GRAPHING CALCULATOR SUPPORTED INSTRUMENTATION SCHEMES FOR THE CONCEPT OF DERIVATIVE: A CASE STUDY

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This paper reports on the role of the graphing calculator (GC) in the learning of derivatives and instantaneous rate of change. In a longitudinal study, we administered task based interviews before and after the introduction of calculus. We analyzed students' use of the GC in these interviews. This paper reports on the case of one student, Andy, who is a resilient user of the GC while he develops into a flexible solver of problems on instantaneous rate of change. His case demonstrates that, although the GC is meant to promote the integration of symbolical, graphical and numerical techniques, it can facilitate a learning process in which symbolical techniques develop separately from other techniques.

INTRODUCTION

Graphing calculators (GC) are widely used in mathematics education because they support a multiple-representational approach to the learning of mathematics. The GC gives opportunities to interactively discover relations between functions and graphs. Burrill et al. (2002) report on evidence that the use of the GC improves the ability to link symbolical, graphical, and numerical representations, in particular for the understanding of functions and algebraic expressions. Also for learning the concept of derivative, the GC can make a possible contribution, as Delos Santos (2006) notes. So, on the one hand there is evidence that using the GC promotes students to develop strong relationships between symbolical and graphical forms of functions and derivatives. On the other hand the question remains: what are effects of handheld technology on students' mathematical thinking (Burrill et al., 2002)?

This study will contribute to this question by zooming in on one particular student during the period, in which he is learning about derivatives at pre-university level. His learning context is Dutch mathematics education, in which the GC is used as a tool during the introduction of derivatives. The GC offers, for example, options to draw the graph of the derivative, such as NDeriv, or to find dy/dx in a point of the graph. Depending on the textbook series and the teacher, different GC-options are used in mathematics lessons.

We will report on student Andy. He was part of a group of ten students in a longitudinal study (Roorda, Vos & Goedhart, in press). In that study general patterns of students' thinking were reported. Andy showed an a-typical pattern, which we left largely unreported as he was an outlier. Unlike the other students, in Andy's thinking the GC

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played an important role. The goal of this paper is to present evidence of how a student's understanding of the concept of derivative can be affected by the use of a GC.

THEORETICAL FRAMEWORK

To study the relationship between the use of digital technology and students' mathematical thinking we use the theoretical framework of *instrumental genesis* (e.g., see Drijvers, Godino, Font & Trouche, 2013; Guin & Trouche, 1999). In this theory, artefacts are distinguished from instruments. The latter refers to a psychological construct, actively constructed by an individual, which consists of the user's mental scheme for using the artefact for a type of tasks. As such, the instrumental scheme integrates technical knowledge of the use of the artefact and the (in our case mathematical) knowledge involved. Instrumental genesis is the process of an (in our case digital, handheld) artefact becoming an instrument; it is a process in which techniques for using the digital tool and mathematical insights co-emerge. The resulting *instrumentation scheme* is the more or less stable way to deal with specific situations or tasks, guided by the opportunities and constraints of the artefact, as well as by the available knowledge.

The theory of instrumental genesis provides a widely applicable framework for investigation of the use of ICT-tools in mathematics education, and avoids an oversimplified separation of mathematical thinking and outsourcing calculations to the artefact. By explicitly describing instrumentation schemes, the instrumental genesis lens may help to identify the relationships between the use of the digital tools and the mathematical knowledge a student develops. This is exactly the way in which we will exploit this theory.

Guin and Trouche (1999) conclude that there is a great diversity in instrumental geneses. However, schemes related to using a GC for studying the derivative so far have hardly been described. In our study, therefore, we will identify such schemes and investigate how these develop over time. In terms of the instrumentation framework, the research question is: how do students' instrumentation schemes develop while studying the concept of derivative with the use of a GC?

METHODS

To gain insight into the development of students with regards to derivatives, we opted for a detailed description and analysis one student's work over a time period of a year. The case study of Andy is part of a longitudinal, multiple case study, in which ten students were followed (Roorda, Vos & Goedhart, in press). The students were in a pre-university science track, which means that they take science and mathematics courses at an advanced level. When we discovered that Andy's development contrasted with the other nine students, we decided to gather additional data on Andy's development.

The data were gathered at four different moments in time, together spanning the period before and after the introduction of calculus at school. In April and November

task-based interviews (Goldin, 2000) were administered. The first interview (TBI-1) was held while Andy was still in grade 10 and the concept of derivative had not yet been introduced in his mathematics classes. The second interview (TBI-2) was held a few weeks after the introduction of differential calculus (difference quotient, differential quotient, derivatives of polynomials) with Andy being in grade 11. Also, we collected his work on two calculus tests, which were set by his teacher (CT-1 and CT-2). CT-1 was immediately after the lesson series, while CT-2 was about two and a half months later for those students with a low mark on the first calculus test. Andy was one of the low performers on the first test.

According to the Dutch curriculum for the pre-university science stream, at the beginning of grade 11 the derivative is introduced in mathematics classes. The introduction starts with the transition from graphs to functions and with the transition from a difference quotient to a differential quotient. Textbooks start with exercises on distance-time graphs to illustrate the meaning of average and instantaneous rate of change. The distance-time situation serves as an example to introduce the mathematical concept of derivative. After this physics-based introduction the slope of the tangent at a graph in the *xy*-plane is approximated by the slope of a line through two points on successively smaller intervals. The rate of change is directly linked to the tangent of the graph. Thereafter, the basic rules of symbolical differentiation are introduced and practiced.

Instruments and analysis

The two task-based interviews were designed to provide in-depth information about students' mathematical thinking while studying the concept of derivative. The tasks offer different representations (graphs, symbols, tables, etc.). Special about the tasks is, that the mathematical terms derivative, slope or differentiation and the symbols f' and dy/dx are explicitly avoided. In the tasks, the concept of derivative is asked for within situated contexts whereby variables have a physical meaning, such as time, volume or distance. The interview protocol prescribed, that a student, after completing a task, was repeatedly asked to check the obtained answer through other techniques. In this way, we were assured to observe a range of Andy's techniques.

In this paper we will focus on two tasks, *Barrel* and *Monopoly* (see Figure 1). These two tasks were selected because they offer students opportunities to use different techniques to solve the tasks, including numerical, graphical and symbolical approaches. The tasks were used in both task-based interviews, and therefore we can compare between the two interviews that were six months apart. We analyzed the interview transcripts and Andy's written answers to the problems, focussing on his techniques and the GC-options used.

The two calculus tests were designed by Andy's mathematics teacher. The tests contained similar tasks, and for this paper we will focus on two tasks: (1) a velocity-task, in which a distance-time formula is given and an instantaneous velocity has to be calculated, and (2) a tangent-task, in which the formula of a function is given

and the tangent has to be calculated for a certain point of the graph. Based on Andy's writings we analyzed his techniques and the GC-options used.

Barrel: A barrel contains a liquid, which runs out through a hole in the bottom. The volume of the liquid in the barrel (V in m^3) decreases over time (t in minutes). The volume of the liquid is expressed with a formula
$V = 10 \left(2 - \frac{1}{60} \cdot t \right)^2$. Also its graph is presented.
a. Calculate the outflow velocity at $t = 40$.
b. When a pump is used, the out-flow velocity can be expressed with the formula
   $V = 40 - 1/3 \cdot t$. When will the out-flow velocity by pumping be equal to the velocity of out-flow through a hole in the bottom?

Monopoly: For a company the revenue function is $R(q) = -0.5q^2 + 12q$ and the cost function is $TK(q) = 0.03q^3 - 0.5q^2 + 4q + 15$.
a. For which amount of sold products do the costs increase at the slowest rate?
b. At what production level will the costs and the revenue increase at the same rate?

Figure 1: Short descriptions, without figures, of the Barrel and Monopoly tasks

RESULTS

We present the results in chronological order. Due to space limitations, Andy's work in TBI-1 and 2 is strongly summarized.

Task-based interview-1 (April, grade 10)

Andy solved the *Barrel-a* task by calculating the volume at $t = 40$ and $t = 41$ and subtract these from each other. For the *Barrel-b* task he plots the linear graph of V into the diagram of the worksheet, and by drawing a parallel tangent to the curved graph (see Figure 2), he estimates that at $t = 60$ the out-flow velocity of both barrels is equal. He checks this estimation by using the trace-option of the GC to move to the volume at $t = 60$ and $t = 61$ and calculate their differences. So, in the *Barrel*-task Andy calculates rates of change on a unit-interval by using his GC as a graph-plotter and value-calculator.
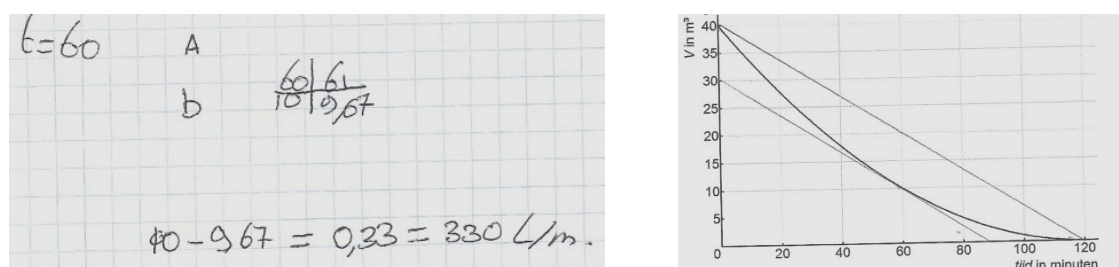

Figure 2: Drawing and calculation of Andy in the *Barrel-b* task

In the task *Monopoly-a* he uses the trace-option of his GC again to move the cursor over the graph (see Figure 3) and to look where the costs increase least. In the task *Monopoly-b* Andy plots the graphs of TK and TO. He uses the option Intersect and calculates the two points of intersection. But then he remarks that this is not correct, because "*the task is about increase*".
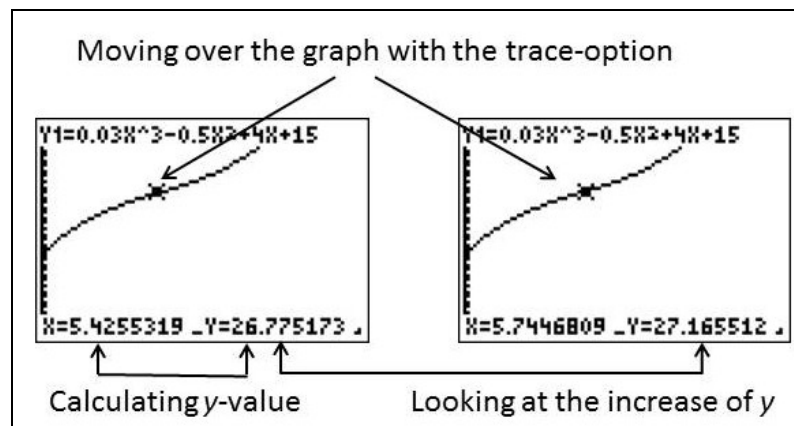
Figure 3: Example of Andy's plot-trace-scheme

Compared to the other nine students, Andy stands out by using his GC for the plot and trace options to explore the given functions. Andy is among the three students (out of ten) who solve the *Barrel*-task correctly. So, although derivatives and instantaneous rate of change have not yet been introduced, Andy is able to give meaning to rate of change in a volume-time situation and in a product-cost situation in terms of steepness of a curved graph. He does this by skilfully using plot, window and trace options of his GC. We refer to this as Andy's *plot-trace-calculate*-scheme (see Figure 3). This scheme reflects a graphical view on instantaneous change as the increase of the function at a small interval on the graph.

CT-1 Test on calculus (October, grade 11)

Andy solves the *velocity*-task about a falling object at $t = 6$ (given a formula for the height) in a remarkable way. Although it is a mathematics test on derivatives, he uses his GC and knowledge of physics to correctly calculate the velocity. Other students use symbolical differentiation for this task. In the *tangent-task*, Andy calculates the derivative function, but then he 'gets stuck' in an incorrect calculation. The test shows that Andy is able to calculate derivatives, but he does not use derivatives, neither to calculate the slope of a tangent, nor to calculate velocity.

TBI-2: Task-based interview (November, grade 11)

In the second task-based interview, six months after the first interview, the tasks *Barrel* and *Monopoly* are used again. To calculate the out-flow velocity in the task Barrel-a, Andy mentions three different procedures. He starts by plotting the graph on his GC and uses the option dy/dx in the CALC-menu to reach a correct answer. When asked to check his answer he mentions two additional techniques: (1) drawing on paper a tangent and calculating its slope, and (2) calculating the difference quotient on a small interval (he puts $t = 40$ and $t = 40.0001$ into his GC to find the corresponding values of V). He remarks about this small-interval technique: "*It is somewhat the same as dx-dy, dy-dx* (option of GC), *but then calculated by hand*." We notice at this point that Andy does not mention the derivative.

In the *Barrel-b* task Andy estimates the answer $t = 60$ by looking at the graph. He checks with the CALC-option dy/dx whether the slope at $t = 60$ is exactly -.333333. The interviewer asks if he is able to calculate the point. Andy says: "*To find this value in a direct way?[…] The line is always 1/3, so you have to find a point on the other graph where it is the same.*"

Andy also uses the dy/dx-option on his GC in the Monopoly-task. By looking at the plotted graphs he estimates the x-value, for which the steepness of both graphs is equal. He makes his cursor jump up-and-down between the two graphs using the dy/dx-option for calculating the steepness (see Figure 4). It is time-consuming and he says: "*I have no idea how to do this in another way*."
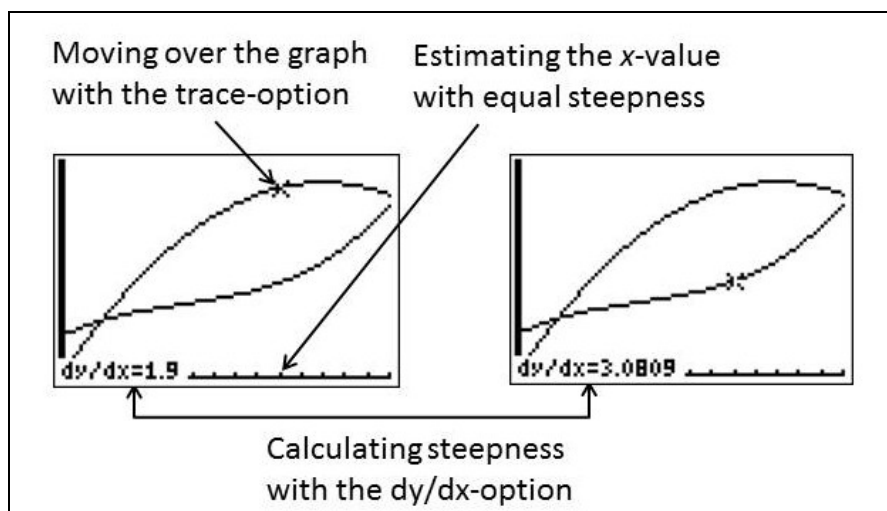

Figure 4: Example of the plot-trace-dy/dx-scheme.

Compared to the other nine students, Andy is the only one who uses the dy/dx-option of the GC. Other students work symbolically with the derivative combined with drawing a tangent.

So, in situated tasks about instantaneous rate of change Andy first explores the situation by plotting and tracing, he proceeds by using the dy/dx-option of his GC. In his explanations he relates the GC-option dy/dx to the tangent and also to the increase at a small interval. We call this the *plot-trace-dy/dx*-scheme. For Andy, this scheme is related to *tangent* and a *difference quotient on a minimal interval*. When asked for other techniques for these tasks, Andy never mentions the derivative. To him, symbolical differentiation apparently is not related to the *plot-trace-dy/dx*-scheme.

CT-2 (15 January, grade 11)

On the second test on calculus Andy solves the velocity-task correctly using the dy-dx-option of his GC. He solves the tangent-task by using derivatives. Thus, in velocity-tasks Andy's *plot-trace-dy/dx*-scheme becomes active, but apparently this scheme is not activated in tangent-tasks in the *xy*-plane.

CONCLUSIONS AND DISCUSSION

Before the introduction of calculus Andy's preferred instrumentation scheme is characterized as *a plot-trace*-scheme: he uses the plot and trace-options of his GC to calculate a rate of change. After the introduction of calculus we observe an uptake of another GC-option, dy/dx. His instrumentation scheme can be characterized as a *plot-trace-dy/dx*-scheme with links to *tangent* and *small interval* procedures. His skill in working with derivatives, which is observed in CT-1 and CT-2, is not used or mentioned by Andy in several situated tasks about velocity and increase. So, options of the GC become part of his instrumentation scheme for situated tasks on rate of change, but this scheme seems to develop separately from the symbolical procedure to calculate derivatives. Compared with nine other students, Andy is unique in his use of the GC. For solving the same tasks, the other students prefer symbolical differentiation combined with the use of a tangent.

The idea that the use of the GC encourages students to create links between graphical and symbolical representations as reported by Burrill et al. (2002) and Delos Santos (2006) does not hold for Andy. Andy's initial, resilient use of plot-options in his GC assimilates the dy/dx-option in situated rate-of-change tasks. Andy does not once mention or use symbolical differentiation in the task-based interviews, despite repeatedly being asked for alternative procedures. Nevertheless, Andy has learnt to use derivatives, as demonstrated in both calculus tests.

It is not clear why Andy does not relate symbolical and GC techniques. Our hypothesis is that Andy's instrumentation scheme is affected by the structure of the textbook. The textbook makes a clear distinction between tasks on the steepness of distance-time graphs, and tasks on tangents in the *xy*-plane. Solutions to the first type of tasks can often be approximations, solutions to the latter type of tasks always have to be exact.

One can wonder if it is a problem that Andy does not relate symbolical techniques and GC-options. An advantage of Andy's approach is his early uptake of graphical and numerical techniques with his *plot-trace*-scheme. A disadvantage is that he has few reasons to replace or supplement his GC-techniques with symbolical differentiation. We surmise that if Andy succeeds in linking symbolical differentiation to his *plot-trace-dy/dx*- scheme, he will have an excellent conceptual understanding of the concept of derivative in all representational aspects.

The theory of instrumental genesis is helpful to identify relationships between the use of the GC and Andy's knowledge about steepness, instantaneous rate of change and velocity in situations. Just as Trouche and Drijvers (2010) point out, the case of Andy shows that the use of technology in education can have complex and subtle effects: instead of being a tool that promotes links between representations, it can facilitate a learning process in which symbolical techniques develop separately from other techniques.

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