & \textit{iff} \quad \neg\phi \wedge \psi \models \phi \circ \psi \end{cases}$$

$$f_\circ(i,0) = \begin{cases} 0 & \textit{iff} \quad \neg\psi \models (\phi \vee \neg\phi) \vee \neg(\phi \circ \psi) \\ i & \textit{iff} \quad \neg\psi, (\phi \circ \psi) \vee \neg(\phi \circ \psi) \models \phi \vee \neg\phi \\ 1 & \textit{iff} \quad \neg\psi \models (\phi \vee \neg\phi) \vee (\phi \circ \psi) \end{cases}$$

$$f_\circ(i,i) = \begin{cases} 0 & \textit{iff} \quad \models (\phi \vee \neg\phi) \vee (\psi \vee \neg\psi) \vee \neg(\phi \circ \psi) \\ i & \textit{iff} \quad (\phi \circ \psi) \vee \neg(\phi \circ \psi) \models (\phi \vee \neg\phi) \vee (\psi \vee \neg\psi) \\ 1 & \textit{iff} \quad \models (\phi \vee \neg\phi) \vee (\psi \vee \neg\psi) \vee (\phi \circ \psi) \end{cases}$$

$$f_\circ(i,1) = \begin{cases} 0 & \textit{iff} \quad \psi \models (\phi \vee \neg\phi) \vee \neg(\phi \circ \psi) \\ i & \textit{iff} \quad \psi, (\phi \circ \psi) \vee \neg(\phi \circ \psi) \models \phi \vee \neg\phi \\ 1 & \textit{iff} \quad \psi \models (\phi \vee \neg\phi) \vee (\phi \circ \psi) \end{cases}$$

$$f_\circ(1,0) = \begin{cases} 0 & \textit{iff} \quad \phi \wedge \neg\psi \models \neg(\phi \circ \psi) \\ i & \textit{iff} \quad \phi \wedge \neg\psi, (\phi \circ \psi) \vee \neg(\phi \circ \psi) \models \chi \\ 1 & \textit{iff} \quad \phi \wedge \neg\psi \models \phi \circ \psi \end{cases}$$

$$f_\circ(1,i) = \begin{cases} 0 & \textit{iff} \quad \phi \models (\psi \vee \neg\psi) \vee \neg(\phi \circ \psi) \\ i & \textit{iff} \quad \phi, (\phi \circ \psi) \vee \neg(\phi \circ \psi) \models \psi \vee \neg\psi \\ 1 & \textit{iff} \quad \phi \models (\psi \vee \neg\psi) \vee (\phi \circ \psi) \end{cases}$$

$$f_\circ(1,1) = \begin{cases} 0 & \textit{iff} \quad \phi \wedge \psi \models \neg(\phi \circ \psi) \\ i & \textit{iff} \quad \phi \wedge \psi, (\phi \circ \psi) \vee \neg(\phi \circ \psi) \models \chi \\ 1 & \textit{iff} \quad \phi \wedge \psi \models \phi \circ \psi. \end{cases}$$

PROOF. Case $f_\circ(0,0) = 0$. ($\Rightarrow$) Suppose that $\neg\phi \wedge \neg\psi \not\models \neg(\phi \circ \psi)$. Then there is a valuation v such that $v(\neg\phi \wedge \neg\psi) = 1$ and $v(\neg(\phi \circ$

$\psi)) \neq 1$. Then $v(\phi) = 0$, $v(\psi) = 0$, and $v(\phi \circ \psi) \neq 0$. Therefore, it must be that $f_\circ(0,0) \neq 0$.

($\Leftarrow$) Suppose that $\neg\phi \wedge \neg\psi \models \neg(\phi \circ \psi)$. Then $\neg p \wedge \neg q \models \neg(p \circ q)$, where p and q are atomic formulas. Then for every valuation v it holds that if $v(\neg p \wedge \neg q) = 1$, then $v(\neg(p \circ q)) = 1$. Then for every valuation v it holds that if $v(p) = 0$ and $v(q) = 0$, then $v(p \circ q) = 0$. Therefore, it must be that $f_\circ(0,0) = 0$.

Case $f_\circ(1,i) = i$. ($\Rightarrow$) Suppose that $\phi, (\phi\circ\psi)\vee\neg(\phi\circ\psi) \not\models \psi\vee\neg\psi$. Then there is a valuation v such that $v(\phi) = 1$, $v((\phi\circ\psi)\vee\neg(\phi\circ\psi)) = 1$ and $v(\psi \vee \neg\psi) \neq 1$. Then $v(\phi) = 1$, $v(\psi) = i$, and $v(\phi \circ \psi) \neq i$. Therefore, it must be that $f_\circ(1,i) \neq i$.

($\Leftarrow$) Suppose that $\phi, (\phi \circ \psi) \vee \neg(\phi \circ \psi) \models \psi \vee \neg\psi$. Then $p, (p \circ q) \vee \neg(p\circ q) \models q\vee\neg q$, where p and q are atomic formulas. Then for every valuation v it holds that if $v(p) = 1$ and $v((p \circ q) \vee \neg(p \circ q)) = 1$, then $v(q \vee \neg q) = 1$. Then for every valuation v it holds that if $v(p) = 1$ and $v(q) = i$, then $v(p \circ q) = i$. Therefore, it must be that $f_\circ(1,i) = i$.

The other cases are proved similarly. □

THEOREM 2. *Let* $\phi, \psi \in \mathcal{L}_{(\sim)_m(\circ)_n}$. *Then*

$$f_\sim(0) = \begin{cases} 0 & \textit{iff} \quad \neg\phi \models \neg \sim \phi \\ i & \textit{iff} \quad \neg\phi, (\sim \phi \vee \neg \sim \phi) \models \psi \\ 1 & \textit{iff} \quad \neg\phi \models \sim \phi \end{cases}$$

$$f_\sim(i) = \begin{cases} 0 & \textit{iff} \quad \models (\phi \vee \neg\phi) \vee \neg \sim \phi \\ i & \textit{iff} \quad (\sim \phi \vee \neg \sim \phi) \models \phi \vee \neg\phi \\ 1 & \textit{iff} \quad \models (\phi \vee \neg\phi) \vee \sim \phi \end{cases}$$

$$f_\sim(1) = \begin{cases} 0 & \textit{iff} \quad \phi \models \neg \sim \phi \\ i & \textit{iff} \quad \phi, (\sim \phi \vee \neg \sim \phi) \models \psi \\ 1 & \textit{iff} \quad \phi \models \sim \phi. \end{cases}$$

PROOF. Adapt the proof of the previous theorem. □

As a result, given K_3's concept of validity and its truth-tables $f_\neg$, $f_\vee$, and $f_\wedge$, each unary operator $\sim_k$ $(1 \leq k \leq m)$ is characterized by the set of three basic inference schemes that characterize the three single entries in its truth-table $f_{\sim_k}$ and each binary operator $\circ_l$ ($1 \leq l \leq n$) is characterized by the set of nine basic inference schemes that characterize the nine single entries in its truth-table $f_{\circ_l}$. The inference schemes that characterize a truth-table are independent.

4 Natural deduction systems

I now use the characterizations of the previous section to construct proof systems for truth-functional extensions of K_3. First, I define a natural deduction system $\mathbf{ND}_{K_3}$ which I later show to be sound and complete with respect to K_3 (this is a corollary of my main theorem). Second, on the basis of $\mathbf{ND}_{K_3}$ and Theorems 1 and 2, I define a natural deduction system for the logic $K_3(\sim)_m(\circ)_n$ as follows: for each unary operator $\sim_k$ ($1 \leq k \leq m$) I add its three characterizing basic inference schemes as derivation rules to $\mathbf{ND}_{K_3}$ and for each binary operator $\circ_l$ ($1 \leq l \leq n$) I add its nine characterizing inference schemes as derivation rules to $\mathbf{ND}_{K_3}$. Third, I show, using a Henkin-style proof, that the resulting natural deduction system is sound and complete with respect to the logic $K_3(\sim)_m(\circ)_n$.

My proof-theoretical study of K_3 closely follows Kooi and Tamminga's (2012) proof-theoretical study of LP. In fact, to construct natural deduction systems for extensions of K_3 and to prove their soundness and completeness, I only slightly adapt Kooi and Tamminga's definitions, lemmas and theorems on extensions of LP.

Let me first define the natural deduction system $\mathbf{ND}_{K_3}$[1].

DEFINITION 2. Derivations in the system $\mathbf{ND}_{K_3}$ are inductively defined as follows:

Basis: The proof tree with a single occurrence of an assumption ϕ is a derivation.

Induction Step: Let $\mathcal{D}$, $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}_3$ be derivations. Then they can be extended by the following rules (double lines indicate that the rules work both ways):

[1] For the notational conventions, see [5].

$$\dfrac{\begin{array}{cc}\mathcal{D}_1 & \mathcal{D}_2\\ \phi & \neg\phi\end{array}}{\psi}\ EFQ$$

$$\dfrac{\begin{array}{cc}\mathcal{D}_1 & \mathcal{D}_2\\ \phi & \psi\end{array}}{\phi\wedge\psi}\ \wedge I \qquad \dfrac{\begin{array}{c}\mathcal{D}\\ \phi\wedge\psi\end{array}}{\phi}\ \wedge E_1 \qquad \dfrac{\begin{array}{c}\mathcal{D}\\ \phi\wedge\psi\end{array}}{\psi}\ \wedge E_2$$

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \phi\end{array}}{\phi\vee\psi}\ \vee I_1 \qquad \dfrac{\begin{array}{c}\mathcal{D}\\ \psi\end{array}}{\phi\vee\psi}\ \vee I_2 \qquad \dfrac{\begin{array}{ccc} & [\phi]^u & [\psi]^v\\ \mathcal{D}_1 & \mathcal{D}_2 & \mathcal{D}_3\\ \phi\vee\psi & \chi & \chi\end{array}}{\chi}\ \vee E^{u,v}$$

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \phi\end{array}}{\neg\neg\phi}\ DN \qquad \dfrac{\begin{array}{c}\mathcal{D}\\ \neg(\phi\vee\psi)\end{array}}{\neg\phi\wedge\neg\psi}\ DeM_{\vee} \qquad \dfrac{\begin{array}{c}\mathcal{D}\\ \neg(\phi\wedge\psi)\end{array}}{\neg\phi\vee\neg\psi}\ DeM_{\wedge}$$

On the basis of $\mathbf{ND}_{K_3}$, I now define a natural deduction system for the logic $K_3(\sim)_m(\circ)_n$. The Theorems 1 and 2 tell me that each truth-table $f_{\sim_k}$ is characterized by three basic inference schemes and that each truth-table $f_{\circ_l}$ is characterized by nine basic inference schemes. I obtain a new natural deduction system for the logic $K_3(\sim)_m(\circ)_n$ by adding to $\mathbf{ND}_{K_3}$ these characterizing basic inference schemes as derivation rules.

More specifically, for each basic inference scheme $\psi_1, \ldots, \psi_j/\phi$ that characterizes an entry $f_{\sim_k}(x) = y$ in the truth-table $f_{\sim_k}$, I add the derivation rule

$$\dfrac{\begin{array}{ccc}\mathcal{D}_1 & & \mathcal{D}_j\\ \psi_1 & \cdots & \psi_j\end{array}}{\phi}\ R_{\sim_k}(x,y)$$

to the natural deduction system $\mathbf{ND}_{K_3}$. Similarly, for each basic inference scheme $\psi_1, \ldots, \psi_j/\phi$ that characterizes an entry $f_{\circ_l}(x,y) = z$ in the truth-table $f_{\circ_l}$, I add the derivation rule

$$\dfrac{\begin{array}{ccc}\mathcal{D}_1 & & \mathcal{D}_j\\ \psi_1 & \cdots & \psi_j\end{array}}{\phi}\ R_{\circ_l}(x,y,z)$$

to the natural deduction system $\mathbf{ND}_{K_3}$.

For instance, assume that $f_\circ(0,0) = 0$ is one of the truth-table entries in $f_\circ$. Then, because Theorem 1 tells me that $f_\circ(0,0) = 0$ is characterized by the basic inference scheme $\neg\phi \wedge \neg\psi / \neg(\phi \circ \psi)$, I add the derivation rule

$$\frac{\begin{array}{c}\mathcal{D}\\ \neg\phi \wedge \neg\psi\end{array}}{\neg(\phi \circ \psi)}\ R_\circ(0,0,0)$$

to the natural deduction system $\mathbf{ND}_{K_3}$.

In this way, I define the system $\mathbf{ND}_{K_3} + \bigcup_{k=1}^{m}\{R_{\sim_k}(x,y) : f_{\sim_k}(x) = y\} + \bigcup_{l=1}^{n}\{R_{\circ_l}(x,y,z) : f_{\circ_l}(x,y) = z\}$, which I refer to as $\mathbf{ND}_{K_3(\sim)_m(\circ)_n}$. I now show that this natural deduction system is sound and complete with respect to the logic $K_3(\sim)_m(\circ)_n$.

4.1 Soundness of $\mathbf{ND}_{K_3(\sim)_m(\circ)_n}$

A conclusion ϕ is *derivable* from a set Π of premises (notation: $\Pi \vdash \phi$) if and only if there is a derivation in the system $\mathbf{ND}_{K_3(\sim)_m(\circ)_n}$ of ϕ from Π.

The system's local soundness is easy to establish:

LEMMA 1 (LOCAL SOUNDNESS). Let $\Pi, \Pi', \Pi'' \subseteq \mathcal{L}_{(\sim)_m(\circ)_n}$ and let $\phi, \psi \in \mathcal{L}_{(\sim)_m(\circ)_n}$. Then

- (i) If $\phi \in \Pi$, then $\Pi \models \phi$
- (ii) If $\Pi \models \phi$ and $\Pi' \models \neg\phi$, then $\Pi, \Pi' \models \psi$
- (iii) If $\Pi \models \phi$ and $\Pi' \models \psi$, then $\Pi, \Pi' \models \phi \wedge \psi$
- (iv) If $\Pi \models \phi \wedge \psi$, then $\Pi \models \phi$
- (v) If $\Pi \models \phi \wedge \psi$, then $\Pi \models \psi$
- (vi) If $\Pi \models \phi$, then $\Pi \models \phi \vee \psi$
- (vii) If $\Pi \models \psi$, then $\Pi \models \phi \vee \psi$
- (viii) If $\Pi \models \phi \vee \psi$ and $\Pi', \phi \models \chi$ and $\Pi'', \psi \models \chi$, then $\Pi, \Pi', \Pi'' \models \chi$
- (ix) $\Pi \models \phi$ if and only if $\Pi \models \neg\neg\phi$
- (x) $\Pi \models \neg(\phi \vee \psi)$ if and only if $\Pi \models \neg\phi \wedge \neg\psi$
- (xi) $\Pi \models \neg(\phi \wedge \psi)$ if and only if $\Pi \models \neg\phi \vee \neg\psi$.

THEOREM 3 (SOUNDNESS). *Let* $\Pi \subseteq \mathcal{L}_{(\sim)_m(\circ)_n}$ *and let* $\phi \in \mathcal{L}_{(\sim)_m(\circ)_n}$. *Then*

$$\textit{If } \Pi \vdash \phi, \textit{ then } \Pi \models \phi.$$

PROOF. By induction on the depth of derivations. The local soundness of the rules of the basic natural deduction system $\mathbf{ND}_{K_3}$ follows from the previous lemma. For each unary operator $\sim_k$ $(1 \leq k \leq m)$ the local soundness of the three derivation rules in $\{R_{\sim_k}(x,y) : f_{\sim_k}(x) = y\}$ follows from Theorem 2. For each binary operator $\circ_l$ $(1 \leq l \leq n)$ the local soundness of the nine derivation rules in $\{R_{\circ_l}(x,y,z) : f_{\circ_l}(x,y) = z\}$ follows from Theorem 1. □

4.2 Completeness of $\mathbf{ND}_{K3(\sim)_m(\circ)_n}$

In my completeness proof, consistent prime theories are the syntactical counterparts of valuations:

DEFINITION 3. Let $\Pi \subseteq \mathcal{L}_{(\sim)_m(\circ)_n}$. Then Π is a *consistent prime theory* (CPT), if

(i)	$\Pi \neq \mathcal{L}_{(\sim)_m(\circ)_n}$	(consistency)
(ii)	If $\Pi \vdash \phi$, then $\phi \in \Pi$	(closure)
(iii)	If $\phi \vee \psi \in \Pi$, then $\phi \in \Pi$ or $\psi \in \Pi$	(primeness).

The syntactical counterpart of the truth-value of a formula under a valuation is a formula's elementhood in a consistent prime theory:

DEFINITION 4. Let $\Pi \subseteq \mathcal{L}_{(\sim)_m(\circ)_n}$ and let $\phi \in \mathcal{L}_{(\sim)_m(\circ)_n}$. Then ϕ's *elementhood in* Π (notation: $e(\phi,\Pi)$) is defined as follows:

$$e(\phi,\Pi) = \begin{cases} \emptyset, & \text{if } \phi \in \Pi \text{ and } \neg\phi \in \Pi \\ 0, & \text{if } \phi \notin \Pi \text{ and } \neg\phi \in \Pi \\ i, & \text{if } \phi \notin \Pi \text{ and } \neg\phi \notin \Pi \\ 1, & \text{if } \phi \in \Pi \text{ and } \neg\phi \notin \Pi. \end{cases}$$

To ensure that in the presence of an operator the notion of elementhood behaves in comformity with the operator's truth-tables, I need the following lemma:

LEMMA 2. Let Π be a CPT and let $\phi, \psi \in \mathcal{L}_{(\sim)_m(\circ)_n}$. Then

(i)	$e(\phi,\Pi) \neq \emptyset$		
(ii)	$f_\neg(e(\phi,\Pi))$	$=$	$e(\neg\phi,\Pi)$
(iii)	$f_\vee(e(\phi,\Pi), e(\psi,\Pi))$	$=$	$e(\phi \vee \psi,\Pi)$
(iv)	$f_\wedge(e(\phi,\Pi), e(\psi,\Pi))$	$=$	$e(\phi \wedge \psi,\Pi)$
(v)	$f_{\sim_k}(e(\phi,\Pi))$	$=$	$e(\sim_k \phi,\Pi)$ for $1 \leq k \leq m$
(vi)	$f_{\circ_l}(e(\phi,\Pi), e(\psi,\Pi))$	$=$	$e(\phi \circ_l \psi,\Pi)$ for $1 \leq l \leq n$.

PROOF.

(i) Suppose $e(\phi, \Pi) = \emptyset$. Then $\phi \in \Pi$ and $\neg\phi \in \Pi$. Then $\Pi \vdash \phi$ and $\Pi \vdash \neg\phi$. By the rule *EFQ*, it must be that $\Pi \vdash \psi$ for all $\psi \in \mathcal{L}_{(\sim)_m(\circ)_n}$. By closure, $\psi \in \Pi$ for all $\psi \in \mathcal{L}_{(\sim)_m(\circ)_n}$. Then $\Pi = \mathcal{L}_{(\sim)_m(\circ)_n}$. Contradiction.

(ii) Suppose $e(\phi, \Pi) = 0$. Then $\phi \notin \Pi$ and $\neg\phi \in \Pi$. By closure and the rule *DN*, $\neg\phi \in \Pi$ and $\neg\neg\phi \notin \Pi$. Hence, $e(\neg\phi, \Pi) = 1 = f_\neg(0) = f_\neg(e(\phi, \Pi))$.

Suppose $e(\phi, \Pi) = i$. Then $\phi \in \Pi$ and $\neg\phi \in \Pi$. By closure and the rule *DN*, $\neg\phi \in \Pi$ and $\neg\neg\phi \in \Pi$. Hence, $e(\neg\phi, \Pi) = i = f_\neg(i) = f_\neg(e(\phi, \Pi))$.

Suppose $e(\phi, \Pi) = 1$. Then $\phi \in \Pi$ and $\neg\phi \notin \Pi$. By closure and the rule *DN*, $\neg\phi \notin \Pi$ and $\neg\neg\phi \in \Pi$. Hence, $e(\neg\phi, \Pi) = 0 = f_\neg(1) = f_\neg(e(\phi, \Pi))$.

(iii) I prove the cases for (1) $e(\phi, \Pi) = 0$ and $e(\psi, \Pi) = 0$, (2) $e(\phi, \Pi) = i$ and $e(\psi, \Pi) = i$, and (3) $e(\phi, \Pi) = 1$ and $e(\psi, \Pi) = i$. The other six cases are proved similarly.

(1) Suppose $e(\phi, \Pi) = 0$ and $e(\psi, \Pi) = 0$. Then $\phi \notin \Pi$, $\psi \notin \Pi$, $\neg\phi \in \Pi$, and $\neg\psi \in \Pi$. By primeness, $\phi \vee \psi \notin \Pi$. By closure and the rules $\wedge I$ and $DeM_\vee$, $\neg(\phi \vee \psi) \in \Pi$. Hence, $e(\phi \vee \psi, \Pi) = 0 = f_\vee(0, 0) = f_\vee(e(\phi, \Pi), e(\psi, \Pi))$.

(2) Suppose $e(\phi, \Pi) = i$ and $e(\psi, \Pi) = i$. Then $\phi \in \Pi$, $\psi \in \Pi$, $\neg\phi \in \Pi$, and $\neg\psi \in \Pi$. By closure and the rule $\vee I_1$, $\phi \vee \psi \in \Pi$. By closure and the rules $\wedge I$ and $DeM_\vee$, $\neg(\phi \vee \psi) \in \Pi$. Hence, $e(\phi \vee \psi, \Pi) = i = f_\vee(i, i) = f_\vee(e(\phi, \Pi), e(\psi, \Pi))$.

(3) Suppose $e(\phi, \Pi) = 1$ and $e(\psi, \Pi) = i$. Then $\phi \in \Pi$, $\psi \in \Pi$, $\neg\phi \notin \Pi$, and $\neg\psi \in \Pi$. By closure and the rule $\vee I_1$, $\phi \vee \psi \in \Pi$. By closure and the rules $\wedge E_1$ and $DeM_\vee$, $\neg(\phi \vee \psi) \notin \Pi$. Hence, $e(\phi \vee \psi, \Pi) = 1 = f_\vee(1, i) = f_\vee(e(\phi, \Pi), e(\psi, \Pi))$.

(iv) Analogous to (iii).

(v) There are three cases for each $\sim_k$ $(1 \leq k \leq n)$. (For readability, the subscript k is dropped in the remainder of this proof.) I prove the case for $e(\phi, \Pi) = 0$. The other two cases are proved similarly.

Suppose $e(\phi, \Pi) = 0$. Then $\phi \notin \Pi$ and $\neg\phi \in \Pi$. There are three cases:

(1) Suppose $R_{\sim}(0, 0)$ is one of the three rules for $\sim$ in $\mathbf{ND}_{K_3(\sim)_m(\circ)_n}$. Then $f_{\sim}(0) = 0$. By closure and the rule $R_{\sim}(0, 0)$, it must be that $\neg \sim \phi \in \Pi$. By (i), it must be that $\sim \phi \notin \Pi$. Therefore, $e(\sim \phi, \Pi) = 0 = f_{\sim}(0) = f_{\sim}(e(\phi, \Pi))$.

(2) Suppose $R_{\sim}(0, i)$ is one of the three rules for $\sim$ in $\mathbf{ND}_{LP(\sim)_m(\circ)_n}$. Then $f_{\sim}(0) = i$. By closure, the fact that Π is a CPT, and the rule $R_{\sim}(0, i)$, it must be that $\sim \phi \vee \neg \sim \phi \notin \Pi$. By closure and the rules $\vee I_1$ and $\vee I_2$, $\sim \phi \notin \Pi$ and $\neg \sim \phi \notin \Pi$. Therefore, $e(\sim \phi, \Pi) = i = f_{\circ}(0) = f_{\sim}(e(\phi, \Pi))$.

(3) Suppose $R_{\sim}(0, 1)$ is one of the three rules for $\sim$ in $\mathbf{ND}_{LP(\sim)_m(\circ)_n}$. Analogous to (1).

(vi) Analogous to (v).

□

Lemma 3 (Truth). Let Π be a CPT. Let v_Π be the function that assigns to each atomic formula p in $\mathcal{P}$ the elementhood of p in Π: $v_\Pi(p) = e(p, \Pi)$ for all p in $\mathcal{P}$. Then for all ϕ in $\mathcal{L}_{(\sim)_m(\circ)_n}$ it holds that

$$v_\Pi(\phi) = e(\phi, \Pi).$$

Proof. By an easy structural induction on ϕ. Use the previous lemma. □

Lemma 4 (Lindenbaum). Let $\Pi \subseteq \mathcal{L}_{(\sim)_m(\circ)_n}$ and let $\phi \in \mathcal{L}_{(\sim)_m(\circ)_n}$. Suppose that $\Pi \nvdash \phi$. Then there is a set $\Pi^* \subseteq \mathcal{L}_{(\sim)_m(\circ)_n}$ such that

(i) $\Pi \subseteq \Pi^*$
(ii) $\Pi^* \not\vdash \phi$
(iii) Π^* is a CPT.

PROOF. Suppose that $\Pi \not\vdash \phi$. Let $\psi_1, \psi_2, \dots$ be an enumeration of $\mathcal{L}_{(\sim)_m(\circ)_n}$. I define the sequence $\Pi_0, \Pi_1, \dots$ of sets of formulas as follows:

$$\begin{aligned} \Pi_0 &= \Pi \\ \Pi_{i+1} &= \begin{cases} \Pi_i \cup \{\psi_{i+1}\}, & \text{if } \Pi_i \cup \{\psi_{i+1}\} \not\vdash \phi \\ \Pi_i, & \text{otherwise.} \end{cases} \end{aligned}$$

Take $\Pi^* = \bigcup_{n\in\mathbb{N}} \Pi_n$. Standard proofs show that (i), (ii), and (iii) hold. □

THEOREM 4 (COMPLETENESS). *Let* $\Pi \subseteq \mathcal{L}_{(\sim)_m(\circ)_n}$ *and let* $\phi \in \mathcal{L}_{(\sim)_m(\circ)_n}$. *Then*

$$\textit{If } \Pi \models \phi, \textit{ then } \Pi \vdash \phi.$$

PROOF. By contraposition. Suppose $\Pi \not\vdash \phi$. By the Lindenbaum lemma, there is a CPT Π^* such that $\Pi \subseteq \Pi^*$ and $\Pi^* \not\vdash \phi$. Let v_{Π^*} be the valuation introduced in the truth lemma. By the truth lemma, it holds that $v_{\Pi^*}(\psi) = 1$ for all ψ in Π and $v_{\Pi^*}(\phi) \neq 1$. Therefore, $\Pi \not\models \phi$. □

COROLLARY 1. *The system* $\mathbf{ND}_{K_3}$ *is sound and complete with respect to* K_3.

PROOF. Consider the logic $K_3\neg$ that is obtained from K_3 by adding K_3's truth-table $f_\neg$ for negation to it. Evidently, $K_3\neg$ is K_3. By the soundness and completeness theorems, $\mathbf{ND}_{K_3\neg}$ is sound and complete with respect to $K_3\neg$. It is easy to see that the rules $R_\neg(0,1)$, $R_\neg(i,i)$, and $R_\neg(1,0)$ are derived rules in $\mathbf{ND}_{K_3}$. □

5 Łukasiewicz's three-valued logic ($Ł_3$)

Let me illustrate this general method for finding natural deduction systems for truth-functional extensions of K_3 with Łukasiewicz's three-valued logic ($Ł_3$). $Ł_3$ evaluates arguments consisting of formulas from a propositional language $\mathcal{L}_\supset$ built from a set $\mathcal{P} = \{p, p', \ldots\}$ of atomic formulas using negation ($\neg$), disjunction ($\vee$), conjunction ($\wedge$), and implication ($\supset$). $Ł_3$ has the same valuations as K_3: in $Ł_3$, a valuation is a function v from the set $\mathcal{P}$ of atomic formulas to the set $\{0, i, 1\}$ of truth-values. A valuation v on $\mathcal{P}$ is extended recursively to a valuation on $\mathcal{L}_\supset$ by the truth-tables for $\neg$, $\vee$, and $\wedge$, and the truth-table for $\supset$:

$f_\supset$	0	i	1
0	1	1	1
i	i	1	1
1	0	i	1

$Ł_3$ has the same concept of validity as K_3: an argument from a set Π of premises to a conclusion ϕ is *valid* (notation: $\Pi \models \phi$) if and only if for each valuation v it holds that if $v(\psi) = 1$ for all ψ in Π, then $v(\phi) = 1$.

Theorem 1 tells me that the truth-table $f_\supset$ is characterized by the following nine basic inference schemes:

$$
\begin{array}{lll}
f_\supset(0,0) = 1 & \text{iff} & \neg\phi \wedge \neg\psi \models \phi \supset \psi \\
f_\supset(0,i) = 1 & \text{iff} & \neg\phi \models (\psi \vee \neg\psi) \vee (\phi \supset \psi) \\
f_\supset(0,1) = 1 & \text{iff} & \neg\phi \wedge \psi \models \phi \supset \psi \\
f_\supset(i,0) = i & \text{iff} & \neg\psi, (\phi \supset \psi) \vee \neg(\phi \supset \psi) \models \phi \vee \neg\phi \\
f_\supset(i,i) = 1 & \text{iff} & \models (\phi \vee \neg\phi) \vee (\psi \vee \neg\psi) \vee (\phi \supset \psi) \\
f_\supset(i,1) = 1 & \text{iff} & \psi \models (\phi \vee \neg\phi) \vee (\phi \supset \psi) \\
f_\supset(1,0) = 0 & \text{iff} & \phi \wedge \neg\psi \models \neg(\phi \supset \psi) \\
f_\supset(1,i) = i & \text{iff} & \phi, (\phi \supset \psi) \vee \neg(\phi \supset \psi) \models \psi \vee \neg\psi \\
f_\supset(1,1) = 1 & \text{iff} & \phi \wedge \psi \models \phi \supset \psi.
\end{array}
$$

From Theorems 3 and 4 it follows that the natural deduction system $\mathbf{ND}_{K_3\supset}$, obtained from adding these nine basic inference schemes as derivation rules to the natural deduction system $\mathbf{ND}_{K_3}$, is sound and complete with respect to $Ł_3$. The general method I presented in this paper, therefore, makes it easy to find natural deduction systems for truth-functional extensions of K_3.

6 Conclusion

Next to Kooi and Tamminga's (2012) proof-theoretical study of LP, the present investigation of K_3 is only a second step in the study of many-valued logics using correspondence analysis. At the current stage of research, the following questions seem pressing. Which many-valued logics can be studied using correspondence analysis? Which many-valued logics cannot? Are there some characteristics a many-valued logic must have to be amenable to correspondence analysis?

References

[1] *Kleene S. C.* On notation for ordinal numbers // Journal of Symbolic Logic. 1938. Vol. 3. P. 150–155.

[2] *Łukasiewicz J.* O logice trójwartościowej // Ruch Filozoficzny. 1920. Vol. 5. P. 170–171. (Translated as: On three-valued logic // Jan Łukasiewicz: Selected Works / Ed. *Borkowski L.* Amsterdam: North-Holland Publishing Company, 1970. P. 87–88).

[3] *Kooi B., Tamminga A.* Completeness via correspondence for extensions of the logic of paradox // The Review of Philosophical Logic. 2012. Vol. 5. P. 720–730.

[4] *Priest G.* The logic of paradox // Journal of Philosophical Logic. 1979. Vol. 8. P. 219–241.

[5] *Troelstra A. S., Schwichtenberg H.* Basic Proof Theory. Cambridge: Cambridge University Press, 1996.