Chow groups and intersection products for tensor triangulated categories

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Chow groups and intersection products for tensor triangulated categories

Chow-groepen en intersectieproducten voor tensor-getrianguleerde categorieën
(met een samenvatting in het Nederlands)

Proefschrift

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Introduction

Chow rings in algebraic geometry. The study of algebraic cycles on an algebraic variety X under the equivalence relation of rational equivalence is a classical topic in algebraic geometry. This is usually formalized by the *Chow group*

$$\mathrm{CH}(X) = \bigoplus_p \mathrm{CH}^p(X)$$

where $\mathrm{CH}^p(X)$ is the group of codimension p cycles on X, i.e. the free abelian group on subvarieties $Y \subset X$ of codimension p, modulo the subgroup of cycles rationally equivalent to zero. The latter is generated by those cycles that appear as the divisor of a rational function on a subvariety of codimension $p-1$ and we can roughly think of the equivalence relation as follows: two cycles are rationally equivalent if one can be continuously deformed into the other along a projective line (see [Ful98, Section 1.6] for a more precise statement). The group $\mathrm{CH}(X)$ can be viewed as a generalization of the (Weil) divisor class group $\mathrm{Cl}(X) = \mathrm{CH}^1(X)$, which in turn often coincides with the Picard group $\mathrm{Pic}(X)$ (e.g. when X is locally factorial). It is named after W.-L. Chow (see [Cho56]).

When X is regular, the group $\mathrm{CH}(X)$ can be made into a graded ring (the *Chow ring*) by defining a product

$$\mathrm{CH}^p(X) \otimes \mathrm{CH}^q(X) \to \mathrm{CH}^{p+q}(X) \ .$$

Geometrically, this is interpreted as taking the intersection of a cycle class of codimension p with one of codimension q, while keeping track of intersection multiplicities. A formal definition of the product requires some work as the codimension of the intersection may not always be right and it is not so easy to define the right notion of intersection multiplicity. There are (at least) three possibilities to overcome these difficulties: the classical way is to use the Moving Lemma (see e.g. [Rob72]), two modern apporaches are given by "deformation to the normal cone" (see [Ful98]) and by using the product in the algebraic K-theory of X (see [Gra78]).

Let us remark that Chow groups and the intersection product are used widely in algebraic geometry, for example for the construction of the category $\mathcal{M}_k$ of motives over a field k. In this category, the morphisms and composition of morphisms are defined using these constructions (see [Sch94]).

Triangulated categories. The question that is addressed in this thesis is how to approach the subject from the point of view of (tensor) triangulated categories. Examples of these arise in algebraic geometry as (several flavors of) derived categories of (quasi)-coherent sheaves on X that can be viewed as an invariant attached to X. In general, derived categories are the natural domain of study for derived functors and historically, the examples just mentioned played a crucial role for the formulation of Grothendieck duality. They are also studied in mathematical physics in the context of "homological mirror symmetry" (see e.g. [Kon95]).

Can one reconstruct the ring (or at least the group) $\mathrm{CH}(X)$ from these triangulated categories "in purely categorical terms"? Can we give a notion of Chow group (or ring) for a triangulated category and transport the existing theory from algebraic geometry to other settings that involve the study of triangulated categories? If $\mathrm{D^b}(X)$ is the bounded derived category of coherent sheaves on X and X is non-singular and has ample canonical or anti-canonical bundle, a well-known result of Bondal and Orlov (see [BO01]) tells us that $\mathrm{D^b}(X)$ is a complete invariant, i.e. we can reconstruct X from the triangulated category $\mathrm{D^b}(X)$. Thus, it is certainly possible to recover $\mathrm{CH}(X)$ from $\mathrm{D^b}(X)$ in that situation. On the other hand, if X is a complex abelian variety of dimension g and $\hat{X}$ denotes its dual, then it is known that there is an equivalence of triangulated categories $\mathrm{D^b}(X) \cong \mathrm{D^b}(\hat{X})$. This equivalence induces an isomorphism

$$\mathrm{CH}_{\mathbb{Q}}(X) \cong \mathrm{CH}_{\mathbb{Q}}(\hat{X})$$

of Chow groups with rational coefficients, but the isomorphism does not preserve the degree of cycles, for example it sends $X \in \mathrm{CH}^0_{\mathbb{Q}}(X)$ to $(-1)^g \cdot (0)_{\hat{X}} \in \mathrm{CH}^g_{\mathbb{Q}}(\hat{X})$ (see e.g. [BL04, Chapter 16]). Hence, we should not expect that the definition of a Chow group $\mathrm{CH}(\mathrm{D^b}(X))$ that depends only on the triangulated structure of $\mathrm{D^b}(X)$ would allow us to talk about subgroups $\mathrm{CH}^p(\mathrm{D^b}(X))$ for $p \in \mathbb{Z}$.

In order to remedy this shortcoming, we allow ourselves to consider more structure than just a triangulation on the category. To be more precise, we consider for any scheme X (regular or not), the derived category of perfect complexes $\mathrm{D^{perf}}(X) \subset \mathrm{D^b}(X)$, which naturally has the structure of a *tensor triangulated category*, i.e. it is equipped with a symmetric monoidal structure induced by the derived tensor product of complexes of sheaves. The inclusion $\mathrm{D^{perf}}(X) \subset \mathrm{D^b}(X)$ is an exact equivalence if X is regular. When X is singular however, the derived tensor product of complexes of sheaves does not extend to $\mathrm{D^b}(X)$ in general and we have to work with $\mathrm{D^{perf}}(X)$ instead if we want a tensor structure. In [Bal05], it is shown that we can associate to every essentially small tensor triangulated category $\mathcal{T}$ a topological space $\mathrm{Spc}(\mathcal{T})$ such that $\mathrm{Spc}(\mathrm{D^{perf}}(X)) \cong X$ as topological spaces. It is also shown that one can reconstruct the whole variety X (i.e. including the structure sheaf) from $\mathrm{D^{perf}}(X)$ considered as a tensor triangulated category. Thus, it is certainly possible to reconstruct $\mathrm{CH}(X)$ from $\mathrm{D^{perf}}(X)$, but we want something more: to construct a functor $\mathrm{CH}^{\Delta}_p(-)$, that takes a tensor triangulated category $\mathcal{T}$ and produces a group $\mathrm{CH}^{\Delta}_p(\mathcal{T})$ such that $\mathrm{CH}^{\Delta}_p(\mathrm{D^{perf}}(X)) \cong \mathrm{CH}^p(X)$.

The Chow groups of a tensor triangulated category. We show that such a construction is realized by a definition of $\mathrm{CH}^{\Delta}_p(-)$ suggested to the author by P. Balmer in 2011 and now available in [Bal13]. One of the characteristic features of this approach is that in the definition of algebraic cycles, one allows for coefficients in certain Grothendieck groups of local categories, instead of taking coefficients in the integers. The definition is constructed in analogy to the situation in the G-theory of a non-singular algebraic variety X, where $\mathrm{CH}^p(X)$ appears in the Brown-Gersten-Quillen coniveau spectral sequence associated to X (see [Qui73]).

For a tensor triangulated category $\mathcal{T}$, the group $\mathrm{CH}^{\Delta}_p(T)$ depends on the choice of a *dimension function* $\dim : \mathrm{Spc}(\mathcal{T}) \to \mathbb{Z} \cup \{\pm\infty\}$ (see Definition 1.4.1), which should behave similarly to the way the Krull (co)dimension on spectral topological spaces does. It gives rise to a filtration

$$\cdots \subset \mathcal{T}_{(p-1)} \subset \mathcal{T}_{(p)} \subset \mathcal{T}_{(p+1)} \subset \cdots \tag{1}$$

of $\mathcal{T}$ that is used to define $\mathrm{CH}^{\Delta}_p(T)$ (see Definitions 2.2.3 and 2.2.4). We prove:

THEOREM (see Theorem 2.3.5). *Let X be a non-singular scheme of finite type over a field. Endow $\mathrm{D}^{\mathrm{perf}}(X)$ with the opposite of the Krull codimension as a dimension function. Then for all $p \in \mathbb{Z}$,*

$$\mathrm{CH}^{\Delta}_p\left(\mathrm{D}^{\mathrm{perf}}(X)\right) \cong \mathrm{CH}^{-p}(X)\ .$$

Apart from reconstructing the classical Chow groups, the definition of $\mathrm{CH}^{\Delta}_p(\mathcal{T})$ also behaves well in its own right, when we consider it as an invariant of $\mathcal{T}$. We show that $\mathrm{CH}^{\Delta}_p(-)$ is functorial for the class of exact functors with a relative dimension $n \in \mathbb{Z}$ (cf. Definition 2.4.1). These are exact (*not* $\otimes$-exact) functors that preserve the filtration that the choice of a dimension function induces on $\mathcal{T}$, up to a shift by n. We show

PROPOSITION (see Proposition 2.4.3). *Let $F : \mathcal{K} \to \mathcal{L}$ be a functor of relative dimension n. Then for all $p \in \mathbb{Z}$, F induces a group homomorphism*

$$\mathrm{CH}^{\Delta}_p(F) : \mathrm{CH}^{\Delta}_p(\mathcal{K}) \to \mathrm{CH}^{\Delta}_{p+n}(\mathcal{L})$$

and we prove that the proper push-forward and flat pull-back morphisms on the classical Chow groups can be interpreted as special cases of the above theorem, at least when the variety X is nice enough, e.g. non-singular, separated of finite type over a field (see Proposition 2.4.13 and Proposition 2.4.15).

Examples from representation theory. Tensor triangulated categories appear in numerous areas of mathematics, and our general definition applies to examples that do not come from algebraic geometry as well. In modular representation theory, for a finite group G and a field k whose characteristic divides $|G|$, one studies the bounded derived category $\mathrm{D}^{\mathrm{b}}(kG\text{-mod})$ of finite-dimensional kG-modules and the stable module category kG-stab. The latter category has the same objects as kG-mod and morphisms

$$\mathrm{Hom}_{kG\text{-stab}}(M,N) = \mathrm{Hom}_{kG\text{-mod}}(M,N)/\mathcal{I}$$

for all finite-dimensional kG-modules M, N, where $\mathcal{I}$ is the subgroup of homomorphisms that factor through a projective module (see Example 1.2.6). Both categories $\mathrm{D}^{\mathrm{b}}(kG\text{-mod})$ and kG-stab are tensor triangulated with tensor product $\otimes_k$ and we show that they have isomorphic tensor triangular Chow groups in almost all degrees, which should not come as a big surprise in view of Rickards equivalence (see [Ric89])

$$kG\text{-stab} \cong \mathrm{D}^{\mathrm{b}}(kG\text{-mod})/\,\mathrm{D}^{\mathrm{perf}}(kG\text{-mod})\ .$$

We prove:

THEOREM (see Theorem 3.2.6). *Consider* kG-stab *and* $\mathrm{D}^{\mathrm{b}}(kG\text{-mod})$ *with the Krull dimension of support as a dimension function on* $\mathrm{Spc}(kG\text{-stab})$ *and* $\mathrm{Spc}(\mathrm{D}^{\mathrm{b}}(kG\text{-mod}))$. *Then for all $p \geq 0$, there are isomorphisms*

$$\mathrm{CH}^{\Delta}_p(kG\text{-stab}) \cong \mathrm{CH}^{\Delta}_{p+1}(\mathrm{D}^{\mathrm{b}}(kG\text{-mod}))\ .$$

We also compute the associated tensor triangular Chow groups for $G=\mathbb{Z}/p^n\mathbb{Z}$ and $G=\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$:

THEOREM (see Propositions 3.3.2, 3.4.7 and 3.4.9). *Let k be a field of characteristic p. For $G=\mathbb{Z}/p^n\mathbb{Z}$, we have*

(i) $\mathrm{CH}_i^\Delta(kG\text{-stab})=0 \quad \forall i\neq 0,$

(ii) $\mathrm{CH}_0^\Delta(kG\text{-stab})\cong\mathbb{Z}/p^n\mathbb{Z},$

and if $p=2$ and $H=\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ then

(iii) $\mathrm{CH}_i^\Delta(kH\text{-stab})=0 \quad \forall i\neq 0,1,$

(iv) $\mathrm{CH}_0^\Delta(kH\text{-stab})\cong\mathbb{Z}/2\mathbb{Z},$

(v) $\mathrm{CH}_1^\Delta(kH\text{-stab})\cong\mathbb{Z}/2\mathbb{Z}$ *if k is algebraically closed,*

when we endow kG-stab, kH-stab *with the Krull dimension as a dimension function on* $\mathrm{Spc}(kG\text{-stab}),\mathrm{Spc}(kH\text{-stab})$.

In the course of the above computations, we also see that it is possible to obtain cycle groups with torsion coefficients (see Proposition 3.3.2), which contrasts with the situation in the algebro-geometric case. This illustrates that we view a general cycle, rather than as a $\mathbb{Z}$-linear combination of irreducible subspaces of codimension p of the spectrum $\mathrm{Spc}(\mathcal{T})$, as an element of a Grothendieck group $\mathrm{K}_0(\mathcal{T}_{(p)}/\mathcal{T}_{(p-1)})$ of a Verdier subquotient of the filtration (1). Only in the non-singular algebro-geometric examples does this produce cycles with coefficients in $\mathbb{Z}$, due to the “coincidence” that the Grothendieck group of the derived category of finite-length modules over a local ring is isomorphic to $\mathbb{Z}$ (see Remark 2.2.5).

Generalization to the relative case and localization. In order to increase the flexibility of our approach, we proceed by extending the definition of tensor triangular Chow groups to the relative case, i.e. we define for each $p\in\mathbb{Z}$ Chow groups $\mathrm{CH}_p^\Delta(\mathcal{T},\mathcal{K})$ of a compactly generated triangulated category $\mathcal{K}$, relative to the action of a tensor triangulated category $\mathcal{T}$ (see Definition 4.2.1 and [Ste13] for the formalism of actions of a tensor triangulated category). Here, both $\mathcal{T}$ and $\mathcal{K}$ are assumed to have set-indexed coproducts and are therefore not essentially small. We show that when one considers the full derived category $D_{\mathrm{Qcoh}}(X)$ of complexes of $\mathcal{O}_X$-modules with quasi-coherent cohomology on a noetherian scheme X, acting on itself via the left-derived tensor product, we recover the tensor triangular Chow groups of $\mathrm{D}^{\mathrm{perf}}(X)$. This is obtained as an immediate consequence of the following more abstract result. Denote by $\mathcal{T}^c\subset\mathcal{T}$ the full subcategory of compact objects, which is an essentially small tensor triangulated category.

THEOREM (see Proposition 4.2.4). *Let $\mathcal{T}$ be a compactly-rigidly generated tensor triangulated category with arbitrary set-indexed coproducts, equipped with a dimension function on $\mathcal{T}^c$ and such that* $\mathrm{Spc}(\mathcal{T}^c)$ *is noetherian. Consider the action of $\mathcal{T}$ on itself via its tensor product, and assume that the local-to-global principle (cf. Definition 4.1.6) holds for this action. Then we have isomorphisms*

$$\mathrm{CH}_p^\Delta(\mathcal{T},\mathcal{T})\cong\mathrm{CH}_p^\Delta(\mathcal{T}^c)$$

for all $p\in\mathbb{Z}$.

The flexibility we gained by extending the original definition allows us to construct localization sequences for our tensor triangular cycle groups and Chow groups.

THEOREM (see Theorem 4.3.9). *Let $\mathcal{T}$ be a compactly-rigidly generated tensor triangulated category with arbitrary set-indexed coproducts such that the local-to-global principle is satisfied for the action of $\mathcal{T}$ on itself. Let $\mathcal{T}^c \subset \mathcal{T}$ denote the full subcategory of compact objects and assume that* $\mathrm{Spc}(\mathcal{T}^c)$ *is a noetherian topological space. Let $U \subset \mathrm{Spc}(\mathcal{T}^c)$ be an open subset with closed complement Z, denote by $\mathcal{T}_Z \subset \mathcal{T}$ the triangulated subcategory of objects with support contained in Z and by $\mathcal{T}_U$ the Verdier quotient $\mathcal{T}/\mathcal{T}_Z$. Then there is an exact sequence*

$$\mathrm{Z}_p^{\Delta}(\mathcal{T},\mathcal{T}_Z) \xrightarrow{i_p} \mathrm{Z}_p^{\Delta}(\mathcal{T}^c) \xrightarrow{l_p} \mathrm{Z}_p^{\Delta}(\mathcal{T}_U^c) \to 0$$

for all $p \in \mathbb{Z}$. Furthermore, if $\mathcal{T}^c/\mathcal{T}_Z^c$ is idempotent complete and $p \geq \dim(Z)$, then we obtain an exact sequence

$$\mathrm{CH}_p^{\Delta}(\mathcal{T},\mathcal{T}_Z) \xrightarrow{\iota_p} \mathrm{CH}_p^{\Delta}(\mathcal{T}^c) \xrightarrow{\ell_p} \mathrm{CH}_p^{\Delta}((\mathcal{T}_U)^c) \to 0\,.$$

Tensor Frobenius pairs and intersection product. The last chapter of the thesis treats the construction of an intersection product on the tensor triangular Chow groups. As their definition was by analogy with the coniveau spectral sequence from algebraic K-theory, one could expect to obtain an intersection product via the higher algebraic K-theory of the category $\mathcal{T}$. It turns out that this is possible under two assumptions.

We first need that $\mathcal{T}$ has an algebraic model, i.e. it arises as the derived category of a tensor Frobenius pair $\boldsymbol{\mathcal{A}}$. A tensor Frobenius pair (see Definition 5.4.2) is a special case of the concept of Frobenius pair from [Sch06] and consists of a pair of Frobenius categories $\mathcal{A}_0 \subset \mathcal{A}$ together with a compatible symmetric monoidal structure on $\mathcal{A}$. The derived category of $\boldsymbol{\mathcal{A}}$ is by definition the Verdier quotient $\underline{\mathcal{A}}/\underline{\mathcal{A}_0}$ of the corresponding stable categories. Frobenius pairs are necessary to be able to define the Waldhausen K-theory of $\mathcal{T}$ (see Schlichting's articles [Sch02, Sch06]) and *tensor* Frobenius pairs make it possible to introduce products in the K-theory of $\mathcal{T}$. The use of the machinery of [Sch06] in conjunction with this new definition requires us to prove that tensor Frobenius pairs are well-behaved with respect to passing to countable envelopes, a result that is proved in Chapter 5, which lays the technical foundations for Chapter 6. As a side effect of assuming that $\mathcal{T}$ arises as the derived category of a tensor Frobenius pair, we exclude a priori some tensor triangulated categories not coming from an algebraic setting (e.g. the stable homotopy category of finite spectra from topology).

Our second and more severe assumption on the category is that the Frobenius pair $\boldsymbol{\mathcal{A}}$ (together with a chosen dimension function for its derived category $\mathcal{T}$) needs to satisfy an analogue of the Gersten conjecture from algebraic geometry (see Definition 6.4.1). This can be interpreted as a "regularity condition" on $\boldsymbol{\mathcal{A}}$.

Under these circumstances we can prove a theorem analogous to the Bloch formula from algebraic K-theory (see [Ful98, Section 20.5]). In order to do this, we make a small adjustment to the definition of $\mathrm{CH}_p^{\Delta}(\mathcal{T})$, and choose to work with subgroups ${}_{\cap}\mathrm{CH}_p^{\Delta}(\mathcal{T}) \subset \mathrm{CH}_p^{\Delta}(\mathcal{T})$ instead (see Definition 6.5.1). As the notation suggests, the group ${}_{\cap}\mathrm{CH}_p^{\Delta}(\mathcal{T})$ does not depend on the choice of tensor Frobenius pair $\boldsymbol{\mathcal{A}}$ but only on its derived category. The tensor Frobenius pair $\boldsymbol{\mathcal{A}}$ *is* used to define a K-theory sheaf $\mathcal{K}_p^0$ on $\mathrm{Spc}(\mathcal{T})$.

THEOREM (see Theorem 6.5.4). *Let $\mathcal{T}$ be an essentially small, rigid, topologically noetherian tensor triangulated category that arises as the derived category of a tensor Frobenius pair. Assume that the triangulated Gersten conjecture holds for $\boldsymbol{\mathcal{A}}$ when we* *equip its derived category with the opposite of the Krull codimension as a dimension function. Then we have isomorphisms*

$$ {}_{\cap}\mathrm{CH}^{\Delta}_{-p}(\mathcal{T}) \cong \mathrm{H}^{p}(\mathrm{Spc}(\mathcal{T}), \mathcal{K}^{0}_{p}) $$

for all $p \in \mathbb{Z}$.

In the light of this result, one can ask if we should work with ${}_{\cap}\mathrm{CH}^{\Delta}_{p}(-)$ instead of $\mathrm{CH}^{\Delta}_{p}(-)$ in general. After all, our result that states agreement with the usual Chow groups from algebraic geometry in the non-singular case also holds true with $\mathrm{CH}^{\Delta}_{p}(-)$ replaced by ${}_{\cap}\mathrm{CH}^{\Delta}_{p}(-)$ (see Lemma 6.7.6). In the end, both definitions may have their own merits.

We exploit the above theorem to construct an intersection product

$$ \alpha : {}_{\cap}\mathrm{CH}^{\Delta}_{p}(\mathcal{T}) \times {}_{\cap}\mathrm{CH}^{\Delta}_{q}(\mathcal{T}) \to {}_{\cap}\mathrm{CH}^{\Delta}_{p+q}(\mathcal{T}) $$

by combining the cup product from sheaf cohomology and the product in the K-theory of $\boldsymbol{\mathcal{A}}$ (see Definition 6.6.3). While the groups ${}_{\cap}\mathrm{CH}^{\Delta}_{*}(\mathcal{T})$ only depend on the derived category $\mathcal{T}$, the product α a priori depends on the full tensor Frobenius pair $\boldsymbol{\mathcal{A}}$.

Using an isomorphism ${}_{\cap}\mathrm{CH}^{\Delta}_{-p}(\mathrm{D}^{\mathrm{perf}}(X)) \cong \mathrm{CH}^{p}(X)$ for a non-singular variety and a result of Grayson (see [Gra78]), we can prove that our construction generalizes the usual intersection product for a specific choice of tensor Frobenius pair, assuming a compatibility condition between the products on Quillen and Waldhausen K-theory.

THEOREM (see Theorem 6.7.7). *Let* X *be a separated, non-singular scheme of finite type over a field. Let* **sPerf** *denote the Frobenius pair of strict perfect complexes on* X *(see Definition 6.7.1) and* $\mathcal{T}$ *the derived category of* **sPerf**. *Assume that diagram (26) commutes for all* $i, j \geq 0$ *and all opens* $U \subset X$. *Let* α *denote the intersection product from Definition 6.6.3 and* α' *the usual intersection product on* X. *Then* $\mathcal{T} \cong \mathrm{D}^{\mathrm{perf}}(X)$ *and the diagram*

$$ \begin{tikzcd} {}_{\cap}\mathrm{CH}^{\Delta}_{-p}(\mathcal{T}) \otimes {}_{\cap}\mathrm{CH}^{\Delta}_{-q}(\mathcal{T}) \arrow[r, "\alpha"] \arrow[d, "\cong"] & {}_{\cap}\mathrm{CH}^{\Delta}_{-p-q}(\mathcal{T}) \arrow[d, "\cong"] \\ \mathrm{CH}^{p}(X) \otimes \mathrm{CH}^{q}(X) \arrow[r, "\alpha'"] & \mathrm{CH}^{p+q}(X) \end{tikzcd} $$

commutes up to a sign $(-1)^{pq}$ *for all* $p, q \geq 0$.

For the reader's convenience, we include a glossary that briefly explains some important notions we use from category theory and algebraic geometry.

CHAPTER 1

A short review of tensor triangular geometry

In this chapter, we review some basic theory of the subject of tensor triangular geometry. For most of the chapter, we follow the treatment in the articles [Bal05, Bal10a, Bal07, Bal10b, BF11]. Before we do this, we need to recall the basic theory of triangulated categories as introduced by Verdier in [Ver96]. For this, we use [Nee01] as our main source, and in part [Kra10] for Bousfield localization. This chapter does not contain new results and for brevity, most proofs will only be referenced. We will, however, sketch the proofs of some results that will be crucial for the development of the theory in the following chapters.

1.1. Triangulated categories

The axioms. We begin with the definition of a triangulated category, as given in Neeman's book [Nee01].

1.1.1. DEFINITION. A *triangulated category* is an additive category $\mathcal{T}$, together with an additive auto-equivalence $\Sigma_{\mathcal{T}} : \mathcal{T} \to \mathcal{T}$ (called the *shift* or *suspension*) and a class of sequences consisting of three composable morphisms

$$A \to B \to C \to \Sigma_{\mathcal{T}}(A)$$

in $\mathcal{T}$ called *distinguished triangles*, satisfying the following axioms:

TR0: The sequence

$$X \xrightarrow{\mathrm{id}} X \longrightarrow 0 \longrightarrow \Sigma_{\mathcal{T}} X$$

is distinguished. All sequences isomorphic to a distinguished triangle are distinguished triangles: if in the commutative diagram in $\mathcal{T}$

$$\begin{tikzcd} X \arrow[r] \arrow[d, "r"] & Y \arrow[r] \arrow[d, "s"] & Z \arrow[r] \arrow[d, "t"] & \Sigma_{\mathcal{T}} X \arrow[d, "\Sigma_{\mathcal{T}} r"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma_{\mathcal{T}} X' \end{tikzcd}$$

the top row is a distinguished triangle and all vertical morphisms are isomorphisms, then the lower row is a distinguished triangle as well.

TR1: For any morphism $f : X \to Y$ in $\mathcal{T}$, there exists a distinguished triangle of the form

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow \Sigma_{\mathcal{T}} X \ .$$

TR2: ("Rotating triangles") Consider the two sequences

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma_{\mathcal{T}} X$$

and

$$Y \xrightarrow{-g} Z \xrightarrow{-h} \Sigma_{\mathcal{T}} X \xrightarrow{-\Sigma_{\mathcal{T}} f} \Sigma Y \ .$$

If one of the two is a distinguished triangle, so is the other.

TR3: For any commutative diagram in $\mathcal{T}$ of the form

$$\begin{tikzcd} X \arrow[r] \arrow[d, "r"] & Y \arrow[r] \arrow[d, "s"] & Z \arrow[r] & \Sigma_{\mathcal{T}} X \arrow[d, "\Sigma_{\mathcal{T}} r"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma_{\mathcal{T}} X' \end{tikzcd}$$

where the rows are distinguished triangles, there exists a morphism

$$t : Z \to Z'$$

such that the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "r"] & Y \arrow[r] \arrow[d, "s"] & Z \arrow[r] \arrow[d, "t"] & \Sigma_{\mathcal{T}} X \arrow[d, "\Sigma_{\mathcal{T}} r"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma_{\mathcal{T}} X' \end{tikzcd}$$

commutes.

TR4: ("The octahedron") Given three distinguished triangles

$$X \xrightarrow{f} Y \xrightarrow{r} Z \xrightarrow{u} \Sigma_{\mathcal{T}} X$$
$$Y \xrightarrow{g} Y' \xrightarrow{s} Y'' \xrightarrow{v} \Sigma_{\mathcal{T}} X$$
$$X \xrightarrow{gf} Y' \xrightarrow{t} Z' \xrightarrow{w} \Sigma_{\mathcal{T}} X$$

we can complete them to a commutative diagram in $\mathcal{T}$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\mathrm{id}"] & Y \arrow[r, "r"] \arrow[d, "g"] & Z \arrow[r, "u"] \arrow[d, "m"] & \Sigma_{\mathcal{T}} X \arrow[d, "\mathrm{id}"] \\ X \arrow[r, "gf"] \arrow[d] & Y' \arrow[r, "t"] \arrow[d, "s"] & Z' \arrow[r, "w"] \arrow[d, "n"] & \Sigma_{\mathcal{T}} X \arrow[d] \\ 0 \arrow[r] \arrow[d] & Y'' \arrow[r, "\mathrm{id}"] \arrow[d, "v"] & Y'' \arrow[r] \arrow[d] & 0 \arrow[d] \\ \Sigma_{\mathcal{T}} X \arrow[r, "\Sigma_{\mathcal{T}} f"] & \Sigma_{\mathcal{T}} Y \arrow[r, "\Sigma_{\mathcal{T}} r"] & \Sigma_{\mathcal{T}} Z \arrow[r, "-\Sigma_{\mathcal{T}} u"] & \Sigma^2_{\mathcal{T}} X \end{tikzcd}$$

where the first two rows and second column are the given triangles and all rows and columns are distinguished triangles. Furthermore, we require the sequence

$$Y \xrightarrow{\begin{pmatrix} g \\ -r \end{pmatrix}} Y' \oplus Z \xrightarrow{\begin{pmatrix} t & m \end{pmatrix}} Z' \xrightarrow{v \circ n} \Sigma_{\mathcal{T}}(Y)$$

to be a distinguished triangle and the compositions $\Sigma_{\mathcal{T}} f \circ w$ and $v \circ n$ to be equal.

If there is no danger of confusion, we will omit the subscript from $\Sigma_{\mathcal{T}}$ and just use the notation Σ instead.

1.1.2. REMARK. It can be shown from the axioms that the composition of any two consecutive arrows in a distinguished triangle is zero, see for example [Nee01, Remark 1.1.3].

1.1.3. REMARK. It is true that the object Z of axiom **TR1** is determined up to isomorphism - this is follows directly from [Nee01, Proposition 1.1.20]. However, as a consequence of the possible non-uniqueness of the morphism t in **TR3**, Z is generally *not determined by f up to unique isomorphism*. Thus Z does not functorially depend on f, which is a well-known shortcoming of triangulated categories. Still, we denote by $\mathrm{cone}(f)$ any object in the isomorphism class of Z.

Triangulated categories are very widespread in the mathematical landscape and usually appear when there is a notion of homotopy involved. The most basic example in the algebraic setting is the derived category of an abelian category.

1.1.4. EXAMPLE. Let $\mathcal{A}$ be an abelian category. Its *derived category* $\mathrm{D}(\mathcal{A})$ is formed by considering the category of chain complexes in $\mathcal{A}$ and formally inverting all morphisms of chain complexes that induce isomorphisms in homology (these morphisms are called "quasi-isomorphisms"). The suspension functor is given by shifting the degree of a complex by one and flipping the sign of the differentials. The distinguished triangles in $\mathrm{D}(\mathcal{A})$ are exactly those diagrams isomorphic (in $\mathrm{D}(\mathcal{A})$) to sequences of chain complexes of the form

$$X^\bullet \xrightarrow{f} Y^\bullet \xrightarrow{i} \mathrm{C}(f) \xrightarrow{p} \Sigma(X^\bullet)$$

where $f : X^\bullet \to Y^\bullet$ is any map of chain complexes, $\mathrm{C}(f)$ is the *mapping cone* of f given as the chain complex

$$\mathrm{C}(f)^i := X^{i+1} \oplus Y^i \qquad d^i_{\mathrm{C}(f)} := \begin{pmatrix} -d_X^{i+1} & 0 \\ f^{i+1} & d_Y^i \end{pmatrix},$$

$i : Y^\bullet \to \mathrm{C}(f)$ is the canonical injection and $p : \mathrm{C}(f) \to \Sigma(X^\bullet)$ is the canonical projection.

A common variant of the theme is to only consider *bounded* chain complexes in $\mathcal{A}$, i.e. those that are zero in high and low enough degrees. This yields the *bounded derived category* $\mathrm{D}^{\mathrm{b}}(\mathcal{A})$ which is triangulated as well. For more details on the construction, see e.g. [GM03].

Another, in a sense more general, construction is the following.

1.1.5. EXAMPLE (see [Hap88, Chapter I.2]). A *Frobenius category* is an exact category $\mathcal{E}$ in the sense of Quillen (see e.g. [Büh10] for a comprehensive treatment of the basic theory of exact categories) that has enough injective and projective objects, and in which the classes of injective and projective objects coincide. The *stable category* $\underline{\mathcal{E}}$ is the category with the same objects as $\mathcal{E}$ and where the morphisms between two objects $A, B \in \underline{\mathcal{E}}$ are given as

$$\mathrm{Hom}_{\underline{\mathcal{E}}}(A, B) := \mathrm{Hom}_{\mathcal{E}}(A, B)/\mathcal{I}$$

where $\mathcal{I}$ is the subgroup of morphisms that factor through a projective-injective object of $\mathcal{E}$. The category $\underline{\mathcal{E}}$ has a natural triangulation, where the suspension functor is given as follows: for each object E of $\mathcal{E}$, choose a fixed conflation

$$0 \to E \to I \to \Sigma(E,I) \to 0 \tag{2}$$

with I injective. We define $\Sigma(E)$ as the object $\Sigma(E,I)$ in $\underline{\mathcal{E}}$, where I comes from the fixed conflation (2). Using Schanuel's Lemma for injective objects, one checks that with a different choice of conflation

$$0 \to E \to J \to \Sigma(E,J) \to 0$$

with J injective, $\Sigma(E,I)$ and $\Sigma(E,I)$ are actually isomorphic in $\underline{\mathcal{E}}$. The functor Σ defines an endofunctor $\underline{\mathcal{E}} \to \underline{\mathcal{E}}$ and it is an equivalence with quasi-inverse defined as follows: for each object F of $\mathcal{E}$, choose a conflation

$$0 \to \Sigma^{-1}F \to P \to F \to 0$$

with P projective.

In order to define the class of distinguished triangles in $\underline{\mathcal{E}}$, we associate to each conflation

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

in $\mathcal{E}$ a *standard triangle*

$$A \xrightarrow{\overline{i}} B \xrightarrow{\overline{p}} C \xrightarrow{\overline{\epsilon}} \Sigma A$$

in the following way: the morphisms $\overline{i}, \overline{p}$ are the images of i, p in $\mathcal{E}$ and $\overline{\epsilon}$ is defined as follows: consider the commutative diagram

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, equal] & B \arrow[r, "p"] \arrow[d, "g"] & C \arrow[d, "\epsilon"] \\
A \arrow[r] & I \arrow[r] & \Sigma A
\end{tikzcd}$$

with exact rows in $\mathcal{E}$, where the lower row is the chosen conflation (2) for $E = A$ and g exists because of the injectivity of I. One checks that the class of ϵ in $\underline{\mathcal{E}}$ is independent of the choice of g and so we take it as the definition of $\overline{\epsilon}$. We define a diagram

$$X \to Y \to Z \to \Sigma X$$

in $\underline{\mathcal{E}}$ to be a distinguished triangle iff it is isomorphic to a standard triangle. This defines the structure of a triangulated category on $\underline{\mathcal{E}}$.

1.1.6. EXAMPLE (cf. [Hap88, Chapter I.3] or [Kel96, Example 6.1]). Let $\mathcal{A}$ be an additive category and consider the category of bounded chain complexes $\mathrm{C}^b(\mathcal{A})$. We endow $\mathrm{C}^b(\mathcal{A})$ with the exact structure where a sequence of morphisms of chain complexes

$$A_\bullet \to B_\bullet \to C_\bullet$$

is a conflation iff $A_i \to B_i \to C_i$ is split-exact for all i. Then one checks that $\mathrm{C}^b(\mathcal{A})$ is a Frobenius category, where the class of projective-injective objects is given by the contractible complexes. The associated stable category is $\mathrm{K}^b(\mathcal{A})$, the bounded homotopy category of $\mathcal{A}$.

1.1.7. Remark. We call a triangulated category that arises as the stable category of a Frobenius category *algebraic*. It can be shown that the derived category of an abelian category is algebraic. There are examples of non-algebraic triangulated categories from topology (e.g. the stable homotopy category of finite spectra, see [Sch10]) but for the rest of this thesis, our examples will always be algebraic.

Before we proceed, let us introduce the appropriate notion of morphism in the world of triangulated categories.

1.1.8. Definition. An additive functor $F : \mathcal{T} \to \mathcal{U}$ between two triangulated categories $\mathcal{T}, \mathcal{U}$ is called *exact* if we have a natural isomorphism

$$F \circ \Sigma_{\mathcal{T}} \cong \Sigma_{\mathcal{U}} \circ F$$

and F sends distinguished triangles to distinguished triangles.

Verdier localization. One of the most useful constructions for triangulated categories is Verdier localization. Given a triangulated category $\mathcal{T}$ and a triangulated subcategory $\mathcal{S} \subset \mathcal{T}$ (see Definition 1.1.9), the basic idea is to construct a triangulated category $\mathcal{T}/\mathcal{S}$ and an exact localization functor $F : \mathcal{T} \to \mathcal{T}/\mathcal{S}$ such that $F(A) = 0$ for all objects $A \in \mathcal{S}$ and the pair $(\mathcal{T}/\mathcal{S}, F)$ is universal for that property.

1.1.9. Definition (See e.g. [Nee01, Definition 1.5.1]). Let $\mathcal{T}$ be a triangulated category and $\mathcal{S} \subset \mathcal{T}$ be a subcategory. The subcategory $\mathcal{S}$ is called *triangulated* if it is a full, additive subcategory such that

- Every object of $\mathcal{T}$ isomorphic to an object of $\mathcal{S}$ is already in $\mathcal{S}$ ($\mathcal{S}$ is a *replete* subcategory).
- $\Sigma(\mathcal{S}) = \mathcal{S}$.
- For any distinguished triangle

$$A \to B \to C \to \Sigma A$$

  such that A, B are objects of $\mathcal{S}$, the object C must also be in $\mathcal{S}$.

We give the basic idea for the construction of $\mathcal{T}/\mathcal{S}$ and F. For a comprehensive treatment in the present context, see [Nee01, Chapter 2]. The objects of $\mathcal{T}/\mathcal{S}$ are the same as the objects of $\mathcal{T}$ and the morphisms are given as follows: given two objects X, Y we consider "fractions", i.e. diagrams of the form

$$X \xleftarrow{f} Z \to Y$$

where $\operatorname{cone}(f)$ is an object of $\mathcal{S}$. We introduce an equivalence relation $\sim$ on the class of fractions $\alpha(X, Y)$. Two fractions

$$X \xleftarrow{f} Z \to Y$$

and

$$X \xleftarrow{g} Z' \to Y$$

in $\alpha(X, Y)$ are considered equivalent if there is a third fraction

$$X \xleftarrow{h} Z'' \to Y$$

in $\alpha(X,Y)$ and morphisms $Z'' \to Z, Z'' \to Z'$ such that the diagram

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[dr] & \\ X & Z'' \arrow[l, "h"] \arrow[r] \arrow[u] \arrow[d] & Y \\ & Z' \arrow[ul, "g"] \arrow[ur] & \end{tikzcd}$$

commutes. We now set $\mathrm{Hom}_{\mathcal{T}/\mathcal{S}}(X,Y) := \alpha(X,Y)/\sim$. Next, one defines a composition of fractions using "homotopy pullbacks" (see [Nee01, Lemma 2.1.16]) and checks that this choice of morphisms makes $\mathcal{T}/\mathcal{S}$ an additive category, and that one obtains a functor $F : \mathcal{T} \to \mathcal{T}/\mathcal{S}$ that is the identity on objects and sends a morphism $f : X \to Y$ to the fraction $X \xleftarrow{\mathrm{id}} X \xrightarrow{f} Y$. The auto-equivalence $\Sigma_{\mathcal{T}}$ induces an auto-equivalence $\Sigma_{\mathcal{T}/\mathcal{S}}$ by componentwise application of $\Sigma_{\mathcal{T}}$ to fractions. If we define the class of distinguished triangles in $\mathcal{T}/\mathcal{S}$ as those diagrams isomorphic to images of distinguished triangles of $\mathcal{T}$ under F, this gives $\mathcal{T}/\mathcal{S}$ the structure of a triangulated category.

1.1.10. THEOREM (see [Nee01, Theorem 2.1.8]). *Let $\mathcal{T}$ be a triangulated category and $\mathcal{S} \subset \mathcal{T}$ a triangulated subcategory. Then the exact functor $F : \mathcal{T} \to \mathcal{T}/\mathcal{S}$ has the property that $F(A) = 0$ for all objects $A \in \mathcal{S}$ and the pair $(\mathcal{T}/\mathcal{S}, F)$ is universal for that property: given any exact functor $G : \mathcal{T} \to \mathcal{U}$ between triangulated categories such that $F(A) = 0$ for all objects $A \in \mathcal{S}$, G must factor as $G = \overline{G} \circ F$ for a unique functor $\overline{G} : \mathcal{T}/\mathcal{S} \to \mathcal{U}$.*

1.1.11. DEFINITION. We call the functor $F : \mathcal{T} \to \mathcal{T}/\mathcal{S}$ of Theorem 1.1.10 the *Verdier quotient functor* or *Verdier localization functor* associated to $\mathcal{S}$.

1.1.12. REMARK. For a morphism $f \in \mathcal{T}$ with $\mathrm{cone}(f) \in \mathcal{S}$, its image $F(f)$ under the localization functor is an isomorphism (see [Nee01, Lemma 2.1.21]). Given an object $S \in \mathcal{S}$, the morphism $0 \to S$ has cone S and is therefore an isomorphism. We see that all objects of $\mathcal{S}$ become isomorphic to 0 in $\mathcal{T}/\mathcal{S}$.

In general, it is *not* true that the full subcategory of objects that F sends to 0 (called the *kernel* of F) is equal to $\mathcal{S}$.

1.1.13. DEFINITION. A triangulated subcategory $\mathcal{S} \subset \mathcal{T}$ is called *thick* if it contains all direct summands of all objects of $\mathcal{S}$. The *thick closure* of a triangulated subcategory $\mathcal{S}$ is the smallest thick triangulated subcategory of $\mathcal{T}$ containing $\mathcal{S}$.

1.1.14. PROPOSITION (see [Nee01, Remark 2.1.39]). *The kernel of F is the thick closure of $\mathcal{S}$.*

PROOF. In [Nee01, Lemma 2.1.33] it is proved that the kernel of F is precisely the full subcategory containing all direct summands of all objects of $\mathcal{S}$. As the thick closure of $\mathcal{S}$ must contain all direct summands of all objects of $\mathcal{S}$, it must therefore contain the kernel of F. But kernels of exact functors are always triangulated subcategories, so the result follows. □

In the case that we are given a chain of triangulated subcategories $\mathcal{R} \subset \mathcal{S} \subset \mathcal{T}$, the following isomorphism theorem holds.

1.1.15. LEMMA. *Let $\mathcal{R} \subset \mathcal{S} \subset \mathcal{T}$ be triangulated subcategories. Then we have an exact equivalence of triangulated categories*

$$(\mathcal{T}/\mathcal{R})/(\mathcal{S}/\mathcal{R}) \cong \mathcal{T}/\mathcal{S}\ .$$

PROOF. Let Q denote the composition of the Verdier quotient functors $Q_1 : \mathcal{T} \to \mathcal{T}/\mathcal{R}$ and $Q_2 : \mathcal{T}/\mathcal{R} \to (\mathcal{T}/\mathcal{R})/(\mathcal{S}/\mathcal{R})$. The functor Q sends all objects of $\mathcal{S}$ to 0 and we will show that the pair $((\mathcal{T}/\mathcal{R})/(\mathcal{S}/\mathcal{R}), Q)$ has the universal property from Theorem 1.1.10.

Let $G : \mathcal{T} \to \mathcal{U}$ be any exact functor that sends all objects of $\mathcal{S}$ to 0. As $\mathcal{R} \subset \mathcal{S}$, the universal property from Theorem 1.1.10 tells us that G factors uniquely as $X \circ Q_1$ with $X : \mathcal{T}/\mathcal{R} \to \mathcal{U}$. But X sends all objects of $\mathcal{S}/\mathcal{R}$ to 0, so another application of Theorem 1.1.10 tells us that it factors uniquely via Q_2. In conclusion, G factors uniquely via $Q_2 \circ Q_1 = Q$, as desired. □

We also record the following lemma.

1.1.16. LEMMA. *The natural functor*

$$I : \mathcal{S}/\mathcal{R} \to \mathcal{T}/\mathcal{R}$$

induced by the inclusion $\mathcal{S} \hookrightarrow \mathcal{T}$ is fully faithful.

PROOF. In order to see that I is full, let $A, B \in \mathcal{S}$ be two objects. Then a morphisms $f : A \to B$ in $\mathcal{T}/\mathcal{R}$ is represented by a fraction

$$A \xleftarrow{h} C \xrightarrow{g} B \tag{3}$$

in $\mathcal{T}$ such that $\mathrm{cone}(h) \in \mathcal{R} \subset \mathcal{S}$. Thus, we have a distinguished triangle

$$C \xrightarrow{h} A \to \mathrm{cone}(h) \to \Sigma C$$

with $A, \mathrm{cone}(h) \in \mathcal{S}$, from which it follows that we must have $C \in \mathcal{S}$ as well, as $\mathcal{S}$ was a triangulated subcategory. Therefore, the fraction (3) also defines a morphism in $\mathcal{S}/\mathcal{R}$ which I maps to f.

To check faithfulness, let $m, m' : D \to E$ be two morphisms in $\mathcal{S}/\mathcal{R}$ represented by fractions in $\mathcal{S}$

$$D \xleftarrow{s} F \xrightarrow{t} E$$

and

$$D \xleftarrow{s'} F' \xrightarrow{t'} E$$

respectively, with $\mathrm{cone}(s), \mathrm{cone}(s') \in \mathcal{R}$. If $I(m) = I(m')$, there exists a fraction

$$D \xleftarrow{s''} F'' \xrightarrow{t''} E$$

in $\mathcal{T}$ with $\operatorname{cone}(s'') \in \mathcal{R}$ and morphisms $u : F'' \to F, v : F'' \to F'$ such that the diagram in $\mathcal{T}$

(4)
$$\begin{tikzcd} & F \arrow[dl, "s"'] \arrow[dr, "t"] & \\ D & F'' \arrow[l, "s''"'] \arrow[r, "t''"] \arrow[u, "u"] \arrow[d, "v"] & E \\ & F' \arrow[ul, "s'"] \arrow[ur, "t'"'] & \end{tikzcd}$$

commutes. As $D, \operatorname{cone}(s'') \in \mathcal{S}$, we conclude again that the same must hold for F'' and therefore, diagram (4) is actually contained in $\mathcal{S}$ and therefore $m = m'$. □

Bousfield localization. Let us introduce a flavor of Verdier localization that will become useful in Chapter 4.

1.1.17. DEFINITION (see [Nee01, Definition 9.1.1]). Let $\mathcal{S} \subset \mathcal{T}$ be a thick triangulated subcategory. We say that a *Bousfield localization functor exists for the pair* $\mathcal{S} \subset \mathcal{T}$ if the Verdier localization functor $F : \mathcal{T} \to \mathcal{T}/\mathcal{S}$ has a right adjoint G.

Bousfield localizations are useful for us, as they let us perform the localization

$$F : \mathcal{T} \to \mathcal{T}/\mathcal{S}$$

inside of $\mathcal{T}$, as in Theorem 1.1.19.

1.1.18. DEFINITION. Let $\mathcal{S} \subset \mathcal{T}$ be a class of objects. Then we define the $\mathcal{S}^{\perp}$ as the full subcategory of $\mathcal{T}$ with objects

$$\{x \in \mathcal{T} : \operatorname{Hom}_{\mathcal{T}}(s, x) = 0 \ \forall s \in \mathcal{S}\} .$$

The category $\mathcal{S}^{\perp}$ is called *the subcategory of $\mathcal{S}$-local objects*.

1.1.19. THEOREM (see [Nee01, Theorem 9.1.16]). *Let $\mathcal{S} \subset \mathcal{T}$ be a thick triangulated subcategory and suppose a Bousfield localization functor exists for the pair $\mathcal{S} \subset \mathcal{T}$. Then the subcategory of $\mathcal{S}$-local objects is equivalent as a triangulated category to the Verdier quotient $\mathcal{T}/\mathcal{S}$. More precisely, the composition*

$$\mathcal{S}^{\perp} \hookrightarrow \mathcal{T} \to \mathcal{T}/\mathcal{S}$$

is an exact equivalence of triangulated categories.

Another useful consequence of the existence of a Bousfield localization functor is the existence of certain functorial triangles. The unit of the adjunction from Definition 1.1.17 gives us for each object $t \in \mathcal{T}$ a morphism $t \to GF(t)$. We denote $GF(t)$ by $t_{\mathcal{S}^{\perp}}$. If we complete this morphism to a distinguished triangle and rotate, we obtain a distinguished triangle $t_{\mathcal{S}} \to t \to t_{\mathcal{S}^{\perp}} \to \Sigma(t_{\mathcal{S}})$, so that $t_{\mathcal{S}} \cong \Sigma^{-1}(\operatorname{cone}(t \to t_{\mathcal{S}^{\perp}}))$. It can be shown (see [Nee01, Proposition 9.1.8]) that $t_{\mathcal{S}} \in \mathcal{S}$.

1.1.20. THEOREM (see [Kra10, Proposition 4.11.2]). *Let $\mathcal{S} \subset \mathcal{T}$ be a thick triangulated subcategory and suppose a Bousfield localization functor exists for the pair $\mathcal{S} \subset \mathcal{T}$. Let $x \to t \to y \to \Sigma(x)$ be a distinguished triangle in $\mathcal{T}$ such that $x \in \mathcal{S}$ and $y \in \mathcal{S}^{\perp}$.*

Then there exist unique isomorphisms $\alpha : x \to t_{\mathcal{S}}, \beta : y \to t_{\mathcal{S}^\perp}$ such that the diagram

$$\begin{tikzcd} x \arrow[r] \arrow[d, "\alpha"] & t \arrow[r] \arrow[d, "\mathrm{id}"] & y \arrow[r] \arrow[d, "\beta"] & \Sigma(x) \arrow[d, "\Sigma(\alpha)"] \\ t_{\mathcal{S}} \arrow[r] & t \arrow[r] & t_{\mathcal{S}^\perp} \arrow[r] & \Sigma(t_{\mathcal{S}}) \end{tikzcd}$$

commutes. In other words, the distinguished triangle $t_{\mathcal{S}} \to t \to t_{\mathcal{S}^\perp} \to \Sigma(t_{\mathcal{S}})$ is unique among triangles $x \to t \to y \to \Sigma(x)$ with $x \in \mathcal{S}$ and $y \in \mathcal{S}^\perp$, up to a unique isomorphism that restricts to the identity on t.

1.1.21. REMARK. The distinguished triangle $t_{\mathcal{S}} \to t \to t_{\mathcal{S}^\perp} \to \Sigma(t_{\mathcal{S}})$ gives rise to two functors $L, \Gamma : \mathcal{T} \to \mathcal{T}$ where $L = GF$ and Γ is defined by $\Gamma(t) := t_{\mathcal{S}}$. We sometimes call L the *localization functor associated to $\mathcal{S}$* and Γ the *acyclization functor associated to $\mathcal{S}$*. For every object $t \in \mathcal{T}$ we then have a distinguished triangle

$$\Gamma(t) \to t \to L(t) \to \Sigma(\Gamma(t)) .$$

1.1.22. REMARK (see [Kra10, Proposition 4.9.1]). The localization functor L associated to $\mathcal{S}$ from Remark 1.1.21 has $\ker L = \mathcal{S}$ and the unit of the adjunction $\eta : \mathrm{Id} \to L$ satisfies the following two properties: the morphism $L\eta : L \to L^2$ is invertible and $L\eta = \eta L$. Giving a pair $\mathcal{S} \subset \mathcal{T}$ for which a Bousfield localization functor exists is equivalent to giving an exact functor $L : \mathcal{T} \to \mathcal{T}$ and a morphism $\eta : \mathrm{Id} \to L$ satisfying these two properties. We call such a functor a *Bousfield localization functor* and, in the terminology of Remark 1.1.21, L is the localization functor associated to $\ker(L)$.

We conclude with an existence statement for Bousfield localizations, for triangulated categories admitting set-indexed coproducts. Recall that a triangulated subcategory of such a category $\mathcal{T}$ is called *localizing* if it closed under the formation of set-indexed coproducts in $\mathcal{T}$.

1.1.23. DEFINITION. Let $\mathcal{T}$ be a triangulated category admitting set-indexed coproducts. An object $t \in \mathcal{T}$ is called *compact* if every morphism $t \to \coprod_{i \in I} x_i$ into a coproduct factors through $\coprod_{i \in J} x_i$, where $J \subset I$ is a finite subset.

1.1.24. DEFINITION. A triangulated category $\mathcal{T}$ admitting set-indexed coproducts is called *compactly generated* if there exists a *set* of compact objects $C \subset \mathcal{T}$ and there is no proper localizing subcategory of $\mathcal{T}$ containing C.

1.1.25. THEOREM (see [Kra10, Proposition 5.2.1]). *Let $\mathcal{T}$ be a triangulated category admitting set-indexed coproducts and $\mathcal{S} \subset \mathcal{T}$ a localizing subcategory that is compactly generated. Then a Bousfield localization functor exists for the pair $\mathcal{S} \subset \mathcal{T}$.*

Grothendieck groups. We often want to associate an abelian group to a triangulated category that reflects its triangulated structure. The Grothendieck group provides such a construction and is defined analogously to the Grothendieck group of an abelian or exact category.

1.1.26. DEFINITION. Let $\mathcal{T}$ be an essentially small triangulated category. We define its *Grothendieck group* $\mathrm{K}_0(\mathcal{T})$ as the free abelian group on the isomorphism classes of $\mathcal{T}$, modulo the subgroup generated by expressions of the form

$$[a]-[b]+[c]$$

for each distinguished triangle $a \to b \to c \to \Sigma(a)$.

1.1.27. PROPOSITION. *Let $F : \mathcal{T} \to \mathcal{U}$ be an exact functor of essentially small triangulated categories. Then F induces a group homomorphism*

$$\mathrm{K}_0(\mathcal{T}) \to \mathrm{K}_0(\mathcal{U})$$

by mapping $[a]$ to $[F(a)]$ and extending linearly.

PROOF. This is a consequence of the fact that F preserves distinguished triangles. $\square$

1.1.28. PROPOSITION. *Let $\mathcal{T}$ be an essentially small triangulated category. Then any element of $\mathrm{K}_0(\mathcal{T})$ is represented by an object of $\mathcal{T}$.*

PROOF. The split-exact sequence

$$a \xrightarrow{i_1} a \oplus b \xrightarrow{p_2} b \xrightarrow{0} \Sigma(a)$$

is a distinguished triangle, where i_1 and p_2 denote the obvious injection and projection morphisms: indeed, it is the coproduct of the distinguished triangle

$$a \xrightarrow{\mathrm{id}} a \to 0 \to \Sigma(a)$$

and the distinguished triangle

$$0 \to b \xrightarrow{\mathrm{id}} b \to 0$$

obtained from

$$b \xrightarrow{\mathrm{id}} b \to 0 \to \Sigma(b)$$

by rotating. The coproduct of two distinguished triangles is a distinguished triangle by [Nee01, Proposition 1.2.1 and Remark 1.2.2]. This tells us that the equality

$$[a]+[b]=[a\oplus b]$$

holds in $\mathrm{K}_0(\mathcal{T})$ for all objects a,b,c of $\mathcal{T}$. This also shows that $[0]=0$ holds in $\mathrm{K}_0(\mathcal{T})$.

Furthermore, if we rotate the triangle $a \xrightarrow{\mathrm{id}} a \to 0 \to \Sigma(a)$ to get the triangle $a \to 0 \to \Sigma(a) \to \Sigma(a)$, we see that

$$[a]+[\Sigma(a)]=0$$

in $\mathrm{K}_0(\mathcal{T})$, which implies that $-[a]=[\Sigma(a)]$ in $\mathcal{T}$.

As any element of $\mathrm{K}_0(\mathcal{T})$ is represented by a (formal) finite sum of isomophism classes of objects in $\mathcal{T}$ and their (formal) inverses, this proves the claim. $\square$

1.1.29. EXAMPLE (see [Ill77, Exposé VIII]). Let $\mathcal{A}$ be an essentially small abelian category. Then $\mathrm{K}_0(\mathrm{D}^{\mathrm{b}}(\mathcal{A})) \cong \mathrm{K}_0(\mathcal{A})$, where $\mathrm{K}_0(\mathcal{A})$ denotes the Grothendieck group of $\mathcal{A}$, i.e. the free abelian group on the set of isomorphism classes of $\mathcal{A}$, modulo the subgroup of generated by expressions of the form

$$[a]-[b]+[c]$$

whenever there is an exact sequence

$$0 \to a \to b \to c \to 0$$

in $\mathcal{A}$.

The isomorphism is explicitly given by

$$[C_\bullet] \mapsto \sum_i (-1)^i [H_i(C_\bullet)] \ .$$

If $F : \mathcal{T} \to \mathcal{U}$ is an exact functor of essentially small triangulated categories, it is sometimes possible to describe the kernel of the induced map $\mathrm{K}_0(\mathcal{T}) \to \mathrm{K}_0(\mathcal{U})$. We collect the following two results which will prove useful throughout the rest of this thesis.

1.1.30. PROPOSITION (see [Tho97, Corollary 2.3]). *Let $\mathcal{T} \hookrightarrow \mathcal{U}$ be the inclusion of a* dense *triangulated subcategory, i.e. every object of $\mathcal{U}$ is a direct summand of an object of $\mathcal{T}$. Then the induced map*

$$\mathrm{K}_0(\mathcal{T}) \to \mathrm{K}_0(\mathcal{U})$$

is injective.

1.1.31. PROPOSITION (see [Ill77, Exposé VIII, Prop. 3.1]). *Let $I : \mathcal{T} \hookrightarrow \mathcal{U}$ be the inclusion of a thick triangulated subcategory and denote by $P : \mathcal{U} \to \mathcal{U}/\mathcal{T}$ the Verdier quotient functor. Then there is an exact sequence*

$$\mathrm{K}_0(\mathcal{T}) \xrightarrow{\mathrm{K}_0(I)} \mathrm{K}_0(\mathcal{U}) \xrightarrow{\mathrm{K}_0(P)} \mathrm{K}_0(\mathcal{U}/\mathcal{T}) \to 0 \ .$$

When $\mathcal{T}$ is too large, its Grothendieck group is not a useful invariant.

1.1.32. PROPOSITION (Eilenberg swindle). *Let $\mathcal{T}$ be an essentially small triangulated category admitting countable coproducts. Then* $\mathrm{K}_0(\mathcal{T}) \cong 0$.

PROOF. Let a be an object of $\mathcal{T}$, then we have a distinguished triangle of the form

$$\bigoplus_{i \in \mathbb{N}} a_i \xrightarrow{\phi} \bigoplus_{i \in \mathbb{N}} a_i \to a \to \Sigma \left(\bigoplus_{i \in \mathbb{N}} a_i \right)$$

where $a_i = a$ for all i and $\phi_{i,j} = 0$ for $j \neq i+1$ and $\phi_{i,i+1} = \mathrm{id}$. Thus,

$$0 = \left[\bigoplus_{i \in \mathbb{N}} a_i \right] - \left[\bigoplus_{i \in \mathbb{N}} a_i \right] + [a] = [a]$$

in $\mathrm{K}_0(\mathcal{T})$, which proves the claim. □

1.2. Tensor triangulated categories and the spectrum

Triangulated categories often have some extra structure. One particular bit of such extra structure is that of a tensor product which is well-behaved with respect to the triangulation.

1.2.1. DEFINITION (see [Bal10b, Definition 3]). A tensor triangulated category is a triangulated category $\mathcal{T}$ endowed with a compatible symmetric monoidal structure. That is, there is a bifunctor

$$\otimes : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$$

and a unit object $\mathbb{I}$, together with associator, unitor and commutator isomorphisms: for all objects X, Y, Z in $\mathcal{T}$, we have natural isomorphisms

$$X \otimes (Y \otimes Z) \cong (X \otimes Y) \otimes Z, \quad X \otimes \mathbb{I} \cong X \cong \mathbb{I} \otimes X, \quad X \otimes Y \cong Y \otimes X$$

that satisfy the coherence conditions of [ML98, Section XI.1] to make $\mathcal{T}$ a symmetric monoidal category. Furthermore, the bifunctor $\otimes$ is exact in each variable.

1.2.2. REMARK. As an addition to the axiomatic of Definition 1.2.1, one can ask that the following coherence condition holds: by the biexactness of $\otimes$, we have natural isomorphisms $(\Sigma X) \otimes - \cong \Sigma(X \otimes -)$ and $- \otimes \Sigma(Y) \cong \Sigma(- \otimes Y)$ for all objects $X, Y \in \mathcal{T}$. These fit into a diagram

$$\begin{tikzcd} (\Sigma X) \otimes (\Sigma Y) \arrow[r, "\sim"] \arrow[d, "\wr"] & \Sigma(X \otimes \Sigma Y) \arrow[d, "\wr"] \\ \Sigma(\Sigma X \otimes Y) \arrow[r, "\sim"] & \Sigma^2(X \otimes Y) \end{tikzcd}$$

which we require to commute up to a sign, i.e. the composition of the upper and right isomorphisms should equal the composition of the left and lower isomorphisms or its additive inverse (see e.g. [Bal10b]). This coherence condition will not be used explicitly in the following, so we don't require it for Definition 1.2.1.

1.2.3. REMARK. We warn the reader that although $\mathcal{T} \times \mathcal{T}$ can be equipped with a component-wise triangulated structure, the functor $\otimes : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ is *not* exact under the assumptions of Definition 1.2.1 if $\mathcal{T} \neq 0$. In this situation, the unit object $\mathbb{I}$ cannot be the zero object (otherwise, $A \cong A \otimes \mathbb{I} \cong 0$ for all objects $A \in \mathcal{T}$, by additivity of $\mathbb{I} \otimes -$) and we consider the two distinguished triangles $\mathbb{I} \xrightarrow{\text{id}} \mathbb{I} \to 0 \to \Sigma\mathbb{I}$ and $\mathbb{I} \xrightarrow{0} \mathbb{I} \to \mathbb{I} \oplus \Sigma\mathbb{I} \to \Sigma\mathbb{I}$. The tensor product of these triangles yields the sequence $\mathbb{I} \xrightarrow{0} \mathbb{I} \to 0 \to \Sigma^2\mathbb{I}$ which cannot be a distinguished triangle unless $\mathbb{I} \cong 0$, which we forbade.

1.2.4. EXAMPLE. Let R be a commutative ring. Then $\mathrm{K}^b(R\text{−proj})$, the bounded homotopy category of finitely generated projective R-modules, is a tensor triangulated category, with the tensor product induced by the usual tensor product of chain complexes.

Generalizing Example 1.2.4 leads to another important example coming from algebraic geometry.

1.2.5. EXAMPLE. A scheme X is called *quasi-separated* if the intersection of any two quasi-compact open subsets of X is again quasi-compact. Let X be a quasi-compact, quasi-separated scheme and consider the derived category of perfect complexes $\mathrm{D}^{\mathrm{perf}}(X)$ on X. This is the triangulated subcategory of $\mathrm{D}_{\mathrm{Qcoh}}(X)$, the derived category of chain complexes of $\mathcal{O}_X$-modules with quasi-coherent homology, that consists of those complexes that are locally quasi-isomorphic to a complex of locally free sheaves of finite rank. The category $\mathrm{D}^{\mathrm{perf}}(X)$ carries the structure of a tensor triangulated category, where the tensor product is given by $\otimes^{\mathrm{L}}$, the left-derived tensor product of sheaves of $\mathcal{O}_X$-modules. The unit object is given by the chain complex $\mathbb{I}_\bullet$ that has $\mathbb{I}_j = 0$ for $j \neq 0$ and $\mathbb{I}_0 = \mathcal{O}_X$.

When X is noetherian, the canonical inclusion $\mathrm{D}^{\mathrm{b}}(\mathrm{Coh}(X)) \to \mathrm{D}(\mathcal{O}_X\text{−mod})$ has essential image $\mathrm{D}^{\mathrm{b}}_{\mathrm{Coh}}(\mathcal{O}_X\text{−mod})$, the bounded derived category of complexes of $\mathcal{O}_X$-modules with coherent homology (see [BGI71, Corollaire 2.2.2.1]). As a perfect complex must be bounded by the quasi-compactness of X and we can check locally that it has coherent cohomology, we can view $\mathrm{D}^{\mathrm{perf}}(X)$ as a triangulated subcategory of $\mathrm{D}^{\mathrm{b}}(X)$ in this case. When X is furthermore separated and regular, every coherent sheaf on X has a finite resolution by locally free sheaves, which implies that the inclusion $\mathrm{D}^{\mathrm{perf}}(X) \hookrightarrow \mathrm{D}^{\mathrm{b}}(X)$ is an equivalence.

In the next chapters, we will sometimes consider examples that do not come from algebraic geometry. The modular representation theory of finite groups provides us with another source of tensor triangulated categories.

1.2.6. EXAMPLE. Let G be a finite group and k be a field such that char(k) divides the order of G. The group algebra kG is self-injective (i.e. injective as a module over itself), which implies that the category kG-mod of finitely generated kG-modules (= finite dimensional representations over k) is a Frobenius category. As we saw in Example 1.1.5, the associated stable category kG-stab is naturally a triangulated category. It is also a tensor triangulated category, where the tensor product of two modules M, N is given by $M \otimes_k N$ (*not* $\otimes_{kG}$; the G-action is diagonal) and the unit is the trivial module k.

It turns out that the extra structure of the tensor product is enough to be able to set up a geometric theory, if we assume that $\mathcal{T}$ is essentially small. The main object one studies in *tensor triangular geometry* is the *spectrum* of a tensor triangulated category, whose construction we describe next.

1.2.7. DEFINITION. Let $\mathcal{T}$ be a tensor triangulated category. A thick triangulated subcategory $\mathcal{J} \subset \mathcal{T}$ is called

- *$\otimes$-ideal* if $\mathcal{T} \otimes \mathcal{J} \subset \mathcal{J}$.
- *prime* if $\mathcal{J}$ is a proper $\otimes$-ideal ($\mathcal{J} \neq \mathcal{T}$) and $A \otimes B \in \mathcal{J}$ implies $A \in \mathcal{J}$ or $B \in \mathcal{J}$ for all objects $A, B \in \mathcal{T}$.

1.2.8. DEFINITION (see [Bal05]). Let $\mathcal{T}$ be an essentially small tensor triangulated category. The *spectrum* of $\mathcal{T}$ is the set

$$\mathrm{Spc}(\mathcal{T}) := \{\mathcal{P} \subset \mathcal{T} : \mathcal{P} \text{ is a prime ideal}\}$$

topologized by the basis of closed sets of the form

$$\mathrm{supp}(A) := \{\mathcal{P} \in \mathrm{Spc}(\mathcal{T}) : A \notin \mathcal{P}\}$$

for objects $A \in \mathcal{T}$. The set $\mathrm{supp}(A)$ is called the *support of A*.

1.2.9. REMARK. Let us stress that $\mathrm{Spc}(\mathcal{T})$ is defined as a topological space, not as a (locally) ringed space. It is possible to equip $\mathrm{Spc}(\mathcal{T})$ with a sheaf of rings (see [Bal05, Section 6]), but we do not use this construction.

1.2.10. REMARK. As we assumed that $\mathcal{T}$ is essentially small, $\mathrm{Spc}(\mathcal{T})$ is a set (i.e. not a proper class). Furthermore, it is always true that $\mathrm{Spc}(\mathcal{T})$ is a *spectral* topological space, i.e. it is homeomorphic to the prime ideal spectrum of some commutative ring (see [BKS07, Proposition 3.5]).

Let us give some computations of $\mathrm{Spc}(\mathcal{T})$ right away:

1.2.11. EXAMPLE (see [BKS07, Theorem 9.5]). Let X be a quasi-compact, quasi-separated scheme, then $\mathrm{Spc}(\mathrm{D}^{\mathrm{perf}}(X)) \cong X$. Moreover, the support $\mathrm{supp}(A^\bullet)$ of a complex $A^\bullet \in \mathrm{D}^{\mathrm{perf}}(X)$ coincides with the support of the total homology of $A^\bullet$ on X under this isomorphism. This was also proved in [Bal05] under the slightly more restrictive assumption that X is topologically noetherian. In both cases, the proof of the statement uses Thomason's classification result from [Tho97].

1.2.12. EXAMPLE (see [Bal05, Corollary 5.10]). Let G be a finite group and k be a field such that char(k) divides the order of G. Then $\mathrm{Spc}(kG\text{-stab}) \cong \mathcal{V}_G(k)$, the *projective support variety* of k. The variety $\mathcal{V}_G(k)$ is defined as $\mathrm{Proj}(\mathrm{H}^*(G,k))$, where $\mathrm{H}^*(G,k)$ denotes the cohomology ring of G over k. The support $\mathrm{supp}(M)$ of a module $M \in kG$-stab coincides with the cohomological support of M in $\mathcal{V}_G(k)$ (see Chapter 3, Definition 3.1.3) under this isomorphism. The proof of the statement uses the classification of thick $\otimes$-ideals in kG-stab from [BCR97].

These examples should already give the reader the impression that the spectrum is an object worth studying. The following universal property reassures us that the definition of $\mathrm{Spc}(\mathcal{T})$ is indeed the right one.

1.2.13. DEFINITION. Let $\mathcal{T}$ be an essentially small tensor triangulated category. A *support datum* on $\mathcal{T}$ is a pair (X,σ), where X is a topological space and σ is a function that assigns to each object $A \in \mathcal{T}$ a closed subset $\sigma(A) \subset X$, such that

(1) $\sigma(0) = \emptyset$ and $\sigma(\mathbb{I}) = X$,
(2) $\sigma(A \oplus B) = \sigma(A) \cup \sigma(B)$ for all objects $A, B \in \mathcal{T}$,
(3) $\sigma(\Sigma A) = \sigma(A)$ for all objects $A \in \mathcal{T}$,
(4) $\sigma(C) \subset \sigma(A) \cup \sigma(B)$ for every distinguished triangle $A \to B \to C \to \Sigma A$,
(5) $\sigma(A \otimes B) = \sigma(A) \cap \sigma(B)$ for all objects $A, B \in \mathcal{T}$.

The pair $(\mathrm{Spc}(\mathcal{T}), \mathrm{supp})$ satisfies the conditions of Definition 1.2.13 as proved in [Bal05, Lemma 2.6] and has a special role among the collection of support data on $\mathcal{T}$.

1.2.14. THEOREM (see [Bal05, Theorem 3.2]). *Let $\mathcal{T}$ be an essentially small tensor triangulated category and (X,σ) be a support datum on $\mathcal{T}$. Then there exists a unique continuous map $f : X \to \mathrm{Spc}(\mathcal{T})$ such that $\sigma(A) = f^{-1}(\mathrm{supp}(A))$ for all objects $A \in \mathcal{T}$, explicitly given as*

$$f(x) = \{a \in \mathcal{T} : x \notin \sigma(a)\} .$$

1.2.15. REMARK. It is an immediate consequence of the properties of σ that

$$\{a \in \mathcal{T} : x \notin \sigma(a)\}$$

is a prime ideal of $\mathcal{T}$.

We can use Theorem 1.2.14 to explicitly compute $\mathrm{Spc}(\mathcal{T})$. Recall from [Bal05] that a $\otimes$-ideal $\mathcal{J} \subset \mathcal{T}$ is called *radical* if $a^{\otimes n} \in \mathcal{J} \Rightarrow a \in \mathcal{J}$ holds for all objects $a \in \mathcal{T}$.

1.2.16. DEFINITION (see [Bal05, Definition 5.1]). A support datum (X,σ) on $\mathcal{T}$ is called *classifying* if

(1) The space X is noetherian and any non-empty irreducible closed subset has a unique generic point.
(2) We have a bijection

$$\{Y \subset X \text{specialization closed subset}\} \overset{1:1}{\longleftrightarrow} \{\mathcal{J} \subset \mathcal{T} \text{ radical } \otimes\text{-ideal}\}$$

given by $Y \mapsto \{a \in \mathcal{T} : \sigma(a) \subset Y\}$ with inverse $\mathcal{J} \mapsto \bigcup_{a\in\mathcal{J}} \sigma(a)$.

Knowing a classifying support datum on $\mathcal{T}$ is enough to compute $\mathrm{Spc}(\mathcal{T})$.

1.2.17. THEOREM (see [Bal05, Theorem 5.2]). *Suppose that (X, σ) is a classifying support datum on $\mathcal{T}$. Then the map $f : X \to \mathrm{Spc}(\mathcal{T})$ of Theorem 1.2.14 is a homeomorphism.*

1.2.18. REMARK. If $\mathrm{Spc}(\mathcal{T})$ is noetherian, then $(\mathrm{Spc}(\mathcal{T}), \mathrm{supp})$ is a classifying support datum on $\mathcal{T}$ (see [Bal05, Theorem 4.10]), and thus computing $\mathrm{Spc}(\mathcal{T})$ is actually equivalent to finding a classifying support datum on $\mathcal{T}$.

1.2.19. EXAMPLE. If $\mathcal{T} = \mathrm{D}^{\mathrm{perf}}(X)$ for X a topologically noetherian scheme, and $\mathrm{supph}(a) \subset X$ denotes the homological support of $a \in \mathcal{T}$, then (X, supph) is a classifying support datum for $\mathcal{T}$ (see [Bal05, Theorem 5.5], [Tho97, Theorem 3.15]), which proves $\mathrm{Spc}(\mathcal{T}) \cong X$ in this case.

1.2.20. EXAMPLE. Let G be a finite group and k a field such that $\mathrm{char}(k)$ divides the order of G. For $M \in kG\text{-stab}$, denote by $\sigma(M) \subset \mathcal{V}_G(k)$ the cohomological support of M. Then $(\mathcal{V}_G(k), \sigma)$ is a classifying support datum on kG-stab (see [Bal05, Theorem 5.9]), which proves $\mathrm{Spc}(kG\text{-stab}) \cong \mathcal{V}_G(k)$.

Note that if $\mathrm{char}(k)$ does not divide $\#G$, the statement becomes trivial, as we have $kG\text{-stab} = 0$ in that case. Indeed, if $\mathrm{char}(k)$ does not divide $\#G$ the ring kG is semi-simple by Maschke's theorem. Thus, every object of kG-mod is projective.

We conclude the section by giving a functoriality property of the spectrum. An exact $\otimes$-functor is by definition an exact functor between tensor triangulated categories that respects the tensor product up to natural isomorphism and preserves the unit.

1.2.21. THEOREM (see [Bal05, Proposition 3.6]). *Let $F : \mathcal{T} \to \mathcal{U}$ be an* exact $\otimes$-functor *of essentially small tensor triangulated categories. Then the map*

$$\begin{aligned} \mathrm{Spc}(F) : \mathrm{Spc}(\mathcal{U}) &\to \mathrm{Spc}(\mathcal{T}) \\ \mathcal{P} &\mapsto F^{-1}(\mathcal{P}) \end{aligned}$$

is well-defined, continuous and for all objects $A \in \mathcal{T}$ we have

$$(\mathrm{Spc}(F))^{-1}(\mathrm{supp}_{\mathcal{T}}(A)) = \mathrm{supp}_{\mathcal{U}}(F(A))$$

PROOF. We give a proof as the statement is only given as an exercise in [Bal05]. It is straightforward to check that $F^{-1}(\mathcal{P})$ is a prime $\otimes$-ideal of $\mathcal{T}$, which gives that $\mathrm{Spc}(F)$ is well-defined. As we can check continuity on a closed basis, this follows from the identity $(\mathrm{Spc}(F))^{-1}(\mathrm{supp}_{\mathcal{T}}(A)) = \mathrm{supp}_{\mathcal{U}}(F(A))$ which we prove now. Let $\mathcal{P} \in \mathrm{supp}_{\mathcal{U}}(F(A))$, i.e. $F(A) \notin \mathcal{P}$. This implies that

$$A \notin F^{-1}(\mathcal{P}) = \mathrm{Spc}(F)(\mathcal{P}) \Leftrightarrow \mathrm{Spc}(F)(\mathcal{P}) \in \mathrm{supp}_{\mathcal{T}}(A) \ .$$

Therefore, $\mathcal{P} \in \mathrm{Spc}(F))^{-1}(\mathrm{supp}_{\mathcal{T}}(A))$. On the other hand, if

$$\mathcal{Q} \in \mathrm{Spc}(F))^{-1}(\mathrm{supp}_{\mathcal{T}}(A)) \ ,$$

then

$$\mathrm{Spc}(F)(\mathcal{Q}) = F^{-1}(\mathcal{Q}) \in \mathrm{supp}_{\mathcal{T}}(A) \Leftrightarrow A \notin F^{-1}(\mathcal{Q}) \ .$$

This implies that $F(A) \notin \mathcal{Q} \Leftrightarrow \mathcal{Q} \in \mathrm{supp}_{\mathcal{U}}(F(A))$ which we wanted to show. $\square$

1.3. Localization and idempotent completion

Next, we turn our attention to localization and idempotent completion of triangulated categories and their role in tensor triangular geometry. Let $\mathcal{T}$ be a tensor triangulated category.

Smashing localizations. Let us first investigate the interaction between Bousfield localizations and the tensor structure on $\mathcal{T}$. *Smashing localizations* first appeared in stable homotopy theory, see e.g. [Rav84].

1.3.1. DEFINITION (cf. [BF11, Definition 2.15]). A Bousfield localization functor $L : \mathcal{T} \to \mathcal{T}$ (see Remark 1.1.22) is called *smashing* if both $\ker(L)$ and $\ker(L)^{\perp}$ are $\otimes$-ideals. In that case we call $\ker(L)$ a *smashing ideal*.

Smashing localizations have a nice description in terms of the tensor product on $\mathcal{T}$.

1.3.2. PROPOSITION. *Let $L : \mathcal{T} \to \mathcal{T}$ be a smashing localization functor. Then $L \cong L(\mathbb{I}) \otimes -$ and $\Gamma \cong \Gamma(\mathbb{I}) \otimes -$.*

PROOF. Consider the triangle

$$\Gamma(\mathbb{I}) \to \mathbb{I} \to L(\mathbb{I}) \to \Sigma(\Gamma(\mathbb{I}))$$

and apply $a \otimes -$ for an object $a \in \mathcal{T}$. By exactness of this functor we obtain a distinguished triangle

$$a \otimes \Gamma(\mathbb{I}) \to a \to a \otimes L(\mathbb{I}) \to \Sigma(a \otimes \Gamma(\mathbb{I}))$$

where $a \otimes \Gamma(\mathbb{I}) \in \ker(L)$ and $a \otimes L(\mathbb{I}) \in \ker(L)^{\perp}$ as we assumed that the localization was smashing and we have $\Gamma(\mathbb{I}) \in \ker(L)$ and $L(\mathbb{I}) \in \ker(L)^{\perp}$. By Theorem 1.1.20, we obtain isomorphisms $a \otimes \Gamma(\mathbb{I}) \cong \Gamma(a)$ and $a \otimes L(\mathbb{I}) \cong L(a)$. $\square$

1.3.3. EXAMPLE. Let X be a quasi-compact, quasi-separated scheme and denote by $\mathcal{T}$ the category $\mathrm{D}_{\mathrm{Qcoh}}(X)$ (see Example 1.2.5). Let $Z \subset X$ be a closed subset with quasi-compact complement U and denote by $\mathrm{D}_Z^{\mathrm{perf}}(X)$ the subcategory of perfect complexes that have the support of their total homology contained in Z. If $\langle \mathrm{D}_Z^{\mathrm{perf}}(X) \rangle$ is the smallest localizing subcategory of $\mathcal{T}$ containing $\mathrm{D}_Z^{\mathrm{perf}}(X)$, then a Bousfield localization functor for the pair $\langle \mathrm{D}_Z^{\mathrm{perf}}(X) \rangle \subset \mathcal{T}$ exists by Theorem 1.1.25 and $\langle \mathrm{D}_Z^{\mathrm{perf}}(X) \rangle \subset \mathcal{T}$ is a smashing ideal by [BF11, Theorem 4.1]. The essential image $\ker(L)^{\perp}$ of L can be identified with $\mathrm{D}_{\mathrm{Qcoh}}(U)$ (see [BF11, Remark 5.13]).

Verdier quotients by tensor ideals.

1.3.4. CONVENTION. We assume for the rest of the chapter that $\mathcal{T}$ is essentially small.

Given a triangulated subcategory $\mathcal{S} \subset \mathcal{T}$, we can form the Verdier quotient $\mathcal{T}/\mathcal{S}$ which will be a triangulated category again. This also works in the context of tensor triangulated categories: if we take $\mathcal{J} \subset \mathcal{T}$ a $\otimes$-ideal then the Verdier quotient $\mathcal{T}/\mathcal{J}$ will inherit the structure of a tensor triangulated category such that the localization functor is an exact $\otimes$-functor and we can give a precise description of the spectrum $\mathrm{Spc}(\mathcal{T}/\mathcal{J})$:

1.3.5. THEOREM (see [Bal05, Proposition 3.11]). *Let* $q : \mathcal{T} \to \mathcal{T}/\mathcal{J}$ *be the Verdier quotient of a tensor triangulated category* $\mathcal{T}$ *by a tensor ideal* $\mathcal{J}$. *Then the map* $\mathrm{Spc}(q) : \mathrm{Spc}(\mathcal{T}/\mathcal{J}) \to \mathrm{Spc}(\mathcal{T})$ *induces a homeomorphism between* $\mathrm{Spc}(\mathcal{T}/\mathcal{J})$ *and the subspace*

$$\{\mathcal{P} \in \mathrm{Spc}(\mathcal{T}) : \mathcal{J} \subset \mathcal{P}\} .$$

1.3.6. EXAMPLE. For $P \in \mathrm{Spc}(\mathcal{T})$, the space $\mathrm{Spc}(\mathcal{T}/P)$ is homeomorphic to the subspace of $\mathrm{Spc}(\mathcal{T})$ consisting of those points containing P, i.e. those that have P in their closure. The category $\mathcal{T}/P$ is *local* i.e. $\mathrm{Spc}(\mathcal{T}/P)$ has a unique closed point (see [Bal10a, Definition 4.1 and Proposition 4.2]).

1.3.7. EXAMPLE. For X a quasi-compact and quasi-separated scheme, consider $\mathcal{T} = \mathrm{D}^{\mathrm{perf}}(X)$ and $Z \subset X$ a closed subset with quasi-compact complement U. Let $\mathrm{D}^{\mathrm{perf}}_Z(X)$ denote the $\otimes$-ideal consisting of those objects with support in Z. We have seen in Example 1.2.11 that $\mathrm{Spc}(\mathrm{D}^{\mathrm{perf}}(X)) \cong X$ and it follows that $\mathrm{Spc}(\mathrm{D}^{\mathrm{perf}}(X)/\mathrm{D}^{\mathrm{perf}}_Z(X)) \cong U$. Note however, that in general we can identify $\mathrm{D}^{\mathrm{perf}}(X)/\mathrm{D}^{\mathrm{perf}}_Z(X)$ only with a dense subcategory of $\mathrm{D}^{\mathrm{perf}}(U)$ (namely with the subcategory of those objects whose class in $\mathrm{K}_0(\mathrm{D}^{\mathrm{perf}}(U))$ belongs to the image of $\mathrm{K}_0(\mathrm{D}^{\mathrm{perf}}(X))$ under the map induced by restriction to U, see [TT90, Chapter 5]). In order to get an equivalence, we therefore need to take idempotent completions which we now introduce.

Idempotent completion. For technical and conceptual reasons (see for example the problem at the end of Example 1.3.7) it is often convenient to work in a setting where idempotent endomorphisms split.

1.3.8. DEFINITION. An additive category $\mathcal{A}$ is called *idempotent complete* if all idempotent endomorphisms split: if A is an object of $\mathcal{A}$ and $e : A \to A$ is such that $e^2 = e$, then there is a decomposition $A \cong \ker(e) \oplus \mathrm{im}(e)$.

1.3.9. EXAMPLE. Any abelian category is idempotent complete as well as any derived category of an abelian category (see [BS01]). A thick triangulated subcategory of an idempotent complete triangulated category is idempotent complete. The full subcategory of the category of finite-dimensional k-vector spaces consisting of the even-dimensional spaces is evidently not idempotent complete.

Given an additive category $\mathcal{A}$, we can always embed it into its *idempotent completion* $\mathcal{A}^\natural$ (also know as its Karoubi envelope or Cauchy completion), an additive category which is idempotent complete. This also works for tensor triangulated categories.

1.3.10. THEOREM (see [BS01] and [Bal05, Remark 3.12]). *Let* $\mathcal{T}$ *be a tensor triangulated category. Then there exists an idempotent complete tensor triangulated category* $\mathcal{T}^\natural$ *and a fully faithful* $\otimes$*-exact functor* $\iota : \mathcal{T} \hookrightarrow \mathcal{T}^\natural$ *such that any exact functor* $\mathcal{T} \to \mathcal{S}$ *to an idempotent complete triangulated category factors via* ι.

SKETCH OF THE PROOF. The category $\mathcal{T}^\natural$ is the idempotent completion of the underlying additive category of $\mathcal{T}$: its objects are given by pairs (A, e) where A is an object of $\mathcal{T}$ and $e : A \to A$ is an idempotent endomorphisms. A morphism $\phi : (A, e) \to (B, f)$ is a morphism $\phi : A \to B$ in $\mathcal{T}$ such that $\phi \circ e = f \circ \phi = \phi$. The functor ι is defined by sending an object A to the pair (A, id_A) and it is easy to see that it is fully faithful.

The category $\mathcal{T}^\natural$ naturally inherits an additive structure from $\mathcal{T}$ and we give it a triangulated structure as follows: The suspension of an object (A,e) is given by $\Sigma(A,e) = (\Sigma(A),\Sigma(e))$ and a diagram

$$\Delta : (A,e) \to (B,f) \to (C,g) \to (\Sigma(A),\Sigma(e))$$

is a distinguished triangle if there is a diagram Δ' of the same form such that $\Delta \oplus \Delta'$ is isomorphic to the image of a distinguished triangle in $\mathcal{T}$ under ι.

The category $\mathcal{T}^\natural$ also inherits a symmetric monoidal structure from $\mathcal{T}$ by setting

$$(A,e)\otimes(B,f) := (A\otimes B, e\otimes f)\ ,$$

which makes $\mathcal{T}^\natural$ tensor triangulated. □

The following theorem says that we can always idempotent complete without changing the spectrum.

1.3.11. THEOREM (see [Bal05, Corollary 3.14]). *Let $\mathcal{T}$ be a tensor triangulated category and $\iota : \mathcal{T} \to \mathcal{T}^\natural$ the inclusion into the idempotent completion. The map*

$$\mathrm{Spc}(\iota) : \mathrm{Spc}(\mathcal{T}^\natural) \to \mathrm{Spc}(\mathcal{T})$$

is a homeomorphism.

1.4. Dimension and decomposition

The spectrum of a tensor triangulated category $\mathcal{T}$ is always a *spectral* topological space, i.e. it is homeomorphic to the spectrum of a commutative ring (see Remark 1.2.10). Therefore, it is sensible to talk about the Krull (co-)dimension of a closed subset. A slightly more general notion is the following:

1.4.1. DEFINITION (see [Bal07]). A *dimension function* on $\mathcal{T}$ is a map

$$\dim : \mathrm{Spc}(\mathcal{T}) \to \mathbb{Z}\cup\{\pm\infty\}$$

such that the following two conditions hold:

(1) If $\mathcal{Q} \subset \mathcal{P}$ are prime tensor ideals of $\mathcal{T}$, then $\dim(\mathcal{Q}) \le \dim(\mathcal{P})$.
(2) If $\mathcal{Q} \subset \mathcal{P}$ and $\dim(\mathcal{Q}) = \dim(\mathcal{P}) \in \mathbb{Z}$, then $\mathcal{Q} = \mathcal{P}$.

For a subset $V \subset \mathrm{Spc}(\mathcal{T})$, we define $\dim(V) := \sup\{\dim(\mathcal{P}) | \mathcal{P} \in V\}$. For every $p \in \mathbb{Z}\cup\{\pm\infty\}$, we define the full subcategory

$$\mathcal{T}_{(p)} := \{a \in \mathcal{T} : \dim(\mathrm{supp}(a)) \le p\}\ .$$

We denote by $\mathrm{Spc}(\mathcal{T})_p$ the set of points $\mathcal{Q}$ of $\mathrm{Spc}(\mathcal{T})$ such that $\dim(\mathcal{Q}) = p$.

1.4.2. REMARK. From the properties of $\mathrm{supp}(-)$, it follows that $\mathcal{T}_{(p)}$ is a thick tensor ideal in $\mathcal{T}$.

1.4.3. EXAMPLE. The main examples of dimension functions we will consider are the Krull dimension and the opposite of the Krull co-dimension. For $\mathcal{P} \in \mathrm{Spc}(\mathcal{T})$, its *Krull dimension* $\dim_{\mathrm{Krull}}(\mathcal{P})$ is the maximal length n of a chain of irreducible closed subsets

$$\emptyset \subsetneq C_0 \subsetneq C_1 \subsetneq \ldots \subsetneq C_n = \overline{\{\mathcal{P}\}}.$$

Dually, we define the *opposite of the Krull co-dimension*

$$-\mathrm{codim}_{\mathrm{Krull}}(\mathcal{P})$$

as follows: if we have a chain of irreducible closed subsets of maximal length

$$\overline{\{\mathcal{P}\}} = C_0 \subsetneq C_1 \ldots \subsetneq C_n = \text{ maximal irred. comp. of } \operatorname{Spc}(\mathcal{T}) \text{ containing } \mathcal{P}$$

we set

$$-\operatorname{codim}_{\text{Krull}}(\mathcal{P}) = -n \ .$$

A dimension function determines a filtration of $\mathcal{T}$. We have a chain of $\otimes$-ideals

$$\mathcal{T}_{(-\infty)} \subset \cdots \subset \mathcal{T}_{(p)} \subset \mathcal{T}_{(p+1)} \subset \cdots \subset \mathcal{T}_{(\infty)} = \mathcal{T} \ .$$

The sub-quotients of this filtration have a local description which we will describe next. First we will introduce another useful property of tensor triangulated categories.

1.4.4. DEFINITION (see [Bal10b, Definition 20]). A tensor triangulated category $\mathcal{T}$ is called *rigid* if there is an exact functor $D : \mathcal{T}^{\mathrm{op}} \to \mathcal{T}$ and a natural isomorphism $\operatorname{Hom}_{\mathcal{T}}(a \otimes b, c) \cong \operatorname{Hom}_{\mathcal{T}}(b, D(a) \otimes c)$ for all objects $a, b, c \in \mathcal{T}$. The object $D(a)$ is called the *dual* of a.

1.4.5. REMARK. From the natural isomorphism

$$\operatorname{Hom}_{\mathcal{T}}(a \otimes b, c) \cong \operatorname{Hom}_{\mathcal{T}}(b, D(a) \otimes c)$$

of Definition 1.4.4, it follows that $a \otimes -$ and $D(a) \otimes -$ form an adjoint pair of functors for all objects $a \in \mathcal{T}$.

When $\mathcal{T}$ is rigid, some useful consequences hold true.

1.4.6. LEMMA (see [Bal07, Corollary 2.5 and Corollary 2.8]). *Let $\mathcal{T}$ be a rigid tensor triangulated category. Then*

(1) $\operatorname{supp}(a) = \emptyset \Leftrightarrow a = 0$ *for all objects* $a \in \mathcal{T}$.
(2) $\operatorname{supp}(a) \cap \operatorname{supp}(b) = \emptyset \Rightarrow \operatorname{Hom}_{\mathcal{T}}(a, b) = 0$ *for all objects* $a \in \mathcal{T}$.

SKETCH OF THE PROOF. In order to prove (1), one first shows that $\operatorname{supp}(a) = \emptyset \Leftrightarrow a^{\otimes n} = 0$ for some $n \geq 1$ (see [Bal05, Corollary 2.4]). The point is then that thick $\otimes$-ideals $\mathcal{J}$ in rigid tensor triangulated categories are always *radical* (see [Bal07, Proposition 2.4]), meaning that $x^{\otimes n} \in \mathcal{J}$ implies $x \in \mathcal{J}$ for all $x \in \mathcal{T}$. Let us prove this statement: it suffices to show that $x \otimes x \in \mathcal{J} \Rightarrow x \in \mathcal{J}$ since $\mathcal{J}$ is a $\otimes$-ideal and we can therefore assume that if $x^{\otimes n} \in \mathcal{J}$, then n is a power of 2.

Next, we use the unit-counit relation of the adjunction from Remark 1.4.5 to obtain two natural transformations

$$(x \otimes -) \to (x \otimes D(x) \otimes x \otimes -) \to (x \otimes -)$$

whose composition is the identity. Applying the functors to $\mathbb{I}$, we obtain two maps

$$x \to x \otimes D(x) \otimes x \to x$$

that compose to the identity on x, and we therefore conclude that x is a direct summand of $x \otimes D(x) \otimes x$. If $\mathcal{J}$ contains $x \otimes x$, it will also contain $x \otimes D(x) \otimes x$ as it is a $\otimes$-ideal. But $\mathcal{J}$ is thick, so it is closed under taking direct summands, hence x is contained in $\mathcal{J}$.

Thus, if $\operatorname{supp}(a) = \emptyset$, there is an $n \geq 1$ such that $a^{\otimes n} = 0$. But $\{0\}$ is a thick $\otimes$-ideal, so it follows that $a \in \{0\}$, i.e. $a = 0$.

Let us also indicate how (2) follows from (1). By [Bal07, Proposition 2.6] we have for an object $x \in \mathcal{T}$ and $\mathcal{P} \in \operatorname{Spc}(\mathcal{T})$ that $x \in \mathcal{P} \Leftrightarrow D(x) \in \mathcal{P}$ from which it follows that $\operatorname{supp}(x) = \operatorname{supp}(D(x))$. But now,

$$\operatorname{Hom}_{\mathcal{T}}(a,b) = \operatorname{Hom}_{\mathcal{T}}(a \otimes \mathbb{I}, b) = \operatorname{Hom}_{\mathcal{T}}(\mathbb{I}, D(a) \otimes b)$$

and we have $\operatorname{supp}(D(a) \otimes b) = \operatorname{supp}(a) \cap \operatorname{supp}(b) = \emptyset$ by assumption. But by (1) we therefore must have $D(a) \otimes b = 0$ which implies that

$$\operatorname{Hom}_{\mathcal{T}}(a,b) = \operatorname{Hom}_{\mathcal{T}}(\mathbb{I}, 0) = 0 \ .$$

□

We now fix a dimension function on a rigid tensor triangulated category $\mathcal{T}$ and look at the sub-quotients of the induced filtration. They have a local description.

1.4.7. THEOREM (see [Bal07, Theorem 3.24]). *Let $\mathcal{T}$ be a rigid tensor triangulated category equipped with a dimension function* dim *such that* $\operatorname{Spc}(\mathcal{T})$ *is a noetherian topological space. Then, for all $p \in \mathbb{Z}$, there is an exact equivalence*

$$\left(\mathcal{T}_{(p)}/\mathcal{T}_{(p-1)}\right)^{\natural} \to \coprod_{\substack{P \in \operatorname{Spc}(\mathcal{T}) \\ \dim(P) = p}} \operatorname{Min}(\mathcal{T}_P) \ .$$

where $\mathcal{T}_P := (\mathcal{T}/P)^{\natural}$ and $\operatorname{Min}(\mathcal{T}_P)$ denotes the full triangulated subcategory of objects with support the unique closed point of $\mathcal{T}_P$ (see Example 1.3.6).

1.4.8. REMARK. The exact equivalence of Theorem 1.4.7 is induced by the functor.

$$\begin{aligned} \mathcal{T}_{(p)}/\mathcal{T}_{(p-1)} &\to \coprod_{\substack{\mathcal{P} \in \operatorname{Spc}(\mathcal{T}) \\ \dim(\mathcal{P}) = p}} \operatorname{Min}(\mathcal{T}/\mathcal{P}) \\ a &\mapsto (Q_{\mathcal{P}}(a)) \end{aligned}$$

where $Q_{\mathcal{P}}$ is the localization functor $\mathcal{T} \to \mathcal{T}/\mathcal{P}$. It is shown in [Bal07] that the image of this functor is dense, so it induces an equivalence after idempotent completion on both sides.

Let us finish the section with the observation that we can restrict a dimension function to the tensor triangulated category associated to an open subset of the spectrum. Let $U \subset \mathcal{T}$ be a quasi-compact open subset with closed complement Z and denote by $\mathcal{T}_Z \subset \mathcal{T}$ the $\otimes$-ideal of objects with support contained in Z. Set

$$\mathcal{T}_U := (\mathcal{T}/\mathcal{T}_Z)^{\natural} \ ,$$

then by Theorem 1.3.5 and Theorem 1.3.11, the spectrum $\operatorname{Spc}(\mathcal{T}_U)$ is homeomorphic to U. The homeomorphism is induced by the functor Res_U given as composition of the Verdier quotient functor $\mathcal{T} \to \mathcal{T}/\mathcal{T}_Z$ and the inclusion functor into the idempotent completion $\mathcal{T}/\mathcal{T}_Z \to (\mathcal{T}/\mathcal{T}_Z)^{\natural}$.

1.4.9. PROPOSITION. *Let* dim *be a dimension function on $\mathcal{T}$. Then*

$$\begin{aligned} \dim|_U :\ \operatorname{Spc}(\mathcal{T}_U) &\to \mathbb{Z} \cup \{\pm\infty\} \\ \mathcal{P} &\mapsto \dim(\operatorname{Spc}(\operatorname{Res}_U)(\mathcal{P})) \end{aligned}$$

is a dimension function on $\mathcal{T}_U$. Furthermore, the restriction of the functor Res_U *to the subcategory $\mathcal{T}_{(p)}$ factors through the subcategory $(\mathcal{T}_U)_{(p)}$ for all $p \in \mathbb{Z} \cup \{\pm\infty\}$, when we equip $\mathcal{T}_U$ with* $\dim|_U$.

PROOF. As $\mathrm{Spc}(\mathrm{Res}_U)$ constitutes an inclusion-preserving bijection between the sets $\{\mathcal{P} \in \mathrm{Spc}(\mathcal{T}_U)\}$ and $\{\mathcal{P} \in U\}$, it follows immediately from the definition of a dimension function that $\dim|_U$ is one. In order to check the second claim, let a be an object of $\mathcal{T}_{(p)}$. Then, by Theorem 1.2.21, we have

$$\mathrm{supp}(\mathrm{Res}_U(a)) = \mathrm{Spc}(\mathrm{Res}_U)^{-1}(\mathrm{supp}(a)) = \mathrm{supp}(a) \cap U \ .$$

Thus,

$$\begin{aligned} \dim|_U(\mathrm{supp}(\mathrm{Res}_U(a))) &= \sup_{\mathcal{P} \in \mathrm{supp}(a) \cap U} \dim(\mathcal{P}) \\ &\leq \sup_{\mathcal{P} \in \mathrm{supp}(a)} \dim(\mathcal{P}) \\ &= \dim(\mathrm{supp}(a)) \leq p \ , \end{aligned}$$

from which it follows that $\mathrm{Res}_U(a)$ is an object of $(\mathcal{T}_U)_{(p)}$. □

CHAPTER 2

Chow groups of tensor triangulated categories

In this chapter we introduce the central object that is studied in this thesis, the Chow groups of an essentially small tensor triangulated category. We recall a definition due to P. Balmer (see [Bal13]), give a proof that it generalizes the classical Chow groups from algebraic geometry and investigate its functoriality properties.

2.1. Chow groups in algebraic geometry

We aim at generalizing the study of cycles on an algebraic variety modulo rational equivalence. The basic setup of this theory is as follows: for an algebraic variety X (by which we shall mean a separated scheme of finite type over a field) one looks for each $p \geq 0$ at the *codimension p cycle group* $\mathrm{Z}^p(X)$, the free abelian group on subvarieties (= closed integral subschemes) of codimension p in X. One now introduces an equivalence relation on $\mathrm{Z}^p(X)$. Two cycles in $\mathrm{Z}^p(X)$ are considered *rationally equivalent* if there exists a finite number of subvarieties $Y_i \subset X$ of codimension $p-1$ and elements of the function fields $f_i \in \mathrm{K}(Y_i)$ such that the difference of the two cycles is equal to the sum of the cycles $\mathrm{div}(f_i)$, the divisors associated to the functions f_i. The divisors $\mathrm{div}(f_i)$ should be thought of as the sum of the zeroes of f_i minus the sum of the poles of f_i, both counted with multiplicities (see [Ful98] for the formal definition). The cycles rationally equivalent to zero form a subgroup of $\mathrm{Z}^p(X)$ and the corresponding quotient is $\mathrm{CH}^p(X)$, the *codimension p Chow group of X*.

2.2. Definitions and conventions

Let us state some basic assumptions that we will use for the rest of the chapter.

2.2.1. CONVENTION. For the rest of the chapter, the term *tensor triangulated category* will mean a category as defined in Definition 1.2.1, with the additional assumption that the category is essentially small.

2.2.2. REMARK. We need to assume that our tensor triangulated categories are essentially small in order to be able to talk about their spectrum (see Remark 1.2.10). We will temporarily drop the assumption in Chapter 4.

We can now give a definition of tensor triangular cycle groups and Chow groups, following the ideas from [Bal13].

2.2.3. DEFINITION. Let $\mathcal{K}$ be a tensor triangulated category as in Convention 2.2.1, equipped with a dimension function. For $p \in \mathbb{Z}$ we define the *p-dimensional cycle group of $\mathcal{K}$* as

$$\mathrm{Z}^{\Delta}_p(\mathcal{K}) := \mathrm{K}_0\left((\mathcal{K}_{(p)}/\mathcal{K}_{(p-1)})^{\natural}\right) ,$$

where $\mathrm{K}_0\left((\mathcal{K}_{(p)}/\mathcal{K}_{(p-1)})^\natural\right)$ is the Grothendieck group (see Definition 1.1.26) of the Verdier quotient $(\mathcal{K}_{(p)}/\mathcal{K}_{(p-1)})^\natural$ and $\mathcal{K}_{(l)} \subset \mathcal{K}$ denotes the full triangulated subcategory of objects with dimension of support $\leq l$ (see Definition 1.4.1), for $l = p, p-1$.

We also need a generalized notion of rational equivalence, which we describe next. Look at the following diagram of subcategories and sub-quotients of $\mathcal{K}$

$$\begin{tikzcd}
\mathcal{K}_{(p)} \arrow[r, hook, "I"] \arrow[d, two heads, "Q"] & \mathcal{K}_{(p+1)} \\
\mathcal{K}_{(p)}/\mathcal{K}_{(p-1)} \arrow[r, hook, "J"] & (\mathcal{K}_{(p)}/\mathcal{K}_{(p-1)})^\natural
\end{tikzcd}$$

where I, J denote the obvious embeddings and Q is the Verdier quotient functor (see Definition 1.1.11). After applying K_0 we get a diagram

$$\begin{tikzcd}
\mathrm{K}_0(\mathcal{K}_{(p)}) \arrow[r, "i"] \arrow[d, two heads, "q"] & \mathrm{K}_0(\mathcal{K}_{(p+1)}) \\
\mathrm{K}_0(\mathcal{K}_{(p)}/\mathcal{K}_{(p-1)}) \arrow[r, hook, "j"] & \mathrm{K}_0\left((\mathcal{K}_{(p)}/\mathcal{K}_{(p-1)})^\natural\right) = \mathrm{Z}^\Delta_p(\mathcal{K})
\end{tikzcd}$$

where the lowercase maps are induced by the uppercase functors (see Proposition 1.1.27).

2.2.4. DEFINITION. Let $\mathcal{K}$ be a tensor triangulated category as in Convention 2.2.1, equipped with a dimension function. For $p \in \mathbb{Z}$ we define the *p-dimensional Chow group of $\mathcal{K}$* as

$$\mathrm{CH}^\Delta_p(\mathcal{K}) := \mathrm{Z}^\Delta_p(\mathcal{K})/j \circ q(\ker(i)).$$

2.2.5. REMARK. It may not be immediately obvious to the reader how the above Definitions are motivated. The following account might remedy the situation for Z^Δ_p: assume that $\mathcal{K}$ is a tensor triangulated category in the sense of Convention 2.2.1 that is rigid, equippped with a dimension function and such that $\mathrm{Spc}(\mathcal{K})$ is a noetherian topological space. By Theorem 1.4.7 the quotient functors $Q_P : \mathcal{K} \to \mathcal{K}/P$ for $P \in \mathrm{Spc}(\mathcal{K})$ induce an exact equivalence

$$(\mathcal{K}_{(p)}/\mathcal{K}_{(p-1)})^\natural \xrightarrow{\sim} \coprod_{P \in \mathrm{Spc}(\mathcal{K})_p} \mathrm{Min}(\mathcal{K}_P) \tag{5}$$

where $\mathrm{Spc}(\mathcal{K})_p$ denotes the set of points P in $\mathrm{Spc}(\mathcal{K})_p$ that have dimension p (Definition 1.4.1) and where $\mathcal{K}_P$ is the local category $(\mathcal{K}/P)^\natural$. The subcategory $\mathrm{Min}(\mathcal{K}_P) \subset \mathcal{K}_P$ is the full subcategory of objects that are supported on the unique closed point of $\mathrm{Spc}(\mathcal{K}_P)$ (see [Bal07], where the subcategory $\mathrm{Min}(\mathcal{K}_P)$ is denoted by $\mathrm{FL}(\mathcal{K}_P)$). The decomposition (5) is in the main reason why we idempotent-complete the Verdier quotient $\mathcal{K}_{(p)}/\mathcal{K}_{(p-1)}$. In analogy with the theory of algebraic cycles, an element of the p-dimensional tensor triangular cycle group of $\mathcal{K}$

$$\mathrm{Z}^\Delta_p(\mathcal{K}) = \mathrm{K}_0\left((\mathcal{K}_{(p)}/\mathcal{K}_{(p-1)})^\natural\right) \cong \coprod_{P \in \mathrm{Spc}(\mathcal{K})_p} \mathrm{K}_0\left(\mathrm{Min}(\mathcal{K}_P)\right)$$

can thus be regarded as a sum of p-dimensional (relative to the dimension function) irreducible closed subsets of $\mathrm{Spc}(\mathcal{K})$ with coefficients in $\mathrm{K}_0\,(\mathrm{Min}(\mathcal{K}_P))$.

In the case that $\mathcal{K} = \mathrm{D}^{\mathrm{perf}}(X)$ for X a non-singular noetherian scheme, we show that the Grothendieck group $\mathrm{K}_0\,(\mathrm{Min}(\mathcal{K}_P))$ group is isomorphic to $\mathbb{Z}$. This will allows us to conclude that Definition 2.2.3 recovers the usual cycle groups of X for $\mathcal{K} = \mathrm{D}^{\mathrm{perf}}(X)$ (see Corollary 2.2.10). Let us first recall the following two auxilliary lemmas.

2.2.6. LEMMA. *Let R be a commutative local noetherian ring with maximal ideal $\mathfrak{m}$ and M be a finitely generated R-module. Then*

$$\mathrm{supp}(M) = \{\mathfrak{m}\} \Leftrightarrow \mathrm{length}(M) < \infty$$

PROOF. Let M be a module over R of length $n < \infty$. We will proceed by induction on n to show that $\mathrm{supp}(M) = \{\mathfrak{m}\}$. For $n = 1$, M is simple and therefore isomorphic to the residue field $R/\mathfrak{m}$ which has support $\{\mathfrak{m}\}$. Indeed, pick any non-zero element $s \in M$, then the image of the map

$$\begin{aligned} m_s : R &\to M \\ r &\mapsto r \cdot s \end{aligned}$$

must be M, so $M \cong R/\ker(m_s)$. But as M was simple, the only possible choice for $\ker(m_s)$ is $\mathfrak{m}$ (otherwise $R/\mathfrak{m}$ would be a proper submodule), proving that $M \cong R/\mathfrak{m}$. Assume now we have proved the statement for all $n \leq n_0$ and let M be a module of length $n_0 + 1$. There is a composition series

$$M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_{n_0} \subset M_{n_0+1} = M$$

with simple subquotients. In particular we have an exact sequence

$$0 \to M_{n_0} \to M_{n_0+1} \to M_{n_0+1}/M_{n_0} \to 0$$

where we know that $M_{n_0+1}/M_{n_0} \cong R/\mathfrak{m}$. Thus

$$\mathrm{supp}(M_{n_0+1}) = \mathrm{supp}(M_{n_0}) \cup \mathrm{supp}(R/\mathfrak{m}) = \{\mathfrak{m}\}\,.$$

For the converse direction, let $\mathrm{supp}(M) = \{\mathfrak{m}\}$. Thus, M is annihilated by $\mathfrak{m}^n$ for some $n \geq 0$, and therefore we obtain a sequence

$$M \supset \mathfrak{m}M \supset \mathfrak{m}^2 M \supset \cdots \supset \mathfrak{m}^{n-1}M \supset \mathfrak{m}^n M = 0\,.$$

For every $i \geq 0$, the module $\mathfrak{m}^i M/\mathfrak{m}^{i+1}M$ is a finite-dimensional $R/\mathfrak{m}$-vector space by Nakayama's Lemma and it therefore has finite length. An induction on n then shows that M must have finite length as well. $\square$

2.2.7. LEMMA. *Let R be a commutative local ring with maximal ideal $\mathfrak{m}$ and denote by R–fl the abelian category of finite length R-modules. Then the map*

$$\begin{aligned} R\text{–fl} &\to \mathbb{Z} \\ M &\mapsto \mathrm{length}(M) \end{aligned}$$

induces an isomoprhism

$$\mathrm{K}_0(R\text{–fl}) \cong \mathbb{Z}\,.$$

PROOF. The length function

$$\begin{aligned} R\text{–fl} &\to \mathbb{Z} \\ M &\mapsto \operatorname{length}(M) \end{aligned}$$

has the property that $\operatorname{length}(M) = \operatorname{length}(M') + \operatorname{length}(M'')$ if there is an exact sequence

$$0 \to M' \to M \to M'' \to 0$$

and therefore induces a group homomorphism

$$\mathrm{K}_0\,(R\text{–fl}) \to \mathbb{Z}\ .$$

This is an isomorphism, as for any module M over $\mathcal{O}_{X,\rho(P)}$ of length n, there is an equality

$$[M] = n \cdot [R/\mathfrak{m}]$$

in $\mathrm{K}_0\,(R\text{–fl})$. Let us prove this by induction: for $n = 1$, this is true as we saw in the proof of Lemma 2.2.6 that any simple R-module must be isomorphic to $R/\mathfrak{m}$.

Assume now we have proved the statement for all $n \le n_0$ and let T be a module of length $n_0 + 1$. There is a composition series

$$T_0 \subset T_1 \subset T_2 \subset \cdots \subset T_{n_0} \subset T_{n_0+1} = T$$

with simple subquotients. In particular we have an exact sequence

$$0 \to T_{n_0} \to T \to T/T_{n_0} \to 0$$

where we know that $T/T_{n_0} \cong R/\mathfrak{m}$. We therefore get the equality

$$[T] = n_0 \cdot [R/\mathfrak{m}] + [R/\mathfrak{m}] = (n_0+1)\cdot[R/\mathfrak{m}]$$

in $\mathrm{K}_0\,(R\text{–fl})$ which proves the statement. □

2.2.8. THEOREM. *Let $\mathcal{K} = \mathrm{D}^{\mathrm{perf}}(X)$ for X a non-singular noetherian scheme. Then*

$$\mathrm{K}_0\,(\mathrm{Min}(\mathcal{K}_P)) \cong \mathbb{Z}$$

for all $P \in \mathrm{Spc}(\mathcal{K}) \cong X$ (see Example 1.2.11).

PROOF. Let ρ denote the isomorphism $\mathrm{Spc}(\mathrm{D}^{\mathrm{perf}}(X)) \to X$. By [Bal07, Chapter 4, Section 1], the category $\mathrm{Min}(\mathcal{K}_P)$ is equivalent to

$$\mathrm{K}^{\mathrm{b}}_{\text{fin.lg.}}(\mathcal{O}_{X,\rho(P)}\text{–free})\ ,$$

the bounded homotopy category of complexes of free $\mathcal{O}_{X,\rho(P)}$-modules of finite-rank with finite length homology. As $\mathcal{O}_{X,\rho(P)}$ is regular by assumption, every bounded complex of finitely generated $\mathcal{O}_{X,\rho(P)}$-modules is quasi-isomorphic to a bounded complex of free $\mathcal{O}_{X,\rho(P)}$-modules of finite rank. Therefore, if $\mathrm{D^b}_{\text{fin.lg.}}(\mathcal{O}_{X,\rho(P)}\text{-mod})$ denotes the bounded derived category of complexes of finitely generated $\mathcal{O}_{X,\rho(P)}$-modules with finite length homology, the natural functor

$$\mathrm{K}^{\mathrm{b}}_{\text{fin.lg.}}(\mathcal{O}_{X,\rho(P)}\text{–free}) \to \mathrm{D^b}_{\text{fin.lg.}}(\mathcal{O}_{X,\rho(P)}\text{-mod})$$

gives rise to an equivalence of categories

$$\mathrm{K}^{\mathrm{b}}_{\text{fin.lg.}}(\mathcal{O}_{X,\rho(P)}\text{–free}) \cong \mathrm{D^b}_{\text{fin.lg.}}(\mathcal{O}_{X,\rho(P)}\text{-mod}).$$

The latter category is in turn equivalent to $\mathrm{D^b}(\mathcal{O}_{X,\rho(P)}\text{–fl})$, the bounded derived category of finite length modules over $\mathcal{O}_{X,\rho(P)}$. Indeed, by Lemma 2.2.6, for a finitely generated module M over $\mathcal{O}_{X,\rho(P)}$, having finite length is the same as being supported on the unique closed point P_0 of $\mathrm{Spec}(\mathcal{O}_{X,\rho(P)})$. The result then follows by [Kel99, Section 1.15, Example b)], where it is shown that for a commutative noetherian ring R, and $\mathcal{A} \subset R$-mod the full abelian subcategory of finitely generated R-modules supported on a closed subscheme Z of $\mathrm{Spec}(R)$, there is an equivalence of categories

$$\mathrm{D^b}(\mathcal{A}) \cong \mathrm{D^b_{\mathcal{A}}}(R\text{-mod}) ,$$

where the latter expression denotes the full subcategory of $\mathrm{D^b}(R\text{-mod})$ consisting of complexes with homology in $\mathcal{A}$.

Summarizing, we have

$$\mathrm{K_0}\left(\mathrm{Min}(\mathcal{K}_P)\right) \cong \mathrm{K_0}\left(\mathrm{D^b}(\mathcal{O}_{X,\rho(P)}\text{–fl})\right) \cong \mathrm{K_0}\left(\mathcal{O}_{X,\rho(P)}\text{–fl}\right) \cong \mathbb{Z} \tag{6}$$

where the penultimate isomorphism is the one from Example 1.1.29 and the last isomorphism is induced by the length function as in Lemma 2.2.7. □

2.2.9. REMARK. We can make the isomorphism $\mathrm{K_0}\left(\mathrm{Min}(\mathcal{K}_P)\right) \cong \mathbb{Z}$ from Theorem 2.2.8 explicit if we identify $\mathrm{K_0}\left(\mathrm{Min}(\mathcal{K}_P)\right)$ with $\mathrm{K^b_{fin.lg.}}(\mathcal{O}_{X,\rho(P)}\text{–free})$ as in the proof of Theorem 2.2.8. We compose the formula from Example 1.1.29 and the length function as in (6) to obtain the following: if $A_\bullet$ is a chain complex in $\mathrm{K^b_{fin.lg.}}(\mathcal{O}_{X,\rho(P)}\text{–free})$, then the map is given as

$$\theta_{\rho(P)} : [A_\bullet] \mapsto \sum_i (-1)^i \,\mathrm{length}(\mathrm{H}^i(A_\bullet)) .$$

2.2.10. COROLLARY. *Let X be a non-singular noetherian scheme. Let $\mathcal{K} = \mathrm{D^{perf}}(X)$ be equipped with the opposite of the Krull codimension as a dimension function (see Example 1.4.3). Then the map*

$$\theta_{-p} := \coprod_{\substack{\rho(P)\\ \dim(\rho(P))=-p}} \theta_{\rho(P)}$$

with $\theta_{\rho(P)}$ as in Remark 2.2.9 induces an isomorphism

$$\mathrm{Z}^{\Delta}_{-p}(\mathcal{K}) \cong \mathrm{Z}^p(X)$$

for all $p \geq 0$.

PROOF. Let $p \geq 0$. Using Remark 2.2.5 and the isomorphism $\mathrm{Spc}(\mathcal{K}) \cong X$ we have a chain of isomorphisms

$$\begin{aligned}
\mathrm{Z}^{\Delta}_{-p}(\mathcal{K}) &\cong \coprod_{P\in \mathrm{Spc}(\mathcal{K})_{-p}} \mathrm{K_0}\left(\mathrm{Min}(\mathcal{K}_P)\right) \\
&\cong \coprod_{\rho(P)\in X^{(p)}} \mathrm{K_0}\left(\mathrm{K^b_{fin.lg.}}(\mathcal{O}_{X,\rho(P)}\text{–free})\right) \\
&\cong \coprod_{P\in X^{(p)}} \mathbb{Z} \\
&\cong \mathrm{Z}^p(X) ,
\end{aligned}$$

where the penultimate map is given by θ_{-p}. □

2.3. Agreement with algebraic geometry

We want to show now that the tensor triangular Chow groups carry their name for a reason. As we will see, they are — at least in the non-singular case — an honest generalization of the classical Chow groups from algebraic geometry.

2.3.1. CONVENTION. We now fix some notation for the rest of the section: if not explicitly stated otherwise, let X denote a separated, non-singular scheme of finite type over a field k, and $\mathrm{D}^{\mathrm{perf}}(X)$ be the derived category of perfect complexes of $\mathcal{O}_X$-modules, which is equivalent to $\mathrm{D}^{\mathrm{b}}(X)$, the bounded derived category of coherent sheaves on X. We will also assume that $\mathrm{D}^{\mathrm{perf}}(X)$ is equipped with $-\operatorname{codim}_{\mathrm{Krull}}$ as a dimension function.

In order to proceed, it is necessary to use some higher algebraic K-theory as developed by Quillen. We recall the following material from [Qui73, §7]: consider the abelian category $\mathrm{Coh}(X)$ of coherent sheaves on X. There is a filtration of this category by codimension of support:

$$\cdots \subset M^i \subset M^{i-1} \subset \cdots \subset M^0 = \mathrm{Coh}(X)$$

where M^p denotes the subcategory of coherent sheaves whose codimension of support is $\geq p$. The subcategory $M^p \subset M^{p+1}$ is a *Serre subcategory*, i.e. a full subcategory such that if $0 \to A \to B \to C \to 0$ is an exact sequence in M^{p+1}, then A, C are objects of M^p if and only if B is one. This property allows us to define the quotient abelian category M^{p+1}/M^p and thus, for every p, there is an exact sequence of abelian categories

$$M^{p+1} \hookrightarrow M^p \twoheadrightarrow M^p/M^{p+1}$$

which induces a long exact localisation sequence of K-groups

(7)
$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{K}_j(M^{p+1}) \arrow[r, "i_j^p"] & \mathrm{K}_j(M^p) \arrow[r, "q_j^p"] & \mathrm{K}_j(M^p/M^{p+1}) \arrow[dlll, rounded corners, "b_j^p"'] & \\
\mathrm{K}_{j-1}(M^{p+1}) \arrow[r, "i_{j-1}^p"] & \mathrm{K}_{j-1}(M^p) \arrow[r, "q_{j-1}^p"] & \mathrm{K}_{j-1}(M^p/M^{p+1}) \arrow[r] & \cdots
\end{tikzcd}$$

Combining these long exact sequences for all p, we can form the associated exact couple and obtain the Quillen coniveau spectral sequence as in [Qui73, §7, Theorem 5.4] with E_1-page

$$E_1^{p,q} = \mathrm{K}_{-p-q}(M^p/M^{p+1}).$$

We are especially interested in the boundary map

(8)
$$d_1 : \mathrm{K}_1(M^{s-1}/M^s) \xrightarrow{b_1^{s-1}} \mathrm{K}_0(M^s) \xrightarrow{q_0^s} \mathrm{K}_0(M^s/M^{s+1})$$

of this spectral sequence. Using that

(9)
$$\mathrm{K}_i(M^s/M^{s+1}) \cong \coprod_{x \in X^{(s)}} \mathrm{K}_i(k(x)) \,,$$

where $X^{(s)}$ denotes the set of points of X whose closure has codimension s in X, Quillen proves the following:

2.3.2. THEOREM (cf. [Qui73, §7, Proposition 5.14]). *The image of*

$$d_1 : \mathrm{K}_1(M^{s-1}/M^s) \longrightarrow \mathrm{K}_0(M^s/M^{s+1}) \cong \coprod_{x\in X^{(s)}} \mathrm{K}_0(k(x)) \cong \coprod_{x\in X^{(s)}} \mathbb{Z} = \mathrm{Z}^s(X)$$

is the subgroup of codimension-p cycles rationally equivalent to zero. In other words, we have $\operatorname{coker}(d_1) \cong \mathrm{CH}^s(X)$.

In our setting, we work with the triangulated category $\mathrm{D}^{\mathrm{perf}}(X) \cong \mathrm{D}^{\mathrm{b}}(X)$ instead of the abelian category $\mathrm{Coh}(X)$. Recall that the defining diagram for the tensor triangular Chow groups in this case is given as follows:

$$\begin{tikzcd}
\mathrm{K}_0(\mathrm{D}^{\mathrm{b}}(X)_{(p)}) \arrow[r, "i"] \arrow[d, two heads, "q"] & \mathrm{K}_0(\mathrm{D}^{\mathrm{b}}(X)_{(p+1)}) \\
\mathrm{K}_0(\mathrm{D}^{\mathrm{b}}(X)_{(p)}/\mathrm{D}^{\mathrm{b}}(X)_{(p-1)}) \arrow[d, hook, "j"] & \\
\underbrace{\mathrm{K}_0\left((\mathrm{D}^{\mathrm{b}}(X)_{(p)}/\mathrm{D}^{\mathrm{b}}(X)_{(p-1)})^{\natural}\right)}_{=\mathrm{Z}^{\Delta}_p(\mathcal{K})} &
\end{tikzcd}$$

This diagram maps to a similar one involving the related abelian categories:

$$\begin{tikzcd}
\mathrm{K}_0\left(\mathrm{D}^{\mathrm{b}}(X)_{(p)}\right) \arrow[rr, "i"] \arrow[d, two heads, "q"] \arrow[drr] & & \mathrm{K}_0\left(\mathrm{D}^{\mathrm{b}}(X)_{(p+1)}\right) \arrow[dr] & \\
\mathrm{K}_0\left(\mathrm{D}^{\mathrm{b}}(X)_{(p)}/\mathrm{D}^{\mathrm{b}}(X)_{(p-1)}\right) \arrow[d, hook, "j"] \arrow[drr] & & \mathrm{K}_0\left(M^{-p}\right) \arrow[r, "i_0"] \arrow[d, "q_0"] & \mathrm{K}_0\left(M^{-p-1}\right) \\
\underbrace{\mathrm{K}_0\left((\mathrm{D}^{\mathrm{b}}(X)_{(p)}/\mathrm{D}^{\mathrm{b}}(X)_{(p-1)})^{\natural}\right)}_{=\mathrm{Z}^{\Delta}_p(\mathcal{K})} & & \mathrm{K}_0\left(M^{-p}/M^{-p+1}\right) &
\end{tikzcd} \tag{10}$$

The diagonal homomorphisms are all given by the formula

$$[C^\bullet] \mapsto \sum_i (-1)^i [H^i(C^\bullet)]\ . \tag{11}$$

We proceed to show that these are actually all isomorphisms, which follows from the fact that there are exact equivalences

$$\mathrm{D}^{\mathrm{b}}(X)_{(q)} \cong \mathrm{D}^{\mathrm{b}}(M^{-q}) \tag{12}$$

and

$$\mathrm{D}^{\mathrm{b}}(X)_{(q)}/\mathrm{D}^{\mathrm{b}}(X)_{(q-1)} \cong \mathrm{D}^{\mathrm{b}}(M^{-q}/M^{-q+1}) \tag{13}$$

for all $q \in \mathbb{Z}$. Indeed, the diagonal maps are then just the usual isomorphisms between $\mathrm{K}_0(\mathrm{D}^{\mathrm{b}}(\mathcal{A}))$ and $\mathrm{K}_0(\mathcal{A})$ for some abelian category $\mathcal{A}$, as in Example 1.1.29. This also proves that j is the identity morphism, as the derived category of an abelian category is idempotent complete [BS01, Corollary 2.10].

The proof of the equivalences (12) and (13) is a consequence of the following theorem:

2.3.3. THEOREM (see [Kel99, Section 1.15]). *Let $\mathcal{B}$ be an abelian category and $\mathcal{A} \subset \mathcal{B}$ a Serre subcategory with quotient $\mathcal{B}/\mathcal{A}$. Assume that the following criterion holds: for each exact sequence*

$$0 \to A \to B \to C \to 0$$

in $\mathcal{B}$ with $A \in \mathcal{A}$, there is a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \\ 0 & \longrightarrow & A & \longrightarrow & A' & \longrightarrow & A'' & \longrightarrow & 0 \end{array}$$

such that A', A'' are objects of $\mathcal{A}$.

Then, there is an exact equivalence of triangulated categories induced by the inclusion

$$\mathrm{D}^{\mathrm{b}}(\mathcal{A}) \to \mathrm{D}^{\mathrm{b}}_{\mathcal{A}}(\mathcal{B}) ,$$

where $\mathrm{D}^{\mathrm{b}}_{\mathcal{A}}(\mathcal{B}) \subset \mathrm{D}^{\mathrm{b}}(\mathcal{B})$ denotes the full subcategory of complexes with homology in $\mathcal{A}$. Furthermore, in the induced sequence of triangulated categories

$$\mathrm{D}^{\mathrm{b}}(\mathcal{A}) \xrightarrow{i} \mathrm{D}^{\mathrm{b}}(\mathcal{B}) \xrightarrow{q} \mathrm{D}^{\mathrm{b}}(\mathcal{B}/\mathcal{A}) ,$$

the functor i is fully faithful and $\mathrm{D}^{\mathrm{b}}(\mathcal{B}/\mathcal{A}) \cong \mathrm{D}^{\mathrm{b}}(\mathcal{B})/\mathrm{D}^{\mathrm{b}}(\mathcal{A})$ (i.e. the sequence is exact*).*

Let us verify that the conditions for Theorem 2.3.3 are satisfied in our case.

2.3.4. LEMMA. *Let $0 \to A \to B \to C \to 0$ be an exact sequence in $\mathrm{Coh}(X)$. Then there exist coherent sheaves A', A'' on X with $\mathrm{supp}(A'), \mathrm{supp}(A'') \subset \mathrm{supp}(A)$ that fit into a commutative diagram with exact rows*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \\ 0 & \longrightarrow & A & \longrightarrow & A' & \longrightarrow & A'' & \longrightarrow & 0 \end{array}$$

PROOF. Suppose that A is supported on a closed subscheme with associated ideal sheaf I. As X is noetherian, we can use the sheaf-theoretic version of the Artin-Rees lemma (cf. [Sta14, Lemma 29.10.3]) which says that there exists a $c > 0$ such that for all $n > c$ we have $I^n B \cap A = I^{n-c}(I^c B \cap A)$. Now take some n_0 such that $I^{n_0-c} A = 0$, then we get the diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "\mathrm{id}"] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] & B/(I^{n_0}B) \arrow[r] & C/(I^{n_0}C) \arrow[r] & 0
\end{tikzcd}$$

where the vertical arrows are given by the canonical projections. It is easy to see that the diagram commutes and that all sheaves in the lower row have their support contained in $\mathrm{supp}(A)$. □

Since we have checked the conditions of Theorem 2.3.3, its first statement tells us that the equivalence (12) holds, because we can write $\mathrm{D^b}(X)_{(q)}$ as $\mathrm{D}^{\mathrm{b}}_{M^{-q}}(\mathrm{Coh}(X))$. The equivalence (13) holds by the second statement of Theorem 2.3.3, which says that

$$\mathrm{D^b}(M^{-q}/M^{-q+1}) \cong \mathrm{D^b}(M^{-q})/\mathrm{D^b}(M^{-q+1}) \ ,$$

where the latter expression is equivalent to $\mathrm{D^b}(X)_{(q)}/\mathrm{D^b}(X)_{(q-1)}$ by the first part of Theorem 2.3.3.

As we know that the diagonal maps in diagram (10) are isomorphisms and that j is the identity morphism we see that

$$j \circ q(\ker(i)) \cong q_0(\ker(i_0)) = q_0(\mathrm{im}(b_0)) = \mathrm{im}(d_1)$$

(see Theorem 2.3.2). We have thus proved the following:

2.3.5. THEOREM. *Let X be a separated, non-singular scheme of finite type over a field and assume that the tensor triangulated category* $\mathrm{D^{perf}}(X)$ *is equipped with the dimension function* $-\mathrm{codim}_{\mathrm{Krull}}$. *Then there are isomorphisms*

$$\mathrm{Z}^{\Delta}_{p}\left(\mathrm{D^{perf}}(X)\right) \cong Z^{-p}(X) \qquad \textit{and} \qquad \mathrm{CH}^{\Delta}_{p}\left(\mathrm{D^{perf}}(X)\right) \cong \mathrm{CH}^{-p}(X)$$

for all $p \in \mathbb{Z}$. □

A couple of remarks are in order:

2.3.6. REMARK. Let us sketch the argument for a more "high-level" proof of the above theorem using Waldhausen models for the categories $\mathrm{D^{perf}}(X)_{(p)}$: for $p \in \mathbb{Z}$, we consider the category $\mathrm{Perf}_{(p)}(X)$ of perfect complexes on X with codimension of homological support $\geq -p$. This category is a *Waldhausen category*, i.e. a category with two classes of morphisms called the *cofibrations* and the *weak equivalences*, which both have to satisfy a list of axioms (see [Wal85]). For a Waldhausen category W, we can define higher algebraic K-groups $\mathrm{K}_i(W)$ for $i \geq 0$ as in [Wal85]. For $\mathrm{Perf}_{(p)}(X)$, the cofibrations are given by the degree-wise split monomorphisms of complexes, and the weak equivalences are given as the quasi-isomorphisms. If we define the Waldhausen category $\mathrm{Perf}_{(p/p-1)}(X)$ as the category $\mathrm{Perf}_{(p)}(X)$ with the same class of cofibrations but with the weak equivalences those morphisms whose mapping cone is quasi-isomorphic to an object of $\mathrm{Perf}_{(p-1)}(X)$, we obtain a sequence of Waldhausen categories

$$\mathrm{Perf}_{(p-1)}(X) \to \mathrm{Perf}_{(p)}(X) \to \mathrm{Perf}_{(p/p-1)}(X)$$

where both functors are given by inclusion. From the localization theorem of [TT90, Theorem 1.8.2], we obtain a long exact localization sequence

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[r] & \mathrm{K}_j(\mathrm{Perf}_{(p-1)}(X)) \arrow[r] & \mathrm{K}_j(\mathrm{Perf}_{(p)}(X)) \arrow[r] & \mathrm{K}_j(\mathrm{Perf}_{(p/p-1)}(X)) \arrow[dll, rounded corners] \\
& \mathrm{K}_{j-1}(\mathrm{Perf}_{(p-1)}(X)) \arrow[r] & \mathrm{K}_{j-1}(\mathrm{Perf}_{(p)}(X)) \arrow[r] & \cdots
\end{tikzcd} \tag{14}$$

By the regularity of X, $\mathrm{Perf}_{p-1}(X)$ and $\mathrm{Perf}_p(X)$ coincide with $\mathrm{C}^b(\mathrm{Coh}(X))_{(p-1)}$ and $\mathrm{C}^b(\mathrm{Coh}(X))_{(p)}$, the categories of bounded complexes of coherent sheaves on X with codimension of homological support $\geq -p+1$ and $\geq -p$, respectively. Their Waldhausen K-theory is in turn isomorphic to the Waldhausen K-theory of $\mathrm{C}^b(\mathrm{Coh}(X)^{(-p+1)})$ and $\mathrm{C}^b(\mathrm{Coh}(X)^{(-p)})$ respectively by [TT90, Theorem 1.9.8], as the natural inclusions induce equivalences on the corresponding derived categories. The natural functor

$$\mathrm{Perf}_{(p/p-1)}(X) \to \mathrm{C}^b(\mathrm{Coh}(X)^{(-p)}/\mathrm{Coh}(X)^{(-p+1)})$$

also induces an equivalence on the level of derived categories and thus we apply [TT90, Theorem 1.9.8] again to obtain that the corresponding Waldhausen K-theories of the involved categories coincide. Finally, the comparison to Quillen K-theory of [TT90, Theorem 1.11.2 and Theorem 1.11.7] yields that the sequences (14) are isomorphic to the sequences (7) and by forming the associated exact couple, we get a new spectral sequence which is isomorphic to Quillen's coniveau spectral sequence. In particular, we can talk about the cokernel of the map d_1 (as in (8)) in this new spectral sequence which is then isomorphic to the cokernel of d_1 in Quillen's coniveau spectral sequence which is in turn isomorphic to $\mathrm{CH}^{-p}(X)$.

2.3.7. REMARK. As we have already seen in Corollary 2.2.10, we don't need the isomorphisms

$$\mathrm{K}_0\left(\mathrm{D}^{\mathrm{perf}}(X)_{(p)}/\mathrm{D}^{\mathrm{perf}}(X)_{(p-1)}\right) \cong \mathrm{K}_0\left(M^{-p}/M^{-p+1}\right) \cong \coprod_{x \in X^{(s)}} \mathrm{K}_0(k(x))$$

to show $\mathrm{Z}_p^{\Delta}\left(\mathrm{D}^{\mathrm{perf}}(X)\right) \cong \mathrm{Z}^{-p}(X)$.

2.3.8. PROPOSITION. *The isomorphism*

$$\rho_X : \mathrm{Z}_p^{\Delta}(\mathrm{D}^{\mathrm{perf}}(X)) \xrightarrow{\sim} \mathrm{Z}^{-p}(X)$$

is explicitly given as follows: if $C^\bullet$ is an object of $\mathrm{D}^{\mathrm{perf}}(X)_{(p)}/\mathrm{D}^{\mathrm{perf}}(X)_{(p-1)}$*, then*

$$\rho_X([C^\bullet]) = \sum_i \sum_{x \in X^{(-p)}} (-1)^i \, \mathrm{length}_{\mathcal{O}_{X,x}}\left(\mathrm{H}^i\left(C^\bullet\right)_x\right) \cdot \overline{\{x\}}\,.$$

PROOF. The isomorphism (9)

$$\mathrm{K}_i(M^p/M^{p+1}) \xrightarrow{\sim} \coprod_{x \in X^{(p)}} \mathrm{K}_i(k(x))$$

is explicitly given as follows: first note that we have an equivalence of categories

$$M^p/M^{p+1} \xrightarrow{\sim} \coprod_{x \in X^{(p)}} \mathcal{O}_{X,x}\text{–fl}$$

induced by the functor

$$\begin{aligned} M^p &\to \coprod_{x\in X^{(p)}} \mathcal{O}_{X,x}\text{–fl} \\ a &\mapsto (a_x)_{x\in X^{(p)}} \end{aligned}$$

(see e.g. [Wei13, Chapter V, §9]) which in turn induces an isomorphism

$$\begin{aligned} \mathrm{K_i}(M^p/M^{p+1}) &\xrightarrow{\sim} \coprod_{x\in X^{(p)}} \mathrm{K}_i\,(\mathcal{O}_{X,x}\text{–fl}) \\ [a] &\mapsto ([a_x])_{x\in X^{(p)}}\ . \end{aligned}$$

Then we have an isomorphism

$$\coprod_{x\in X^{(p)}} \mathrm{K}_i\,(\mathcal{O}_{X,x}\text{–fl}) \xleftarrow{\sim} \coprod_{x\in X^{(p)}} \mathrm{K}_i\,(k(x))$$

given by componentwise *dévissage* (see [Qui73]), i.e. the inclusion of the category of finite-dimensional $k(x)$-vector spaces into $\mathcal{O}_{X,x}$–fl induces an isomorphism in K-theory. For $i = 0$, we have already seen this in Lemma 2.2.7: any element $[a_x] \in \mathrm{K}_0\,(\mathcal{O}_{X,x}\text{–fl})$ can be written as $n \cdot [k(x)]$, where $n = \mathrm{length}(a_x)$. We conclude that for $i = 0$, the isomorphism (9) is given explicitly as

$$[a] \mapsto \sum_{x\in X^{(p)}} \mathrm{length}_{\mathcal{O}_{X,x}}(a_x)\cdot\overline{\{x\}}$$

Precomposing with formula (11), we obtain the explicit description

$$\rho_X([C^\bullet]) = \sum_i \sum_{x\in X^{(-p)}} (-1)^i\,\mathrm{length}_{\mathcal{O}_{X,x}}\left(\mathrm{H}^i\left(C^\bullet\right)_x\right)\cdot\overline{\{x\}}\ .$$

as desired. □

The proof of Theorem 2.3.5 shows that ρ_X factors through $\mathrm{CH}^\Delta_p(\mathrm{D}^{\mathrm{perf}}(X))$ and by abuse of notation, we shall denote the induced isomorphism

$$\mathrm{CH}^\Delta_p(\mathrm{D}^{\mathrm{perf}}(X)) \to \mathrm{CH}^{-p}(X)$$

by ρ_X as well.

2.4. Functoriality

As we now have a reasonable definition of tensor triangular Chow groups at hand, we would like to check that it has the functoriality properties one would expect it to have from the algebro-geometric Chow groups.

Functors with a relative dimension. We first have to define which class of functors we allow. In this section, $\mathcal{K}$ and $\mathcal{L}$ will always denote tensor triangulated categories as in Convention 2.2.1, and we assume that both are equipped with a dimension function.

2.4.1. Definition. Let $F : \mathcal{K} \to \mathcal{L}$ be an exact functor. We say that F has *relative dimension* n if there exists some $n \in \mathbb{Z}$ such that $F(\mathcal{K}_{(p)}) \subset \mathcal{L}_{(p+n)}$ for all p, and n is the smallest integer such that this relation holds.

2.4.2. REMARK. Note that we *do not* require that F is a tensor functor (cf. Proposition 2.4.6, Example 2.4). The composition of two functors of relative dimension n and m is a functor of relative dimension at most $n+m$. In all of the examples that follow, $n=0$. However, the extra flexibility of having $n \neq 0$ might be useful for future applications.

2.4.3. THEOREM. *Let $F : \mathcal{K} \to \mathcal{L}$ be a functor of relative dimension $\leq n$. Then F induces group homomorphisms*

$$\mathrm{z}_p^n(F) : \mathrm{Z}_p^{\Delta}(\mathcal{K}) \to \mathrm{Z}_{p+n}^{\Delta}(\mathcal{L}) \qquad \textit{and} \qquad \mathrm{c}_p^n(F) : \mathrm{CH}_p^{\Delta}(\mathcal{K}) \to \mathrm{CH}_{p+n}^{\Delta}(\mathcal{L})$$

for all $p \in \mathbb{Z}$. If F has relative dimension $< n$, then $\mathrm{z}_p^n(F)$ and $\mathrm{c}_p^n(F)$ are both trivial.

PROOF. We have the following commutative diagram

$$\begin{tikzcd}
\mathcal{K}_{(p)} \arrow[r, hook, "J_{\mathcal{K}}"] \arrow[d, two heads, "Q_{\mathcal{K}}"] \arrow[dr, "F_p"] & \mathcal{K}_{(p+1)} \arrow[dr, "F_{p+1}"] & \\
\mathcal{K}_{(p)}/\mathcal{K}_{(p-1)} \arrow[d, hook, "I_{\mathcal{K}}"] \arrow[dr, "\overline{F}"] & \mathcal{L}_{(p+n)} \arrow[r, hook, "J_{\mathcal{L}}"] \arrow[d, two heads, "Q_{\mathcal{L}}"] & \mathcal{L}_{(p+n+1)} \\
(\mathcal{K}_{(p)}/\mathcal{K}_{(p-1)})^{\natural} \arrow[dr, "\hat{F}"] & \mathcal{L}_{(p+n)}/\mathcal{L}_{(p+n-1)} \arrow[d, hook, "I_{\mathcal{L}}"] & \\
 & (\mathcal{L}_{(p+n)}/\mathcal{L}_{(p+n-1)})^{\natural} &
\end{tikzcd}$$

where F_i is the restriction of F to $\mathcal{K}_{(i)}$ for $i = p, p+1$, $\overline{F}$ exists because

$$F(\mathcal{K}_{p-1}) \subset \mathcal{L}_{p+n-1} = \ker(Q_{\mathcal{L}})$$

and $\hat{F}$ exists as $I_{\mathcal{L}} \circ \overline{F}$ is a functor to an idempotent complete category. Applying the functor $\mathrm{K}_0(-)$ yields the diagram

$$\begin{tikzcd}
\mathrm{K}_0\left(\mathcal{K}_{(p)}\right) \arrow[r, "j_{\mathcal{K}}"] \arrow[d, two heads, "q_{\mathcal{K}}"] \arrow[dr, "f_p"] & \mathrm{K}_0\left(\mathcal{K}_{(p+1)}\right) \arrow[dr, "f_{p+1}"] & \\
\mathrm{K}_0\left(\mathcal{K}_{(p)}/\mathcal{K}_{(p-1)}\right) \arrow[d, hook, "i_{\mathcal{K}}"] \arrow[dr, "\overline{f}"] & \mathrm{K}_0\left(\mathcal{L}_{(p+n)}\right) \arrow[r, "j_{\mathcal{L}}"] \arrow[d, two heads, "q_{\mathcal{L}}"] & \mathrm{K}_0\left(\mathcal{L}_{(p+n+1)}\right) \\
\mathrm{K}_0\left((\mathcal{K}_{(p)}/\mathcal{K}_{(p-1)})^{\natural}\right) \arrow[dr, "\hat{f}"] & \mathrm{K}_0\left(\mathcal{L}_{(p+n)}/\mathcal{L}_{(p+n-1)}\right) \arrow[d, hook, "i_{\mathcal{L}}"] & \\
 & \mathrm{K}_0\left((\mathcal{L}_{(p+n)}/\mathcal{L}_{(p+n-1)})^{\natural}\right) &
\end{tikzcd}$$

where the lowercase arrows are induced by the corresponding uppercase ones. We set $\mathrm{z}_p^n(F) := \hat{f}$. From the diagram, we also deduce that

$$\hat{f} \circ i_{\mathcal{K}} \circ q_{\mathcal{K}}(\ker(j_{\mathcal{K}})) \subset i_{\mathcal{L}} \circ q_{\mathcal{L}}(\ker(j_{\mathcal{L}}))$$

which implies that $\hat{f}$ also induces a homomorphism $\mathrm{c}_p^n(F)$ between the factor groups.

If the relative dimension of F is $m = n - r$ for some $r \geq 1$, then

$$F(\mathcal{K}_{(p)}) \subset \mathcal{L}_{(p+m)} = \mathcal{L}_{(p+n-r)} \subset \mathcal{L}_{(p+n-1)} \ .$$

Therefore $\mathrm{z}_p^n(F)$ and $\mathrm{c}_p^n(F)$ are both 0 in this case. $\square$

2.4.4. NOTATION. If $F : \mathcal{K} \to \mathcal{L}$ has relative dimension m, then we will denote the induced homomorphisms $\mathrm{z}_p^m(F)$ and $\mathrm{c}_p^m(F)$ from Theorem 2.4.3 by $\mathrm{Z}_p^\Delta(F)$ and $\mathrm{CH}_p^\Delta(F)$, respectively.

2.4.5. REMARK. Theorem 2.4.3 and Remark 2.4.2 show that for all p, $\mathrm{Z}_p^\Delta(-)$ and $\mathrm{CH}_p^\Delta(-)$ are functors from the category of essentially small tensor triangulated categories equipped with a dimension function to the category of abelian groups, with respect to the class of functors with a relative dimension.

Let us finish the discussion with a general example of a functor with relative dimension 0.

2.4.6. PROPOSITION. *Let $a \in \mathcal{K}$ be an object such that* $\dim(\mathrm{supp}(a)) \neq \pm\infty$. *Then the functor*

$$a \otimes - : \mathcal{K} \to \mathcal{K}$$

has relative dimension 0.

PROOF. For any object $b \in \mathcal{K}$, we have

$$\mathrm{supp}(a \otimes b) = \mathrm{supp}(a) \cap \mathrm{supp}(b) \subset \mathrm{supp}(b) \ ,$$

from which it follows that $\dim(\mathrm{supp}(a \otimes b)) \leq \dim(\mathrm{supp}(b))$. Thus $a \otimes -$ has relative dimension ≤ 0. But $\mathrm{supp}(a \otimes a) = \mathrm{supp}(a)$ and therefore $\dim(\mathrm{supp}(a \otimes a)) = \dim(\mathrm{supp}(a))$, which shows that $a \otimes -$ leaves the dimension of support of the object a fixed and finite. We conclude that $a \otimes -$ has relative dimension 0. $\square$

Projection formulas and relative dimension. For a pair of adjoint functors (f_*, f^*) with relative dimensions $\dim(f_*)$ and $\dim(f^*)$ that behave similarly as the derived direct image and inverse image functor in the derived projection formula from algebraic geometry (see e.g. [Huy06, p. 83]), we can give a relation between $\dim(f_*)$ and $\dim(f^*)$.

2.4.7. DEFINITION. Let $\mathcal{C}, \mathcal{D}$ be tensor triangulated categories as in Convention 2.2.1, that are both equipped with a dimension function. Assume we are given an adjoint pair of exact functors (f^*, f_*) between $\mathcal{C}$ and $\mathcal{D}$

$$\begin{tikzcd} \mathcal{C} \arrow[d, bend right, "f^*"'] \\ \mathcal{D} \arrow[u, bend right, "f_*"'] \end{tikzcd}$$

where f^* is also a tensor functor. We say that the pair (f^*, f_*) *satisfies the projection formula* if for all $D \in \mathcal{D}, C \in \mathcal{C}$ there are isomorphisms

$$C \otimes_{\mathcal{C}} f_*(D) \cong f_*(f^*(C) \otimes_{\mathcal{D}} D)$$

which are natural in both variables.

2.4.8. REMARK. The situation of Definition 2.4.7 is not restricted to algebraic geometry, see e.g. Theorem 3.5.6.

2.4.9. LEMMA. *Let (f^*, f_*) be a pair of functors between $\mathcal{C}$ and $\mathcal{D}$ that satisfies the projection formula. Assume that* $\dim(\operatorname{supp}(f_*(\mathbb{I}_{\mathcal{D}}))) \neq \pm\infty$. *Then the functor $f_* \circ f^*$ has relative dimension* 0.

PROOF. Using that (f^*, f_*) satisfies the projection formula, we have an isomorphism

$$f_* \circ f^*(C) \cong f_*(\mathbb{I}_{\mathcal{D}}) \otimes C$$

for all objects C of $\mathcal{C}$. The result then follows by Proposition 2.4.6. □

2.4.10. COROLLARY. *Let (f^*, f_*) be a pair of functors between $\mathcal{C}$ and $\mathcal{D}$ that satisfies the projection formula and assume* $\dim(\operatorname{supp}(f_*(\mathbb{I}_{\mathcal{D}}))) \neq \pm\infty$. *Furthermore assume f^* and f_* have relative dimensions* $\dim(f^*)$ *and* $\dim(f_*)$ *respectively. Then*

$$\dim(f^*) + \dim(f_*) \geq 0$$

PROOF. This is an immediate consequence of the fact that

$$\dim(f_* \circ f^*) \leq \dim(f^*) + \dim(f_*)$$

(see Remark 2.4.2) and Lemma 2.4.9. □

Let us give two examples from algebraic geometry, which show that functors with a relative dimension occur naturally.

Example: flat pullback. We fix X, Y integral, separated schemes of finite type over a field. We consider $\mathrm{D}^{\mathrm{perf}}(X)$ and $\mathrm{D}^{\mathrm{perf}}(Y)$ with the standard structure of tensor triangulated categories and assume that they are equipped with the opposite of the Krull codimension function $-\mathrm{codim}_{\mathrm{Krull}}$.

We say that a flat morphism $f : X \to Y$ *has relative dimension r*, if for all closed subvarieties $V \subset Y$, we have that $\dim(f^{-1}(V)) = \dim(V) + r$. For such a morphism f, Fulton defines in [Ful98, Chapter 1.7] a pullback homomorphism

$$f^* : \mathrm{CH}_n(Y) \to \mathrm{CH}_{n+r}(X)$$

for $0 \leq n \leq \dim(Y)$ by sending $[V] \in \mathrm{CH}_n(Y)$ to $[f^{-1}(V)] \in \mathrm{CH}_{n+r}(X)$, the class of the scheme-theoretic inverse image of V under f.

We now fix a flat morphism $f : X \to Y$ that has relative dimension r.

2.4.11. LEMMA. *For all closed subsets $Z \subset Y$, we have*

$$\mathrm{codim}(Z) = \mathrm{codim}(f^{-1}(Z)) \ .$$

PROOF. Assume that $\dim(Z) = c$, then, since f has relative dimension r, we have

$$\dim(f^{-1}(Z)) = c + r \ .$$

As X, Y are integral of finite type over a field, it follows that

$$\mathrm{codim}(Z) = \dim(Y) - \dim(Z) = \dim(Y) - c,$$

and similarly that

$$\mathrm{codim}(f^{-1}(Z)) = \dim(X) - c - r \ .$$

As $f^{-1}(Y) = X$, we must have that $\dim(X) = \dim(Y) + r$, from which the desired equality $\operatorname{codim}(Z) = \operatorname{codim}(f^{-1}(Z))$ follows. $\square$

2.4.12. LEMMA. *The functor* $\mathrm{L}f^* : \mathrm{D}^{\mathrm{perf}}(Y) \to \mathrm{D}^{\mathrm{perf}}(X)$ *has relative dimension* 0.

PROOF. We need to check that for every $A^\bullet \in \mathrm{D}^{\mathrm{perf}}(Y)_{(p)}$, the complex $\mathrm{L}f^*(A^\bullet)$ is contained in $\mathrm{D}^{\mathrm{perf}}(X)_{(p)}$. Thus, assume that

$$-\operatorname{codim}\left(\operatorname{supp}\left(\bigoplus_i \mathrm{H}^i(A^\bullet)\right)\right) = q \leq p\ .$$

As f is flat, f^* is exact, and so we have

$$\bigoplus_i \mathrm{H}^i\left(\mathrm{L}f^*\left(A^\bullet\right)\right) = \bigoplus_i f^*\left(\mathrm{H}^i\left(A^\bullet\right)\right)$$

This implies that

$$\begin{aligned} -\operatorname{codim}\left(\operatorname{supp}\left(\bigoplus_i \mathrm{H}^i(\mathrm{L}f^*(A^\bullet))\right)\right) &= -\operatorname{codim}\left(\bigcup_i \operatorname{supp}\left(f^*\left(\mathrm{H}^i(A^\bullet)\right)\right)\right) \\ &= -\operatorname{codim}\left(f^{-1}\left(\bigcup_i \operatorname{supp}\left(\mathrm{H}^i(A^\bullet)\right)\right)\right) \\ &= q \leq p \end{aligned}$$

where the last equality follows from Lemma 2.4.11. This proves the statement. $\square$

Using the previous results, we know now that $\mathrm{L}f^*$ induces morphisms between the tensor triangular cycle and Chow groups of $\mathrm{D}^{\mathrm{perf}}(Y)$ and $\mathrm{D}^{\mathrm{perf}}(X)$. The following theorem shows that these are the expected ones.

2.4.13. THEOREM. *Assume that X, Y are non-singular and for $S = X, Y$, let*

$$\rho_S : \mathrm{CH}^\Delta_p(\mathrm{D}^{\mathrm{perf}}(S)) \to \mathrm{CH}^{-p}(S)$$

be the isomorphisms from Proposition 2.3.8. Then for all p, there is a commutative diagram

$$\begin{tikzcd} \mathrm{CH}^\Delta_p(\mathrm{D}^{\mathrm{perf}}(Y)) \arrow[r, "\mathrm{CH}^\Delta_p(\mathrm{L}f^*)"] \arrow[d, "\rho_Y"] & \mathrm{CH}^\Delta_p(\mathrm{D}^{\mathrm{perf}}(X)) \arrow[d, "\rho_X"] \\ \mathrm{CH}^{-p}(Y) \arrow[r, "f^*"] & \mathrm{CH}^{-p}(X) \end{tikzcd}$$

where f^* denotes the flat pullback homomorphism on the usual Chow group. (cf. [Ful98, Chapter 1.7]*).*

PROOF. As both f^* and $\mathrm{CH}^\Delta_p(\mathrm{L}f^*)$ are induced by the corresponding morphisms on the cycle level, it is enough to check that the diagram

$$\begin{tikzcd} \mathrm{Z}^\Delta_p(\mathrm{D}^{\mathrm{perf}}(Y)) \arrow[r, "\mathrm{Z}^\Delta_p(\mathrm{L}f^*)"] \arrow[d, "\rho_Y"] & \mathrm{Z}^\Delta_p(\mathrm{D}^{\mathrm{perf}}(X)) \arrow[d, "\rho_X"] \\ Z^{-p}(Y) \arrow[r, "f^*"] & Z^{-p}(X) \end{tikzcd}$$

commutes. In order to do this, let $Z \subset Y$ be a subvariety (=reduced and irreducible subscheme) of Y of codimension $-p$, with associated ideal sheaf I_Z and cycle $[Z] \in Z^{-p}(Y)$. Consider the class $[W^\bullet]$ in

$$\begin{aligned} Z^{\Delta}_p(\mathrm{D}^{\mathrm{perf}}(Y)) &= \mathrm{K}_0\left((\mathrm{D}^{\mathrm{perf}}(Y)_{(p)}/\,\mathrm{D}^{\mathrm{perf}}(Y)_{(p-1)})^{\natural}\right) \\ &= \mathrm{K}_0(\mathrm{D}^{\mathrm{perf}}(Y)_{(p)}/\,\mathrm{D}^{\mathrm{perf}}(Y)_{(p-1)}) \end{aligned}$$

where $W^\bullet$ is the complex concentrated in degree zero with $\mathrm{H}^0(W^\bullet) = \mathcal{O}_Y/I_Z =: \mathcal{O}_Z$. Then, $\rho_Y([W^\bullet]) = Z \in Z^{-p}(Y)$: indeed, using Proposition 2.3.8 we calculate

$$\rho_Y([W^\bullet]) = \sum_i \sum_{P \in Y^{(-p)}} (-1)^i \operatorname{length}_{\mathcal{O}_{Y,P}}\left(\mathrm{H}^i(W^\bullet)_P\right) \cdot \overline{\{P\}}$$

where $H^i(W^\bullet)_P$ is the stalk of the i-th cohomology sheaf of the complex $W^\bullet$ at the point P. Using that $W^\bullet$ is concentrated in degree zero and that $\operatorname{length}_{\mathcal{O}_{Y,P_Z}}(\mathcal{O}_{Z,P_Z})$ is equal to 1, where P_Z is the generic point of Z, we see that $\rho_Y([W^\bullet]) = Z$.

Furthermore, using that f is flat, we compute that $Z^{\Delta}_p(\mathrm{L}f^*)([W^\bullet]) = [U^\bullet]$, where $U^\bullet$ is the complex of sheaves concentrated in degree zero with $\mathrm{H}^0(U^\bullet) = \mathcal{O}_X/I_{f^{-1}(Z)}$ and $f^{-1}(Z)$ denotes the scheme-theoretic inverse image of Z under f. Clearly we have $\rho_X([U^\bullet]) = [f^{-1}(Z)]$, the cycle associated to the scheme-theoretic inverse image of Z, and so we conclude that

$$\rho_X \circ Z^{\Delta}_p(\mathrm{L}f^*) \circ \rho_Y^{-1}([Z]) = [f^{-1}(Z)] = f^*[Z]$$

By additivity of the four maps in the diagram the theorem follows. □

Example: proper push-forward. Let X and Y be integral, non-singular, separated schemes of finite type over an algebraically closed field. (The latter assumption will be needed in order to use [Ser65, Proposition V.C.6.2]). Let $f : X \to Y$ denote a proper morphism. We consider $\mathrm{D}^{\mathrm{perf}}(X), \mathrm{D}^{\mathrm{perf}}(Y)$ with the standard structure of tensor triangulated categories, *but this time we choose* $\dim_{\mathrm{Krull}}$ *as a dimension function*. Note that this implies $\mathrm{CH}^{\Delta}_p(\mathrm{D}^{\mathrm{perf}}(S)) \cong \mathrm{CH}^{\dim(S)-p}(S)$ for $S = X, Y$.

As f is proper, we obtain a functor $\mathrm{R}f_* : \mathrm{D}^{\mathrm{b}}(\mathrm{Coh}(X)) \to \mathrm{D}^{\mathrm{b}}(\mathrm{Coh}(Y))$ (see e.g. [Huy06, Theorem 3.23]), and so our regularity assumptions on X and Y imply that we also get a functor $\mathrm{R}f_* : \mathrm{D}^{\mathrm{perf}}(X) \to \mathrm{D}^{\mathrm{perf}}(Y)$.

2.4.14. LEMMA. *The functor* $\mathrm{R}f_* : \mathrm{D}^{\mathrm{perf}}(X) \to \mathrm{D}^{\mathrm{perf}}(Y)$ *has relative dimension* 0.

PROOF. Let $A^\bullet$ be a complex in $\mathrm{D}^{\mathrm{perf}}(X)$ such that $\dim\left(\operatorname{supp}\left(\bigoplus_i \mathrm{H}^i(A^\bullet)\right)\right) \leq d$. There is a spectral sequence

$$E_2^{p,q} = \mathrm{R}^p f_*(\mathrm{H}^q(A^\bullet)) \Longrightarrow \mathrm{H}^{p+q}(\mathrm{R}f_*(A^\bullet))$$

(see for example [Huy06, p.74 (3.4)]) that converges, as $A^\bullet$ is bounded. By assumption, all the cohomology sheaves $\mathrm{H}^q(A^\bullet)$ are supported in dimension $\leq d$, and by [Ser65, Proposition V.C.6.2 (a)], we therefore have $\dim(\operatorname{supp}(\mathrm{R}^p f_*(\mathrm{H}^q(A^\bullet)))) \leq d$ for all p. This implies that the terms $E_\infty^{p,q}$ are supported in dimension $\leq d$ as well. Therefore, all objects $\mathrm{H}^{p+q}(\mathrm{R}f_*(A^\bullet))$ admit a finite filtration such that the subquotients are supported in dimension $\leq d$. An induction argument then shows that the same must hold for

$\mathrm{H}^{p+q}(\mathrm{R}f_*(A^\bullet))$. We conclude that

$$\dim\left(\operatorname{supp}\left(\bigoplus_i \mathrm{H}^i(\mathrm{R}f_*(A^\bullet))\right)\right) \leq d$$

which shows that $\mathrm{R}f_*(A^\bullet) \in \mathrm{D}^{\mathrm{perf}}(Y)_{(d)}$. In order to show that the relative dimension of $\mathrm{R}f_*$ is 0, we need to show that there is a $B^\bullet \in \mathrm{D}^{\mathrm{perf}}(X)$ such that $\dim(\operatorname{supp}(B^\bullet)) = \dim(\operatorname{supp}(\mathrm{R}f_*(B^\bullet)))$. If P is any closed point of X with associated ideal sheaf I_P, then the complex $C_P^\bullet$ concentrated in degree 0 with $\mathcal{O}_X/I_P$ has $\dim(\operatorname{supp}(C_P^\bullet)) = 0$. By the result we just proved, $\mathrm{R}f_*(C_P^\bullet) \in \mathrm{D}^{\mathrm{perf}}(Y)_{(0)}$, which implies that either $\mathrm{R}f_*(C_P^\bullet) = 0$ or $\dim(\operatorname{supp}(\mathrm{R}f_*(C_P^\bullet))) = 0$. If $\mathrm{R}f_*(C_P^\bullet) = 0$, we would certainly have $\mathrm{H}^0(\mathrm{R}f_*(C_P^\bullet)) = 0$, but this is impossible by the spectral sequence we used above: indeed, it is easy to see that $E_\infty^{0,0} = E_2^{0,0}$ as $\mathrm{H}^i(C_P^\bullet) = 0$ for $i \neq 0$. But we have $E_2^{0,0} = \mathrm{R}^0 f_*(C_P^\bullet) = f_*(\mathcal{O}_X/I_P) \neq 0$. Thus $\mathrm{H}^0(\mathrm{R}f_*(C_P^\bullet))$ has a non-zero subquotient from which we deduce that $\mathrm{R}f_*(C_P^\bullet) \neq 0$. We conclude that $\dim(\operatorname{supp}(\mathrm{R}f_*(C_P^\bullet))) = 0$ which completes the proof. □

The previous lemma establishes that $\mathrm{R}f_*$ induces homomorphisms

$$\mathrm{CH}_p^\Delta(\mathrm{D}^{\mathrm{perf}}(X)) \to \mathrm{CH}_p^\Delta(\mathrm{D}^{\mathrm{perf}}(Y))$$

for all p. Again, we can show that these are exactly the ones we would expect.

2.4.15. PROPOSITION. *Denote by $\rho_S : \mathrm{CH}_p^\Delta(\mathrm{D}^{\mathrm{perf}}(S)) \to \mathrm{CH}^{\dim(S)-p}(S)$ for $S = X, Y$ the isomorphisms from Proposition 2.3.8. Then for all p, there is a commutative diagram*

$$\begin{tikzcd}
\mathrm{CH}_p^\Delta(\mathrm{D}^{\mathrm{perf}}(X)) \arrow[r, "\mathrm{CH}_p^\Delta(\mathrm{R}f_*)"] \arrow[d, "\rho_X"] & \mathrm{CH}_p^\Delta(\mathrm{D}^{\mathrm{perf}}(Y)) \arrow[d, "\rho_Y"] \\
\mathrm{CH}^{\dim(X)-p}(X) \arrow[r, "f_*"] & \mathrm{CH}^{\dim(Y)-p}(Y)
\end{tikzcd}$$

where f_* denotes the proper push-forward homomorphism on the usual Chow group (cf. [Ful98, Chapter 1.4]*).*

PROOF. Again, it suffices to show the statement for the maps on the cycle groups, as the maps on the Chow groups are induced by those. By additivity of the four maps in the diagram it is enough to check that for an (integral) subvariety $V \subset X$ of dimension p and an element $v \in Z_p^\Delta(\mathrm{D}^{\mathrm{b}}(X))$ with $\rho_X(v) = [V]$, we have $\rho_Y \circ \mathrm{Z}_p^\Delta(\mathrm{R}f_*)(v) = f_*([V])$. So, fix V as above and consider the complex of coherent sheaves $W^\bullet$ that is concentrated in degree 0 and has $\mathrm{H}^0(W^\bullet) = \mathcal{O}_V$, where $\mathcal{O}_V = \mathcal{O}_X/\mathcal{I}_V$ and $\mathcal{I}_V$ is the ideal sheaf associated to V. The complex $W^\bullet$ represents a class $[W^\bullet]$ in

$$\mathrm{Z}_p^\Delta(\mathrm{D}^{\mathrm{b}}(X)) = \mathrm{K}_0\left(\mathrm{D}^{\mathrm{b}}(X)_{(p)}/\mathrm{D}^{\mathrm{b}}(X)_{(p-1)}\right) = \mathrm{K}_0\left(\left(\mathrm{D}^{\mathrm{b}}(X)_{(p)}/\mathrm{D}^{\mathrm{b}}(X)_{(p-1)}\right)^\natural\right)$$

and similarly to the calculation in Theorem 2.4.13 we see that $\rho_X([W^\bullet]) = V$.

For the next step, we compute

$$\begin{aligned}\rho_Y \circ \mathrm{Z}_p^{\Delta}(\mathrm{R}f_*)([W^\bullet]) &= \sum_i \sum_{Q \in Y_{(p)}} (-1)^i \operatorname{length}_{\mathcal{O}_{Y,Q}} \left(\mathrm{H}^i(\mathrm{R}f_*(W^\bullet))_Q\right) \cdot \overline{\{Q\}} \\ &= \sum_i \sum_{Q \in Y_{(p)}} (-1)^i \operatorname{length}_{\mathcal{O}_{Y,Q}} \left(\mathrm{R}^i f_*(\mathcal{O}_V)_Q\right) \cdot \overline{\{Q\}} \\ &= \sum_i (-1)^i \sum_{Q \in Y_{(p)}} \operatorname{length}_{\mathcal{O}_{Y,Q}} \left(\mathrm{R}^i f_*(\mathcal{O}_V)_Q\right) \cdot \overline{\{Q\}}.\end{aligned}$$

Using [Ser65, Proposition V.C.6.2 (b)], we see that this is equal to $f_*(V)$, which means that we have shown $\rho_Y \circ \mathrm{Z}_p^{\Delta}(Rf_*)([W^\bullet]) = f_*(V)$ and thus have finished the proof of the theorem. □

2.5. An alternative definition of rational equivalence

Instead of choosing the K-theoretic approach of Definition 2.2.4 in order to obtain a notion of rational equivalence, one can try to imitate the original construction from algebraic geometry of taking divisors of functions on subvarieties. Following [Bal13], we can define "divisors of functions" in the categorical context.

2.5.1. CONVENTION. For the rest of the section, $\mathcal{K}$ denotes a tensor triangulated category in the sense of Convention 2.2.1 that is rigid and such that $\operatorname{Spc}(\mathcal{K})$ is a noetherian topological space. We also fix a dimension function on $\mathcal{K}$.

Let $Q^\natural$ denote the composition of the Verdier quotient functor

$$\mathcal{K}_{(p)} \to \mathcal{K}_{(p)}/\mathcal{K}_{(p-1)}$$

and the inclusion into the idempotent completion

$$\mathcal{K}_{(p)}/\mathcal{K}_{(p-1)} \to (\mathcal{K}_{(p)}/\mathcal{K}_{(p-1)})^\natural .$$

The functor $Q^\natural$ induces a group homomorphism

$$\begin{aligned} q^\natural : \mathrm{K}_0(\mathcal{K}_{(p)}) &\to \mathrm{K}_0\left((\mathcal{K}_{(p)}/\mathcal{K}_{(p-1)})^\natural\right) = \mathrm{Z}_p^{\Delta}(\mathcal{K}) \\ [a] &\mapsto [Q^\natural(a)] \end{aligned}$$

For an object a in the Verdier quotient $\mathcal{K}_{(p+1)}/\mathcal{K}_{(p)}$ and an automorphism

$$f : a \to a \ ,$$

choose a fraction $a \xleftarrow{\beta} b \xrightarrow{\alpha} a$ in $\mathcal{K}_{(p+1)}$ representing f. We will then have $\operatorname{cone}(\beta) \in \mathcal{K}_{(p)}$ by definition of the Verdier quotient. We also must have $\operatorname{cone}(\alpha) \in \mathcal{K}_{(p)}$: indeed, α must be an isomorphism in $\mathcal{K}_{(p+1)}/\mathcal{K}_{(p)}$ as the composition $\alpha \circ \beta^{-1} = f$ is one, and thus its cone must be zero in $\mathcal{K}_{(p+1)}/\mathcal{K}_{(p)}$. This implies that it is in $\mathcal{K}_{(p)}$, as the latter is a thick subcategory.

We then define the *divisor of f* (cf. [Bal13]) as

$$\operatorname{div}^{\Delta}(f) := q^\natural([\operatorname{cone}(\alpha)] - [\operatorname{cone}(\beta)]) = [Q^\natural(\operatorname{cone}(\alpha))] - [Q^\natural(\operatorname{cone}(\beta))]$$

The following shows that $\operatorname{div}^{\Delta}(f)$ is well-defined.

2.5.2. PROPOSITION. *The expression* $\operatorname{div}^{\Delta}(f)$ *does not depend on the choice of* α *and* β.

PROOF. If we have an equivalent fraction $a \xleftarrow{\beta'} c \xrightarrow{\alpha'} a$, there is by definition a commutative diagram in $\mathcal{K}_{(p+1)}$

$$\begin{tikzcd} & b \arrow[dl, "\beta"'] \arrow[dr, "\alpha"] & \\ a & d \arrow[l, "y"] \arrow[r, "x"'] \arrow[u, "f"] \arrow[d, "g"'] & a \\ & c \arrow[ul, "\beta'"] \arrow[ur, "\alpha'"'] & \end{tikzcd} .$$

Using the octahedral axiom, we obtain the following distinguished triangles in $\mathcal{K}_{(p)}$:

$$\mathrm{cone}(f) \to \mathrm{cone}(y) \to \mathrm{cone}(\beta) \to \Sigma(\mathrm{cone}(f))$$
$$\mathrm{cone}(g) \to \mathrm{cone}(y) \to \mathrm{cone}(\beta') \to \Sigma(\mathrm{cone}(g))$$
$$\mathrm{cone}(f) \to \mathrm{cone}(x) \to \mathrm{cone}(\alpha) \to \Sigma(\mathrm{cone}(f))$$
$$\mathrm{cone}(g) \to \mathrm{cone}(x) \to \mathrm{cone}(\alpha') \to \Sigma(\mathrm{cone}(g))$$

These show that $[\mathrm{cone}(\alpha)] - [\mathrm{cone}(\alpha')]$ and $[\mathrm{cone}(\beta)] - [\mathrm{cone}(\beta')]$ are both equal to the element $[\mathrm{cone}(g)] - [\mathrm{cone}(f)]$ in $\mathrm{K}_0(\mathcal{K}_{(p)})$. Thus, we have

$$[\mathrm{cone}(\alpha)] - [\mathrm{cone}(\beta)] = [\mathrm{cone}(\alpha')] - [\mathrm{cone}(\beta')]$$

in $\mathrm{K}_0(\mathcal{K}_{(p)})$. Applying the homomorphism $q^\natural$ on both sides of the equation yields the statement. $\square$

We now define alternative Chow groups as "cycles modulo divisors of functions".

2.5.3. DEFINITION. Let $\mathfrak{I}$ denote the subgroup of $\mathrm{Z}^\Delta_p(\mathcal{K})$ generated by all expressions $\mathrm{div}^\Delta(f)$, where f runs over all automorphisms of all objects of $\mathcal{K}_{(p+1)}/\mathcal{K}_{(p)}$. Then define

$$\mathrm{ch}^\Delta_p(\mathcal{K}) := \mathrm{Z}^\Delta_p(\mathcal{K})/\mathfrak{I} .$$

Let us now investigate the relation between $\mathrm{ch}^\Delta_p(\mathcal{K})$ and $\mathrm{CH}^\Delta_p(\mathcal{K})$. Recall from Definition 2.2.4 that $\mathrm{CH}^\Delta_p(\mathcal{K}) = \mathrm{Z}^\Delta_p(\mathcal{K})/j \circ q(\ker(i))$ where i, q, j are taken from the diagram

$$\begin{tikzcd} \mathrm{K}_0(\mathcal{K}_{(p)}) \arrow[r, "i"] \arrow[d, two heads, "q"] & \mathrm{K}_0(\mathcal{K}_{(p+1)}) \\ \mathrm{K}_0(\mathcal{K}_{(p)}/\mathcal{K}_{(p-1)}) \arrow[r, hook, "j"] & \mathrm{K}_0\left((\mathcal{K}_{(p)}/\mathcal{K}_{(p-1)})^\natural\right) = \mathrm{Z}^\Delta_p(\mathcal{K}) . \end{tikzcd}$$

2.5.4. PROPOSITION. *We have an inclusion* $\mathfrak{I} \subset j \circ q(\ker(i))$.

PROOF. If $f : a \to a$ is an isomorphism in $\mathcal{K}_{(p+1)}/\mathcal{K}_{(p)}$ represented by a fraction

$$a \xleftarrow{\beta} b \xrightarrow{\alpha} a$$

in $\mathcal{K}_{(p+1)}$, then in $\mathrm{K}_0(\mathcal{K}_{(p+1)})$, we have

$$[\mathrm{cone}(\alpha)] = [\mathrm{cone}(\beta)] = [b] - [a]$$

and thus $[\mathrm{cone}(\alpha)] - [\mathrm{cone}(\beta)] = 0$. Therefore, $[\mathrm{cone}(\alpha)] - [\mathrm{cone}(\beta)]$ will certainly be in $\ker(i)$. The statement then follows as $q^{\natural} = j \circ q$. □

2.5.5. COROLLARY. *For all $p \in \mathbb{Z}$, there is an epimorphism*

$$\mathrm{ch}^{\Delta}_{p}(\mathcal{K}) \to \mathrm{CH}^{\Delta}_{p}(\mathcal{K}) \ .$$

PROOF. This is an immediate consequence of Proposition 2.5.4. □

It is not clear to the author if the inclusion $\mathfrak{I} \supset j \circ q(\ker(i))$ holds in general, so $\mathrm{ch}^{\Delta}_{p}(\mathcal{K})$ and $\mathrm{CH}^{\Delta}_{p}(\mathcal{K})$ are a priori different. We will now prove that the two groups coincide when we are dealing with separated, non-singular schemes of finite type over a field that have an ample line bundle.

2.5.6. THEOREM. *Let $X, \mathrm{D}^{\mathrm{perf}}(X)$ be as in Convention 2.3.1 and assume furthermore that X has an ample line bundle $\mathcal{L}$. Then there are isomorphisms*

$$\mathrm{ch}^{\Delta}_{p}(\mathrm{D}^{\mathrm{perf}}(X)) \cong \mathrm{CH}^{\Delta}_{p}(\mathrm{D}^{\mathrm{perf}}(X)) \cong \mathrm{CH}^{-p}(X)$$

for all $p \in \mathbb{Z}$.

PROOF. Using Theorem 2.3.5 and Proposition 2.5.4, we already know that the subgroup $\mathfrak{I}$ is contained in the subgroup of cycles rationally equivalent to zero. Thus, it suffices to show that any cycle rationally equivalent to zero can be obtained as $\mathrm{div}^{\Delta}(f)$ for some object $a \in \mathrm{D}^{\mathrm{b}}(X)_{(p+1)}/\mathrm{D}^{\mathrm{b}}(X)_{(p)}$ and morphism $f \in \mathrm{Aut}(a)$. The essential point is that for a subvariety $V \subset X$ of codimension $-(p+1)$ we can write the function field of V as

$$k(V) = \left(\bigoplus_{i \geq 0} \Gamma\left(X, \mathcal{O}_V \otimes \mathcal{L}^{\otimes i}\right) \right)_{((0))} ,$$

where $\mathcal{O}_V := \mathcal{O}_X/\mathcal{I}_V$ and $\mathcal{I}_V$ is the ideal sheaf associated to V. Indeed, this is a consequence of [Gro61, Théorème 4.5.2] and the fact that the restriction of an ample line bundle to a closed subscheme is ample.

Thus, for $h \in k(V)$, we can write $h = f/g$ with $f, g \in \Gamma(X, \mathcal{O}_V \otimes \mathcal{L}^{\otimes n})$ for some $n \in \mathbb{N}$. From this, we obtain exact sequences

$$0 \to \mathcal{O}_V \xrightarrow{m_f} \mathcal{O}_V \otimes \mathcal{L}^{\otimes n} \to \mathrm{coker}(m_f) \to 0$$

and

$$0 \to \mathcal{O}_V \xrightarrow{m_g} \mathcal{O}_V \otimes \mathcal{L}^{\otimes n} \to \mathrm{coker}(m_g) \to 0$$

where m_f, m_g are the obvious multiplication maps. By using the local isomorphisms $\mathcal{L}^{\otimes n}|_{U_i} \cong \mathcal{O}_X|_{U_i}$ for some open cover $\{U_i\}_{i \in I}$, we obtain that

$$\mathrm{supp}(\mathrm{coker}(m_f)) = V(f) \subset V$$

and

$$\mathrm{supp}(\mathrm{coker}(m_g)) = V(g) \subset V.$$

If we interpret the above exact sequences as distinguished triangles in the Verdier quotient $\mathrm{D}^{\mathrm{perf}}(X)_{(p+1)}/\mathrm{D}^{\mathrm{perf}}(X)_{(p)}$, we therefore see that both m_f and m_g are isomorphisms in this category, as

$$\mathrm{codim}(V(f)) = \mathrm{codim}(V(g)) = -p$$

CHAPTER 3

Tensor triangular Chow groups in modular representation theory

So far we have mostly considered examples from algebraic geometry. However, tensor triangulated categories also occur in different contexts. One of these is modular representation theory, where one studies kG-modules for a finite group G and a field k such that char(k) divides $|G|$. A useful tool in this context is the stable category kG-stab, which is obtained as the stable category of kG-mod, the Frobenius category (see Example 1.1.5) of finitely generated left kG-modules. The category kG-stab is a tensor triangulated category. By a theorem of Rickard (see [Ric89]), it is closely related to $\mathrm{D}^{\mathrm{b}}(kG\text{-mod})$, the bounded derived category of finitely generated kG-modules, which is also tensor triangulated. Using the theory from the previous chapter, we therefore have a notion of tensor triangular Chow groups for these categories. In this chapter we compare the tensor triangular Chow groups of kG-stab and $\mathrm{D}^{\mathrm{b}}(kG\text{-mod})$, compute concrete examples of these groups and show that stable induction and restriction functors fit in the framework of functors with a relative dimension.

3.1. Basic definitions and results

We recall some basic definitions and results that we will need. All of them can be found in the books by Carlson [Car96] and Benson [Ben98a, Ben98b] or in Balmer's article [Bal05]. For the rest of the chapter, G will denote a finite group, k is a field of characteristic p dividing $|G|$, and kG is the corresponding group algebra. Associated to this algebra is the abelian category kG-mod consisting of the finitely-generated left kG-modules. Given two modules $M, N \in kG$-mod, we can form their tensor product $M \otimes_k N$, which is again a finitely-generated left kG-module when we consider it with the diagonal action

$$g(m \otimes n) := gm \otimes gn$$

for $g \in G, m \in M$ and $n \in N$ and extend linearly. Furthermore, $\mathrm{Hom}_k(M, N)$, the set of k-linear maps from M to N can be made a finitely generated kG-module by setting

$$(gf)(m) := f(g^{-1}m)$$

for $g \in G, m \in M$ and $n \in N$ and extending linearly.

The category kG-mod is a Frobenius category (see Example 1.1.5), and so we can form the associated stable category kG-stab which is naturally triangulated. It can be given a symmetric-monoidal structure with the tensor product induced by $- \otimes_k -$ with unit object k, the trivial kG-module. Thus, kG-stab is an essentially small, tensor triangulated category. It is also rigid, where the dual of an object M is given as $\mathrm{Hom}_k(M, k)$.

3.1.1. DEFINITION. The *cohomology ring* of kG is defined as the graded ring

$$\mathrm{H}^*(G,k) := \bigoplus_{i\geq 0} \mathrm{Ext}^i_{kG}(k,k)\ .$$

The *projective support variety* of kG is defined as

$$\mathcal{V}_G(k) := \mathrm{Proj}(\mathrm{H}^*(G,k))\ .$$

3.1.2. REMARK. When p is odd, $\mathrm{H}^*(G,k)$ is in general only a *graded* commutative ring, so when we write $\mathrm{Proj}(\mathrm{H}^*(G,k))$ we really mean $\mathrm{Proj}(\mathrm{H}^{\mathrm{ev}}(G,k))$ in this case, where $\mathrm{H}^{\mathrm{ev}}(G,k)$ is the subring of all elements of even degree. Another way to deal with this difficulty is to extend the definition of Proj to graded-commutative k-algebras (cf. [BBC09, Section 1]).

Suppose we are given any two finite-dimensional kG-modules M,N. Then the Evens-Venkov theorem (see [Car96, Theorem 9.1]) shows that $\bigoplus_{i\geq 0} \mathrm{Ext}^i_{kG}(M,N)$ is a finitely generated graded module over $\mathrm{H}^*(G,k)$.

3.1.3. DEFINITION. For a kG-module $M \neq 0$, define $\mathrm{J}(M) \subset \mathrm{H}^*(G,k)$ as the annihilator ideal of $\mathrm{Ext}^*_{kG}(M,M)$ in $\mathrm{H}^*(G,k)$. The *variety of M* is the subvariety of $\mathcal{V}_G(k)$ associated to $\mathrm{J}(M)$.

3.1.4. DEFINITION. Let M be in kG-mod. A *minimal projective resolution of M* is a projective resolution $P_\bullet \to M$ such that for every other projective resolution $Q_\bullet \to M$ there exists an injective chain map $(P_\bullet \to M) \to (Q_\bullet \to M)$ and a surjective chain map $(Q_\bullet \to M) \to (P_\bullet \to M)$ that both lift the identity on M.

3.1.5. THEOREM (see [Car96, Theorem 4.3]). *Let M be a module in kG-mod. Then M has a minimal projective resolution.*

3.1.6. DEFINITION. Let M be in kG-mod and let $P_\bullet \to M$ be a minimal projective resolution. The *complexity* $\mathrm{c}_G(M)$ of M is defined as the least integer s such that there is a constant $\kappa > 0$ with

$$\dim_k(P_n) \leq \kappa \cdot n^{s-1} \quad \text{for } n > 0$$

The complexity of a module can be read off from its variety:

3.1.7. THEOREM (cf. [Ben98b, Prop. 5.7.2]). *If M is a finitely generated kG-module, then*

$$\dim(\mathcal{V}_G(M)) = \mathrm{c}_G(M) - 1.$$

The projective support variety of kG can be reconstructed from kG-stab:

3.1.8. THEOREM (cf. [Bal05, Corollary 5.10]). *There is a homeomorphism*

$$\phi : \mathcal{V}_G(k) \longrightarrow \mathrm{Spc}(kG\text{-stab})\ .$$

Furthermore, the support of a module $M \in kG$-stab corresponds to $\mathcal{V}_G(M)$ under this map.

For the rest of the chapter, we will take $\dim_{\mathrm{Krull}}$ (cf. Example 1.4.3) as a dimension function for kG-stab. By Theorem 3.1.8 this coincides with the usual Krull dimension on $\mathcal{V}_G(k)$ under the homeomorphism ρ.

3.2. Derived category vs. stable category

We consider $\mathrm{D^b}(kG\text{-mod})$, the bounded derived category of finitely generated kG-modules with its natural triangulation. It becomes a tensor triangulated category with the usual extension to chain complexes of the tensor product $\otimes_k$ of kG-modules *over* k.

Let us immediately state that $\mathrm{D^b}(kG\text{-mod})$ and kG-stab are closely related: the category kG-stab arises as a Verdier quotient of $\mathrm{D^b}(kG\text{-mod})$. Let $\mathrm{K}^b(kG\text{-proj})$ denote the bounded homotopy category of finitely generated projective kG-modules. Since quasi-isomorphisms between bounded complexes of projective modules are the same as homotopy equivalences, $\mathrm{K}^b(kG\text{-proj})$ embeds into $\mathrm{D^b}(kG\text{-mod})$ as a full triangulated subcategory.

3.2.1. THEOREM (see [Ric89]). *The natural functor*

$$kG\text{-stab} \to \mathrm{D^b}(kG\text{-mod})/\mathrm{K}^b(kG\text{-proj})$$

induced by the inclusion $kG\text{-mod} \to \mathrm{D^b}(kG\text{-mod})$ *is an exact equivalence of tensor triangulated categories.*

The following theorem tells us that the spectra of $\mathrm{D^b}(kG\text{-mod})$ and kG-stab differ in one point only.

3.2.2. THEOREM (see [Bal10a, Theorem 8.5]). *We have an isomorphism*

$$\rho : \mathrm{Spc}(\mathrm{D^b}(kG\text{-mod})) \longrightarrow \mathrm{Spec^h}(\mathrm{H}^*(G,k))$$

where $\mathrm{Spec^h}(\mathrm{H}^*(G,k))$ *is the spectrum of homogeneous prime ideals in* $\mathrm{H}^*(G,k)$. *Furthermore the diagram*

$$\begin{tikzcd}
\mathrm{Spc}(kG\text{-stab}) \arrow[r, "\mathrm{Spc}(q)"] & \mathrm{Spc}(\mathrm{D^b}(kG\text{-mod})) \arrow[d, "\rho"] \\
\mathrm{Proj}(\mathrm{H}^*(G,k)) \arrow[u, "\varphi"] \arrow[r, hook] & \mathrm{Spec^h}(\mathrm{H}^*(G,k))
\end{tikzcd}$$

commutes, where φ *is the isomorphism from Theorem 3.1.8,* $\mathrm{Spc}(q)$ *is the map associated to the quotient functor*

$$q : \mathrm{D^b}(kG\text{-mod}) \to \mathrm{D^b}(kG\text{-mod})/\mathrm{K}^b(kG\text{-proj}) \cong kG\text{-stab}\ ,$$

and the lower arrow is the inclusion of the open subset with complement the unique closed point of $\mathrm{Spec^h}(\mathrm{H}^*(G,k))$ *corresponding to the irrelevant ideal.*

3.2.3. REMARK. It is crucial here that we consider $\mathrm{D^b}(kG\text{-mod})$ with the tensor product $\otimes_k$, as opposed to $\otimes_{kG}$: there is no natural left-module structure on $M \otimes_{kG} N$ for two left kG-modules M, N. If G is commutative, $\otimes_{kG}$ makes $\mathrm{K}^b(kG\text{-proj}) \subset \mathrm{D^b}(kG\text{-mod})$ a tensor triangulated category, but its spectrum is much less interesting, as it is homeomorphic to the usual prime ideal spectrum $\mathrm{Spec}(kG)$.

We start to compare $\mathrm{CH}^{\Delta}_p(\mathrm{D^b}(kG\text{-mod}))$ and $\mathrm{CH}^{\Delta}_p(kG\text{-stab})$.

3.2.4. PROPOSITION. *Consider* kG-stab *and* $\mathrm{D^b}(kG\text{-mod})$ *with the Krull dimension of support as a dimension function. Then for all* $p \geq 0$, *the Verdier quotient functor*

$$q : \mathrm{D^b}(kG\text{-mod}) \to \mathrm{D^b}(kG\text{-mod})/\mathrm{K}^b(kG\text{-proj}) \cong kG\text{-stab}$$

induces isomorphisms

$$\mathrm{Z}^{\Delta}_{p+1}(\mathrm{D^b}(kG\text{-mod})) \cong \mathrm{Z}^{\Delta}_{p}(kG\text{-stab}) .$$

PROOF. First, remark that the functor q sends an object with dimension of support $p+1$ to an object with dimension of support p for $p \geq 0$. This follows as we have

$$\mathrm{supp}(q(a))) = \mathrm{Spc}(q)^{-1}(\mathrm{supp}(a)) = \mathrm{supp}(a) \cap \mathrm{Spc}(kG\text{-stab}) \subset \mathrm{Spc}(\mathrm{D^b}(kG\text{-mod}))$$

and the space $\mathrm{Spc}(\mathrm{D^b}(kG\text{-mod}))$ has exactly one closed point $\{0\} \subset \mathrm{D^b}(kG\text{-mod})$ more than $\mathrm{Spc}(kG\text{-stab})$, which is contained in the closure of every point of $\mathrm{Spc}(\mathrm{D^b}(kG\text{-mod}))$.

If $\mathcal{K} = \mathrm{D^b}(kG\text{-mod})$ and $\mathcal{J} = \mathrm{K}^b(kG\text{-proj})$, we use Lemma 1.1.15 to see that

$$\begin{aligned}\mathcal{K}_{(p+1)}/\mathcal{K}_{(p)} &\cong (\mathcal{K}_{(p+1)}/\mathcal{J})/(\mathcal{K}_{(p)}/\mathcal{J}) \\ &\cong kG\text{-stab}_{(p)}\,/\,kG\text{-stab}_{(p-1)} ,\end{aligned}$$

and the equivalence induces one on the idempotent completions. By applying $\mathrm{K}_0(-)$, we get the desired result. □

In order to prove Proposition 3.2.4 for Chow groups instead of cycle groups, we need the following elementary lemma about abelian groups.

3.2.5. LEMMA. *Let $f : A \to B$ be a morphism of abelian groups, $S \subset A$ be a subgroup and $\hat{f} : A/S \to B/f(S)$ be the induced morphism. Then* $\ker \hat{f} = p(\ker f)$*, where $p : A \to A/S$ is the canonical projection.*

PROOF. Let $[x] \in \ker \hat{f}$, then $0 = \hat{f}([x]) = [f(x)]$, which implies that $f(x) \in f(S)$. Let $s \in S$ be such that $f(s) = f(x)$. Then

$$f(x-s) = f(x) - f(s) = f(x) - f(x) = 0$$

and

$$p(x-s) = p(x) - p(s) = [x]$$

which proves that $\ker \hat{f} \subset p(\ker f)$. The other inclusion is trivial. □

3.2.6. THEOREM. *Consider kG-stab and $\mathrm{D^b}(kG\text{-mod})$ with the Krull dimension of support as a dimension function. Then for all $p \geq 0$, there are isomorphisms*

$$\mathrm{CH}^{\Delta}_{p}(kG\text{-stab}) \cong \mathrm{CH}^{\Delta}_{p+1}(\mathrm{D^b}(kG\text{-mod})) .$$

PROOF. Let $\mathcal{K} := \mathrm{D^b}(kG\text{-mod}), \mathcal{T} := kG$-stab and consider the commutative diagram

$$\begin{tikzcd}
\mathcal{K}_{(p+1)} \arrow[r, "\pi"] \arrow[d, "\iota"] \arrow[dr, "\kappa"] & \left(\mathcal{K}_{(p+1)}/\mathcal{K}_{(p)}\right)^{\natural} \arrow[dr, "\psi"] & \\
\mathcal{K}_{(p+2)} \arrow[dr, "\lambda"] & \mathcal{T}_{(p)} \arrow[r, "\underline{\pi}"] \arrow[d, "\underline{\iota}"] & \left(\mathcal{T}_{(p)}/\mathcal{T}_{(p-1)}\right)^{\natural} \\
& \mathcal{T}_{(p+1)} &
\end{tikzcd}$$

where the diagonal functors κ, λ are restrictions of the Verdier quotient

$$q : \mathrm{D}^{\mathrm{b}}(kG\text{-mod}) \to kG\text{-stab}$$

and ψ is the equivalence from the proof of Proposition 3.2.4. We have

$$\ker(\mathrm{K}_0(\underline{\iota})) = \mathrm{K}_0(\kappa)(\ker(\mathrm{K}_0(\iota)))$$

as we are in the situation of Lemma 3.2.5. This shows that

$$\mathrm{K}_0(\underline{\pi})(\ker(\mathrm{K}_0(\underline{\iota}))) = \mathrm{K}_0(\underline{\pi}) \circ \mathrm{K}_0(\kappa)(\ker(\mathrm{K}_0(\iota))) = \mathrm{K}_0(\psi) \circ \mathrm{K}_0(\pi)(\ker(\mathrm{K}_0(\iota)))$$

which gives the desired result. □

We now proceed to compute some examples of tensor triangular Chow groups coming from kG-stab.

3.3. The case $G = \mathbb{Z}/p^n\mathbb{Z}$

We begin with the case where $G = \mathbb{Z}/p^n\mathbb{Z}$ for some prime p and $n \in \mathbb{N}$. In the following, k will be any field of characteristic p. It follows from [Car96, Theorem 7.3] that $\mathcal{V}_G(k)$ is a point, and so a finitely generated kG-module has complexity 1 if and only if it is non-projective.

Computing the tensor triangular cycle groups for kG-stab amounts to calculating

$$\mathrm{K}_0\left((kG\text{-stab}_{(i)}/kG\text{-stab}_{(i-1)})^{\natural}\right) .$$

One immediately sees that the only non-trivial case is when $i = 0$. Then

$$\mathrm{Z}_0^{\Delta}(kG\text{-stab}) = \mathrm{K}_0(kG\text{-stab})) ,$$

as kG-stab is idempotent complete. In order to compute this, we use the following result:

3.3.1. PROPOSITION (see [TW91, Proposition 1]). *Let B be a Frobenius k-algebra, let B-mod be the category of finitely generated left B-modules and B-stab the corresponding stable category. Then it holds that*

$$\mathrm{K}_0(B\text{-stab}) \cong \mathrm{K}_0(B\text{-mod})/\langle \mathrm{proj} \rangle ,$$

where $\langle \mathrm{proj} \rangle$ is the subgroup generated by the isomorphism classes of projective modules.

Note that $\mathrm{K}_0(kG\text{-mod})) \cong \mathbb{Z}$, as kG is a commutative local artinian ring: indeed, for modules over artinian rings, being finitely generated and having finite length are equivalent, and then the result follows by Lemma 2.2.7. For local rings, projective and free modules coincide, and thus it follows from Proposition 3.3.1 that

$$\mathrm{Z}_0^{\Delta}(kG\text{-stab}) \cong \mathbb{Z}/p^n\mathbb{Z} .$$

We also see that this group coincides with $\mathrm{CH}_0^{\Delta}(kG\text{-stab})$, as we are in the top dimension. Summarizing, we have the following:

3.3.2. PROPOSITION. *Let $G = \mathbb{Z}/p^n\mathbb{Z}$ for some prime p and $n \in \mathbb{N}$ and k any field of characteristic p. Then*

$$\mathrm{Z}_i^{\Delta}(kG\text{-stab}) = \mathrm{CH}_i^{\Delta}(kG\text{-stab}) = 0 \quad \textit{for all } i \neq 0$$

and

$$\mathrm{Z}_0^{\Delta}(kG\text{-stab}) = \mathrm{CH}_0^{\Delta}(kG\text{-stab}) = \mathbb{Z}/p^n\mathbb{Z}.$$

3.4. The case $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

If $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} = \langle x, y | x^2 = y^2 = 1, xy = yx \rangle$ and k is a field of characteristic 2, the computations become more involved.

As a consequence of [Car96, Theorem 7.6], we have that $\mathcal{V}_G(k) = \mathbb{P}^1$. Therefore there is a proper subcategory of kG-stab consisting of the modules of complexity ≤ 1. In order to work with those, we need the following classification:

3.4.1. LEMMA. *All finite-dimensional indecomposable kG-modules of odd dimension have complexity 2.*

PROOF. Let M be a odd-dimensional indecomposable module. If we assume that M has complexity 1, then by [Ben98b, Theorem 5.10.4 and Corollary 5.10.7], M must be periodic, with period 1. In other words, if $\epsilon : P \twoheadrightarrow M$ is a projective cover of M, then we must have $M \cong \ker(\epsilon)$. However, since G is a 2-group, the only indecomposable projective module is the free module of rank 1 (see [Ben98a, Section 3.14]), which has k-dimension 4. Thus, if M has dimension $2n+1$ and P has dimension $4m$, then using that ϵ is surjective and the dimension formula, we get $\dim_k(\ker(\epsilon)) = 4m - 2n - 1$. We see immediately that $\ker(\epsilon)$ cannot have dimension $2n+1$, and thus M cannot have complexity 1. As it is non-projective it therefore must have complexity 2. $\square$

We also see that a complementary result holds for the even-dimensional representations:

3.4.2. LEMMA. *All finite-dimensional, non-projective indecomposable kG-modules of even dimension have complexity 1.*

PROOF. It follows from [CM12, Proposition 3.1] that a non-projective indecomposable kG-module of even dimension is periodic with period 1. As an immediate consequence, those modules have complexity 1. $\square$

3.4.3. REMARK. Lemma 3.4.2 also follows from the following explicit calculation: using the classification of all indecomposable kG-modules (cf. [Ben98a, Theorem 4.3.3]), one sees that any non-projective, indecomposable even-dimensional kG-module is isomorphic to one of the form

$$x \mapsto \begin{pmatrix} I & I \\ 0 & I \end{pmatrix} \qquad y \mapsto \begin{pmatrix} I & J \\ 0 & I \end{pmatrix}$$

where I is the $n \times n$ identity matrix and J is some $n \times n$ matrix over k. Note that in this presentation, the above modules may fail to be mutually non-isomorphic for different J. This type of module will from now on be denoted by $\mathrm{M}_n(J)$ and we proceed to find the first term of a projective resolution for it. In order to do so, fix the basis $(1, x+1, y+1, xy+x+y+1)$ for kG and consider for $1 \leq i \leq n$ the $n \times 4$ matrices

$$B_i := \begin{pmatrix} \vdots & \vdots & \vdots & \vdots \\ 0 & f_i & J_i & 0 \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \quad \text{and} \quad E_i := \begin{pmatrix} \vdots & \vdots & \vdots & \vdots \\ f_i & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

where f_i is the i-th standard basis vector of length n and J_i is the i-th column vector of J. One now verifies the following statement by an explicit computation:

3.4.4. LEMMA. *The linear map $\epsilon : kG^n \to \mathrm{M}_n(J)$ given by the $2n \times 4n$ matrix*

$$\begin{pmatrix} B_1 & \cdots & B_n \\ E_1 & \cdots & E_n \end{pmatrix}$$

is a surjective kG-module homomorphism. Furthermore we have $\ker(\epsilon) \cong \mathrm{M}_n(J)$. □

From this it follows that the complexity of $\mathrm{M}_n(J)$ is ≤ 1. As it is not projective, it must therefore have complexity 1.

The following is a direct consequence of Lemma 3.4.1 and Lemma 3.4.2:

3.4.5. COROLLARY. *The indecomposable kG-modules of odd dimension are exactly the indecomposable modules of complexity 2. The non-projective indecomposable kG-modules of even dimension are exactly the indecomposable modules of complexity 1.* □

Using this classification, we can calculate the zero-dimensional Chow group.

3.4.6. LEMMA. *The map*

$$[M] \mapsto \dim_k(M) \mod 4$$

defines an isomorphism $\mathrm{K}_0(kG\text{-stab}) \to \mathbb{Z}/4\mathbb{Z}$. *Furthermore, if*

$$\alpha : \mathrm{K}_0(kG\text{-stab}_{(0)}) \to \mathrm{K}_0(kG\text{-stab}) \cong \mathbb{Z}/4\mathbb{Z}$$

denotes the map induced by the inclusion functor $kG\text{-stab}_{(0)}) \to kG\text{-stab}$, *then*

$$\mathrm{im}(\alpha) \cong \mathbb{Z}/2\mathbb{Z} \subset \mathbb{Z}/4\mathbb{Z} \ .$$

PROOF. The ring kG is local and artinian, and thus it follows from Lemma 2.2.7 that the map

$$[M] \mapsto \mathrm{length}(M) = \dim_k(M)$$

defines an isomorphism $\mathrm{K}_0(kG\text{-mod}) \to \mathbb{Z}$. Therefore, the map

$$[M] \mapsto \dim_k(M) \mod 4$$

defines an isomorphism $\mathrm{K}_0(kG\text{-stab}) \to \mathbb{Z}/4\mathbb{Z}$ by Lemma 3.3.1, as every finitely generated projective module over a local ring is free.

By Corollary 3.4.5, the image of α in $\mathrm{K}_0(kG\text{-stab})$ consists of exactly those classes $[M]$ where M has even dimension, i.e.

$$\dim_k(M) = 0 \mod 4$$

or

$$\dim_k(M) = 2 \mod 4 \ .$$

Thus, $\mathrm{im}(\alpha) \cong \mathbb{Z}/2\mathbb{Z}$. □

3.4.7. PROPOSITION. *There is an isomorphism*

$$\mathrm{CH}_0^\Delta(kG\text{-stab}) \cong \mathbb{Z}/2\mathbb{Z}.$$

PROOF. By definition,

$$\mathrm{Z}_0^\Delta(kG\text{-stab}) = \mathrm{K}_0\left((kG\text{-stab}_{(0)})^\natural\right) \cong \mathrm{K}_0(kG\text{-stab}_{(0)}) ,$$

as $kG\text{-stab}_{(-1)} = 0$ and thick subcategories of idempotent complete categories are idempotent complete themselves. Using this, we have that

$$\mathrm{CH}_0^\Delta(kG\text{-stab}) \cong \mathrm{Z}_0^\Delta(kG\text{-stab})/\ker(\alpha) ,$$

where

$$\alpha : \mathrm{K}_0(kG\text{-stab}_{(0)}) \to \mathrm{K}_0(kG\text{-stab}_{(1)}) = \mathrm{K}_0(kG\text{-stab})$$

is the map from Lemma 3.4.6. Using the isomorphism theorem for abelian groups, we conclude that

$$\mathrm{CH}_0^\Delta(kG\text{-stab}) \cong \mathrm{im}(\alpha) \cong \mathbb{Z}/2\mathbb{Z}$$

by Lemma 3.4.6. □

For the one-dimensional Chow group we need to work a bit harder. We first take a closer look at the quotient $\mathcal{L} := kG\text{-stab}/kG\text{-stab}_{(0)}$.

3.4.8. LEMMA. *Assume k is algebraically closed. The category $\mathcal{L}$ is idempotent complete.*

PROOF. Under the additional hypothesis, it is shown in [CDW94, Example 5.1] that up to isomorphism, the only indecomposable object in $\mathcal{L}$ is k, which has endomorphism ring $K := k(\zeta)$, a transcendental field extension of k. It follows that $\mathcal{L}$ is equivalent to the category of finite-dimensional vector spaces over K, which is idempotent complete. □

This enables us to prove the following:

3.4.9. PROPOSITION. *Assume k is algebraically closed. There is an isomorphism*

$$\mathrm{CH}_1^\Delta(kG\text{-stab}) \cong \mathbb{Z}/2\mathbb{Z}.$$

PROOF. The sequence of triangulated categories

$$kG\text{-stab}_{(0)} \hookrightarrow kG\text{-stab} \to kG\text{-stab}/kG\text{-stab}_{(0)}$$

induces an exact sequence

$$\mathrm{K}_0(kG\text{-stab}_{(0)}) \xrightarrow{\alpha} \mathrm{K}_0(kG\text{-stab}) \to \mathrm{K}_0(kG\text{-stab}/kG\text{-stab}_{(0)}) \to 0$$

where α is the map from Lemma 3.4.6. Therefore,

$$\mathrm{CH}_1^\Delta(kG\text{-stab}) \cong \mathrm{K}_0\left((kG\text{-stab}/kG\text{-stab}_{(0)})^\natural\right) \cong \mathrm{K}_0(kG\text{-stab})/\mathrm{im}(\alpha) \cong \mathbb{Z}/2\mathbb{Z}$$

as follows from Lemma 3.4.8 and Lemma 3.4.6. □

3.5. Relative dimension of restriction and induction

We finish the chapter by showing that stable induction and restriction functors from modular representation theory have a relative dimension as defined in Definition 2.4.1.

Some auxilliary results from representation theory. Let us first give some well-known representation-theoretic results. Let G be a finite group and k a field such that $\mathrm{char}(k) = p$ divides $|G|$.

3.5.1. LEMMA. *Let $H < G$ be a subgroup. There is an isomorphism of left kH-modules*

$$kG \cong \bigoplus_{H\backslash G} kH$$

PROOF. Let $x_1, \dots, x_n \in G$ be a complete set of representatives for $H\backslash G$. First we see that $x_1, \dots, x_n$ span kG as a left kH-module: let $a_1 y_1 + \dots a_m y_m \in kG$ with $a_i \in k$ and $y_i \in G$. Then each y_i is contained in exactly one coset Hx_{j_i}, i.e. there is an element $h_i \in Hx_{j_i}$ such that $y_i = h_i x_{j_i}$. Therefore,

$$a_1 y_1 + \dots a_m y_m = (a_1 h_1) x_{j_1} + \dots + (a_m h_m) x_{j_m} .$$

In order to check linear independence, assume

$$b_1 x_1 + \dots + b_n x_n = 0$$

for $b_i \in kH$. We see that this can only happen if $b_i = 0$ for all i as the cosets Hx_i and Hx_j are mutually disjoint for $i \neq j$. □

3.5.2. LEMMA. *Let $H < G$ be a subgroup. The functors*

$$\mathrm{Ind}_H^G : kH\text{-mod} \to kG\text{-mod}$$

and

$$\mathrm{Res}_H^G : kG\text{-mod} \to kH\text{-mod}$$

are exact.

PROOF. Using Lemma 3.5.1 we see that kG is a free left kH module, and therefore $\mathrm{Ind}_H^G = kG \otimes_{kH} -$ is exact. As Res_H^G acts as the identity on morphisms, it preserves injectivity and surjectivity and therefore is exact. □

3.5.3. LEMMA. *A functor between abelian categories with an exact right-adjoint preserves projective objects. Dually, a functor between abelian categories with an exact left-adjoint preserves injective objects.*

PROOF. Let $F : \mathcal{A} \to \mathcal{B}$ be a functor with an exact right-adjoint G and $P \in \mathcal{A}$ a projective object. By definition, this means that the functor $\mathrm{Hom}_{\mathcal{A}}(P, -)$ is exact. By the adjointness property of G, the functors $\mathrm{Hom}_{\mathcal{B}}(F(P), -)$ and $\mathrm{Hom}_{\mathcal{A}}(P, G(-))$ are naturally isomorphic. But the latter one is a composition of the exact functors G and $\mathrm{Hom}_{\mathcal{A}}(P, -)$ and thus is exact. Therefore $\mathrm{Hom}_{\mathcal{B}}(F(P), -)$ is exact and it follows that $F(P)$ is projective. The argument for the second statement is dual. □

3.5.4. COROLLARY. *The functors Ind_H^G and Res_H^G preserve projective modules.*

PROOF. This follows immediately from the fact that Ind_H^G and Res_H^G are mutually adjoint on both sides (see e.g. [Car96, Proposition 3.2]) and Lemma 3.5.3. □

3.5.5. COROLLARY. *The functors* Ind_H^G *and* Res_H^G *induce exact functors*

$$\underline{\mathrm{Ind}}_H^G : kH\text{-stab} \to kG\text{-stab}$$

and

$$\underline{\mathrm{Res}}_H^G : kG\text{-stab} \to kH\text{-stab}$$

□

3.5.6. THEOREM. *The functors* $\underline{\mathrm{Ind}}_H^G$ *and* $\underline{\mathrm{Res}}_H^G$ *form an adjoint pair that satisfies the projection formula, in the sense of Definition 2.4.7.*

PROOF. The adjunction of the functors $\underline{\mathrm{Ind}}_H^G$ and $\underline{\mathrm{Res}}_H^G$ is obtained from the adjunction of their non-stable counterparts: for modules $M \in kG$-mod, $L \in kH$-mod, the natural isomorphism

$$\Psi : \mathrm{Hom}_{kG}(\mathrm{Ind}_H^G(L), M) \to \mathrm{Hom}_{kH}(L, \mathrm{Res}_H^G(M))$$

is given as

$$\sigma \mapsto \sigma\eta$$

where $\eta : L \to \mathrm{Ind}_H^G(L) = kG \otimes_{kH} L$ is given as the map $l \mapsto 1 \otimes l$ (see [Car96, Proof of Proposition 3.2]). Now if σ factors through a projective module kG module, $\sigma\eta$ will also factor through the restriction of the same projective module to H, which is projective again by Corollary 3.5.4. Thus Ψ induces an isomorphism between the stable homomorphism sets.

Furthermore, Frobenius reciprocity (see e.g. [Car96, Theorem 3.1]) tells us that there are natural isomorphisms in kG-mod

$$\mathrm{Ind}_H^G(L) \otimes M \cong \mathrm{Ind}_H^G(L \otimes \mathrm{Res}_H^G(M))$$

and these descend to the stable category to give us natural isomorphisms

$$\underline{\mathrm{Ind}}_H^G(L) \otimes M \cong \underline{\mathrm{Ind}}_H^G(L \otimes \underline{\mathrm{Res}}_H^G(M)) .$$

This shows that the pair $(\underline{\mathrm{Ind}}_H^G, \underline{\mathrm{Res}}_H^G)$ satisfies the projection formula as desired. □

3.5.7. REMARK. The adjunction between $\underline{\mathrm{Ind}}_H^G$ and $\underline{\mathrm{Res}}_H^G$ can also be deduced from the following more general result: let $F : \mathcal{S} \to \mathcal{T}$ and $G : \mathcal{T} \to \mathcal{S}$ be a pair of adjoint exact functors between triangulated categories $\mathcal{S}, \mathcal{T}$ and let $\mathcal{S}' \subset \mathcal{S}, \mathcal{T}' \subset \mathcal{T}$ be thick triangulated subcategories such that $F(\mathcal{S}') \subset \mathcal{T}'$ and $G(\mathcal{T}') \subset \mathcal{S}'$. Then the induced functors $\overline{F} : \mathcal{S}/\mathcal{S}' \to \mathcal{T}/\mathcal{T}'$ and $\overline{G} : \mathcal{T}/\mathcal{T}' \to \mathcal{S}/\mathcal{S}'$ are adjoint as well. This statement is proved by showing that the unit and the counit of the desired adjunction are given by the images of the unit and counit of the adjunction between F and G under the corresponding localization functors.

In the case of $\underline{\mathrm{Ind}}_H^G$ and $\underline{\mathrm{Res}}_H^G$, the adjunction between Ind_H^G and Res_H^G induces an exact adjunction between the bounded derived categories $\mathrm{D^b}(kH\text{-mod}) =: \mathcal{S}$ and $\mathrm{D^b}(kG\text{-mod}) =: \mathcal{T}$. The roles of $\mathcal{S}'$ and $\mathcal{T}'$ are then played by the thick triangulated subcategories $\mathrm{K}^b(kH\text{-proj})$ and $\mathrm{K}^b(kG\text{-proj})$, respectively.

The following Lemma will be useful when we discuss the relative dimension of the induction functor.

3.5.8. LEMMA. *Let $M \in kH$-mod. Then*

$$\dim_k(\mathrm{Ind}_H^G(M)) = [G : H] \cdot \dim_k(M).$$

PROOF. By Lemma 3.5.1 there is an isomorphism of kH-modules

$$\mathrm{Res}_H^G\left(\mathrm{Ind}_H^G(M)\right) \cong \mathrm{Res}_H^G(kG \otimes_{kH} M) \cong \left(\bigoplus_{G/H} kH\right) \otimes_{kH} M \cong \bigoplus_{G/H} M$$

which proves the lemma as Res_H^G leaves dimensions intact. □

Relative dimension of restriction. We now consider the stable restriction functor

$$\underline{\mathrm{Res}}_H^G : kG\text{-stab} \to kH\text{-stab} \ .$$

We fix the Krull dimension as a dimension function for kG-stab and kH-stab. Recall that for $M \in kX$-stab we have $\dim(\mathrm{supp}(M)) = c_X(M) - 1$ for $X = G, H$. We begin with the following easy observation:

3.5.9. LEMMA. *Let $M \in kG$-mod. Then*

$$\mathrm{c}_H\left(\mathrm{Res}_H^G(M)\right) \leq \mathrm{c}_G(M)$$

PROOF. Assume that $M \in kG$-mod has complexity s. The functor Res_H^G sends a minimal projective resolution $P_\bullet \to M$ to a projective resolution $\mathrm{Res}_H^G(P_\bullet) \to \mathrm{Res}_H^G(M)$ of $\mathrm{Res}_H^G(M)$ by Lemma 3.5.3. A minimal projective resolution $Q_\bullet \to \mathrm{Res}_H^G(M)$ admits an injective chain map to $\mathrm{Res}_H^G(P_\bullet) \to \mathrm{Res}_H^G(M)$ by definition, and therefore we must have

$$\dim_k(Q_n) \leq \dim_k(\mathrm{Res}_H^G(P_n)) = \dim_k(P_n) \leq \kappa \cdot n^{s-1}$$

as $M \in kG$-mod had complexity s. This implies that $\mathrm{Res}_H^G(M)$ has complexity $\leq s$. □

With a little more work we can now compute the relative dimension of $\underline{\mathrm{Res}}_H^G$:

3.5.10. THEOREM. *Let $H \subset G$ be a subgroup such that p divides $|H|$. Then $\underline{\mathrm{Res}}_H^G$ has relative dimension 0.*

PROOF. It follows from Lemma 3.5.9 that for all objects $M \in kG$-stab, we have the inequality $\dim(\mathrm{supp}(\underline{\mathrm{Res}}_H^G)) \leq \dim(\mathrm{supp}(M))$ and thus, if $\underline{\mathrm{Res}}_H^G$ has a relative dimension, it must be ≤ 0. In order to prove the statement, it therefore suffices to show that there is an object M_0 of kG-stab such that $\dim(\mathrm{supp}(\underline{\mathrm{Res}}_H^G(M_0))) \geq \dim(\mathrm{supp}(M_0))$ and therefore $\dim(\mathrm{supp}(\underline{\mathrm{Res}}_H^G(M_0))) = \dim(\mathrm{supp}(M_0))$.

Let $P \in \mathcal{V}_H(k)$ be a closed point (which exists as p divides $|H|$) and look at $Q = \mathrm{Spc}\left(\underline{\mathrm{Res}}_H^G\right)(P) \in \mathcal{V}_G(k)$ which is closed as well since the map $\mathrm{Spc}\left(\underline{\mathrm{Res}}_H^G\right)$ is closed (see [Bal14, Theorem 2.4 (b)]). Take $M_0 \in kG$-stab such that $\mathrm{supp}(M_0) = \{Q\}$. This is possible as we can realize any subvariety as the support of a module, see [Ben98b, Chapter 5.9], or more abstractly [Bal05, Corollary 2.17]. Note that this means that $\dim(\mathrm{supp}(M_0)) = 0$. We know that

$$\mathrm{supp}(\underline{\mathrm{Res}}_H^G(M_0)) = \left(\mathrm{Spc}\left(\underline{\mathrm{Res}}_H^G\right)\right)^{-1}(\mathrm{supp}(M_0))$$

which must contain the closed point P. Thus,

$$\dim(\mathrm{supp}(\underline{\mathrm{Res}}^G_H(M_0))) \geq 0 = \dim(\mathrm{supp}(M_0)) ,$$

which finishes the proof. $\square$

3.5.11. REMARK. Assume that $p \nmid |H|$. Then by Maschke's theorem kH is semi-simple (see e.g. [Car96, Theorem 1.7]), which implies that every finitely generated left kH-module is projective. Consequently, kH-stab $= 0$ and $\underline{\mathrm{Res}}^G_H(M) = 0$ for all modules $M \in kG$-stab. As $\dim(\mathrm{supp}(0)) = \dim(\emptyset) = -\infty$, the functor $\underline{\mathrm{Res}}^G_H$ does not have a relative dimension in this case.

Relative dimension of induction. Let us consider the stable induction functor

$$\underline{\mathrm{Ind}}^G_H : kH\text{-stab} \to kG\text{-stab}$$

next. Again, we fix the Krull dimension as a dimension function for kH-stab and kG-stab.

3.5.12. LEMMA. *Let $M \in kH$-mod. Then*

$$\mathrm{c}_G\left(\mathrm{Ind}^G_H(M)\right) \leq \mathrm{c}_H(M)$$

PROOF. Let $M \in kH$-mod have minimal projective resolution $P_\bullet \to M$. Assume that M has complexity s, then $\dim_k(P_n) \leq \kappa \cdot n^{s-1}$ for all n and some constant κ. As Ind^G_H is an exact functor that preserves projectives, $\mathrm{Ind}^G_H(P_\bullet) \to \mathrm{Ind}^G_H(M)$ is a projective resolution of $\mathrm{Ind}^G_H(M)$. By Lemma 3.5.8, we have that

$$\dim_k \mathrm{Ind}^G_H(P_n) = [G:H]\dim_k(P_n) \leq [G:H]\kappa \cdot n^{s-1}$$

from which it follows that a minimal projective resolution of $\mathrm{Ind}^G_H(M)$ has growth rate at most $s-1$, as it admits an injective chain map to $\mathrm{Ind}^G_H(P_\bullet) \to \mathrm{Ind}^G_H(M)$. Thus $\mathrm{Ind}^G_H(M)$ has complexity at most s. $\square$

We now need two easy auxilliary lemmas concerning projective kG-modules. Recall that $p = \mathrm{char}(k)$.

3.5.13. LEMMA (see [Car96, Corollary 1.6]). *Let p^a be the exact power of p dividing $|G|$ and P a projective kG-module. Then p^a divides* $\dim_k(P)$.

PROOF. Let $S \leq G$ be a Sylow p-subgroup of order p^a. Then $\mathrm{Res}^G_S(P)$ is a projective kS-module. As S is a p-group, projectivity and freeness of kS-modules coincide, so $\dim_k(\mathrm{Res}^G_S(P)) = \dim_k(P)$ is a multiple of $|S| = p^a$. $\square$

3.5.14. LEMMA. *Let $H \leq G$ be a subgroup such that p divides $|H|$. Then we have $\underline{\mathrm{Ind}}^G_H(k) \neq 0$, so in particular* $\mathrm{supp}(\underline{\mathrm{Ind}}^G_H(k))$ *is a non-empty closed subset of $V_G(k)$.*

PROOF. The module $\underline{\mathrm{Ind}}^G_H(k)$ being non-zero is equivalent to $\mathrm{Ind}^G_H(k)$ being non-projective. We know that $\mathrm{Ind}^G_H(k)$ is the permutation representation on the cosets G/H, which has dimension $[G:H]$. If p^a is the exact power of p dividing $|G|$, then assuming that p divides $|H|$ tells us that $p^a \nmid [G:H] = \dim_k(\mathrm{Ind}^G_H(k))$ which implies that $\mathrm{Ind}^G_H(k)$ cannot be projective by Lemma 3.5.13. $\square$

3.5.15. COROLLARY. *Let $H \leq G$ be a subgroup such that p divides $|H|$. Then $\underline{\mathrm{Ind}}^G_H$ has a relative dimension and it is ≤ 0.*

PROOF. From Lemma 3.5.12, we see that $\underline{\mathrm{Ind}}_H^G(kH\text{-stab}_{(n)}) \subset kG\text{-stab}_{(n)}$ for all $n \in \mathbb{Z}_{\geq 0}$. Since $\dim(\mathrm{supp}(\underline{\mathrm{Ind}}_H^G(k))) \geq 0$ by Lemma 3.5.14, we have $\dim\left(\underline{\mathrm{Ind}}_H^G\right) > -\infty$. □

3.5.16. THEOREM. *Let $H \leq G$ be a subgroup such that p divides $|H|$. Then the functor $\underline{\mathrm{Ind}}_H^G$ has relative dimension* 0.

PROOF. By Theorem 3.5.6, Theorem 3.5.10, Lemma 3.5.14 and Corollary 3.5.15, the assumptions of Corollary 2.4.10 are satisfied. Therefore

$$0 \leq \dim\left(\underline{\mathrm{Res}}_H^G\right) + \dim\left(\underline{\mathrm{Ind}}_H^G\right) = \dim\left(\underline{\mathrm{Ind}}_H^G\right)$$

since we already know that the relative dimension of $\underline{\mathrm{Res}}_H^G$ is zero. Together with Corollary 3.5.15 this yields that the relative dimension of $\underline{\mathrm{Ind}}_H^G$ is 0. □

3.5.17. REMARK. If $p \nmid |H|$, then $kH\text{-stab} = 0$ and $\underline{\mathrm{Ind}}_H^G$ is the inclusion of 0 into kG-stab. This functor does not have a relative dimension as

$$\dim(\mathrm{supp}(0)) = \dim(\emptyset) = -\infty\,.$$

CHAPTER 4

Relative tensor triangular Chow groups

So far we have considered tensor triangular Chow groups only for essentially small tensor triangulated categories. For tensor triangulated categories that are not essentially small we run into problems: for example, for such categories $\mathcal{T}$ we have no definition of $\mathrm{Spc}(\mathcal{T})$, so it does not even make sense to define subcategories like $\mathcal{T}_{(p)}$. The situation changes when we assume that $\mathcal{T}$ is compactly rigidly generated, i.e. the compact objects $\mathcal{T}^c \subset \mathcal{T}$ form a set, coincide with the rigid ones, and they generate $\mathcal{T}$. In this case, it is shown by Balmer-Favi in [BF11] that there is a notion of support for objects of $\mathcal{T}$ which assigns to an object $A \in \mathcal{T}$ a (not necessarily closed) subset $\mathrm{supp}(A) \subset \mathrm{Spc}(\mathcal{T}^c)$. This is generalized by Stevenson in [Ste13], which introduces the concept of an action of a compactly-rigidly generated tensor triangulated category $\mathcal{T}$ on a triangulated category $\mathcal{K}$ (which need not have a symmetric monoidal structure). In this setting it is possible to define a notion of relative support for objects of $\mathcal{K}$, which assigns to an object $A \in \mathcal{K}$ a subset $\mathrm{supp}(A) \subset \mathrm{Spc}(\mathcal{T}^c)$. It recovers the notion of [BF11] mentioned before, when we set $\mathcal{K} = \mathcal{T}$ and act via the tensor product of $\mathcal{T}$. This construction is the starting point for our definition of tensor triangular Chow groups for $\mathcal{K}$, relative to the action of $\mathcal{T}$.

In the following section, a lot of notation will be introduced. For clarity and reference, we include an overview below.

4.1. Preliminaries

Let $\mathcal{T}$ be a triangulated category.

4.1.1. DEFINITION. The category $\mathcal{T}$ is called a *compactly-rigidly generated tensor triangulated category* if

(i) *$\mathcal{T}$ is compactly generated* (see Definition 1.1.24).

(ii) *$\mathcal{T}$ is equipped with a compatible closed symmetric monoidal structure*

$$\otimes : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$$

with unit object $\mathbb{I}$. Here, a symmetric monoidal structure on $\mathcal{T}$ is *closed* if for all objects $A \in \mathcal{T}$ the functor $A \otimes -$ has a right adjoint $\underline{\mathrm{hom}}(A,-)$. A *compatible* closed symmetric monoidal structure on $\mathcal{T}$ is one such that $\otimes$ satisfies the conditions of Definition 1.2.1 and Remark 1.2.2 and such that $\underline{\mathrm{hom}}(A,-)$ is exact for all objects $A \in \mathcal{T}$. (This last condition is actually redundant since adjoints of exact functors are automatically exact, see [Nee01, Lemma 5.3.6].)

(iii) *$\mathbb{I}$ is compact and all compact objects of $\mathcal{T}$ are rigid.* Let $\mathcal{T}^c \subset \mathcal{T}$ denote the full subcategory of compact objects of $\mathcal{T}$. Then we require that $\mathbb{I} \in \mathcal{T}^c$ and that all objects A of $\mathcal{T}^c$ are rigid, i.e. for every object $B \in \mathcal{T}$ the natural map

$$\underline{\circ} : \underline{\mathrm{hom}}(A,\mathbb{I}) \otimes B \cong \underline{\mathrm{hom}}(A,\mathbb{I}) \otimes \underline{\mathrm{hom}}(\mathbb{I},B) \to \underline{\mathrm{hom}}(A,B) \ ,$$

Table of notations $x \in \mathrm{Spc}(\mathcal{T}^c), A \in \mathcal{K}, p \in \mathbb{Z}$		
Γ_V, L_V for $V \subset \mathrm{Spc}(\mathcal{T}^c)$	Acyclization and localization functor associated to a specialization-closed set $V \subset \mathrm{Spc}(\mathcal{T}^c)$	p. 61
Y_x	$\{\mathcal{P} \in \mathrm{Spc}(\mathcal{T}^c) : x \not\subseteq \mathcal{P}\}$	p. 62
$\Gamma_x A$	$(\Gamma_{\overline{\{x\}}} L_{Y_x}(\mathbb{I})) * A$	p. 62
$V_{\leq p}$	$\{x \in \mathrm{Spc}(\mathcal{T}^c) \| \dim(x) \leq p\}$	p. 63
V_p	$\{x \in \mathrm{Spc}(\mathcal{T}^c) \| \dim(x) = p\}$	p. 63
$\Gamma_p A$	$(\Gamma_{V_{\leq p}} L_{V_{\leq p-1}}(\mathbb{I})) * A$	p. 63
$\mathcal{K}_{(p)}$	$\tau_{\mathcal{K}}(V_{\leq p}) = \{A \in \mathcal{K} : \mathrm{supp}(A) \subset V_{\leq p}\}$	p. 63
$\mathcal{K}_x$	Essential image of $L_{\langle x \rangle}(\mathbb{I}) * -$	p. 64

is an isomorphism.

4.1.2. CONVENTION. Throughout this chapter we assume that $\mathcal{T}$ is a compactly-rigidly generated tensor triangulated category that satisfies the following conditions (cf. [BF11, Hypothesis 1.1]:

(i) *$\mathcal{T}^c$ is equipped with a dimension function* dim *and* $\mathrm{Spc}(\mathcal{T}^c)$ *is a noetherian topological space*. The subcategory $\mathcal{T}^c$ is a tensor triangulated category in the sense of Convention 2.2.1 that is also rigid (see [BF11, Hypothesis 1.1]). Thus it makes sense to talk about $\mathrm{Spc}(\mathcal{T}^c)$ and dimension functions on $\mathcal{T}^c$.

(ii) *$\mathcal{T}$ acts on a (fixed) triangulated category $\mathcal{K}$ via an action $*$ in the sense of* [Ste13]. We assume $\mathcal{K}$ to be compactly generated as well.

Note that $\mathcal{T}$ is not essentially small since it has set-indexed coproducts.

Let us quickly recall from [Ste13, Definition 3.2] what it means for $\mathcal{T}$ to have an action $*$ on $\mathcal{K}$. We are given a biexact bifunctor

$$* : \mathcal{T} \times \mathcal{K} \to \mathcal{K}$$

that commutes with coproducts in both variables, whenever they exist. Furthermore we are given natural isomorphisms

$$\alpha_{X,Y,A} : (X \otimes Y) * A \xrightarrow{\sim} X * (Y * A)$$
$$l_A : \mathbb{I} * A \xrightarrow{\sim} A$$

for all objects $X, Y \in \mathcal{T}, A \in \mathcal{K}$. These natural isomorphisms should satisfy a list of coherence relations. For X, Y, Z objects of $\mathcal{T}$ and A an object of $\mathcal{K}$, the following diagrams need to be commutative:

(1)

$$\begin{tikzcd}
 & X * (Y * (Z * A)) & \\
X * ((Y \otimes Z) * A) \arrow[ur, "X * \alpha_{Y,Z,A}"] & & (X \otimes Y) * (Z * A) \arrow[ul, "\alpha_{X,Y,Z*A}"'] \\
(X \otimes (Y \otimes Z)) * A \arrow[u, "\alpha_{X,Y\otimes Z,A}"] & & ((X \otimes Y) \otimes Z) * A \arrow[ll] \arrow[u, "\alpha_{X\otimes Y,Z,A}"']
\end{tikzcd}$$

where the lower unlabeled arrow is the associator isomorphism from the symmetric monoidal structure on $\mathcal{T}$.

(2)

$$\begin{tikzcd} X*(\mathbb{I}*A) \arrow[r, "X*l_A"] & X*A \\ (X\otimes\mathbb{I})*A \arrow[u, "\alpha_{X,\mathbb{I},A}"] \arrow[ur, "u*A"'] & \end{tikzcd} \qquad \begin{tikzcd} \mathbb{I}*(X*A) \arrow[r, "l_{X*A}"] & X*A \\ (\mathbb{I}\otimes X)*A \arrow[u, "\alpha_{\mathbb{I},X,A}"] \arrow[ur, "v*A"'] & \end{tikzcd}$$

where u, v are the respective left and right unitor isomorphisms from the symmetric monoidal structure on $\mathcal{T}$.

(3)

$$\begin{tikzcd} \Sigma^r\mathbb{I}*\Sigma^s A \arrow[r, "f"] \arrow[d, "e"] & \Sigma^{r+s}A \arrow[d, "(-1)^{rs}"] \\ \Sigma^r(\mathbb{I}*\Sigma^s A) \arrow[r, "\Sigma^r(l_{\Sigma^s A})"] & \Sigma^{r+s}A \end{tikzcd}$$

where e comes from the exactness of $*$ in the first variable, and f is the composition

$$\Sigma^r\mathbb{I}*\Sigma^s A \to \Sigma^s(\Sigma^r\mathbb{I}*A) \to \Sigma^{r+s}(\mathbb{I}*A) \xrightarrow{\Sigma^{r+s}(l_A)} \Sigma^{r+s}A$$

with the first two isomorphisms coming from the biexactness of $*$.

4.1.3. REMARK. With this definition, a tensor triangulated category $\mathcal{T}$ as in Convention 4.1.2 has an action on itself via $\otimes$. The natural isomorphisms $\alpha_{X,Y,A}, l_A$ are then given by the associator and unitor of the symmetric monoidal structure on $\mathcal{T}$ and one checks that all the required coherence conditions hold as the coherence conditions for the monoidal structure on $\mathcal{T}$ are satsified and the bifunctor $\otimes$ is compatible with the triangulated structure on $\mathcal{T}$. The functor $\otimes$ always preserves coproducts in both variables for any closed symmetric monoidal structure on any category, as the functor $a\otimes -$ has a right adjoint for all objects $a \in \mathcal{T}$ (see e.g. [HPS97, Remark A.2.2]).

Following [BF11], we can assign to every object $a \in \mathcal{T}$ a support $\operatorname{supp}(a) \subset \operatorname{Spc}(\mathcal{T}^c)$: given a specialization-closed subset $V \subset \operatorname{Spc}(\mathcal{T}^c)$, we have a distinguished triangle

$$\Gamma_V(\mathbb{I}) \to \mathbb{I} \to L_V(\mathbb{I}) \to \Sigma\Gamma_V(\mathbb{I})$$

where Γ_V and L_V are the acyclization and localization functors associated to the smashing ideal that is generated by the compact objects with support in V (see Remark 1.1.21 and Definition 1.3.1). For objects $A \in \mathcal{K}$, we set $\Gamma_V A := \Gamma_V(\mathbb{I}) * A$ and $L_V A := L_V(\mathbb{I}) * A$. Note that if $\mathcal{K} = \mathcal{T}$ and $*$ is given by $\otimes$ as in Remark 4.1.3, then this definition yields the similar expressions $\Gamma_V(A), \Gamma_V A$ and $L_V(A), L_V A$. However, by Proposition 1.3.2, these actually give isomorphic objects, so there should be no room for confusion.

For a point $x \in \operatorname{Spc}(\mathcal{T}^c)$ the subsets $\overline{\{x\}}$ and $Y_x := \{\mathcal{P} \in \operatorname{Spc}(\mathcal{T}^c) : x \not\subseteq \mathcal{P}\}$ are specialization-closed and so we define the "residue object" $\Gamma_x \mathbb{I} \in \mathcal{T}$ as $\Gamma_{\overline{\{x\}}} L_{Y_x}(\mathbb{I})$. For $A \in \mathcal{T}$, we now define the *support of an object* $A \in \mathcal{T}$ as

$$\operatorname{supp}(A) := \{x \in \operatorname{Spc}(\mathcal{T}^c) | \Gamma_x \mathbb{I} \otimes A \neq 0\}.$$

In [Ste13], the same residue objects are used to define supports for objects of $\mathcal{K}$. For an object $B \in \mathcal{K}$, set $\Gamma_x B := \Gamma_x \mathbb{I} * B$, then we define the *support of* B as

$$\operatorname{supp}(B) := \{x \in \operatorname{Spc}(\mathcal{T}^c) | \Gamma_x B \neq 0\}.$$

This notion of support gives us a way to describe certain subcategories of $\mathcal{K}$. A triangulated subcategory $\mathcal{M} \subset \mathcal{K}$ is called *localizing* $\mathcal{T}$*-submodule* if it is a localizing subcategory (see Section 1.1) such that $\mathcal{T} * \mathcal{M} \subset \mathcal{M}$. We obtain order-preserving maps

$$\begin{aligned} \{\text{subsets of } \operatorname{Spc}(\mathcal{T}^c)\} &\underset{\sigma_{\mathcal{K}}}{\overset{\tau_{\mathcal{K}}}{\rightleftarrows}} \{\text{localizing } \mathcal{T}\text{-submodules of } \mathcal{K}\} \\ S &\mapsto \{t \in \mathcal{K} : \operatorname{supp}(t) \subset S\} \\ \bigcup_{t \in \mathcal{M}} \operatorname{supp}(t) &\mapsfrom \mathcal{M} \end{aligned}$$

where the ordering on both sides is given by inclusion (see [Ste13, Definition 5.9]).

We record the following properties of the support that will be very useful for the sequel.

4.1.4. PROPOSITION (cf. [Ste13, Proposition 5.7]). *Let $V \subset \operatorname{Spc}(\mathcal{T}^c)$ be a subset closed under specialization and A be an object of $\mathcal{K}$. Then*

$$\operatorname{supp}(\Gamma_V(\mathbb{I}) * A) = \operatorname{supp}(A) \cap V$$

and

$$\operatorname{supp}(L_V(\mathbb{I}) * A) = \operatorname{supp}(A) \cap (\operatorname{Spc}(\mathcal{T}^c) \setminus V)\ .$$

4.1.5. REMARK. In [Ste13, Proposition 5.7], Proposition 4.1.4 is proved for those specialization-closed subsets V which are contained in the subset $\operatorname{Vis}(\mathcal{T}^c) \subset \operatorname{Spc}(\mathcal{T}^c)$ of so-called *visible* points of the spectrum. The set $\operatorname{Vis}(\mathcal{T}^c)$ coincides with $\operatorname{Spc}(\mathcal{T})$ in our case, since we assumed the latter space to be noetherian (see e.g. [Ste13, Section 5]).

4.1.6. DEFINITION (cf. [Ste13, Definition 6.1]). We say that the action $*$ of $\mathcal{T}$ on $\mathcal{K}$ satisfies the *local-to-global principle* if for each A in $\mathcal{K}$

$$\langle A \rangle_* = \langle \Gamma_x A | x \in \operatorname{Spc}(\mathcal{T}^c) \rangle_*$$

where for a collection of objects $S \subset \mathcal{K}$ we denote by $\langle S \rangle_*$ the smallest localizing $\mathcal{T}$-submodule of $\mathcal{K}$ containing S.

4.1.7. REMARK. The local-to-global principle holds very often, e.g. when $\mathcal{T}$ arises as the homotopy category of a monoidal model category (cf. [Ste13, Proposition 6.8]). If the local-to-global principle holds, it has the useful consequence that supp detects the vanishing of an object: if $\operatorname{supp}(A) = \emptyset$ then

$$\langle A \rangle_* = \langle \Gamma_x A | x \in \operatorname{Spc}(\mathcal{T}^c) \rangle_* = \langle 0 \rangle_* = 0$$

which implies that $A = 0$.

For $p \in \mathbb{Z}$, let

$$V_{\leq p} := \{x \in \operatorname{Spc}(\mathcal{T}^c) \,|\, \dim(x) \leq p\}, \quad V_p := \{x \in \operatorname{Spc}(\mathcal{T}^c) \,|\, \dim(x) = p\}$$

and set $\Gamma_p A := \Gamma_{V_{\leq p}} L_{V_{\leq p-1}} A$. In [Ste12], a decomposition theorem analogous to Theorem 1.4.7 is proved. Let us first fix some notation:

4.1.8. NOTATION (cf. [Ste12, Notation 3.6]). Let $\mathcal{L}_i$ be a collection of localizing subcategories of $\mathcal{K}$, indexed by a set I. Then $\prod_{i \in I} \mathcal{L}_i$ denotes the subcategory of $\mathcal{K}$ whose objects are coproducts of the objects of $\mathcal{L}_i$, where the morphisms and the triangulated structure are defined componentwise with respect to I.

4.1.9. PROPOSITION (cf. [Ste12, Lemma 4.3]). *Suppose that the action of $\mathcal{T}$ on $\mathcal{K}$ satisfies the local-to-global principle and let $p \in \mathbb{Z}$. There is an equality of subcategories*

$$\Gamma_p \mathcal{K} = \tau_{\mathcal{K}}(V_p) = \prod_{x \in V_p} \Gamma_x \mathcal{K}$$

where $\Gamma_x \mathcal{K}$ denotes the essential image of the functor $\Gamma_x(\mathbb{I}) * -$. □

We give another description of $\Gamma_p \mathcal{K}$ that bears more resemblance to what we have seen for essentially small tensor triangulated categories. For $p \in \mathbb{Z}$, define

$$\mathcal{K}_{(p)} := \tau_{\mathcal{K}}(V_{\leq p}) \,.$$

4.1.10. LEMMA. *Assume the local-to-global principle holds for the action of $\mathcal{T}$ on $\mathcal{K}$. Then, for all $p \in \mathbb{Z}$, there is an equality of subcategories*

$$\mathcal{K}_{(p)} = \Gamma_{V_{\leq p}} \mathcal{K} = \{A \in \mathcal{K} \,|\, \exists A' : A \cong \Gamma_{V_{\leq p}}(\mathbb{I}) * A'\}$$

PROOF. Let $A \in \Gamma_{V_{\leq p}} \mathcal{K}$, then we have an isomorphism $A \cong \Gamma_{V_{\leq p}}(\mathbb{I}) * A'$. By Proposition 4.1.4, we know that $\operatorname{supp}(A) = \operatorname{supp}(\Gamma_{V_{\leq p}}(\mathbb{I}) * A') = \operatorname{supp}(A') \cap V_{\leq p}$, from which it follows that A is supported in dimension $\leq p$. Thus, $A \in \mathcal{K}_{(p)}$.

Conversely, assume that $A \in \mathcal{K}_{(p)}$. We apply the functor $- * A$ to the localisation triangle

$$\Gamma_{V_{\leq p}}(\mathbb{I}) \to \mathbb{I} \to L_{V_{\leq p}}(\mathbb{I}) \to \Sigma \Gamma_{V_{\leq p}}(\mathbb{I})$$

to obtain the triangle

$$\Gamma_{V_{\leq p}}(\mathbb{I}) * A \to A \to L_{V_{\leq p}}(\mathbb{I}) * A \to \Sigma \Gamma_{V_{\leq p}}(\mathbb{I}) * A$$

As A is supported in dimension $\leq p$, it follows again from Proposition 4.1.4 that

$$\operatorname{supp}(L_{V_{\leq p}}(\mathbb{I}) * A) = \emptyset$$

and therefore $L_{V_{\leq p}}(\mathbb{I}) * A = 0$ by the local-to-global principle, as shown in Remark 4.1.7. Therefore $\Gamma_{V_{\leq p}}(\mathbb{I}) * A \cong A$ which implies that $A \in \Gamma_{V_{\leq p}} \mathcal{K}$. □

The following statement is the desired description of $\Gamma_p \mathcal{K}$.

4.1.11. LEMMA. *Suppose that the action of $\mathcal{T}$ on $\mathcal{K}$ satisfies the local-to-global principle and let $p \in \mathbb{Z}$. There is an equality of subcategories*

$$\Gamma_p \mathcal{K} = \mathcal{K}_{(p)} / \mathcal{K}_{(p-1)}$$

where we view the latter quotient as the essential image of the functor $L_{V_{\leq p-1}}(\mathbb{I}) * -$ restricted to $\mathcal{K}_{(p)}$.

PROOF. If A is an object of $\Gamma_p\mathcal{K}$, we have $\mathrm{supp}(A) \subset V_p \subset V_{\leq p}$, so we certainly have $A \in \mathcal{K}_{(p)}$. If we apply $- * A$ to the localization triangle

$$\Gamma_{V_{\leq p-1}}(\mathbb{I}) \to \mathbb{I} \to L_{V_{\leq p-1}}(\mathbb{I}) \to \Sigma\Gamma_{V_{\leq p-1}}(\mathbb{I})$$

we obtain the distinguished triangle

$$\Gamma_{V_{\leq p-1}}(\mathbb{I}) * A \to A \to L_{V_{\leq p-1}}(\mathbb{I}) * A \to \Sigma\Gamma_{V_{\leq p-1}} * A(\mathbb{I})$$

where $\mathrm{supp}(\Gamma_{V_{\leq p-1}}(\mathbb{I}) * A) = V_{\leq p-1} \cap \mathrm{supp}(A) = \emptyset$ by Proposition 4.1.4. Thus, we have $\Gamma_{V_{\leq p-1}}(\mathbb{I}) * A \cong 0$ and we obtain an isomorphism $A \cong L_{V_{\leq p-1}}(\mathbb{I}) * A$, which proves that A is in the essential image of $L_{V_{\leq p-1}}(\mathbb{I}) * -$ restricted to $\mathcal{K}_{(p)}$.

On the other hand, if A is an object of the essential image of $L_{V_{\leq p-1}}(\mathbb{I}) * -$ restricted to $\mathcal{K}_{(p)}$, there exists an object A' of $\mathcal{K}_{(p)}$ such that $A \cong L_{V_{\leq p-1}}(\mathbb{I}) * A'$. By Lemma 4.1.10, we know that $\mathrm{supp}(A') \subset V_{\leq p}$. But then by Proposition 4.1.4,

$$\mathrm{supp}(A) = \mathrm{supp}(A') \cap (\mathrm{Spc}(\mathcal{T}^c) \setminus V_{\leq p-1}) \subset V_{\leq p} \cap (\mathrm{Spc}(\mathcal{T}^c) \setminus V_{\leq p-1}) = V_p$$

which proves that $A \in \Gamma_p\mathcal{K}$. □

We can push the analogy with Theorem 1.4.7 even further: for $x \in \mathrm{Spc}(\mathcal{T}^c)$, define $\mathcal{K}_x$ as the essential image of the localisation functor

$$L_{\langle x\rangle}(\mathbb{I}) * -$$

associated to the localizing subcategory $\langle x\rangle \subset \mathcal{T}$ and denote by $\mathrm{Min}(\mathcal{K}_x) \subset \mathcal{K}_x$ the subcategory of objects with support contained in $\{x\}$.

4.1.12. PROPOSITION. *Suppose that the action of $\mathcal{T}$ on $\mathcal{K}$ satisfies the local-to-global principle and let $x \in \mathrm{Spc}(\mathcal{T}^c)$. Then there is an equality of subcategories*

$$\mathrm{Min}(\mathcal{K}_x) = \tau_{\mathcal{K}}(\{x\}) = \Gamma_x\mathcal{K} .$$

PROOF. For the first equality, if A is an object of $\mathrm{Min}(\mathcal{K}_x)$, then by definition we have $\mathrm{supp}(A) \subset \{x\}$ which implies that A is contained in $\tau_{\mathcal{K}}(\{x\})$.

If A is in $\tau_{\mathcal{K}}(\{x\})$, we need to prove that it is in the essential image of the localisation functor $L_{\langle x\rangle}(\mathbb{I}) * -$. In order to see this, notice that

$$x = \{B \in \mathcal{T}^c : \mathrm{supp}(B) \subset Y_x\}$$

which implies that $L_{\langle x\rangle}(\mathbb{I}) = L_{Y_x}(\mathbb{I})$. In the corresponding localization triangle

$$\Gamma_{Y_x}(\mathbb{I}) * A \to A \to L_{Y_x}(\mathbb{I}) * A \to \Sigma\Gamma_{Y_x}(\mathbb{I}) * A$$

we obtain by Proposition 4.1.4 that

$$\mathrm{supp}(\Gamma_{Y_x}(\mathbb{I}) * A) \subset Y_x \cap \{x\} = \emptyset ,$$

which implies that $\Gamma_{Y_x}(\mathbb{I}) * A \cong 0$ and $A \cong L_{Y_x}(\mathbb{I}) * A \cong L_{\langle x\rangle}(\mathbb{I}) * A$. This shows that A is in the essential image of $L_{\langle x\rangle}(\mathbb{I}) * -$.

For the second equality, if A is an object of $\tau_{\mathcal{K}}(\{x\})$, we have $\mathrm{supp}(A) \subset \{x\}$. Using Proposition 4.1.4 and the localization triangles

$$\Gamma_{Y_x}(\mathbb{I}) * A \to A \to L_{Y_x}(\mathbb{I}) * A \to \Sigma\Gamma_{Y_x}(\mathbb{I}) * A$$

and

$$\Gamma_{\overline{\{x\}}}(\mathbb{I}) * A \to A \to L_{\overline{\{x\}}}(\mathbb{I}) * A \to \Sigma\Gamma_{\overline{\{x\}}}(\mathbb{I}) * A$$

we see that $A \cong L_{Y_x}(\mathbb{I}) * A$ and $A \cong \Gamma_{\overline{\{x\}}}(\mathbb{I}) * A$. Combining these isomorphisms, we get that

$$A \cong (\Gamma_{\overline{\{x\}}}(\mathbb{I}) \otimes L_{Y_x}(\mathbb{I})) * A = \Gamma_x A,$$

which shows that A is contained in the essential image of $\Gamma_x(\mathbb{I}) * -$.

If A is in the essential image of $\Gamma_x(\mathbb{I}) * -$, then there is an A' such that

$$A \cong (\Gamma_{\overline{\{x\}}}(\mathbb{I}) \otimes L_{Y_x}(\mathbb{I})) * A' .$$

Applying Proposition 4.1.4, we get that

$$\mathrm{supp}(A) = \overline{\{x\}} \cap (\mathrm{Spc}(\mathcal{T}^c) \setminus Y_x) \cap \mathrm{supp}(A') \subset \overline{\{x\}} \cap (\mathrm{Spc}(\mathcal{T}^c) \setminus Y_x) = \{x\} ,$$

which proves that A is an object of $\tau_{\mathcal{K}}(\{x\})$. □

Proposition 4.1.9 and Lemma 4.1.11 serve as a motivation for the definition of relative tensor triangular cycle groups (see Definition 4.2.1), in the same way that Theorem 1.4.7 motivated Definition 2.2.3.

We finish the section with a useful result about the subcategories $\mathcal{K}_{(p)}$.

4.1.13. PROPOSITION. *Suppose that the action of $\mathcal{T}$ on $\mathcal{K}$ satisfies the local-to-global principle. Then the subcategories $\mathcal{K}_{(p)}$ are compactly generated for all p and $(\mathcal{K}_{(p)})^c = (\mathcal{K}^c)_{(p)}$.*

PROOF. As the set $V_{\leq p}$ is specialization-closed for all $p \in \mathbb{Z}$, it follows from [Ste13, Corollary 4.11] that $\Gamma_{V_{\leq p}}\mathcal{K}$ is compactly generated for all $p \in \mathbb{Z}$. But $\Gamma_{V_{\leq p}}\mathcal{K}$ is equal to $\mathcal{K}_{(p)}$ by Lemma 4.1.10, and so $\mathcal{K}_{(p)}$ is compactly generated for all $p \in \mathbb{Z}$.

The subcategory $\mathcal{K}_{(p)}$ is precisely the kernel of the coproduct-preserving localization functor $L_{V_{\leq p}}(\mathbb{I}) * -$: indeed, if A is an object of $\mathcal{K}_{(p)}$, then by Proposition 4.1.4

$$\mathrm{supp}(L_{V_{\leq p}}(\mathbb{I}) * A) = (\mathrm{Spc}(\mathcal{T}) \setminus V_{\leq p}) \cap \mathrm{supp}(A) = \emptyset ,$$

which implies that $L_{V_{\leq p}}(\mathbb{I}) * A = 0$ as we have assumed the local-to-global principle, and hence $A \in \ker(L_{V_{\leq p}})$. On the other hand, if we assume that

$$A \in \ker(L_{V_{\leq p}}(\mathbb{I}) * -) ,$$

then from the localization triangle

$$\Gamma_{V_{\leq p}}(\mathbb{I}) * A \to A \to L_{V_{\leq p}}(\mathbb{I}) * A \to \Sigma\Gamma_{V_{\leq p}}(\mathbb{I}) * A$$

we obtain that $A \cong \Gamma_{V_{\leq p}}(\mathbb{I}) * A$. But $\mathrm{supp}(\Gamma_{V_{\leq p}}(\mathbb{I}) * A) = \mathrm{supp}(A) \cap V_{\leq p} \subset V_{\leq p}$ by Proposition 4.1.4, which implies that $A \in \mathcal{K}_{(p)}$.

By [Kra10, Proposition 5.5.1] the right adjoint of the inclusion

$$\iota : \mathcal{K}_{(p)} = \ker(L_{V_{\leq p}}(\mathbb{I}) * -) \to \mathcal{T}$$

preserves coproducts and by [Kra10, Lemma 5.4.1] it follows that ι preserves compactness. Therefore, $(\mathcal{K}_{(p)})^c \subset (\mathcal{K}^c)_{(p)}$. The converse inclusion is an immediate consequence of the definition of compactness. □

In the light of the equality of subcategories of Proposition 4.1.13, we will simply use the notation $\mathcal{K}^c_{(p)}$ for $(\mathcal{K}_{(p)})^c = (\mathcal{K}^c)_{(p)}$ if the local-to-global principle holds.

4.2. Relative tensor triangular Chow groups

In addition to the hypotheses from Convention 4.1.2, we will assume that the local-to-global principle holds for the action of $\mathcal{T}$ on $\mathcal{K}$ for the rest of the section. For clarity, we will still explicitly mention this hypothesis in the formulation of the results.

The category $\mathcal{K}_{(p)}$ is compactly generated for all $p \in \mathbb{Z}$ by Proposition 4.1.13. We therefore have that $(\mathcal{K}_{(p)}/\mathcal{K}_{(p-1)})^c$ is the thick closure of $\mathcal{K}^c_{(p)}/\mathcal{K}^c_{(p-1)}$ in $\mathcal{K}_{(p)}/\mathcal{K}_{(p-1)}$ (see [Kra10, Theorem 5.6.1]). Thus, we get an injection

$$j : \mathrm{K}_0\left(\mathcal{K}^c_{(p)}/\mathcal{K}^c_{(p-1)}\right) \hookrightarrow \mathrm{K}_0\left((\mathcal{K}_{(p)}/\mathcal{K}_{(p-1)})^c\right) .$$

Furthermore, the quotient functor $\mathcal{K}^c_{(p)} \to \mathcal{K}^c_{(p)}/\mathcal{K}^c_{(p-1)}$ and the embedding $\mathcal{K}^c_{(p)} \to \mathcal{K}^c_{(p+1)}$ induce maps

$$q : \mathrm{K}_0\left(\mathcal{K}^c_{(p)}\right) \to \mathrm{K}_0\left(\mathcal{K}^c_{(p)}/\mathcal{K}^c_{(p-1)}\right)$$

and

$$i : \mathrm{K}_0\left(\mathcal{K}^c_{(p)}\right) \to \mathrm{K}_0\left(\mathcal{K}^c_{(p+1)}\right) .$$

4.2.1. DEFINITION. We define the *p-dimensional tensor triangular cycle groups of $\mathcal{K}$, relative to the action of $\mathcal{T}$* and the *p-dimensional tensor triangular Chow groups of $\mathcal{K}$, relative to the action of $\mathcal{T}$* as follows:

$$\mathrm{Z}^\Delta_p(\mathcal{T},\mathcal{K}) := \mathrm{K}_0\left((\mathcal{K}_{(p)}/\mathcal{K}_{(p-1)})^c\right)$$

and

$$\mathrm{CH}^\Delta_p(\mathcal{T},\mathcal{K}) := \mathrm{Z}^\Delta_{(p)}(\mathcal{T},\mathcal{K})/j \circ q(\ker(i)).$$

4.2.2. REMARK. As we assumed that the local-to-global principle is satisfied, we can view an element of $\mathrm{Z}^\Delta_p(\mathcal{T},\mathcal{K})$ as a formal sum of p-dimensional points x_i of $\mathrm{Spc}(\mathcal{T}^c)$, with coefficients in $\mathrm{K}_0\left((\Gamma_x\mathcal{K})^c\right)$ for $x \in V_p$, by Proposition 4.1.9 and Lemma 4.1.11.

4.2.3. REMARK. The category $\mathcal{K}_{(p)}/\mathcal{K}_{(p-1)}$ has arbitrary coproducts and is therefore idempotent complete (cf. [Nee01, Proposition 1.6.8]). Since $(\mathcal{K}_{(p)}/\mathcal{K}_{(p-1)})^c$ is the thick closure of $\mathcal{K}^c_{(p)}/\mathcal{K}^c_{(p-1)}$ in $\mathcal{K}_{(p)}/\mathcal{K}_{(p-1)}$, we obtain that $(\mathcal{K}_{(p)}/\mathcal{K}_{(p-1)})^c$ is equivalent to the idempotent completion $(\mathcal{K}^c_{(p)}/\mathcal{K}^c_{(p-1)})^\natural$.

Next, we compare $\mathrm{CH}^\Delta_p(\mathcal{T},\mathcal{T})$ to $\mathrm{CH}^\Delta_p(\mathcal{T}^c)$.

4.2.4. PROPOSITION. *Consider the action of $\mathcal{T}$ on itself via the tensor product $\otimes$ and assume that the local-to-global principle holds for this action. Then we have isomorphisms*

$$\mathrm{Z}^\Delta_p(\mathcal{T},\mathcal{T}) \cong \mathrm{Z}^\Delta_p(\mathcal{T}^c) \quad \textit{and} \quad \mathrm{CH}^\Delta_p(\mathcal{T},\mathcal{T}) \cong \mathrm{CH}^\Delta_p(\mathcal{T}^c) .$$

PROOF. By definition we have $\mathrm{Z}^\Delta_p(\mathcal{T},\mathcal{T}) = \mathrm{K}_0\left((\mathcal{T}_{(p)}/\mathcal{T}_{(p-1)})^c\right)$ and Remark 4.2.3 shows that $(\mathcal{T}_{(p)}/\mathcal{T}_{(p-1)})^c$ is equivalent to $\left(\mathcal{T}^c_{(p)}/\mathcal{T}^c_{(p-1)}\right)^\natural$. We conclude that

$$\mathrm{Z}^\Delta_p(\mathcal{T},\mathcal{T}) = \mathrm{K}_0\left((\mathcal{T}_{(p)}/\mathcal{T}_{(p-1)})^c\right) \cong \mathrm{K}_0\left(\left(\mathcal{T}^c_{(p)}/\mathcal{T}^c_{(p-1)}\right)^\natural\right)$$

which is equal to $\mathrm{Z}_p^{\Delta}(\mathcal{T}^c)$ by definition. The notions of rational equivalence agree as well as the maps j, q, i from Definition 4.2.1 are equal to the corresponding maps from Definition 2.2.4. □

Now, let X be a noetherian separated scheme and let $\mathrm{D}(X) := \mathrm{D}(\mathrm{Qcoh}(X))$ be the full derived category of complexes of quasi-coherent $\mathcal{O}_X$-modules. The category $\mathrm{D}(X)$ is a compactly-rigidly generated tensor triangulated category with arbitrary coproducts (see [BF11, Example 1.2]), and we have $\mathrm{D}(X)^c = \mathrm{D}^{\mathrm{perf}}(X)$ (cf. [BvdB03, Theorem 3.1.1]).

4.2.5. COROLLARY. *We have isomorphisms*

$$\mathrm{Z}_p^{\Delta}(\mathrm{D}(X), \mathrm{D}(X)) \cong \mathrm{Z}_p^{\Delta}(\mathrm{D}^{\mathrm{perf}}(X))$$

and

$$\mathrm{CH}_p^{\Delta}(\mathrm{D}(X), \mathrm{D}(X)) \cong \mathrm{CH}_p^{\Delta}(\mathrm{D}^{\mathrm{perf}}(X))$$

for all $p \in \mathbb{Z}$. In particular, if X is non-singular, of finite type over a field and we equip $\mathrm{D}(X)^c$ with the opposite of the Krull codimension as a dimension function, we have

$$\mathrm{Z}_p^{\Delta}(\mathrm{D}(X), \mathrm{D}(X)) \cong \mathrm{Z}^{-p}(X) \quad \textit{and} \quad \mathrm{CH}_p^{\Delta}(\mathrm{D}(X), \mathrm{D}(X)) \cong \mathrm{CH}^{-p}(X).$$

PROOF. This is an immediate consequence of Proposition 4.2.4 and Theorem 2.3.5. The local-to-global prinicple holds for the action of $\mathrm{D}(X)$ on itself as it arises as the homotopy category of a monoidal model category by the main result of [Gil07] and therefore the criterion of [Ste13, Proposition 6.8] applies. □

4.3. Application: restriction to open subsets

Let $U \subset \mathrm{Spc}(\mathcal{T}^c)$ be an open subset with complement Z. If we denote by $\mathcal{T}_Z$ the smashing ideal in $\mathcal{T}$ generated by the subcategory $(\mathcal{T}^c)_Z \subset \mathcal{T}^c$ of all objects with support contained in Z, then the quotient category $\mathcal{T}_U := \mathcal{T}/\mathcal{T}_Z$ is a tensor triangulated category satisfying all the assumptions made at the beginning of this chapter, whose spectrum $\mathrm{Spc}(\mathcal{T}_U^c)$ can be identified with U. We will show that the localization functor induces surjective maps

$$\mathrm{Z}_p^{\Delta}(\mathcal{T}^c) \to \mathrm{Z}_p^{\Delta}((\mathcal{T}_U)^c)$$

and

$$\mathrm{CH}_p^{\Delta}(\mathcal{T}^c) \to \mathrm{CH}_p^{\Delta}((\mathcal{T}_U)^c)$$

for all p. The kernels of these maps can be described with the help of the relative cycle and Chow groups that we introduced in the previous section.

4.3.1. LEMMA. *Let $\mathcal{T}$ be a compactly-rigidly generated tensor triangulated category (see Definition 4.1.1) such that $\mathrm{Spc}(\mathcal{T}^c)$ is a noetherian topological space. Let $U \subset \mathrm{Spc}(\mathcal{T}^c)$ be an open subset with complement Z. Then $\mathcal{T}_U$ is a compactly-rigidly generated tensor triangulated category and $\mathrm{Spc}(\mathcal{T}_U^c) \cong U$ is a noetherian topological space. Furthermore, if the local-to-global principle holds for the action of $\mathcal{T}$ on itself, it also holds for the action of $\mathcal{T}_U$ on itself.*

PROOF. The category $\mathcal{T}_U$ is compactly-rigidly generated by [Ste13, Beginning of Section 8] and $\mathrm{Spc}(\mathcal{T}_U^c) \cong U$, as $\mathcal{T}_U^c = (\mathcal{T}_c/(\mathcal{T}_c)_Z)^\natural$ by [BF11, Theorem 4.1]. The space U is noetherian as it is a subspace of a noetherian topological space by [Sta14, Lemma 5.8.2]. For the second statement, first note that any localizing $\otimes$-ideal $I \subset \mathcal{T}_U$ is one of $\mathcal{T}$ as well. Indeed, for objects $A \in I, S \in \mathcal{T}$, we have $S \otimes A \cong S \otimes (L_Z(\mathbb{I}) \otimes A) \cong (S \otimes L_Z(\mathbb{I})) \otimes A \in I$ as $S \otimes L_Z(\mathbb{I}) \in \mathcal{T}_U$. Thus it suffices to show that for $A \in \mathcal{T}_U$, we have that $\langle A \rangle_\otimes = \langle \Gamma_x(A) | x \in \mathrm{Spc}(\mathcal{T}_U^c) \rangle_\otimes$, where we interpret $\langle - \rangle_\otimes$ as the smallest localizing $\otimes$-ideal *in* $\mathcal{T}$ containing $-$. Then we have

$$\begin{aligned} \langle A \rangle_\otimes &= \langle \Gamma_x(A) | x \in \mathrm{Spc}(\mathcal{T}^c) \rangle_\otimes \\ &= \langle \Gamma_x(\mathbb{I}) \otimes A | x \in \mathrm{Spc}(\mathcal{T}^c) \rangle_\otimes \end{aligned}$$

by the local-to-global principle. But $\Gamma_x(\mathbb{I}) \otimes L_Z(\mathbb{I}) = 0$ if $x \notin U$ and $\Gamma_x(\mathbb{I}) \otimes L_Z(\mathbb{I}) \cong \Gamma_x(\mathbb{I})$ if $x \in U$ (see [Ste13, Proposition 8.3]), so

$$\begin{aligned} \langle \Gamma_x(\mathbb{I}) \otimes A | x \in \mathrm{Spc}(\mathcal{T}^c) \rangle_\otimes &= \langle \Gamma_x(\mathbb{I}) \otimes L_Z(\mathbb{I}) \otimes A | x \in \mathrm{Spc}(\mathcal{T}^c) \rangle_\otimes \\ &= \langle \Gamma_x(A) | x \in U \rangle_\otimes \ . \end{aligned}$$

Finally, as $\mathrm{Spc}(\mathcal{T}_U^c) \cong U$, we have

$$\langle \Gamma_x(A) | x \in U \rangle_\otimes = \langle \Gamma_x(A) | x \in \mathrm{Spc}(\mathcal{T}_U^c) \rangle \ ,$$

which finishes the proof. □

For the rest of the section, we assume that $\mathcal{T}$ is a tensor triangulated category in the sense of Convention 4.1.2, acting on itself via $\otimes$ (i.e. $\mathcal{K} = \mathcal{T}$) and that the local-to-global principle holds for this action. For any open subset $U \subset \mathrm{Spc}(\mathcal{T}^c)$, we will equip $\mathcal{T}_U^c$ with the dimension function obtained as the restriction of the dimension function on $\mathcal{T}^c$ (see Proposition 1.4.9).

4.3.2. LEMMA. *The localization functor $L_Z : \mathcal{T} \to \mathcal{T}_U$ induces group homomorphisms*

$$l_p : \mathrm{Z}_p^\Delta(\mathcal{T}^c) \to \mathrm{Z}_p^\Delta((\mathcal{T}_U)^c)$$

and

$$\ell_p : \mathrm{CH}_p^\Delta(\mathcal{T}^c) \to \mathrm{CH}_p^\Delta((\mathcal{T}_U)^c)$$

for all $p \in \mathbb{Z}$.

PROOF. By [Kra10, Proposition 5.5.1 and Lemma 5.4.1], the localization functor L_Z restricts to

$$\mathcal{T}^c \xrightarrow{L_Z} \mathcal{T}_U^c$$

on the level of compact objects. By Proposition 1.4.9, we have that $L_Z(\mathcal{T}_{(p)}^c) \subset \left(\mathcal{T}_U^c\right)_{(p)}$ for all $p \in \mathbb{Z}$. Thus the restriction of the functor L_Z has relative dimension ≤ 0 and therefore induces maps

$$l_p : \mathrm{Z}_p^\Delta(\mathcal{T}^c) \to \mathrm{Z}_p^\Delta(\mathcal{T}_U^c)$$

and

$$\ell_p : \mathrm{CH}_p^\Delta(\mathcal{T}^c) \to \mathrm{CH}_p^\Delta(\mathcal{T}_U^c)$$

for all $p \in \mathbb{Z}$ by Theorem 2.4.3. □

4.3.3. PROPOSITION. *The maps l_p, ℓ_p from Lemma 4.3.2 are surjective.*

PROOF. Given an essentially small, rigid tensor triangulated category $\mathcal{K}$ with noetherian spectrum and equipped with a dimension function, recall that we have a decomposition

$$\big(\mathcal{K}_{(p)}/\mathcal{K}_{(p-1)}\big)^{\natural} \cong \coprod_{\substack{\mathcal{Q}\in\mathrm{Spc}(\mathcal{K})\\ \dim(\mathcal{Q})=p}} \mathrm{Min}(\mathcal{K}_{\mathcal{Q}})$$

according to Theorem 1.4.7. In the situation of Lemma 4.3.2 we obtain a commutative diagram for each $p \in \mathbb{Z}$

$$\begin{tikzcd}
\Big(\mathcal{T}^c_{(p)}/\mathcal{T}^c_{(p-1)}\Big)^{\natural} \arrow[r, "\widehat{L_Z}"] \arrow[d, "\wr"] & \big((\mathcal{T}^c_U)_{(p)}/(\mathcal{T}^c_U)_{(p-1)}\big)^{\natural} \arrow[d, "\wr"] \\
\coprod\limits_{\substack{\mathcal{Q}\in\mathrm{Spc}(\mathcal{T}^c)\\ \dim(\mathcal{Q})=p}} \mathrm{Min}(\mathcal{T}^c_{\mathcal{Q}}) \arrow[dr, "\underline{L_Z}"] & \coprod\limits_{\substack{\mathcal{Q}\in\mathrm{Spc}(\mathcal{T}^c_U)\\ \dim(\mathcal{Q})=p}} \mathrm{Min}((\mathcal{T}^c_U)_{\mathcal{Q}}) \arrow[d, "\wr"] \\
& \coprod\limits_{\substack{\mathcal{Q}\in U\\ \dim(\mathcal{Q})=p}} \mathrm{Min}(\mathcal{T}^c_{\mathcal{Q}})\,.
\end{tikzcd}$$

Here, the equivalence

$$\coprod_{\substack{\mathcal{Q}\in\mathrm{Spc}(\mathcal{T}^c_U)\\ \dim(\mathcal{Q})=p}} \mathrm{Min}((\mathcal{T}^c_U)_{\mathcal{Q}}) \cong \coprod_{\substack{\mathcal{Q}\in U\\ \dim(\mathcal{Q})=p}} \mathrm{Min}(\mathcal{T}^c_{\mathcal{Q}})$$

follows as $\mathcal{T}^c_Z \subset \mathcal{Q}$ for all $\mathcal{Q} \in U$. One now checks that the functor $\underline{L_Z}$ is given on objects as the canonical projection

$$(a_{\mathcal{Q}})_{\mathcal{Q}\in\mathrm{Spc}(\mathcal{T}^c)} \mapsto (a_{\mathcal{Q}})_{\mathcal{Q}\in U}$$

which induces a surjection of abelian groups upon applying K_0. But considering that

$$\mathrm{K}_0\left(\Big(\mathcal{T}^c_{(p)}/\mathcal{T}^c_{(p-1)}\Big)^{\natural}\right) = \mathrm{Z}^{\Delta}_p(\mathcal{T},\mathcal{T})$$

and

$$\mathrm{K}_0\left(\Big((\mathcal{T}_U)^c_{(p)}/(\mathcal{T}_U)^c_{(p-1)}\Big)^{\natural}\right) = \mathrm{Z}^{\Delta}_p(\mathcal{T}_U,\mathcal{T}_U)$$

this shows that the induced map $\mathrm{z}^0_p(L_Z) = l_p$ on the tensor triangular cycle groups is a surjection. This implies that the same must be true for ℓ_p, which is induced by l_p on a quotient of these. □

Next we want to identify the kernels of l_p and ℓ_p, with the help of our relative Chow groups.

4.3.4. LEMMA. *The category $\mathcal{T}_Z$ is compactly generated with $(\mathcal{T}_Z)^c = \mathcal{T}^c_Z$, it has arbitrary (set-indexed) coproducts and carries a natural action of $\mathcal{T}$. Furthermore the local-to-global principle holds for the action of $\mathcal{T}$ on $\mathcal{T}_Z$ under our assumption that it holds for the action of $\mathcal{T}$ on itself.*

PROOF. By definition $\mathcal{T}_Z$ is the smallest localizing triangulated subcategory of $\mathcal{T}$ containing $\mathcal{T}_Z^c$ and thus it must be compactly generated and have set-indexed coproducts by definition. By [BF11, Theorem 4.1] one has that $(\mathcal{T}_Z)^c = \mathcal{T}_Z^c$ and that $\mathcal{T}_Z$ is a $\otimes$-ideal, i.e. the action of $\mathcal{T}$ on itself restricts to an action of $\mathcal{T}$ on $\mathcal{T}_Z$. In order to check that the local-to-global principle holds for the action of $\mathcal{T}$ on $\mathcal{T}_Z$, we need to check that for an object $A \in \mathcal{T}_Z$, we have

$$\langle A \rangle_* = \langle \Gamma_x A | x \in \mathrm{Spc}(\mathcal{T}^c) \rangle_* \ .$$

But this is a direct consequence of the fact that the local-to-global principle holds for the action of $\mathcal{T}$ on itself, since this is true when we consider A as an object of $\mathcal{T}$ and a localizing submodule of $\mathcal{T}_Z$ is also one of $\mathcal{T}$. □

4.3.5. LEMMA. *The inclusion functor*

$$\mathcal{T}_Z \hookrightarrow \mathcal{T}$$

induces group homomorphisms

$$i_p : \mathrm{Z}_p^{\Delta}(\mathcal{T}, \mathcal{T}_Z) \to \mathrm{Z}_p^{\Delta}(\mathcal{T}^c)$$

and

$$\iota_p : \mathrm{CH}_p^{\Delta}(\mathcal{T}, \mathcal{T}_Z) \to \mathrm{CH}_p^{\Delta}(\mathcal{T}^c)$$

for all $p \in \mathbb{Z}$.

PROOF. Again, it follows from [Kra10, Proposition 5.5.1 and Lemma 5.4.1] that the inclusion functor restricts to the level of compact objects:

$$I : \mathcal{T}_Z^c \hookrightarrow \mathcal{T}^c$$

By the universal property of Verdier localization and idempotent completion, one obtains induced functors

$$\underline{I}_p : \left((\mathcal{T}_Z^c)_{(p)}/(\mathcal{T}_Z^c)_{(p-1)}\right)^{\natural} \longrightarrow \left((\mathcal{T}^c)_{(p)}/(\mathcal{T}^c)_{(p-1)}\right)^{\natural} \ .$$

As we saw in Lemma 4.3.5 the category $\mathcal{T}_Z$ is compactly generated, has set-indexed coproducts and a natural action by $\mathcal{T}$ that satisfies the local-to-global principle. Thus it makes sense to talk about the relative cycle groups $\mathrm{Z}_p^{\Delta}(\mathcal{T}, \mathcal{T}_Z)$ and by the discussion at the beginning of Section 4.2 we have that

$$\mathrm{Z}_p^{\Delta}(\mathcal{T}, \mathcal{T}_Z) \cong \mathrm{K}_0\left(\left((\mathcal{T}_Z^c)_{(p)}/(\mathcal{T}_Z^c)_{(p-1)}\right)^{\natural}\right) \ .$$

We see that after applying K_0, the functor $\underline{I}_p$ induces a map

$$i_p : \mathrm{Z}_p^{\Delta}(\mathcal{T}, \mathcal{T}_Z) \to \mathrm{Z}_p^{\Delta}(\mathcal{T}^c)$$

and this map respects rational equivalence as I sends

$$\ker\left(\mathrm{K}_0\left(\mathcal{T}_Z^c)_{(p)}\right) \to \mathrm{K}_0\left((\mathcal{T}_Z^c)_{(p+1)}\right)\right)$$

to

$$\ker\left(\mathrm{K}_0\left(\mathcal{T}_{(p)}^c\right) \to \mathrm{K}_0\left(\mathcal{T}_{(p+1)}^c\right)\right) \ .$$

Thus we also get an induced map

$$\iota_p : \mathrm{CH}_p^{\Delta}(\mathcal{T}, \mathcal{T}_Z) \to \mathrm{CH}_p^{\Delta}(\mathcal{T}^c)$$

as desired. □

4.3.6. PROPOSITION. *We have an equality of abelian groups*

$$\mathrm{im}(i_p) = \ker(l_p)$$

for all $p \in \mathbb{Z}$.

PROOF. Recall from the proof of Proposition 4.3.3 that l_p is induced by the projection functor

$$\coprod_{\substack{Q\in \mathrm{Spc}(\mathcal{T}^c)\\ \dim(Q)=p}} \mathrm{Min}(\mathcal{T}^c_Q) \longrightarrow \coprod_{\substack{Q\in U\\ \dim(Q)=p}} \mathrm{Min}(\mathcal{T}^c_Q)$$

$$(a_Q)_{\substack{Q\in \mathrm{Spc}(\mathcal{T}^c)\\ \dim(Q)=p}} \mapsto (a_Q)_{\substack{Q\in U\\ \dim(Q)=p}}$$

from which we see that $\ker(l_p)$ is given as the subgroup

$$\coprod_{\substack{Q\in Z\\ \dim(Q)=p}} \mathrm{K}_0(\mathrm{Min}(\mathcal{T}^c_Q)) \subset \coprod_{\substack{Q\in \mathrm{Spc}(\mathcal{T}^c)\\ \dim(Q)=p}} \mathrm{K}_0(\mathrm{Min}(\mathcal{T}^c_Q)) = \mathrm{K}_0\left(\mathcal{T}^c_{(p)}/\mathcal{T}^c_{(p-1)}\right) .$$

Recall from the proof of Lemma 4.3.5 that i_p is obtained as the map on K_0 induced by the functor

$$\underline{I}_p : \left((\mathcal{T}^c_Z)_{(p)}/(\mathcal{T}^c_Z)_{(p-1)}\right)^\natural \longrightarrow \left((\mathcal{T}^c)_{(p)}/(\mathcal{T}^c)_{(p-1)}\right)^\natural ,$$

which in turn is induced by by the inclusion $\mathcal{T}^c_Z \hookrightarrow \mathcal{T}^c$. The essential image of the functor $\underline{I}_p$ is precisely the subcategory

$$\coprod_{\substack{Q\in Z\\ \dim(Q)=p}} \mathrm{Min}(\mathcal{T}^c_Q) ,$$

which proves that i_p has image equal to $\ker(l_p)$. □

Note that Proposition 4.3.6 implies that $\mathrm{im}(\iota_p) \subset \ker(\ell_p)$. For the other inclusion, the situation seems more subtle and we only obtain the following weaker statement.

4.3.7. PROPOSITION. *Assume that* $\mathcal{T}^c/\mathcal{T}^c_Z$ *is idempotent complete, that is,* $\mathcal{T}^c/\mathcal{T}^c_Z = (\mathcal{T}_U)^c$. *Then we have an equality of abelian groups*

$$\mathrm{im}(\iota_p) = \ker(\ell_p)$$

for all $p \geq \dim(Z)$.

PROOF. We need to check that $\ker(\ell_p)$ is the image of $\ker(l_p)$ under the quotient map $\mathrm{Z}^\Delta_p(\mathcal{T}^c) \to \mathrm{CH}^\Delta_p(\mathcal{T}^c)$. To prove this, we use the fact that if $p \geq \dim(Z)$, then $\mathcal{T}^c_Z \subset \mathcal{T}^c_{(p)}$.

Let us first show that $\mathcal{T}^c_{(p)}/\mathcal{T}^c_Z \cong (\mathcal{T}_U)^c_{(p)}$. We know that $(\mathcal{T}_U)^c$ is given as $(\mathcal{T}^c/\mathcal{T}^c_Z)^\natural$, which is equal to $\mathcal{T}^c/\mathcal{T}^c_Z$ by assumption. By Lemma 1.1.16, $\mathcal{T}^c_{(p)}/\mathcal{T}^c_Z$ embeds fully faithfully into $\mathcal{T}^c/\mathcal{T}^c_Z$, so all there is left to show is that any object of $(\mathcal{T}_U)^c_{(p)} = (\mathcal{T}^c/\mathcal{T}^c_Z)_{(p)}$ is in the essential image of this embedding. In order to see this, let b be an object of $(\mathcal{T}^c/\mathcal{T}^c_Z)_{(p)}$ and let $a \in \mathcal{T}^c$ be an object that the localization functor $\mathcal{T}^c \to \mathcal{T}^c/\mathcal{T}^c_Z$ sends to b. As we have $\mathrm{supp}(b) = \mathrm{supp}(a) \cap U$ and V only contains points of dimension $\leq p$, we must have $\dim(\mathrm{supp}(a)) \leq p$, i.e. $a \in \mathcal{T}^c_{(p)}$, which completes the argument.

Next, consider the commutative diagram

$$\begin{tikzcd}
\mathrm{K}_0(\mathcal{T}^c_{(p)}) \arrow[r, "i"] \arrow[d, "q^\natural"] \arrow[dr, "\pi_p"] & \mathrm{K}_0(\mathcal{T}^c_{(p+1)}) \arrow[dr, "\pi_{p+1}"] & \\
\underbrace{\mathrm{K}_0\left((\mathcal{T}^c_{(p)}/\mathcal{T}^c_{(p-1)})^\natural\right)}_{=\mathrm{Z}^\Delta_p(\mathcal{T}^c)} \arrow[dr, "l_p"] & \mathrm{K}_0((\mathcal{T}_U)^c_{(p)}) \arrow[r, "i_U"] \arrow[d, "q_U^\natural"] & \mathrm{K}_0((\mathcal{T}_U)^c_{(p+1)}) \\
& \underbrace{\mathrm{K}_0\left(((\mathcal{T}_U)^c_{(p)}/(\mathcal{T}_U)^c_{(p-1)})^\natural\right)}_{=\mathrm{Z}^\Delta_p((\mathcal{T}_U)^c)} &
\end{tikzcd}$$

where i, i_u are induced by the respective inclusion functors, $q^\natural, q_U^\natural$ are induced by the composition of the Verdier quotient functor and the inclusion into the respective idempotent completions and π_p, π_{p+1} are induced by taking the Verdier quotient by $\mathcal{T}^c_Z$. By Lemma 3.2.5, we have $\ker(i_U) = \pi_p(\ker(i))$ and therefore

$$l_p \circ q^\natural(\ker(i)) = q_U^\natural \circ \pi_p(\ker(i)) = q_U^\natural(\ker(i_U))$$

As $\mathrm{CH}^\Delta_p(\mathcal{T}^c) = \mathrm{Z}^\Delta_p(\mathcal{T}^c)/q^\natural(\ker(i))$ and $\mathrm{CH}^\Delta_p((\mathcal{T}_U)^c) = \mathrm{Z}^\Delta_p((\mathcal{T}_U)^c)/q_U^\natural(\ker(i_U))$, another application of Lemma 3.2.5 yields that $\ker(\ell_p)$ is the image of $\ker(l_p)$ under the quotient map $\mathrm{Z}^\Delta_p(\mathcal{T}^c) \to \mathrm{CH}^\Delta_p(\mathcal{T}^c)$, as desired. □

4.3.8. EXAMPLE. If X is a non-singular, separated scheme of finite type over a field and $\mathcal{T} = D(X)$, then $\mathcal{T}^c = \mathrm{D}^{\mathrm{perf}}(X) = \mathrm{D}^{\mathrm{b}}(\mathrm{Coh}(X))$ and $\mathcal{T}^c/\mathcal{T}^c_Z = \mathrm{D}^{\mathrm{b}}(\mathrm{Coh}(U))$ for all open subsets $U \subset X$. Indeed, if we look at the Serre subcategory

$$\mathcal{A} := \mathrm{Coh}_Z(X) \subset \mathrm{Coh}(X) =: \mathcal{B}$$

of coherent sheaves with support contained in Z, then Lemma 2.3.4 shows that the conditions of Theorem 2.3.3 are satisfied in this case. It follows that

$$\mathcal{T}^c/\mathcal{T}^c_Z = \mathrm{D}^{\mathrm{b}}(\mathrm{Coh}(X))/\,\mathrm{D}^{\mathrm{b}}(\mathrm{Coh}(X))_Z \cong \mathrm{D}^{\mathrm{b}}(\mathrm{Coh}(X)/\mathrm{Coh}_Z(X)) \cong \mathrm{D}^{\mathrm{b}}(\mathrm{Coh}(U))\ ,$$

where we used the well-known equivalence $\mathrm{Coh}(X)/\mathrm{Coh}_Z(X) \cong \mathrm{Coh}(U)$ (see for example [Rou10, Prop. 3.1]). As $\mathrm{D}^{\mathrm{b}}(\mathrm{Coh}(U))$ is idempotent complete, the conditions of Proposition 4.3.7 are thus met in this case.

The following theorem summarizes the results of the section.

4.3.9. THEOREM. *Let $\mathcal{T}$ be a tensor triangulated category in the sense of Convention 4.1.2 such that the local-to-global principle is satisfied for the action of $\mathcal{T}$ on itself. Let $U \subset \mathrm{Spc}(\mathcal{T}^c)$ be an open subset with closed complement Z. Then there is an exact sequence*

$$\mathrm{Z}^\Delta_p(\mathcal{T}, \mathcal{T}_Z) \xrightarrow{i_p} \mathrm{Z}^\Delta_p(\mathcal{T}^c) \xrightarrow{l_p} \mathrm{Z}^\Delta_p((\mathcal{T}_U)^c) \to 0$$

for all $p \in \mathbb{Z}$. *Furthermore, if* $\mathcal{T}^c/\mathcal{T}^c_Z$ *is idempotent complete and* $p \geq \dim(Z)$, *then we have an exact sequence*

$$\mathrm{CH}^{\Delta}_p(\mathcal{T},\mathcal{T}_Z) \xrightarrow{\iota_p} \mathrm{CH}^{\Delta}_p(\mathcal{T}^c) \xrightarrow{\ell_p} \mathrm{CH}^{\Delta}_p((\mathcal{T}_U)^c) \to 0\ .$$

4.3.10. Remark. The exact sequences from Theorem 4.3.9 should be compared to the corresponding ones for cycle and Chow groups of algebraic varieties (see [Ful98, Proposition 1.8]): if X is an algebraic variety, $Z \subset X$ a closed subscheme with open complement U, then we get exact sequences of cycle groups

$$\mathrm{Z}_p(Z) \to \mathrm{Z}_p(X) \to \mathrm{Z}_p(U) \to 0$$

and Chow groups

$$\mathrm{CH}_p(Z) \to \mathrm{CH}_p(X) \to \mathrm{CH}_p(U) \to 0$$

for all $p \in \mathbb{Z}$.

Note that we don't have a suitable category $\mathcal{R}$ at our disposal such that $\mathrm{Z}^{\Delta}_p(\mathcal{R}^c)$ or $\mathrm{CH}^{\Delta}_p(\mathcal{R}^c)$ maps to $\ker(l_p)$ or $\ker(\ell_p)$, respectively. This is why we need the relative groups here. We do *not* know whether for a non-singular X and $\mathcal{T} = D(X)$, we have

$$\mathrm{Z}^{\Delta}_p(\mathcal{T},\mathcal{T}_Z) \cong \mathrm{Z}_p(Z)$$

or

$$\mathrm{CH}^{\Delta}_p(\mathcal{T},\mathcal{T}_Z) \cong \mathrm{CH}_p(Z)\ .$$

CHAPTER 5

The countable envelope of a tensor Frobenius pair

In this chapter, we lay parts of the technical foundations for Chapter 6 by showing that the countable envelope of a tensor Frobenius pair (see Definition 5.4.2) naturally inherits the structure of a tensor Frobenius pair. This is an extension of work of Keller [Kel90, Appendix B] and Schlichting [Sch06, Section 4] to a symmetric monoidal setting. It will be used in Chapter 6 in order to define products in Schlichting's construction of algebraic K-theory of a Frobenius pair (see [Sch06]).

5.1. Ind-objects in an additive category

In this section we recall some of the theory of ind-objects in the additive setting. We heavily rely on the exposition in [KS06].

Let $\mathcal{E}$ be a small additive category and denote by $\hat{\mathcal{E}}^{\mathrm{add}} := \mathrm{Funct}_{\mathrm{add}}(\mathcal{E}^{\mathrm{op}}, \mathbf{Ab})$ the abelian category of additive functors from $\mathcal{E}$ to the category of Abelian groups. By composition with the forgetful functor, it can be considered as a full subcategory of the category of all functors $\hat{\mathcal{E}} := \mathrm{Funct}(\mathcal{E}^{\mathrm{op}}, \mathbf{Set})$ from $\mathcal{E}$ to the category of sets (see [KS06, Proposition 8.2.12]).

The Yoneda functor gives an a priori embedding $\mathcal{E} \to \hat{\mathcal{E}}$, but as Hom-sets are abelian groups and Hom-functors are additive in our setting, it factors through an embedding $h_{\mathcal{E}} : \mathcal{E} \to \hat{\mathcal{E}}^{\mathrm{add}}$. Given a small filtered category I and a functor $\alpha : I \to \mathcal{E}$ in $\mathcal{E}$, its colimit in $\mathcal{E}$ might not exist. We denote by "$\varinjlim$" α the colimit of the inductive system $h_{\mathcal{E}} \circ F$ in $\hat{\mathcal{E}}$, which is also in $\hat{\mathcal{E}}^{\mathrm{add}}$.

5.1.1. DEFINITION (cf. [KS06, Definition 6.1.1]). An *ind-object* in $\mathcal{E}$ is by definition an object of $\hat{\mathcal{E}}$ that is isomorphic in $\hat{\mathcal{E}}$ to "$\varinjlim$" α for some small filtered category I and a functor $\alpha : I \to \mathcal{E}$. We denote by $\mathrm{Ind}(\mathcal{E})$ the full subcategory of $\hat{\mathcal{E}}$ consisting of the ind-objects in $\mathcal{E}$. The functor $h_{\mathcal{E}}$ induces a full embedding $\iota_{\mathcal{E}} : \mathcal{E} \to \mathrm{Ind}(\mathcal{E})$.

5.1.2. REMARK. In the literature, the category of ind-objects in $\mathcal{E}$ is often defined as the full subcategory of $\hat{\mathcal{E}}$ consisting of filtered colimits of representable functors (see e.g. [AGV71]). The resulting category $\mathrm{Ind}'(\mathcal{E})$ is equivalent to $\mathrm{Ind}(\mathcal{E})$ from Definition 5.1.1 and it is also possible to construct an explicit quasi-inverse to the inclusion $\mathrm{Ind}'(\mathcal{E}) \hookrightarrow \mathrm{Ind}(\mathcal{E})$ as follows: for any object $A \in \mathrm{Ind}(\mathcal{E})$, denote by $\mathcal{E}_A$ the category with objects arrows $s_U : U \to A$ in $\mathrm{Ind}(\mathcal{E})$ with $U \in \mathcal{E}$ (we identify $\mathcal{E}$ with a subcategory of $\mathrm{Ind}(\mathcal{E})$ via $\iota_{\mathcal{E}}$). A morphism $f : s_U \to s_V$ in $\mathcal{E}_A$ is a morphism in $\mathcal{E}$ that makes the diagram in $\mathrm{Ind}(\mathcal{E})$

$$\begin{tikzcd} U \arrow[r, "s_U"] \arrow[d, "f"'] & A \\ V \arrow[ur, "s_V"'] & \end{tikzcd}$$

commute. The category $\mathcal{E}_A$ is cofinally small and filtered by [KS06, Proposition 6.1.5] and thus we can define a functor

$$\begin{aligned} \mathrm{Ind}(\mathcal{E}) &\to \mathrm{Ind}'(\mathcal{E}) \subset \mathrm{Ind}(\mathcal{E}) \\ A &\mapsto \underset{(U\to A)\in\mathcal{E}_A}{\text{“}\varinjlim\text{”}} U \end{aligned} \tag{15}$$

which has image in $\mathrm{Ind}'(\mathcal{E})$. By [KS06, Proposition 2.6.3 (i)], the natural map

$$\underset{(U\to A)\in\mathcal{E}_A}{\text{“}\varinjlim\text{”}} U \to A$$

is an isomorphism. If $A = \text{“}\varinjlim\text{”}\alpha$ for some functor $\alpha : I \to \mathcal{E}$, then there is an associated functor $I \to \mathcal{E}_A$ which maps $i \in I$ to the canonical morphism $\alpha(i) \to A$. This functor is cofinal by [KS06, Proposition 2.6.3 (ii)] and we see that the functor (15) is indeed the desired quasi-inverse.

Under our assumptions, $\mathrm{Ind}(\mathcal{E})$ carries the expected additional structure.

5.1.3. LEMMA. *The category* $\mathrm{Ind}(\mathcal{E})$ *is additive.*

PROOF. It is immediate from the definition of $\mathrm{Ind}(\mathcal{E})$ as a full subcategory of $\hat{\mathcal{E}}$ that the category $\mathrm{Ind}(\mathcal{E})$ is pre-additive, i.e. the morphism sets are abelian groups and composition is bilinear. As $\mathcal{E}$ is additive it has finite coproducts and by [KS06, Proposition 6.1.18], it follows that $\mathrm{Ind}(\mathcal{E})$ admits small (and in particular finite) coproducts. As finite coproducts and products coincide in a pre-additive category (see [KS06, Corollary 8.2.4]), it follows by [KS06, Lemma 8.2.9] that $\mathrm{Ind}(\mathcal{E})$ is additive. □

We finish the section with two statements about the indization of symmetric monoidal categories.

5.1.4. PROPOSITION. *Let $\mathcal{E}$ be endowed with a symmetric monoidal structure such that the functor $a \otimes -$ is additive for all objects $a \in \mathcal{E}$. Then* $\mathrm{Ind}(\mathcal{E})$ *naturally inherits a symmetric monoidal structure such that the inclusion $h_{\mathcal{E}} : \mathcal{E} \to$* $\mathrm{Ind}(\mathcal{E})$ *preserves the tensor product.*

PROOF. The statement seems to be well-known for $\mathrm{Ind}'(\mathcal{E})$, at least in the context of abelian monoidal categories (see e.g. [Del90, Section 7] or [Hái02, Section 3.4]), where one sets

$$\underset{I}{\text{“}\varinjlim\text{”}}\alpha \otimes_{\mathrm{I}} \underset{J}{\text{“}\varinjlim\text{”}}\beta := \underset{I\times J}{\text{“}\varinjlim\text{”}}\alpha \otimes \beta$$

with $\alpha : I \to \mathcal{E}$ and $\beta : J \to \mathcal{E}$ functors from small filtered categories I, J to $\mathcal{E}$ and $\otimes$ the tensor product on $\mathcal{E}$. Thus, we can define a symmetric monoidal structure on $\mathrm{Ind}(\mathcal{E})$ by pulling back along the equivalence (15). Explicitly, we set for two objects $A, B \in \mathrm{Ind}(\mathcal{E})$

$$A \otimes_{\mathrm{I}} B := \underset{((U\to A),(V\to B))\in\mathcal{E}_A\times\mathcal{E}_B}{\text{“}\varinjlim\text{”}} U \otimes V\ .$$

The unit object of $\mathrm{Ind}(\mathcal{E})$ is given as the image of the unit object of $\mathcal{E}$ under $h_{\mathcal{E}}$ and the associativity, commutativity and unit isomorphisms are all induced by the ones of $\mathcal{E}$. □

5.1.5. REMARK. The product $\otimes_I$ is naturally isomorphic to the restriction of the *Day convolution product* on $\hat{\mathcal{E}}$ (see [Day70]) to $\mathrm{Ind}(\mathcal{E})$. This product commutes with colimits in both arguments and the Yoneda embedding takes the tensor product on $\mathcal{E}$ to the convolution product on $\hat{\mathcal{E}}$. Therefore, it must be isomorphic to $\otimes_I$.

5.1.6. LEMMA. *In the situation of Proposition 5.1.4, the functor $A \otimes_{\mathrm{I}} -$ is additive for all objects $A \in \mathrm{Ind}(\mathcal{E})$.*

PROOF. By [KS06, Proposition 8.2.15], in order to prove additivity, it suffices to show that $A \otimes_{\mathrm{I}} -$ preserves binary products. Assume we are given functors $\alpha : I \to \mathcal{E}, \beta : J \to \mathcal{E}, \gamma : K \to \mathcal{E}$ from small filtered categories I, J, K to $\mathcal{E}$ such that $\text{“}\varinjlim_I\text{”}\alpha \cong A, \text{“}\varinjlim_J\text{”}\beta \cong B, \text{“}\varinjlim_K\text{”}\gamma \cong C$. Then

$$\begin{aligned}
A \otimes_{\mathrm{I}} (B \times C) &\cong \left(\text{“}\varinjlim_I\text{”}\alpha\right) \otimes_{\mathrm{I}} \left(\left(\text{“}\varinjlim_J\text{”}\beta\right) \times \left(\text{“}\varinjlim_K\text{”}\gamma\right)\right) \\
&\cong \left(\text{“}\varinjlim_I\text{”}\alpha\right) \otimes_{\mathrm{I}} \left(\text{“}\varinjlim_{J\times K}\text{”}\beta \times \gamma\right) \\
&\cong \text{“}\varinjlim_{I\times J\times K}\text{”}\, \alpha \otimes (\beta \times \gamma) \\
&\cong \text{“}\varinjlim_{I\times J\times K}\text{”}\, \alpha \otimes \beta \times \alpha \otimes \gamma \ ,
\end{aligned}$$

where we used that “$\varinjlim$” commutes with finite products and that $\otimes$ is additive in each variable. As the diagonal functor $I \to I \times I$ is cofinal (see [KS06, Corollary 3.2.3]), we obtain

$$\begin{aligned}
\text{“}\varinjlim_{I\times J\times K}\text{”}\, \alpha \otimes \beta \times \alpha \otimes \gamma &\cong \text{“}\varinjlim_{I\times J\times I\times K}\text{”}\ \alpha \otimes \beta \times \alpha \otimes \gamma \\
&\cong \left(\text{“}\varinjlim_{I\times J}\text{”}\alpha \otimes \beta\right) \times \left(\text{“}\varinjlim_{I\times K}\text{”}\alpha \otimes \gamma\right) \\
&\cong A \otimes_{\mathrm{I}} B \times A \otimes_{\mathrm{I}} C
\end{aligned}$$

as desired. □

5.2. The countable envelope of an exact category

From now on, we endow $\mathcal{E}$ with the structure of an exact category (in the sense of Quillen). We are interested in the countable evelope $\mathrm{C}\mathcal{E}$, which is defined as a full subcategory of $\mathrm{Ind}(\mathcal{E})$. Let I_0 denote the category

$$\bullet \to \bullet \to \bullet \to \bullet \to \cdots$$

where we omit identities and compositions of morphisms.

5.2.1. DEFINITION (cf. [Kel90, Appendix B]). The *countable envelope* $\mathrm{C}\mathcal{E}$ *of* $\mathcal{E}$ is defined as the full subcategory of $\mathrm{Ind}(\mathcal{E})$ consisting of all those objects isomorphic to one of the form "$\varinjlim$"α, where $\alpha : I_0 \to \mathcal{E}$ is a functor that maps all arrows of I_0 to inflations in $\mathcal{E}$.

5.2.2. REMARK. The embedding $\mathcal{E} \to \mathrm{Ind}(\mathcal{E})$ factors via $\mathrm{C}\mathcal{E}$ by choosing for an object $E \in \mathcal{E}$ the functor α_E that maps I_0 to the constant diagram

$$E \xrightarrow{\mathrm{id}} E \xrightarrow{\mathrm{id}} E \xrightarrow{\mathrm{id}} E \to \cdots$$

in $\mathcal{E}$.

Keller shows in [Kel90, Appendix B] that $\mathrm{C}\mathcal{E}$ can be endowed with an exact structure as follows:

5.2.3. THEOREM ([Kel90, Appendix B]). *The following defines an exact structure on $\mathrm{C}\mathcal{E}$: a sequence of maps $A \to B \to C$ is a conflation if and only if it is isomorphic to a sequence*

$$\text{“}\varinjlim\text{”}\alpha \xrightarrow{\text{“}\varinjlim\text{”} f} \text{“}\varinjlim\text{”}\beta \xrightarrow{\text{“}\varinjlim\text{”} g} \text{“}\varinjlim\text{”}\gamma$$

where $\alpha, \beta, \gamma : I_0 \to \mathcal{E}$ are functors that send all maps of I_0 to inflations, and $f : \alpha \to \beta, g : \beta \to \gamma$ are morphisms of functors such that $\alpha(i) \xrightarrow{f(i)} \beta(i) \xrightarrow{g(i)} \gamma(i)$ is a conflation in $\mathcal{E}$ for all $i \in I_0$.

5.2.4. REMARK. It follows that the embedding $\mathcal{E} \to \mathrm{C}\mathcal{E}$ is exact.

5.2.5. REMARK. In [Kel90, Appendix B], the exact structure is actually defined on the category $\mathrm{Ind}'(\mathcal{E})$, but it defines an exact structure on the equivalent category $\mathrm{Ind}(\mathcal{E})$ as well.

5.3. Tensor exact categories

5.3.1. DEFINITION. A *tensor exact category* is an exact category $\mathcal{E}$ equipped with a compatible symmetric monoidal structure $\otimes_{\mathcal{E}}$, i.e. the functors

$$a \mapsto a \otimes_{\mathcal{E}} b$$

are exact for all objects $b \in \mathcal{E}$.

5.3.2. PROPOSITION. *For a tensor exact category $\mathcal{E}$, the countable envelope $\mathrm{C}\mathcal{E}$ naturally inherits the structure of a tensor exact category such that the embedding $\mathcal{E} \to \mathrm{C}\mathcal{E}$ is tensor exact.*

PROOF. The symmetric monoidal structure on $\mathrm{C}\mathcal{E}$ is the restriction of the one on $\mathrm{Ind}(\mathcal{E})$ (see Proposition 5.1.4). For two functors $\alpha, \beta : I_0 \to \mathcal{E}$ with

$$\text{“}\varinjlim\text{”}\alpha = A, \quad \text{“}\varinjlim\text{”}\beta = B$$

we have by definition

$$A \otimes_{\mathrm{I}} B = \underset{((U\to A),(V\to B))\in\mathcal{E}_A\times\mathcal{E}_B}{\text{“}\varinjlim\text{”}} U \otimes V \cong \underset{I_0\times I_0}{\text{“}\varinjlim\text{”}}\alpha\otimes\beta \cong \underset{I_0}{\text{“}\varinjlim\text{”}}\alpha\otimes\beta$$

where the first isomorphism follows from [KS06, Proposition 2.6.3 (ii)] and the second one follows as the diagonal functor $I_0 \to I_0 \times I_0$ is cofinal (see [KS06, Corollary 3.2.3]). This proves that the tensor product of two objects in $\mathrm{C}\mathcal{E}$ is in $\mathrm{C}\mathcal{E}$ again. Indeed, the morphisms $\alpha(i) \otimes \beta(i) \to \alpha(j) \otimes \beta(j)$ are inflations for all objects $i, j \in I_0$ by the exactness property of $\otimes$.

It remains to show that for $A \in \mathrm{C}\mathcal{E}$, the functor $A \otimes_{\mathrm{I}} -$ is exact. Let $\alpha, \beta, \gamma : I_0 \to \mathcal{E}$ be functors and $f : \alpha \to \beta, g : \beta \to \gamma$ be natural transformations such that

$$\alpha(i) \xrightarrow{f(i)} \beta(i) \xrightarrow{g(i)} \gamma(i)$$

is a conflation for all objects $i \in I_0$. If $A \cong$ "$\varinjlim$"δ, then applying $A \otimes_{\mathrm{I}} -$ to the conflation

$$\text{“}\varinjlim\text{”}\alpha \xrightarrow{\text{“}\varinjlim\text{”} f} \text{“}\varinjlim\text{”}\beta \xrightarrow{\text{“}\varinjlim\text{”} g} \text{“}\varinjlim\text{”}\gamma$$

yields a sequence isomorphic to

$$\text{“}\varinjlim\text{”}\alpha \otimes \delta \xrightarrow{\text{“}\varinjlim\text{”} f \otimes \mathrm{id}} \text{“}\varinjlim\text{”}\beta \otimes \delta \xrightarrow{\text{“}\varinjlim\text{”} g \otimes \mathrm{id}} \text{“}\varinjlim\text{”}\gamma \otimes \delta\ .$$

As for all $i \in I_0$, the sequence

$$\alpha(i) \otimes \delta(i) \xrightarrow{f(i) \otimes \mathrm{id}} \beta(i) \otimes \delta(i) \xrightarrow{g(i) \otimes \mathrm{id}} \gamma(i) \otimes \delta(i)$$

is a conflation by the exactness of the tensor product on $\mathcal{E}$, it follows that $A \otimes_{\mathrm{I}} -$ is isomorphic to an exact functor and therefore exact itself. $\square$

5.3.3. DEFINITION. We say that a tensor exact category $\mathcal{E}$ satisfies *the pushout product axiom* if for every two inflations $f : A \to B, g : C \to D$ in $\mathcal{E}$, the canonical morphism

$$A \otimes D \coprod_{A \otimes C} B \otimes C \to B \otimes D$$

is an inflation.

Recall from [Büh10, Example 13.11] that for any category $\mathcal{D}$ and an exact category $\mathcal{E}$, the category $\mathcal{E}^{\mathcal{D}}$ of functors $\mathcal{D} \to \mathcal{E}$ inherits an exact structure, where a sequence of natural transformations

$$F \to G \to H$$

is defined to be exact if $F(d) \to G(d) \to H(d)$ is exact in $\mathcal{E}$ for all objects $d \in \mathcal{D}$. We call this the *pointwise exact structure on* $\mathcal{E}^{\mathcal{D}}$.

5.3.4. LEMMA. *Let $\mathcal{E}$ be a tensor exact category with tensor product $\otimes_{\mathcal{E}}$ and denote by $\tilde{\mathrm{C}}(\mathcal{E})$ the category of functors $\alpha : I_0 \to \mathcal{E}$, such that α maps all morphisms of I_0 to inflations, with the pointwise exact structure (see* [Kel90]*). Then $\tilde{\mathrm{C}}(\mathcal{E})$ with the pointwise tensor product $\otimes_{\tilde{\mathrm{C}}(\mathcal{E})}$ makes $\tilde{\mathrm{C}}(\mathcal{E})$ a tensor exact category. Furthermore, if $\mathcal{E}$ satisfies the pushout product axiom, then so does $\tilde{\mathrm{C}}(\mathcal{E})$.*

PROOF. It is clear that $\tilde{\mathrm{C}}(\mathcal{E})$ inherits a symmetric monoidal structure from $\mathcal{E}$: the associator, unitor and commutator isomorphisms are all given pointwise by the symmetric monoidal structure on $\mathcal{E}$ and they satisfy the required coherence conditions as they are satisfied for $\otimes_{\mathcal{E}}$. Note that the exactness properties of $\otimes_{\mathcal{E}}$ show that for two functors

$\alpha, \beta \in \tilde{\mathrm{C}}(\mathcal{E})$, their tensor product $a \otimes_{\tilde{\mathrm{C}}(\mathcal{E})} b$ is again a functor that maps all morphisms of I_0 to inflations. The exactness properties of $\otimes_{\mathcal{E}}$ also imply that $\otimes_{\tilde{\mathrm{C}}(\mathcal{E})}$ has them as well and thus $\tilde{\mathrm{C}}(\mathcal{E})$ together with $\otimes_{\tilde{\mathrm{C}}(\mathcal{E})}$ is indeed a tensor exact category.

Now let us assume that $\mathcal{E}$ satisfies the pushout product axiom. As we can compute pushouts in $\tilde{\mathrm{C}}(\mathcal{E})$ pointwise, it follows that the map in question from Definition 5.3.3 is pointwise an inflation and therefore an inflation in $\tilde{\mathrm{C}}(\mathcal{E})$ by definition of the exact structure. □

5.3.5. Proposition. *Assume $\mathcal{E}$ satisfies the pushout product axiom. Then the same holds true for* $\mathrm{C}\mathcal{E}$.

Proof. Let us first remark that by Lemma 5.3.4, the functor category $\tilde{\mathrm{C}}(\mathcal{E})$ satisfies the pushout-product axiom. Furthermore, it is an immediate consequence of the definition of the exact structure on $\mathrm{C}\mathcal{E}$ and the tensor product $\otimes_{\mathrm{C}\mathcal{E}}$ that the functor

$$\text{“}\varinjlim\text{”} : \tilde{\mathrm{C}}(\mathcal{E}) \to \mathrm{C}\mathcal{E}$$

is exact and preserves tensor products.

Now, let $f : A \to A'$, $g : B \to B'$ be two inflations in $\mathrm{C}\mathcal{E}$. This means that there exist inflations $f' : \alpha \to \alpha'$ and $g' : \beta \to \beta'$ in $\tilde{\mathrm{C}}\mathcal{E}$ such that $f \cong$ “$\varinjlim$”(f') and $g \cong$ “$\varinjlim$”(g') (see Theorem 5.2.3). Look at the pushout diagram in $\tilde{\mathrm{C}}(\mathcal{E})$

$$\begin{tikzcd}
\alpha \otimes_{\tilde{\mathrm{C}}\mathcal{E}} \beta \arrow[r, "\mathrm{id}\otimes g'"] \arrow[d, "f'\otimes \mathrm{id}"'] & \alpha \otimes_{\tilde{\mathrm{C}}\mathcal{E}} \beta' \arrow[d] \arrow[ddr, bend left, "\mathrm{id}\otimes g'"] & \\
\alpha' \otimes_{\tilde{\mathrm{C}}\mathcal{E}} \beta \arrow[r] \arrow[drr, bend right, "f'\otimes \mathrm{id}"'] & \alpha' \otimes_{\tilde{\mathrm{C}}\mathcal{E}} \beta \coprod_{\alpha \otimes_{\tilde{\mathrm{C}}\mathcal{E}} \beta} \alpha \otimes_{\tilde{\mathrm{C}}\mathcal{E}} \beta' \arrow[dr, dotted, "h'"] & \\
 & & \alpha' \otimes_{\tilde{\mathrm{C}}\mathcal{E}} \beta'
\end{tikzcd}$$

where h' is an inflation as $\tilde{\mathrm{C}}\mathcal{E}$ satisfies the pushout product axiom. We now apply the functor “$\varinjlim$” to this diagram. As exact functors preserve pushouts along inflations (see [Büh10, Proposition 5.2]) and “$\varinjlim$” commutes with the tensor products, we obtain a pushout diagram isomorphic to

$$\begin{tikzcd}
A \otimes_{\mathrm{C}\mathcal{E}} B \arrow[r, "\mathrm{id}\otimes g"] \arrow[d, "f\otimes \mathrm{id}"'] & A \otimes_{\mathrm{C}\mathcal{E}} B' \arrow[d] \arrow[ddr, bend left, "\mathrm{id}\otimes g"] & \\
A' \otimes_{\mathrm{C}\mathcal{E}} B \arrow[r] \arrow[drr, bend right, "f\otimes \mathrm{id}"'] & A' \otimes_{\mathrm{C}\mathcal{E}} B \coprod_{A \otimes_{\mathrm{C}\mathcal{E}} B} A \otimes_{\mathrm{C}\mathcal{E}} B' \arrow[dr, dotted, "h"] & \\
 & & A' \otimes_{\mathrm{C}\mathcal{E}} B'
\end{tikzcd}$$

where h is an inflation as “$\varinjlim$” is exact. This finishes the proof. □

5.4. Tensor Frobenius pairs

Recall that a Frobenius category is an exact category with enough injective objects, such that the class of injective and projective objects coincide.

5.4.1. DEFINITION (see [Sch06, Section 3.4]). A *Frobenius pair* $\boldsymbol{\mathcal{E}} = (\mathcal{E}, \mathcal{E}_0)$ is a strictly full, faithful and exact inclusion of Frobenius categories $\mathcal{E}_0 \hookrightarrow \mathcal{E}$ such that the projective-injective objects of $\mathcal{E}_0$ are mapped to the projective-injective objects of $\mathcal{E}$.

We now give a symmetric monoidal version of Definition 5.4.1.

5.4.2. DEFINITION. A *tensor Frobenius pair* $\boldsymbol{\mathcal{E}} = (\mathcal{E}, \mathcal{E}_0, \otimes)$ consists of a Frobenius pair $(\mathcal{E}, \mathcal{E}_0)$ and a symmetric monoidal structure on $\mathcal{E}$ with tensor product $\otimes$, that makes $\mathcal{E}$ a tensor exact category and satisfies the following properties:

(i) For all objects $A \in \mathcal{E}$, the functor $A \otimes -$ preserves the projective/injective objects of $\mathcal{E}$.
(ii) $\mathcal{E}_0$ is a $\otimes$-ideal in $\mathcal{E}$, i.e. it is stable under tensoring with any object of $\mathcal{E}$.
(iii) The tensor exact category $\mathcal{E}$ satisfies the pushout product axiom.

5.4.3. REMARK. In many examples, $\mathcal{E}$ will be a category of chain complexes over some exact category and $\mathcal{E}_0$ the subcategory of acyclic complexes. From this point of view, requiring that $\mathcal{E}_0$ is a $\otimes$-ideal says that $\otimes$ passes directly to the corresponding derived category.

The pushout product axiom is there to make sure that $\otimes$ induces a product in the Waldhausen K-theory of the Frobenius pair (see Lemma 6.6.2).

5.4.4. REMARK. Here is an example where the axiom of Definition 5.4.2 requiring that $\mathcal{E}_0$ is a $\otimes$-ideal is *not* satisfied: let R-mod be the abelian category of finitely generated modules over a commutative noetherian ring R and consider $\mathrm{C^b}(R\text{-mod})$, the exact category of bounded chain complexes of finitely generated R-modules, with conflations the degree-wise split ones and $\mathrm{aC^b}(\mathrm{R\text{-}mod})$, the exact subcategory of acyclic complexes. Then $(\mathrm{C^b}(R\text{-mod}), \mathrm{aC^b}(\mathrm{R\text{-}mod}))$ is a Frobenius pair and the tensor product of chain complexes $\otimes_R$ makes this example almost a tensor Frobenius pair. However, $\mathrm{aC^b}(\mathrm{R\text{-}mod})$ is not a tensor ideal as $\otimes_R$ is not an exact functor in general.

If $\mathcal{E}$ is a Frobenius category, $\mathrm{C}\mathcal{E}$ is one as well, with the exact structure from Theorem 5.2.3, according to [Sch06, Section 4]. It follows that for a Frobenius pair $\boldsymbol{\mathcal{E}} = (\mathcal{E}, \mathcal{E}_0)$, its countable envelope $\mathrm{C}\boldsymbol{\mathcal{E}} := (\mathrm{C}\mathcal{E}, \mathrm{C}\mathcal{E}_0)$ is again a Frobenius pair. We want to prove an analogous statement for tensor Frobenius pairs.

5.4.5. THEOREM. *Let $\boldsymbol{\mathcal{E}} = (\mathcal{E}, \mathcal{E}_0, \otimes)$ be a tensor Frobenius pair. Then its* countable envelope $\mathrm{C}\boldsymbol{\mathcal{E}} := (\mathrm{C}\mathcal{E}, \mathrm{C}\mathcal{E}_0, \otimes_{\mathrm{I}})$ *is a tensor Frobenius pair.*

PROOF. We know that $(\mathrm{C}\mathcal{E}, \mathrm{C}\mathcal{E}_0)$ is a Frobenius pair and Proposition 5.3.2 gives a symmetric monoidal structure on $\mathrm{C}\mathcal{E}$ with tensor product $\otimes_{\mathrm{I}}$ that makes $\mathrm{C}\mathcal{E}$ a tensor exact category. Furthermore, $\mathrm{C}\mathcal{E}$ will satisfy the pushout product axiom by Proposition 5.3.5.

In order to show that $\mathrm{C}\mathcal{E}_0$ is a $\otimes_{\mathrm{I}}$-ideal in $\mathrm{C}\mathcal{E}$, let $A \cong \text{“}\varinjlim\text{”}\alpha, B \cong \text{“}\varinjlim\text{”}\beta$ for two functors $\alpha : I_0 \to \mathcal{E}, \beta : I_0 \to \mathcal{E}_0$. Then

$$A \otimes_{\mathrm{I}} B \cong \text{“}\varinjlim\text{”}\alpha \otimes \beta$$

and as $\mathcal{E}_0$ is a $\otimes$-ideal in $\mathcal{E}$, it follows that $\alpha \otimes \beta$ has image $\mathcal{E}_0$ and thus $A \otimes_{\mathrm{I}} B \in \mathrm{C}\mathcal{E}_0$.

It remains to prove that $A \otimes_{\mathrm{I}} -$ preserves the projective-injective objects of $\mathrm{C}\mathcal{E}$ which are given as direct summands of objects isomorphic to "$\varinjlim$"ι where $\iota : I_0 \to \mathcal{E}-\mathrm{prinj}$ takes values in the full subcategory of projective-injective objects of $\mathcal{E}$ (see [Sch06, Definition 4.3]). For such ι and any "$\varinjlim$"$\alpha \in \mathrm{C}\mathcal{E}$ we have

$$\left(\text{“}\varinjlim\text{”}\alpha\right) \otimes_{\mathrm{I}} \left(\text{“}\varinjlim\text{”}\iota\right) \cong \left(\text{“}\varinjlim\text{”}\alpha \otimes \iota\right)$$

and as $\mathcal{E}$ is a tensor Frobenius pair we see that the functor $\alpha \otimes \iota$ takes values in $\mathcal{E}-\mathrm{prinj}$. Thus for any $A \in \mathrm{C}\mathcal{E}$, $A \otimes_I -$ preserves objects isomorphic to "$\varinjlim$"α where $\alpha : I_0 \to \mathcal{E}-\mathrm{prinj}$. As it is an additive functor it also preserves their direct summands. We conclude that $A \otimes_{\mathrm{I}} -$ preserves the projective-injective objects of $\mathrm{C}\mathcal{E}$ which finishes the proof. $\square$

CHAPTER 6

Intersection products via higher K-theory

In the previous chapters we introduced Chow groups for tensor triangulated categories and showed that they have a lot of desirable properties, in analogy with the situation in algebraic geometry. The intersection product, one of the most important operations on the Chow groups of a non-singular algebraic variety, however, does not have an analogue in the tensor triangular world yet. In this chapter, we give a construction that provides us — under favorable circumstances — with an intersection product for a tensor triangulated category $\mathcal{T}$, that is defined on groups ${}_{\cap}\mathrm{CH}^{\Delta}_{p}(\mathcal{T})$ (see Definition 6.5.1) which turn out to be subgroups of the tensor triangular Chow groups $\mathrm{CH}^{\Delta}_{p}(\mathcal{T})$ from Chapter 2. In the case that $\mathcal{T} = \mathrm{D}^{\mathrm{perf}}(X)$ for a separated, non-singular scheme X of finite type over a field, the groups ${}_{\cap}\mathrm{CH}^{\Delta}_{p}(\mathcal{T})$ coincide with $\mathrm{CH}^{\Delta}_{p}(\mathcal{T})$ (see Lemma 6.7.6) and thus recover the usual Chow groups of X as well. Thus, we may consider them as another useful generalization of the usual Chow groups of a scheme, competing with $\mathrm{CH}^{\Delta}_{p}(\mathcal{T})$.

In order to define the intersection product, the category $\mathcal{T}$ should satisfy two conditions: Firstly, $\mathcal{T}$ should have an "algebraic model" in the sense that there should exist a tensor Frobenius pair (see Chapter 5) with derived category $\mathcal{T}$. Following Schlichting [Sch06], the assumption that $\mathcal{T}$ has a Frobenius pair as a model gives us the tools of the higher and negative algebraic K-theory of the model. Our second assumption concerns the behavior of a localization sequence arising from the K-theory of the Frobenius models associated to certain sub-quotients of $\mathcal{T}$, and states that an analogue of the Gersten conjecture from algebraic geometry should hold (see Definition 6.4.1).

6.1. Algebraic models

For the rest of the chapter, let $\mathcal{T}$ denote an essentially small tensor triangulated category as in Definition 1.2.1. It is well-known that there is no K-theory functor from the category of small triangulated categories to the category of spaces, if we require that it satisfies some natural axioms (see [Sch02]). In order to be able to talk about the higher and negative K-theory of $\mathcal{T}$, we therefore work with an algebraic model of $\mathcal{T}$, rather than $\mathcal{T}$ itself. The primary aim of this section is, given a tensor triangulated category $\mathcal{T}$ with an algebraic model, to produce algebraic models for certain triangulated subquotients of $\mathcal{T}$, as well as for their idempotent completions.

Monoidal models. Recall from Chapter 5 the notions of Frobenius pair and tensor Frobenius pair, and from Example 1.1.5 that the *stable category* of a Frobenius category $\mathcal{A}$ is the category $\underline{\mathcal{A}}$ with objects the same as $\mathcal{A}$ and morphisms those of $\mathcal{A}$ modulo the subgroup of maps that factor through a projective-injective object. The category $\underline{\mathcal{A}}$

is a triangulated category and for a Frobenius pair $\boldsymbol{\mathcal{A}} = (\mathcal{A}, \mathcal{A}_0)$, $\underline{\mathcal{A}_0}$ is a triangulated subcategory of $\underline{\mathcal{A}}$. The *derived category* of a Frobenius pair $\boldsymbol{\mathcal{A}} = (\mathcal{A}, \mathcal{A}_0)$ is the Verdier quotient of the stable categories $\mathrm{D}(\boldsymbol{\mathcal{A}}) := \underline{\mathcal{A}}/\underline{\mathcal{A}_0}$.

6.1.1. LEMMA. *Let $\boldsymbol{\mathcal{A}} = (\mathcal{A}, \mathcal{A}_0, \otimes)$ be a tensor Frobenius pair. Then $\mathrm{D}(\boldsymbol{\mathcal{A}})$ inherits the structure of a tensor triangulated category and the localization functor $q : \mathcal{A} \to \mathrm{D}(\boldsymbol{\mathcal{A}})$ is a tensor functor.*

PROOF. As $\mathcal{A}_0$ is a tensor ideal, the triangulated subcategory $\underline{\mathcal{A}_0}$ is a tensor ideal in $\underline{\mathcal{A}}$ and thus the quotient $\underline{\mathcal{A}}/\underline{\mathcal{A}_0}$ is a tensor triangulated category where the tensor product $\otimes^L$ is induced from the one on $\mathcal{A}$. Indeed, $\otimes^L$ makes $\mathrm{D}(\boldsymbol{\mathcal{A}})$ a symmetric monoidal category, where the associativity, commutativity and unit natural isomorphisms are given as the images of the ones of $(\mathcal{A}, \otimes)$ under the functor $\mathcal{A} \to \mathrm{D}(\boldsymbol{\mathcal{A}})$. The functors $a \otimes^L -$ are exact for all objects a of $\mathrm{D}(\boldsymbol{\mathcal{A}})$ since the definition of tensor Frobenius pair guarantees that $a \otimes -$ is a map of Frobenius pairs for all objects a of $\mathcal{A}$. These maps always induce exact functors on the derived categories (cf. [Sch06, Section 3.5]). □

6.1.2. EXAMPLE. Let X be a non-singular, separated scheme of finite type over a field. Consider the Frobenius pair $(\mathrm{sPerf}(X), \mathrm{asPerf}(X))$, where $\mathrm{sPerf}(X)$ denotes the exact category of strict perfect complexes on X with conflations the degree-wise split ones and $\mathrm{asPerf}(X)$ is the subcategory of acyclic complexes (see Definition 6.7.1). In Section 6.7 we will see that this is a tensor Frobenius pair with respect to the usual tensor product of chain complexes, with derived category $\mathrm{D}^{\mathrm{perf}}(X)$.

6.1.3. EXAMPLE. Let G be a finite group, k be a field such that $\mathrm{char}(k)$ divides $|G|$ and let kG-mod be the category of finitely generated kG-modules, which is a Frobenius category (see Chapter 3). Denote by kG-proj the subcategory of projective modules, then $(kG\text{-mod}, kG\text{-proj})$ is a Frobenius pair. It is also a tensor Frobenius pair with respect to the tensor product of modules $\otimes_k$ and its derived category is kG-stab, the stable category of the Frobenius category kG-mod.

6.1.4. COROLLARY. *Let $\mathcal{J} \subset \mathrm{D}(\boldsymbol{\mathcal{A}})$ be a tensor ideal and let $\mathcal{B} \subset \mathcal{A}$ be the full subcategory of those objects that become isomorphic to an object of $\mathcal{J}$ after passing to $\mathrm{D}(\boldsymbol{\mathcal{A}})$. Then $\boldsymbol{\mathcal{B}} = (\mathcal{B}, \mathcal{A}_0)$ is a Frobenius pair and $\boldsymbol{\mathcal{C}} = (\mathcal{A}, \mathcal{B}, \otimes)$ is a tensor Frobenius pair, with derived categories $\mathrm{D}(\boldsymbol{\mathcal{B}}) = \mathcal{J}$ and $\mathrm{D}(\boldsymbol{\mathcal{C}}) = \mathrm{D}(\boldsymbol{\mathcal{A}})/\mathcal{J}$.*

PROOF. From [Sch06, Section 5.2], we already know that $(\mathcal{B}, \mathcal{A}_0)$ and $(\mathcal{A}, \mathcal{B})$ are Frobenius pairs with corresponding derived categories $\mathcal{J}$ and $\mathrm{D}(\boldsymbol{\mathcal{A}})/\mathcal{J}$. The fact that $\boldsymbol{\mathcal{C}}$ is a *tensor* Frobenius pair follows since the localization functor $\mathcal{A} \to \mathrm{D}(\boldsymbol{\mathcal{A}})$ is a tensor functor and the preimage of a tensor ideal under such a functor is again a tensor ideal. □

Models for idempotent completion. If $\mathcal{T} = \mathrm{D}(\boldsymbol{\mathcal{A}})$ for a given tensor Frobenius pair $\boldsymbol{\mathcal{A}} = (\mathcal{A}, \mathcal{A}_0)$, we would like to find a tensor Frobenius pair that models the idempotent completion $\mathcal{T}^{\natural}$. The idea is to first embed $\mathrm{D}(\boldsymbol{\mathcal{A}})$ into $\mathrm{D}(\mathrm{C}\boldsymbol{\mathcal{A}})$, the derived category of the countable enevelope of $\boldsymbol{\mathcal{A}}$ (see Theorem 5.4.5), which is idempotent complete, and then to take thick closures. Let us give some more details.

The embedding $\boldsymbol{\mathcal{A}} \to \mathrm{C}\boldsymbol{\mathcal{A}}$ (see Remark 5.2.2) induces a fully faithful embedding

$$\mathrm{D}(\boldsymbol{\mathcal{A}}) \to \mathrm{D}(\mathrm{C}\boldsymbol{\mathcal{A}})$$

(see [Sch06, Proposition 4.4]). In particular we can view $\mathrm{D}(\boldsymbol{\mathcal{A}})$ as a triangulated subcategory of $\mathrm{D}(\mathrm{C}\boldsymbol{\mathcal{A}})$ and consider its thick closure $\overline{\mathrm{D}(\boldsymbol{\mathcal{A}})} \subset \mathrm{D}(\mathrm{C}\boldsymbol{\mathcal{A}})$ which is a triangulated subcategory as well. By [Sch06, Section 5.2], $\overline{\mathrm{D}(\boldsymbol{\mathcal{A}})}$ admits a Frobenius model $\boldsymbol{\mathcal{A}}^{\natural}$ given as follows: if $\mathcal{B}$ is the full subcategory of $\mathrm{C}\mathcal{A}$ that consists of objects that are isomorphic to objects of $\overline{\mathrm{D}(\boldsymbol{\mathcal{A}})}$ in $\mathrm{D}(\mathrm{C}\boldsymbol{\mathcal{A}})$, then $\boldsymbol{\mathcal{A}}^{\natural} = (\mathcal{B}, \mathrm{C}\mathcal{A}_0)$.

6.1.5. LEMMA. *Assume that $\boldsymbol{\mathcal{A}}$ is a tensor Frobenius pair. Then the Frobenius pair $\boldsymbol{\mathcal{A}}^{\natural}$ is a tensor Frobenius pair, with the tensor structure inherited from the one of* $\mathrm{C}\boldsymbol{\mathcal{A}}$.

PROOF. According to Theorem 5.4.5, $\mathrm{C}\boldsymbol{\mathcal{A}}$ is naturally a tensor Frobenius pair. The Frobenius pair $\boldsymbol{\mathcal{A}}^{\natural}$ is given as $(\mathcal{B}, \mathcal{A}_0)$, where $\mathcal{B}$ is the full subcategory of $\mathcal{F}\mathcal{A}$ that consists of objects that are isomorphic to objects of $\overline{\mathrm{D}(\boldsymbol{\mathcal{A}})}$ in $\mathrm{D}(\mathrm{C}\boldsymbol{\mathcal{A}})$. From this perspective, it is clear that all we have to prove is that $\mathcal{B}$ is closed under taking $\otimes_{\mathrm{C}\mathcal{A}}$-products.

To do this, notice that by Proposition 5.3.2, the embedding $\mathrm{D}(\boldsymbol{\mathcal{A}}) \to \mathrm{D}(\mathrm{C}\boldsymbol{\mathcal{A}})$ preserves tensor products, and therefore $\mathrm{D}(\boldsymbol{\mathcal{A}})$ is closed under $\otimes_{\mathrm{C}\boldsymbol{\mathcal{A}}}$-products when we consider it as a triangulated subcategory of $\mathrm{D}(\mathrm{C}\boldsymbol{\mathcal{A}})$. Now, take two objects A, B of $\mathcal{B} \subset \mathcal{A}$ and denote by $L : \mathrm{C}\mathcal{A} \to \mathrm{D}(\mathrm{C}\boldsymbol{\mathcal{A}})$ the localization functor given as the composition

$$\mathrm{C}\mathcal{A} \to \underline{\mathrm{C}\mathcal{A}} \to \underline{\mathrm{C}\mathcal{A}}/\underline{\mathrm{C}\mathcal{A}_0} = \mathrm{D}(\mathrm{C}\boldsymbol{\mathcal{A}}) \; .$$

The functor L preserves tensor products since both functors in the composition do. By definition of thick closure there exist two objects $A', B' \in \mathcal{B}$ such that $L(A) \oplus L(A') \in \mathrm{D}(\boldsymbol{\mathcal{A}})$ and $L(B) \oplus L(B) \in \mathrm{D}(\boldsymbol{\mathcal{A}})$. Thus

$$\begin{aligned}
&\big(L(A) \oplus L(A')\big) \otimes_{\mathrm{D}(\mathcal{F}\boldsymbol{\mathcal{A}})} \big(L(B) \oplus L(B')\big) \cong \\
&\cong \big(L(A) \otimes_{\mathrm{D}(\mathcal{F}\boldsymbol{\mathcal{A}})} L(B)\big) \oplus \big(L(A) \otimes_{\mathrm{D}(\mathcal{F}\boldsymbol{\mathcal{A}})} L(B')\big) \oplus \big(L(B) \otimes_{\mathrm{D}(\mathcal{F}\boldsymbol{\mathcal{A}})} L(A')\big) \\
&\quad \oplus \big(L(B) \otimes_{\mathrm{D}(\mathcal{F}\boldsymbol{\mathcal{A}})} L(B')\big)
\end{aligned}$$

which shows that $L(A) \otimes_{\mathrm{D}(\mathrm{C}\boldsymbol{\mathcal{A}})} L(B) = L(A \otimes_{\mathrm{C}\mathcal{A}} B)$ is isomorphic to a direct summand of an object in $\mathrm{D}(\boldsymbol{\mathcal{A}})$ and proves that $A \otimes_{\mathrm{C}\mathcal{A}} B \in \mathcal{B}$. □

6.1.6. LEMMA. *The category $\mathrm{D}\left(\boldsymbol{\mathcal{A}}^{\natural}\right)$ realizes the idempotent completion $(\mathrm{D}(\boldsymbol{\mathcal{A}}))^{\natural}$ as a tensor triangulated category.*

PROOF. This follows as $\mathrm{D}(\mathrm{C}\boldsymbol{\mathcal{A}})$ is idempotent complete (since it has countable coproducts by [Sch06, Proposition 4.4]) and $\mathrm{D}\left(\boldsymbol{\mathcal{A}}^{\natural}\right)$ is the thick closure of $\mathrm{D}(\boldsymbol{\mathcal{A}})$ in $\mathrm{D}(\mathrm{C}\boldsymbol{\mathcal{A}})$. The equivalence is explicitly given by sending a pair (a, e) in $\mathrm{D}(\boldsymbol{\mathcal{A}})^{\natural}$, with a an object of $\mathrm{D}(\boldsymbol{\mathcal{A}})$ and $e : a \to a$ an idempotent endomorphism, to $\mathrm{im}(e) \in \mathrm{D}\left(\boldsymbol{\mathcal{A}}^{\natural}\right)$. We see that this equivalence preserves the tensor product, as the embedding $\mathrm{D}(\boldsymbol{\mathcal{A}}) \to \mathrm{D}\left(\boldsymbol{\mathcal{A}}^{\natural}\right)$ preserves tensor products by Proposition 5.3.2. □

6.1.7. LEMMA. *The assignment $\boldsymbol{\mathcal{A}} \mapsto \boldsymbol{\mathcal{A}}^{\natural}$ is functorial for maps of Frobenius pairs.*

PROOF. The assignment $\boldsymbol{\mathcal{A}} \mapsto \mathrm{C}\boldsymbol{\mathcal{A}}$ is functorial (see [Sch06, Definition 4.3]) and so a map of Frobenius pairs $m : \boldsymbol{\mathcal{A}} \to \boldsymbol{\mathcal{B}}$ gives a map $\mathrm{C}m : \mathrm{C}\boldsymbol{\mathcal{A}} \to \mathrm{C}\boldsymbol{\mathcal{B}}$. By the additivity of $\mathrm{C}m$ it follows that its restriction to $\boldsymbol{\mathcal{A}}^{\natural}$ maps into $\boldsymbol{\mathcal{B}}^{\natural}$ which proves the lemma. □

As a consequence of Lemma 6.1.6, we now have a Frobenius model for $(\mathrm{D}(\boldsymbol{\mathcal{A}}))^{\natural}$.

6.2. Higher and negative algebraic K-theory of a Frobenius pair

Let $\boldsymbol{\mathcal{A}} = (\mathcal{A}, \mathcal{A}_0)$ be a Frobenius pair. In [Sch06, Section 11], Schlichting defines a K-theory spectrum $\mathbb{K}(\boldsymbol{\mathcal{A}})$ for $\boldsymbol{\mathcal{A}}$ that we will use in the following. The associated K-groups of $\boldsymbol{\mathcal{A}}$ are given as follows (see [Sch06, Theorem 11.7]):

- For $i > 0$, the groups $\mathbb{K}_i(\boldsymbol{\mathcal{A}})$ are the Waldhausen K-groups of $\boldsymbol{\mathcal{A}}$. That is, we make $\mathcal{A}$ into a category with cofibrations and weak equivalences by declaring the cofibrations to be the inflations of $\mathcal{A}$ and the weak equivalences those morphisms that become isomorphisms in $\mathcal{T}$. Then $\mathbb{K}_i(\boldsymbol{\mathcal{A}})$ is the i-th Waldhausen K-group $\mathrm{K}_i^w(\mathcal{A})$ of the category with cofibrations and weak equivalences $\mathcal{A}$.
- $\mathbb{K}_0(\boldsymbol{\mathcal{A}}) = \mathrm{K}_0\left(\mathrm{D}(\boldsymbol{\mathcal{A}})^\natural\right)$.
- For $i < 0$ one defines $\mathbb{K}_i(\boldsymbol{\mathcal{A}})$ as follows: Let $\mathrm{S}_0\boldsymbol{\mathcal{A}}$ denote the full subcategory of $\mathrm{C}\mathcal{A}$ consisting of all objects in the kernel of the Verdier quotient functor
$$\mathrm{D}(\mathrm{C}\boldsymbol{\mathcal{A}}) \to \mathrm{D}(\mathrm{C}\boldsymbol{\mathcal{A}})/\mathrm{D}(\boldsymbol{\mathcal{A}}) .$$
The suspension $\mathrm{S}\boldsymbol{\mathcal{A}}$ of $\boldsymbol{\mathcal{A}}$ is defined as the Frobenius pair $(\mathrm{C}\mathcal{A}, \mathrm{S}_0\boldsymbol{\mathcal{A}})$, and for $n \geq 1$, $\mathrm{S}^n\boldsymbol{\mathcal{A}}$ denotes the Frobenius pair obtained from $\boldsymbol{\mathcal{A}}$ by applying the suspension construction n times. For $i < 0$, Schlichting (see [Sch06, Definition 4.7]) defines
$$\mathbb{K}_i(\boldsymbol{\mathcal{A}}) := \mathrm{K}_0(\mathrm{S}^{-i}\boldsymbol{\mathcal{A}}) .$$

One then obtains long exact localization sequences. Let

$$\boldsymbol{\mathcal{B}} \to \boldsymbol{\mathcal{A}} \to \boldsymbol{\mathcal{C}}$$

be an exact sequence of Frobenius pairs, i.e. one such that the induced sequence

$$\mathrm{D}(\boldsymbol{\mathcal{B}}) \to \mathrm{D}(\boldsymbol{\mathcal{A}}) \to \mathrm{D}(\boldsymbol{\mathcal{C}})$$

is exact up to factors: the composition is zero, the functor $\mathrm{D}(\boldsymbol{\mathcal{B}}) \to \mathrm{D}(\boldsymbol{\mathcal{A}})$ is fully faithful and the induced functor

$$\mathrm{D}(\boldsymbol{\mathcal{A}})/\mathrm{D}(\boldsymbol{\mathcal{B}}) \to \mathrm{D}(\boldsymbol{\mathcal{C}})$$

is cofinal. Then we obtain a long exact localization sequence

$$\cdots \to \mathbb{K}_p(\boldsymbol{\mathcal{B}}) \to \mathbb{K}_p(\boldsymbol{\mathcal{A}}) \to \mathbb{K}_p(\boldsymbol{\mathcal{C}}) \to \mathbb{K}_{p-1}(\boldsymbol{\mathcal{B}}) \to \cdots$$

for all $p \in \mathbb{Z}$ (see [Sch06, Theorem 11.10]).

6.2.1. REMARK. Assume that $\mathcal{T} = \mathrm{D}(\boldsymbol{\mathcal{A}})$ for a *tensor* Frobenius pair $\boldsymbol{\mathcal{A}}$, such that $\mathcal{T}$ is a tensor triangulated category. Let $\mathcal{J} \subset \mathcal{T}$ be a tensor ideal. Corollary 6.1.4 and Lemmas 6.1.5 and 6.1.6 provide models for $\boldsymbol{\mathcal{B}}$ and $\boldsymbol{\mathcal{C}}$ for $\mathcal{J}$ and $(\mathcal{T}/\mathcal{J})^\natural$, respectively. The sequence of Frobenius pairs

$$\boldsymbol{\mathcal{B}} \to \boldsymbol{\mathcal{A}} \to \boldsymbol{\mathcal{C}}$$

induces the sequence of derived categories

$$\mathcal{J} \to \mathcal{T} \to (\mathcal{T}/\mathcal{J})^\natural$$

which is exact up to factors. This gives us a long exact sequence in K-theory

$$\cdots \to \mathbb{K}_p(\boldsymbol{\mathcal{B}}) \to \mathbb{K}_p(\boldsymbol{\mathcal{A}}) \to \mathbb{K}_p(\boldsymbol{\mathcal{C}}) \to \mathbb{K}_{p-1}(\boldsymbol{\mathcal{B}}) \to \cdots .$$

6.2.2. REMARK. An application of the localization sequence implies the following: if we are given two Frobenius pairs with equivalent derived categories, and the equivalence is induced by a functor on the level of Frobenius pairs (which need *not* be an equivalence), then the K-groups arising from the two different models will be isomorphic. This is why we informally think of the K-theory of the Frobenius pair $\boldsymbol{\mathcal{A}}$ as the K-theory of the triangulated category $\mathrm{D}(\boldsymbol{\mathcal{A}})$. One must be careful though: it is not true that any model of $\mathrm{D}(\boldsymbol{\mathcal{A}})$ yields the same K-theory (see [Sch02]).

6.3. K-theory sheaves on $\mathrm{Spc}(\mathcal{T})$

Let us start by proving a basic but useful lemma.

6.3.1. LEMMA. *Let $\mathcal{T}$ be an essentially small tensor triangulated category that is equipped with a dimension function* dim. *Then for all $l \in \mathbb{Z}$, the subcategory*

$$\mathcal{T}_{(l)} \subset \left(\mathcal{T}^{\natural}\right)_{(l)}$$

is dense. Therefore the inclusion induces an equivalence

$$\left(\mathcal{T}_{(l)}\right)^{\natural} \cong \left(\mathcal{T}^{\natural}\right)_{(l)} .$$

PROOF. As $\mathcal{T}$ is dense in $\mathcal{T}^{\natural}$, for every object $a \in \mathcal{T}^{\natural}$, $a \oplus \Sigma(a) \in \mathcal{T}$. Indeed, this follows by Thomason's classification of dense subcategories (see [Tho97]) which gives

$$\mathcal{T} = \left\{a \in \mathcal{T}^{\natural} : [a] \in \mathrm{K}_0(\mathcal{T}) \subset \mathrm{K}_0\left(\mathcal{T}^{\natural}\right)\right\} .$$

Given $b \in \left(\mathcal{T}^{\natural}\right)_{(l)}$, we have $\Sigma(b) \in \left(\mathcal{T}^{\natural}\right)_{(l)}$ as well and by our previous argument $b \oplus \Sigma(b) \in \mathcal{T}$. As

$$\dim(\mathrm{supp}(b \oplus \Sigma(b))) = \dim(\mathrm{supp}(b) \cup \mathrm{supp}(\Sigma(b))) = \dim(\mathrm{supp}(b)) \le l ,$$

it follows that $b \oplus \Sigma(b) \in \mathcal{T}_{(l)}$. This shows that every object of $\left(\mathcal{T}^{\natural}\right)_{(l)}$ is a direct summand of an object of $\mathcal{T}_{(l)}$ and therefore proves the claim. □

Before we define K-theory sheaves on $\mathrm{Spc}(\mathcal{T})$ we fix some assumptions on $\mathcal{T}$ that we will need for the rest of this chapter.

6.3.2. CONVENTION. For the rest of the chapter, we fix a tensor Frobenius pair $\boldsymbol{\mathcal{A}} = (\mathcal{A}, \mathcal{A}_0, \otimes)$ and let $\mathcal{T} = \mathrm{D}(\boldsymbol{\mathcal{A}})$. We assume $\mathcal{T}$ to be essentially small, rigid, equipped with a dimension function dim and such that $\mathrm{Spc}(\mathcal{T})$ is noetherian.

6.3.3. DEFINITION. For any $p \in \mathbb{Z}_{\ge 0}, l \in \mathbb{Z}$, the sheaf $\mathcal{K}_p^{(l)}$ on $\mathrm{Spc}\,\mathcal{T}$ is defined as the sheaf associated to the presheaf

$$U \mapsto \mathbb{K}_p\left((\boldsymbol{\mathcal{A}}_U)_{(l)}\right)$$

for an open $U \subset \mathrm{Spc}(\mathcal{T})$ with complement Z. Here, $(\boldsymbol{\mathcal{A}}_U)_{(l)}$ is the Frobenius pair obtained from $\boldsymbol{\mathcal{A}}$ by subsequently taking models for the Verdier quotient $\mathcal{T}/\mathcal{T}_Z$, then for the triangulated subcategory $(\mathcal{T}/\mathcal{T}_Z)_{(l)}$ and finally for the idempotent completion $\left((\mathcal{T}/\mathcal{T}_Z)_{(l)}\right)^{\natural} \cong (\mathcal{T}_U)_{(l)}$ by Lemma 6.3.1, as described in Section 6.1. By construction, we then have

$$\mathrm{D}((\boldsymbol{\mathcal{A}}_U)_{(l)}) = (\mathcal{T}_U)_{(l)} .$$

The restriction map $\mathcal{K}_p^{(l)}(U) \to \mathcal{K}_p^{(l)}(V)$ for two opens $V \subset U \subset \operatorname{Spc}(\mathcal{T})$ with complements $W \supset Z$ respectively is induced in the following way: the Frobenius pair that models $(\mathcal{T}/\mathcal{T}_Z)_{(l)}$ is given by $(\mathcal{A}^U_{(l)}, \mathcal{A}_Z)$, where $\mathcal{A}_Z$ is the full subcategory of $\mathcal{A}$ consisting of those objects that become isomorphic to objects of $\mathcal{T}_Z$ in $\mathrm{D}(\boldsymbol{\mathcal{A}})$ and $\mathcal{A}^U_{(l)}$ is the full subcategory of $\mathcal{A}$ consisting of those objects that become isomorphic to objects of $(\mathcal{T}/\mathcal{T}_Z)_{(l)}$ in $\mathrm{D}((\mathcal{A}, \mathcal{A}_Z)) = \mathcal{T}/\mathcal{T}_Z$. Using Proposition 1.4.9, we see that there is a map of Frobenius pairs

$$(\mathcal{A}^U_{(l)}, \mathcal{A}_Z) \to (\mathcal{A}^V_{(l)}, \mathcal{A}_W)$$

given by inclusion. After applying idempotent completion as in Lemma 6.1.7 we obtain a map of Frobenius pairs

$$(\boldsymbol{\mathcal{A}}_U)_{(l)} \to (\boldsymbol{\mathcal{A}}_V)_{(l)}$$

which induces the restriction map.

Similarly for any $p \in \mathbb{Z}_{\geq 0}, l \in \mathbb{Z}$, we define the sheaves $\mathcal{K}_p^{(l/l-1)}$ on $\operatorname{Spc}(\mathcal{T})$ as the sheaves associated to the presheaves

$$U \mapsto \mathbb{K}_p\left((\boldsymbol{\mathcal{A}}_U)_{(l)/(l-1)}\right)$$

for an open $U \subset \operatorname{Spc}(\mathcal{T})$ with complement Z. Here, $(\boldsymbol{\mathcal{A}}_U)_{(l)/(l-1)}$ is the Frobenius pair associated to the subquotient $\left((\mathcal{T}_U)_{(l)}/(\mathcal{T}_U)_{(l-1)}\right)^\natural$ of $\mathcal{T}$, given as $(\mathcal{A}^U_{(l)}, \mathcal{A}^U_{(l-1)})^\natural$. By construction,

$$\mathrm{D}\left(\left(\mathcal{A}^U_{(l)}, \mathcal{A}^U_{(l-1)}\right)\right) \cong (\mathcal{T}/\mathcal{T}_Z)_{(l)} / (\mathcal{T}/\mathcal{T}_Z)_{(l-1)}$$

and thus we indeed have

$$\begin{aligned}
\mathrm{D}\left(\left(\mathcal{A}^U_{(l)}, \mathcal{A}^U_{(l-1)}\right)^\natural\right) &\cong \left((\mathcal{T}/\mathcal{T}_Z)_{(l)} / (\mathcal{T}/\mathcal{T}_Z)_{(l-1)}\right)^\natural \\
&\cong \left((\mathcal{T}/\mathcal{T}_Z)^\natural_{(l)} / (\mathcal{T}/\mathcal{T}_Z)^\natural_{(l-1)}\right)^\natural \\
&\cong \left((\mathcal{T}_U)_{(l)}/(\mathcal{T}_U)_{(l-1)}\right)^\natural
\end{aligned}$$

by [Bal07, Proposition 1.13] and Lemma 6.3.1. For an open $V \subset U$, there is a map of Frobenius pairs

$$\left(\mathcal{A}^U_{(l)}, \mathcal{A}^U_{(l-1)}\right) \to \left(\mathcal{A}^V_{(l)}, \mathcal{A}^V_{(l-1)}\right)$$

given by inclusion. Again, after applying idempotent completion as in Lemma 6.1.7 we obtain a map of Frobenius pairs

$$(\boldsymbol{\mathcal{A}}_U)_{(l)/(l-1)} \to (\boldsymbol{\mathcal{A}}_V)_{(l)/(l-1)}$$

which induces the restriction map.

The next result is a key instrument for the constructions of the following sections, as it shows that we can use the sheaves $\mathcal{K}_p^{(l/l-1)}$ to calculate cohomology.

6.3.4. PROPOSITION. *For any $p \in \mathbb{Z}_{\geq 0}, l \in \mathbb{Z}$, the sheaves $\mathcal{K}_p^{(l/l-1)}$ are flasque.*

PROOF. We show that the presheaf

$$U \mapsto \mathbb{K}_p\left((\boldsymbol{\mathcal{A}}_U)_{(l)/(l-1)}\right)$$

is already a sheaf and that it is flasque. The main point here is that the equivalence

$$\big((\mathcal{T}_U)_{(l)}/(\mathcal{T}_U)_{(l-1)}\big)^\natural \cong \coprod_{\substack{Q\in U\\ \dim(Q)=l}} \mathrm{Min}(\mathcal{T}_Q) \tag{16}$$

from Theorem 1.4.7 is induced on the level of Frobenius models.

For $Q \in U, \dim(P) = l$, the Frobenius pair associated to $\mathrm{Min}(\mathcal{T}_Q)$ is constructed as follows: we let $\mathcal{A}_Q \subset \mathcal{A}$ be the full subcategory of those objects becoming isomorphic to objects of $Q \subset \mathcal{T}$ in $\mathrm{D}(\boldsymbol{\mathcal{A}}) = \mathcal{T}$. Let $\mathcal{A}_{\mathrm{Min}} \subset \mathcal{A}$ be the full subcategory of objects becoming isomorphic to objects with minimal support in $\mathrm{D}\big((\mathcal{A}, \mathcal{A}_Q)\big) = \mathcal{T}/Q$. The Frobenius model we use for $\mathrm{Min}(\mathcal{T}_Q)$ is then given as $(\mathcal{A}_{\mathrm{Min}}, \mathcal{A}_Q)^\natural$ which we will denote by $\mathbf{Min}_Q$. Indeed, by construction we have

$$\mathrm{D}\left(\mathbf{Min}_Q\right) = \mathrm{D}\left((\mathcal{A}_{\mathrm{Min}}, \mathcal{A}_Q)^\natural\right) \cong (\mathrm{Min}(\mathcal{T}/Q))^\natural \cong \mathrm{Min}(\mathcal{T}_Q)\ .$$

The last equivalence follows by Lemma 6.3.1 as $\mathrm{Min}(\mathcal{T}/Q) = (\mathcal{T}/Q)_{(n)}$, where $n \in \mathbb{Z}$ is the dimension of the unique closed point of $\mathcal{T}/Q$.

There is an inclusion $\mathcal{A}^U_{(l-1)} \subset \mathcal{A}_Q$ (see Definition 6.3.3) by [Bal07, Prop. 3.21]. We also have $\mathcal{A}^U_{(l)} \subset \mathcal{A}_{\mathrm{Min}}$ which implies that we get a map of Frobenius pairs

$$(\mathcal{A}^U_{(l)}, \mathcal{A}^U_{(l-1)}) \to (\mathcal{A}_{\mathrm{Min}}, \mathcal{A}_Q)$$

for all $Q \in U$, given by inclusion. After idempotent completion we obtain maps

$$(\boldsymbol{\mathcal{A}}_U)_{(l)/(l-1)} \to \mathbf{Min}_Q$$

and the sum of these maps for all $P \in U$

$$\epsilon_U : (\boldsymbol{\mathcal{A}}_U)_{(l)/(l-1)} \to \coprod_{\substack{Q\in U\\ \dim(Q)=l}} \mathbf{Min}_Q$$

induces the equivalence (16) on the derived categories.

As a consequence, we see that the sheaf $\mathcal{K}_p^{(l/l-1)}$ is given as the sheafification of the presheaf

$$U \mapsto \coprod_{\substack{Q\in U\\ \dim(Q)=l}} \mathbb{K}_p\left(\mathbf{Min}_Q\right)\ .$$

Now, for two opens $V \subset U$ consider the diagram

$$\begin{tikzcd}
(\boldsymbol{\mathcal{A}}_U)_{(l)/(l-1)} \arrow[r, "\mathrm{res}"] \arrow[d, "\epsilon_U"] & (\boldsymbol{\mathcal{A}}_V)_{(l)/(l-1)} \arrow[d, "\epsilon_V"] \\
\coprod\limits_{\substack{Q\in U\\ \dim(Q)=l}} \mathbf{Min}_Q \arrow[r, "\pi"] & \coprod\limits_{\substack{Q\in V\\ \dim(Q)=l}} \mathbf{Min}_Q
\end{tikzcd}$$

where res is the restriction functor from Definition 6.3.3 and π is the canonical projection. One checks that this square is commutative. The maps ϵ_U and ϵ_V become equivalences on the corresponding derived categories and therefore $\mathbb{K}_p(\epsilon_U), \mathbb{K}_p(\epsilon_V)$ become isomorphisms and the square commutes after applying $\mathbb{K}_p(-)$. It follows that the restriction maps of the presheaf

$$U \mapsto \coprod_{\substack{Q\in U\\ \dim(Q)=l}} \mathbb{K}_p\left(\mathbf{Min}_Q\right)$$

are given as the canonical projections.

We now show that this presheaf is already a sheaf (and will therefore coincide with $\mathcal{K}_p^{(l/l-1)}$): from the nature of the restriction maps, it is clear that an element of the group $\mathbb{K}_p\left(\mathcal{A}_U)_{(l)/(l-1)}\right)$ with trivial restriction to an open cover must be trivial on U. Furthermore, if we are given an open covering $U = \bigcup_{i\in I} V_i$ and $s_i \in \mathbb{K}_p\left((\mathcal{A}_{V_i})_{(l)/(l-1)}\right)$ with compatible restrictions to the mutual intersections, we can glue them together to an element $s \in \mathbb{K}_p\left(\mathcal{A}_U)_{(l)/(l-1)}\right)$: from the s_i we know what the germ s_P of s at P should be for every $P \in U$. In order to check that there are only finitely many non-zero s_P's, we use that $\mathrm{Spc}(\mathcal{T})$ was assumed to be noetherian and thus finitely many $V_{i_1}, \ldots, V_{i_n}$ suffice to cover U. By definition, $(s_{i_j})_P = 0$ for all but finitely many $P \in V_{i_j}$ for $j = 1, \ldots, n$. This implies that $s_P = 0$ for all but finitely many $P \in U$ and thus $s \in \mathbb{K}_p\left(\mathcal{A}_U)_{(l)/(l-1)}\right)$ as desired.

The flasqueness of $\mathcal{K}_p^{(l/l-1)}$ now follows directly, as its restriction maps coincide with those of the presheaf, and these are clearly surjective. $\square$

6.4. The triangulated Gersten conjecture

We stick to our assumptions from Convention 6.3.2. For any $l \in \mathbb{Z}$ and $U \subset \mathrm{Spc}(\mathcal{T})$ we have a sequence of Frobenius pairs

$$(\mathcal{A}_U)_{(l-1)} \to (\mathcal{A}_U)_{(l)} \to (\mathcal{A}_U)_{(l)/(l-1)}$$

which induces a sequence of tensor triangulated categories

$$(\mathcal{T}_U)_{(l-1)} \hookrightarrow (\mathcal{T}_U)_{(l)} \to \left((\mathcal{T}_U)_{(l)}/(\mathcal{T}_U)_{(l-1)}\right)^{\natural}$$

that is exact up to factors. Therefore we obtain localization sequences

$$\cdots \to \mathbb{K}_p\left((\mathcal{A}_U)_{(l)}\right) \to \mathbb{K}_p\left((\mathcal{A}_U)_{(l)/(l-1)}\right) \to \mathbb{K}_{p-1}\left((\mathcal{A}_U)_{(l-1)}\right) \to \cdots$$

which, by applying sheafification, give us a long exact sequence of sheaves

$$\cdots \to \mathcal{K}_p^{(l-1)} \to \mathcal{K}_p^{(l)} \to \mathcal{K}_p^{(l/l-1)} \to \mathcal{K}_{p-1}^{(l-1)} \to \cdots . \tag{17}$$

6.4.1. DEFINITION. We say that *the triangulated Gersten conjecture holds for the Frobenius pair $\mathcal{A}$ (see Convention 6.3.2) in bidegree* (l,p) for $(l,p) \in \mathbb{Z}^2$ if in the above long exact sequence (17), the map $\mathcal{K}_p^{(l-1)} \to \mathcal{K}_p^{(l)}$ vanishes.

6.4.2. REMARK. Whether the triangulated Gersten conjecture holds for $\mathcal{A}$ might depend on the choice of dimension function for $\mathcal{T}$.

6.4.3. REMARK. As we will see in Lemma 6.7.5, the triangulated Gersten conjecture can be viewed as a generalization of the usual Gersten conjecture from algebraic K-theory. Let us recall the statement of the usual conjecture.

CONJECTURE (Gersten). *Let* X *be the spectrum of a regular local ring* R. *Let* $\mathrm{M}_l(X)$ *denote the category of coherent sheaves on* X *with codimension of support* $\geq l$ *with associated Quillen* K*-groups* $\mathrm{K}_p(\mathrm{M}_l(X))$ *for* $p \geq 0$. *Then the maps*

$$\mathrm{K}_p(\mathrm{M}_{l+1}(X)) \to \mathrm{K}_p(\mathrm{M}_l(X))$$

induced for all $p \geq 0$ *by the inclusion* $\mathrm{M}_{l+1}(X) \to \mathrm{M}_l(X)$ *vanish.*

The conjecture was proved by Quillen in [Qui73] for the case that R is a finitely generated algebra over a field, and later Panin [Pan03] removed the finite generation hypothesis. Quillen uses his result in [Qui73] to prove the *Bloch formula*, which identifies the Chow groups of a non-singular variety X with certain cohomology groups of K-theory sheaves on X. We will use the triangulated Gersten conjecture for a similar purpose in Theorem 6.5.4.

A more direct relation of the usual Gersten conjecture to Definition 6.4.1 becomes visible as follows: one may check the vanishing of the maps

$$\iota_p^l : \mathcal{K}_p^{(l-1)} \to \mathcal{K}_p^{(l)}$$

on the level of stalks. For $Q \in \mathrm{Spc}(\mathcal{T})$, we have Frobenius pairs $\boldsymbol{\mathcal{X}} := (\mathcal{A}_{l-1}^Q, \mathcal{A}_Q)^\natural$ and $\boldsymbol{\mathcal{Y}} := (\mathcal{A}_l^Q, \mathcal{A}_Q)^\natural$. Here $\mathcal{A}_Q$ is the full subcategory of objects of $\mathcal{A}$ in the kernel of the Verdier localization $\mathrm{D}(\boldsymbol{\mathcal{A}}) = \mathcal{T} \to \mathcal{T}/Q$ and $\mathcal{A}_n^Q$ is the full subcategory of $\mathcal{A}$ of objects that in $\mathrm{D}(\mathcal{A}, \mathcal{A}_Q)$ become isomorphic to an object of the triangulated subcategory $\mathrm{D}(\mathcal{A}, \mathcal{A}_Q)_{(n)} = (\mathcal{T}/Q)_{(n)}$, for $n = l-1, l$. The derived categories $\mathrm{D}(\boldsymbol{\mathcal{X}}), \mathrm{D}(\boldsymbol{\mathcal{Y}})$ are given as $(\mathcal{T}_Q)_{(l-1)}$ and $(\mathcal{T}_Q)_{(l)}$, respectively. Furthermore, we have a map of Frobenius pairs $\boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ given by inclusion which induces the map ι_p^l on the stalks at Q:

$$(\iota_p^l)_Q : \mathbb{K}_p(\boldsymbol{\mathcal{X}}) \to \mathbb{K}_p(\boldsymbol{\mathcal{Y}}) \,.$$

The triangulated Gersten conjecture holds in bidegree (l, p), if the maps $(\iota_p^l)_Q$ vanish for all points $Q \in \mathrm{Spc}(\mathcal{T})$.

If $\boldsymbol{\mathcal{A}}$ satisfies the triangulated Gersten conjecture in bidegrees (l, p) and $(l, p-1)$, then the long exact sequence (17) contains the short exact sequence

$$0 \to \mathcal{K}_i^{(l)} \to \mathcal{K}_i^{(l/l-1)} \to \mathcal{K}_{i-1}^{(l-1)} \to 0 \,. \tag{18}$$

6.5. The triangulated Bloch formula

For any essentially small tensor triangulated category $\mathcal{L}$ equipped with a dimension function and $l \in \mathbb{Z}$, we can define sheaves of Grothendieck groups on $\mathrm{Spc}(\mathcal{L})$ as follows: let $\mathcal{F}^l(\mathcal{L})$ denote the sheaf associated to the presheaf

$$U \mapsto \mathrm{K}_0\left((\mathcal{L}_U)_{(l)}\right))$$

and let $\mathcal{F}^{l/l-1}(\mathcal{L})$ denote the sheaf

$$U \mapsto \mathrm{K}_0\left(\left((\mathcal{L}_U)_{(l)}/(\mathcal{L}_U)_{(l-1)}\right)^\natural\right)$$

so that we have $\mathcal{F}^l(\mathrm{D}(\boldsymbol{\mathcal{A}})) = \mathcal{K}_0^{(l)}$ and $\mathcal{F}^{l/l-1}(\mathrm{D}(\boldsymbol{\mathcal{A}})) = \mathcal{K}_0^{(l/l-1)}$ as special cases (see Definition 6.3.3). Note that for $\mathcal{F}^{l/l-1}(\mathcal{L})$, we don't need to sheafify by Proposition 6.3.4. There is also a map of sheaves

$$\beta : \mathcal{F}^l(\mathcal{L}) \to \mathcal{F}^{l/l-1}(\mathcal{L}) \tag{19}$$

which is obtained as the sheafification of a map of presheaves β' induced by the composition of the Verdier localization functor and the inclusion into the idempotent completion:

$$\beta'(U) : \mathrm{K}_0\left((\mathcal{L}_U)_{(l)})\right) \to \mathrm{K}_0\left((\mathcal{L}_U)_{(l)}/(\mathcal{L}_U)_{(l-1)}\right) \hookrightarrow \mathrm{K}_0\left(\left((\mathcal{L}_U)_{(l)}/(\mathcal{L}_U)_{(l-1)}\right)^{\natural}\right) .$$

For $\mathcal{L} = \mathrm{D}(\mathcal{A})$, the map β is the one of the localization sequence (17). We will be interested in the group of global sections

$$\Gamma(\mathrm{im}(\beta)) \subset \Gamma\left(\mathcal{F}^{l/l-1}(\mathcal{L})\right) = \mathrm{K}_0\left((\mathcal{L}_{(l)}/\mathcal{L}_{(l-1)})^{\natural}\right) = \mathrm{Z}^{\Delta}_l(\mathcal{L}) , \tag{20}$$

where $\mathrm{Z}^{\Delta}_l(\mathcal{L})$ is the dimension l tensor triangular cycle group of $\mathcal{L}$ from Chapter 2. The image of the map of presheaves β' on the level of global sections is the subgroup

$$\Gamma(\mathrm{im}(\beta')) = \mathrm{K}_0\left(\mathcal{L}^{\natural}_{(l)}/\mathcal{L}^{\natural}_{(l-1)}\right) \subset \mathrm{K}_0\left((\mathcal{L}_{(l)}/\mathcal{L}_{(l-1)})^{\natural}\right) .$$

As the presheaf $\mathrm{im}(\beta')$ is separated (it is, after all, a sub-presheaf of a sheaf), the natural map $\mathrm{im}(\beta') \to \mathrm{im}(\beta)$ from presheaf to sheafification is injective and thus we have an inclusion

$$j : \Gamma(\mathrm{im}(\beta')) = \mathrm{K}_0\left(\mathcal{L}^{\natural}_{(l)}/\mathcal{L}^{\natural}_{(l-1)}\right) \hookrightarrow \Gamma(\mathrm{im}(\beta)) \tag{21}$$

as well. Let $i : \mathrm{K}_0(\mathcal{L}^{\natural}_{(l)}) \to \mathrm{K}_0(\mathcal{L}^{\natural}_{(l+1)})$ be the map induced by the inclusion and $\phi : \mathrm{K}_0(\mathcal{L}^{\natural}_{(l)}) \to \mathrm{K}_0(\mathcal{L}^{\natural}_{(l)}/\mathcal{L}^{\natural}_{(l-1)})$ be the map induced by the Verdier quotient functor.

6.5.1. DEFINITION. The *l-dimensional $\cap$-cycle group* of $\mathcal{L}$ is defined as the group

$${}_{\cap}\mathrm{Z}^{\Delta}_l(\mathcal{L}) := \Gamma(\mathrm{im}(\beta)) \subset \mathrm{Z}^{\Delta}_l(\mathcal{L}) .$$

The *l-dimensional $\cap$-Chow group* $\mathcal{L}$ is defined as the quotient

$${}_{\cap}\mathrm{CH}^{\Delta}_l(\mathcal{L}) := {}_{\cap}\mathrm{Z}^{\Delta}_l(\mathcal{L})/j \circ \phi(\ker(i)) .$$

6.5.2. REMARK. We will see in Theorem 6.5.4 that these $\cap$-Chow groups show up in the cohomology of the sheaf $\mathcal{K}^{(0)}_p$ (see Definition 6.3.3). From Definition 2.2.4, it also follows that

$${}_{\cap}\mathrm{CH}^{\Delta}_l(\mathcal{L}) \subset \mathrm{CH}^{\Delta}_l(\mathcal{L}) .$$

When $\mathcal{L}^{\natural}_{(l)}/\mathcal{L}^{\natural}_{(l-1)}$ is idempotent complete already, it follows from (21) that

$${}_{\cap}\mathrm{Z}^{\Delta}_l(\mathcal{L}) = \mathrm{Z}^{\Delta}_l(\mathcal{L}) \quad \text{and} \quad {}_{\cap}\mathrm{CH}^{\Delta}_l(\mathcal{L}) = \mathrm{CH}^{\Delta}_l(\mathcal{L}) .$$

This is true for the cases we considered in the example computations of Theorem 2.3.5 and Propositions 3.3.2, 3.4.7 and 3.4.9.

6.5.3. EXAMPLE. Let X be a non-singular separated scheme of finite type over a field and $\mathcal{L} = \mathrm{D}^{\mathrm{perf}}(X)$, the derived category of perfect complexes equipped with the opposite of the codimension of support as a dimension function. In Theorem 2.3.5, it is proved that $\mathrm{Z}^{\Delta}_{-n}(\mathcal{L}) \cong \mathrm{Z}^n(X)$ and $\mathrm{CH}^{\Delta}_{-n}(\mathcal{L}) \cong \mathrm{CH}^n(X)$ for all $n \in \mathbb{Z}$. In this case we also have isomorphisms ${}_{\cap}\mathrm{Z}^{\Delta}_{-n}(\mathcal{L}) \cong \mathrm{Z}^{\Delta}_{-n}(\mathcal{L})$ and ${}_{\cap}\mathrm{CH}^{\Delta}_{-n}(\mathcal{L}) \cong \mathrm{CH}^{\Delta}_{-n}(\mathcal{L})$ by Remark 6.5.2 (see also Lemma 6.7.6).

We now assume that the dimension function dim for $\mathcal{T} = \mathrm{D}(\boldsymbol{\mathcal{A}})$ is given as the opposite of the Krull codimension and furthermore, that the triangulated Gersten conjecture holds for $\boldsymbol{\mathcal{A}}$ and for this choice of dimension function in bidegrees (i, j) with $-p-2 \leq i \leq 0$ and $-1 \leq j \leq p$. Splicing the short exact sequences (18) together yields a partial flasque resolution of the sheaf $\mathcal{K}_p^{(0)}$

$$\mathcal{K}_p^{(0)} \to \mathcal{K}_p^{(0/-1)} \to \cdots \to \mathcal{K}_1^{(-p+1/-p)} \xrightarrow{\delta_1} \mathcal{K}_0^{(-p/-p-1)} \xrightarrow{\delta_0} \mathcal{K}_{-1}^{(-p-1/-p-2)} \tag{22}$$

that we can use to calculate its cohomology.

6.5.4. THEOREM (Triangulated Bloch formula). *Assume that the dimension function* dim *for $\mathcal{T}$ is given as the opposite of the codimension and that the triangulated Gersten conjecture holds for $\boldsymbol{\mathcal{A}}$ and for this choice of dimension function in bidegrees (i, j) with $-p-2 \leq i \leq 0$ and $-1 \leq j \leq p$. Then we have isomorphisms*

$$_{\cap}\mathrm{CH}^{\Delta}_{-p}(\mathcal{T}) \cong \mathrm{H}^p(\mathrm{Spc}(\mathcal{T}), \mathcal{K}_p^{(0)})$$

for all $p \in \mathbb{Z}$.

PROOF. We will use the partial flasque resolution (22) of $\mathcal{K}_p^{(0)}$ to calculate the group $\mathrm{H}^p(\mathrm{Spc}(\mathcal{T}), \mathcal{K}_p^{(0)})$. The maps

$$\mathcal{K}_1^{(-p+1/-p)} \to \mathcal{K}_0^{(-p/-p-1)} \to \mathcal{K}_{-1}^{(-p-1/-p-2)}$$

are spliced together from the exact sequences (18) in the following way:

$$\begin{tikzcd}
& & & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{K}_1^{(-p+1)} \arrow[r] & \mathcal{K}_1^{(-p+1/-p)} \arrow[r, "\alpha"] \arrow[dr, "\delta_1"'] & \mathcal{K}_0^{(-p)} \arrow[r] \arrow[d, "\beta"] & 0 \\
& & & \mathcal{K}_0^{(-p/-p-1)} \arrow[d, "\gamma"'] \arrow[dr, "\delta_0"] & \\
& & 0 \arrow[r] & \mathcal{K}_{-1}^{(-p-1)} \arrow[r, "\epsilon"'] \arrow[d] & \mathcal{K}_{-1}^{(-p-1/-p-2)} \\
& & & 0 &
\end{tikzcd}$$

In order to calculate cohomology, we apply the global section functor. As taking global sections is a left-exact functor, $\Gamma(\epsilon)$ is injective and so we have that

$$\ker(\Gamma(\delta_0)) = \ker(\Gamma(\gamma)) = \Gamma(\ker(\gamma)) = \Gamma(\mathrm{im}(\beta)) = {}_{\cap}\mathrm{Z}^{\Delta}_{-p}(\mathcal{T}) \ ,$$

again by left-exactness of the global section functor.

Recall that the maps α, β are given as sheafifications of maps α', β' between the corresponding presheaves. By the functoriality of sheafification it follows that $\beta \circ \alpha$ is given as the sheafification of the composition $\beta' \circ \alpha'$. But $\beta' \circ \alpha'$ is already a map of sheaves and we therefore have that $\beta \circ \alpha = \beta' \circ \alpha'$. The map $\Gamma(\beta \circ \alpha)$ is therefore given as the composition of the maps

$$\psi : \mathbb{K}_1\left((\mathcal{A}_X)_{(-p+1)/(-p)}\right) \to \mathrm{K}_0\left(\mathcal{T}^{\natural}_{(-p)}\right)$$

with $X = \mathrm{Spc}(\mathcal{T})$ and

$$\phi : \mathrm{K}_0\left(\mathcal{T}^{\natural}_{(-p)}\right) \to \mathrm{K}_0\left(\mathcal{T}^{\natural}_{(-p)}/\mathcal{T}^{\natural}_{(-p-1)}\right)$$

from the corresponding localization sequences. By the exactness of the localization sequence, $\mathrm{im}(\psi) = \ker(i)$ with

$$i : \mathrm{K}_0\left(\mathcal{T}^{\natural}_{(-p)}\right) \to \mathrm{K}_0\left(\mathcal{T}^{\natural}_{(-p+1)}\right)$$

as in Definition 6.5.1. Thus, we obtain $\mathrm{im}(\Gamma(\beta \circ \alpha)) = \phi(\ker(i))$.

By our previous calculations we conclude that

$$\begin{aligned} \mathrm{H}^p(\mathrm{Spc}(\mathcal{T}), \mathcal{K}^{(0)}_p) &= \ker(\Gamma(\delta_0))/\,\mathrm{im}(\Gamma(\delta_1)) \\ &= {}_{\cap}\mathrm{Z}^{\Delta}_{-p}(\mathcal{T})/j \circ \phi(\ker(i)) \\ &= {}_{\cap}\mathrm{CH}^{\Delta}_{-p}(\mathcal{T}) \end{aligned}$$

which was to be shown. □

6.5.5. REMARK. From the proof of Theorem 6.5.4, we can get a simpler definition of ${}_{\cap}\mathrm{Z}^{\Delta}_{-p}(\mathcal{T})$, not using K-theory sheaves. Namely, we see that the map of sheaves

$$\epsilon \circ \gamma : \mathcal{K}_0^{(-p/-p-1)} \to \mathcal{K}_{-1}^{(-p-1/-p-2)}$$

can be computed on global sections as the composition of the maps

$$\gamma' : \mathrm{K}_0\left(\mathcal{T}_{(-p)}/\mathcal{T}_{(-p-1)}\right)^{\natural} \to \mathbb{K}_{-1}\left((\mathcal{A}_X)_{(-p-1)}\right)$$

with $X = \mathrm{Spc}(\mathcal{T})$ and

$$\epsilon' : \mathbb{K}_{-1}\left((\mathcal{A}_X)_{(-p-1)}\right) \to \mathbb{K}_{-1}\left((\mathcal{A}_X)_{(-p-1)/(-p-2)}\right) ,$$

both coming from the corresponding long exact localization sequences. We therefore see that

$${}_{\cap}\mathrm{Z}^{\Delta}_{-p}(\mathcal{T}) = \Gamma(\mathrm{im}(\beta)) = (\gamma')^{-1}(\ker(\epsilon')) .$$

This reformulation of Definition 6.5.1 has the disadvantage that it needs tensor Frobenius pairs in order to talk about $\mathbb{K}_{-1}$ and it is not obvious that it is actually independent of a choice of such a tensor Frobenius pair.

6.6. The intersection product

Recall our assumptions for $\mathcal{T}$ from Convention 6.3.2. We now let dim be the opposite of the Krull codimension and require furthermore that the triangulated Gersten conjecture holds for $\mathcal{A}$ in bidegrees (i, j) with $-p-2 \le i \le 0$ and $-1 \le j \le p$.

First, let us recall a general well-known fact about cup products in sheaf cohomology (see [Bre67, Theorem 7.1 and Proposition 7.2]). Let X be a topological space and $\mathcal{F}, \mathcal{G}$ be sheaves of abelian groups on X. Then there exists a unique associative bilinear product

$$\cup : \mathrm{H}^p(X, \mathcal{F}) \times \mathrm{H}^q(X, \mathcal{G}) \to \mathrm{H}^{p+q}(X, \mathcal{F} \otimes_{\mathbb{Z}} \mathcal{G})$$

for all $p,q \in \mathbb{Z}_{\geq 0}$ such that for $p = q = 0$, the product is the one induced by the tensor product $\Gamma(X,\mathcal{F}) \times \Gamma(X,\mathcal{G}) \to \Gamma(X,\mathcal{F} \otimes \mathcal{G})$ and the axioms of [Bre67, Theorem 7.1] are satisfied. The product $\cup$ is called the *cup product*.

An application of Theorem 6.5.4 then yields the following:

6.6.1. COROLLARY. *Under the assumptions of Theorem 6.5.4 and for $p,q \in \mathbb{Z}_{\geq 0}$ there are bilinear maps*

$$_{\cap}\mathrm{CH}^{\Delta}_{-p}(\mathcal{T}) \times {_{\cap}\mathrm{CH}^{\Delta}_{-q}}(\mathcal{T}) \to \mathrm{H}^{p+q}\left(\mathrm{Spc}(\mathcal{T}), \mathcal{K}^{(0)}_{p} \otimes_{\mathbb{Z}} \mathcal{K}^{(0)}_{q}\right) .$$

In order to construct the intersection product, we need a map $\mathcal{K}^{(0)}_{p} \otimes_{\mathbb{Z}} \mathcal{K}^{(0)}_{q} \to \mathcal{K}^{(0)}_{p+q}$, which will then induce the product map

$$\mathrm{H}^{p+q}(\mathrm{Spc}(\mathcal{T}), \mathcal{K}^{(0)}_{p} \otimes_{\mathbb{Z}} \mathcal{K}^{(0)}_{q}) \to \mathrm{H}^{p+q}(\mathrm{Spc}(\mathcal{T}), \mathcal{K}^{(0)}_{p+q}) = {_{\cap}\mathrm{CH}^{\Delta}_{-p-q}}(\mathcal{T})$$

It will be derived from a bilinear map on Waldhausen K-theory induced by the tensor product.

6.6.2. LEMMA. *Let $\boldsymbol{\mathcal{A}} = (\mathcal{A}, \mathcal{A}_0, \otimes)$ be a tensor Frobenius pair. If we consider $\mathcal{A}$ as a Waldhausen category, then $\otimes$ is a biexact functor $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ in the sense of* [Wal85, Section 1.5].

PROOF. By assumption, $a \otimes -$ is an exact functor for all objects a of $\mathcal{A}$ which implies that it preserves cofibrations, as those are just the inflations. Let $f : x \to y$ be a weak equivalence, i.e. a map that becomes an isomorphism after passing to $\mathrm{D}(\boldsymbol{\mathcal{A}})$. This means that the object $\mathrm{cone}(f)$ of $\underline{\mathcal{A}}$ is in $\underline{\mathcal{A}_0}$. As $\underline{\mathcal{A}_0}$ is a tensor ideal in $\underline{\mathcal{A}}$ and passing from $\mathcal{A}$ to the stable category $\underline{\mathcal{A}}$ preserves tensor products, it follows that $\mathrm{id}_a \otimes f$ is an isomorphism in $\mathrm{D}(\boldsymbol{\mathcal{A}})$ as well. Therefore $a \otimes -$ preserves weak equivalences. Finally, it is proved in [Büh10, Proposition 5.2] that exact functors of exact categories preserve pushouts along inflations, which in our case means that pushouts along weak equivalences are preserved. Therefore, the functors $a \otimes -$ (and by symmetry $- \otimes a$) are exact in the sense of Waldhausen (see [Wal85, Section 1.5]).

It remains to check that $\otimes$ satisfies the "more technical condition" of [Wal85, Section 1.5]. This asserts that for two cofibrations $\alpha : a \rightarrowtail a', \beta : b \rightarrowtail b'$ in the diagram

$$\begin{tikzcd}
a \otimes b \arrow[r, tail, "\alpha \otimes \mathrm{id}_b"] \arrow[d, tail, "\mathrm{id}_a \otimes \beta"'] & a' \otimes b \arrow[d] \arrow[ddr, bend left, "\mathrm{id}_{a'} \otimes \beta"] & \\
a \otimes b' \arrow[r] \arrow[drr, bend right, "\alpha \otimes \mathrm{id}_{b'}"'] & a' \otimes b \coprod_{a \otimes b} a \otimes b' \arrow[dr, dotted, "\phi"] & \\
 & & a' \otimes b'
\end{tikzcd}$$

the arrow ϕ is a cofibration, i.e. an inflation. This is exactly the pushout product axiom of Definition 5.4.2. $\square$

A biexact functor $\otimes : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ in the above sense gives rise to bilinear maps

$$\mathbb{K}_p(\boldsymbol{\mathcal{A}}) \otimes_{\mathbb{Z}} \mathbb{K}_q(\boldsymbol{\mathcal{A}}) \to \mathbb{K}_{p+q}(\boldsymbol{\mathcal{A}})$$

for all $p,q \geq 0$ (see [Wal85, Section 1.5]). In particular, we obtain maps

$$\mathbb{K}_p(\mathcal{A}_U) \otimes_{\mathbb{Z}} \mathbb{K}_q(\mathcal{A}_U) \to \mathbb{K}_{p+q}(\mathcal{A}_U)$$

as $\mathcal{A}_U := (\mathcal{A}_U)_{(0)}$ inherits the structure of a tensor Frobenius pair from $\mathcal{A}$ by Corollary 6.1.4 and Lemma 6.1.5. These maps sheafify to

$$\mathcal{K}_p^{(0)} \otimes_{\mathbb{Z}} \mathcal{K}_q^{(0)} \to \mathcal{K}_{p+q}^{(0)}.$$

and give us

$$\mathrm{H}^{p+q}\left(\mathrm{Spc}(\mathcal{T}), \mathcal{K}_p^{(0)} \otimes_{\mathbb{Z}} \mathcal{K}_q^{(0)}\right) \to \mathrm{H}^{p+q}\left(\mathrm{Spc}(\mathcal{T}), \mathcal{K}_{p+q}^{(0)}\right) = {}_{\cap}\mathrm{CH}^{\Delta}_{-p-q}(\mathcal{T}) \tag{23}$$

for all $p,q \geq 0$.

6.6.3. Definition. Let $\mathcal{A}$ be a tensor Frobenius pair with derived category $\mathcal{T}$ that satisfies the assumptions of Theorem 6.5.4. For $p,q \in \mathbb{Z}_{\geq 0}$, the *intersection product* is the bilinear map

$$\alpha : {}_{\cap}\mathrm{CH}^{\Delta}_{-p}(\mathcal{T}) \otimes {}_{\cap}\mathrm{CH}^{\Delta}_{-q}(\mathcal{T}) \to {}_{\cap}\mathrm{CH}^{\Delta}_{-p-q}(\mathcal{T})$$

that arises as the composition of the map in Corollary 6.6.1 and in (23).

6.6.4. Remark. While the groups ${}_{\cap}\mathrm{CH}^{\Delta}_{n}(\mathcal{T})$ only depend on $\mathrm{D}(\mathcal{A}) = \mathcal{T}$, the product α of Definition 6.6.3 might depend on the whole model $\mathcal{A}$.

6.6.5. Remark. Let $\mathcal{A}, \mathcal{B}$ be two tensor Frobenius pairs satsifying the assumptions of Theorem 6.5.4 and $F : \mathcal{A} \to \mathcal{B}$ a map of tensor Frobenius pairs (i.e. a map of Frobenius pairs that respects the tensor products up to natural isomorphism) such that the induced maps on the derived categories has relative dimension 0 (see Definition 2.4.1). Then F induces maps

$${}_{\cap}\mathrm{CH}(F)_{-p} : {}_{\cap}\mathrm{CH}^{\Delta}_{-p}(\mathrm{D}(\mathcal{A})) \to {}_{\cap}\mathrm{CH}^{\Delta}_{-p}(\mathrm{D}(\mathcal{B}))$$

for all $p \in \mathbb{Z}_{\geq 0}$ and there is a commutative diagram

$$\begin{tikzcd}
{}_{\cap}\mathrm{CH}^{\Delta}_{-p}(\mathrm{D}(\mathcal{A})) \times {}_{\cap}\mathrm{CH}^{\Delta}_{-q}(\mathrm{D}(\mathcal{A})) \arrow[r, "\alpha_{\mathcal{A}}"] \arrow[d, "{}_{\cap}\mathrm{CH}(F)_{-p} \times {}_{\cap}\mathrm{CH}(F)_{-q}"'] & {}_{\cap}\mathrm{CH}^{\Delta}_{-p-q}(\mathrm{D}(\mathcal{A})) \arrow[d, "{}_{\cap}\mathrm{CH}(F)_{-p-q}"] \\
{}_{\cap}\mathrm{CH}^{\Delta}_{-p}(\mathrm{D}(\mathcal{B})) \times {}_{\cap}\mathrm{CH}^{\Delta}_{-q}(\mathrm{D}(\mathcal{B})) \arrow[r, "\alpha_{\mathcal{B}}"] & {}_{\cap}\mathrm{CH}^{\Delta}_{-p-q}(\mathrm{D}(\mathcal{B}))
\end{tikzcd}$$

with $\alpha_{\mathcal{A}}, \alpha_{\mathcal{B}}$ the respective products from Definition 6.6.3. In this sense, the construction is functorial.

6.6.6. Remark. The author expects that properties of the product in Waldhausen K-theory and the cup product in sheaf cohomology as in Corollary 6.6.1 imply that the intersection product from Definition 6.6.3 makes

$$\bigoplus_{p \geq 0} {}_{\cap}\mathrm{CH}^{\Delta}_{-p}(\mathcal{T})$$

a graded-commutative ring with unit the class of $\mathbb{I}$ in ${}_{\cap}\mathrm{CH}^{\Delta}_{0}(\mathcal{T})$.

If $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and k is an algebraically closed field of characteristic 2, the results of Chapter 3 show that

$$\bigoplus_{p \geq 0} {}_{\cap}\mathrm{CH}^{\Delta}_{-p}(kG\text{-stab}) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}\ .$$

The only possible commutative unital ring structure on this group, that also has a nilpotent element is $(\mathbb{Z}/2\mathbb{Z})[\epsilon]/(\epsilon^2)$. Thus, if the above assumption holds true, any choice of tensor Frobenius pair with derived category kG-stab that satisfies the triangulated Gersten conjecture in the relevant degrees (if it exists) must yield the same intersection product.

6.7. Example: strict perfect complexes on a non-singular algebraic variety

We now introduce the main example of Definition 6.6.3 which justifies the name "intersection product". Let X be a non-singular separated scheme of finite type over a field. Recall that a *strict perfect complex* on X is a bounded complex of locally free $\mathcal{O}_X$-modules of finite rank.

6.7.1. DEFINITION. Let sPerf denote the category of strict perfect complexes on X endowed with the following structure of exact category: a sequence of strict perfect complexes

$$\mathcal{F}^\bullet \to \mathcal{G}^\bullet \to \mathcal{H}^\bullet$$

is a conflation if it is degree-wise a split exact sequence. We denote the full subcategory of acyclic strict perfect complexes by asPerf.

6.7.2. LEMMA. *The triple* $\mathbf{sPerf} = (\mathrm{sPerf}, \mathrm{asPerf}, \otimes_{\mathcal{O}_X})$ *is a tensor Frobenius pair.*

PROOF. For an exact category $\mathcal{E}$, let $\mathrm{Ch}^b(\mathcal{E})$ denote the exact category of all bounded chain complexes over $\mathcal{E}$, with the conflations defined as the degree-wise split exact sequences. Let $\mathrm{Ac}^b(\mathcal{E}) \subset \mathrm{Ch}^b(\mathcal{E})$ denote full subcategory of acyclic complexes. In [Sch06, Section 5.3], it is shown that $(\mathrm{Ch}^b(\mathcal{E}), \mathrm{Ac}^b(\mathcal{E}))$ is a Frobenius pair. Thus, when we consider the full subcategory of locally free sheaves of finite rank in $\mathrm{Coh}(X)$ as an exact category, it follows that $(\mathrm{sPerf}, \mathrm{asPerf})$ is a Frobenius pair.

It is clear that the tensor product of two strict perfect complexes is again strict perfect and as tensoring with a strict perfect complex is an exact functor, it follows that asPerf is a $\otimes_{\mathcal{O}_X}$-ideal. It remains to check that the pushout product axiom of Definition 5.4.2 holds true. Thus, let $f : A_\bullet \rightarrowtail X_\bullet$ and $g : B_\bullet \rightarrowtail Y_\bullet$ be two inflations in sPerf. This means that for each $i \in \mathbb{Z}$ we have automorphisms $\alpha_i : X_i \to X_i$ and $\beta_i : Y_i \to Y_i$ such that $\alpha_i \circ f_i$ is a split injection $A_i \hookrightarrow A_i \oplus C_i$ and $\beta_i \circ g_i$ is a split injection $B_i \hookrightarrow B_i \oplus D_i$. The maps $f \otimes \mathrm{id}_{B_\bullet}$ and $\mathrm{id}_{A_\bullet} \otimes g$ are given componentwise as

$$(f \otimes \mathrm{id}_{B_\bullet})_k : \bigoplus_{i+j=k} A_i \otimes B_j \to \bigoplus_{i+j=k} X_i \otimes B_j$$

$$(\mathrm{id}_{A_\bullet} \otimes g)_k : \bigoplus_{i+j=k} A_i \otimes B_j \to \bigoplus_{i+j=k} A_i \otimes Y_j$$

and after post-composing with the isomorphisms consisting of diagonal matrices with entries $\alpha_i \otimes \mathrm{id}_{B_j}$ and $\mathrm{id}_{A_i} \otimes \beta_j$ respectively, we obtain split injections

$$\bigoplus_{i+j=k} A_i \otimes B_j \to \bigoplus_{i+j=k} (A_i \otimes B_j) \oplus (C_i \otimes B_j)$$

$$\bigoplus_{i+j=k} A_i \otimes B_j \to \bigoplus_{i+j=k} (A_i \otimes B_j) \oplus (A_i \otimes D_j)\,.$$

We see that therefore

$$\left((A_\bullet \otimes Y_\bullet) \coprod_{A_\bullet \otimes B_\bullet} (X_\bullet \otimes B_\bullet) \right)_k \cong \bigoplus_{i+j=k} (A_i \otimes B_j) \oplus (A_i \otimes D_j) \oplus (C_i \otimes B_j)$$

Similarly, we see that

$$(X_\bullet \otimes Y_\bullet)_k \cong \bigoplus_{i+j=k} (A_i \otimes B_j) \oplus (A_i \otimes D_j) \oplus (C_i \otimes B_j) \oplus (C_i \otimes D_j)$$

and the induced map

$$\left((A_\bullet \otimes Y_\bullet) \coprod_{A_\bullet \otimes B_\bullet} (X_\bullet \otimes B_\bullet) \right)_k \longrightarrow (X_\bullet \otimes Y_\bullet)_k$$

is given as the canonical inclusion, which is split. This shows that the pushout product axiom holds in sPerf and finishes the proof of the lemma. □

6.7.3. LEMMA. *The category* $\mathrm{D}(\mathbf{sPerf})$ *is equivalent to* $\mathrm{D}^{\mathrm{perf}}(X)$ *as a tensor triangulated category.*

PROOF. The inclusion functor from the exact category of strict perfect complexes into the exact category of perfect complexes induces an exact equivalence of derived categories between $\mathrm{D}(\mathbf{sPerf})$ and $\mathrm{D}^{\mathrm{perf}}(X)$ if X has an ample family of line bundles, as follows from [TT90, Proposition 2.3.1], as mentioned in the proof of [TT90, Lemma 3.8]. As being noetherian, separated and regular already implies that X admits an ample family of line bundles (see [BGI71, Corollaire 2.2.7.1]), our assumptions on X guarantee that the inclusion is an equivalence. It is also a tensor functor as we can compute the derived tensor product by tensoring with a quasi-isomorphic strict perfect complex. □

6.7.4. CONVENTION. For the remaining part of the chapter, we set $\boldsymbol{\mathcal{T}} := \mathbf{sPerf}$ and $\mathcal{T} := \mathrm{D}(\mathbf{sPerf}) \cong \mathrm{D}^{\mathrm{perf}}(X)$. We fix the opposite of the codimension of support as a dimension function on $\mathcal{T}$.

6.7.5. LEMMA. *The Frobenius pair* **sPerf** *satisfies the triangulated Gersten conjecture in bidegrees* (l, p) *for* $l \leq 0$ *and* $p \geq -1$.

PROOF. First, let us introduce some maps of exact sequences of Frobenius pairs, which will allow us to ged rid of idempotent completions and work with complexes of coherent sheaves instead of perfect ones.

For $U \subset X$ open with complement Z, we start with

$$\begin{tikzcd} (\boldsymbol{\mathcal{T}}_U)_{(l-1)} \arrow[r] & (\boldsymbol{\mathcal{T}}_U)_{(l)} \arrow[r] & (\boldsymbol{\mathcal{T}}_U)_{(l)/(l-1)} \\ \left(\mathbf{sPerf}^U_{(l-1)}, \mathbf{sPerf}_Z\right) \arrow[r] \arrow[u] & \left(\mathbf{sPerf}^U_{(l)}, \mathbf{sPerf}_Z\right) \arrow[r] \arrow[u] & \left(\mathbf{sPerf}^U_{(l)}, \mathbf{sPerf}^U_{(l-1)}\right) \arrow[u] \end{tikzcd} \tag{24}$$

in the notation of Definition 6.3.3, where the vertical arrows are given as the inclusion into the countable envelope. The vertical arrows induce induce equivalences of the corresponding derived categories as X is regular (see Section 2.3) and thus they induce isomorphisms in $\mathbb{K}$-theory.

For an abelian category $\mathcal{A}$, define the Frobenius pair

$$\mathbf{Ch}^b(\mathcal{A}) := (\mathrm{Ch}^b(\mathcal{A}), \mathrm{Ac}^b(\mathcal{A})) \ ,$$

where $\mathrm{Ch}^b(\mathcal{A})$ is the category of bounded chain complexes in $\mathcal{A}$ and $\mathrm{Ac}^b(\mathcal{A})$ is the full subcategory of complexes homotopy equivalent to an acyclic chain complex. The conflations in $\mathbf{Ch}^b(\mathcal{A})$ are by definition the degree-wise split exact sequences. There is a map of exact sequences of Frobenius pairs

(25)

$$\begin{tikzcd}
\left(\mathbf{sPerf}^U_{(l-1)}, \mathbf{sPerf}_Z\right) \arrow[r] \arrow[d] & \left(\mathbf{sPerf}^U_{(l)}, \mathbf{sPerf}_Z\right) \arrow[r] \arrow[d] & \left(\mathbf{sPerf}^U_{(l)}, \mathbf{sPerf}^U_{(l-1)}\right) \arrow[d] \\
\mathbf{Ch}^b\left((\mathrm{Coh}(U)_{(l-1)}\right) \arrow[r] & \mathbf{Ch}^b\left(\mathrm{Coh}(U)_{(l)}\right) \arrow[r] & \mathbf{Ch}^b\left(\mathrm{Coh}(U)_{(l)}/\mathrm{Coh}(U)_{(l-1)}\right)
\end{tikzcd}$$

where the vertical maps are given by restriction to U. Again, we check that they induce equivalences of the corresponding derived categories and therefore induce isomorphisms in $\mathbb{K}$-theory.

Using the maps (24) and (25) and [Sch06, Theorem 11.10], we see that the localization sequences corresponding to

$$(\boldsymbol{\mathcal{T}}_U)_{(l-1)} \to (\boldsymbol{\mathcal{T}}_U)_{(l)} \to (\boldsymbol{\mathcal{T}}_U)_{(l)/(l-1)}$$

and

$$\mathbf{Ch}^b\left((\mathrm{Coh}(U)_{(l-1)}\right) \to \mathbf{Ch}^b\left(\mathrm{Coh}(U)_{(l)}\right) \to \mathbf{Ch}^b\left(\mathrm{Coh}(U)_{(l)}/\mathrm{Coh}(U)_{(l-1)}\right)$$

are isomorphic. This proves the lemma for $p = -1$ by [Sch06, Theorem 9.1], which shows that $\mathbb{K}_{-1}(\mathbf{Ch}^b(\mathcal{A})) = 0$ for any abelian category $\mathcal{A}$. For $p \geq 0$, [TT90, Theorem 1.11.7] shows that both localization sequences are in turn isomorphic to the localization sequence

$$\cdots \to \mathrm{K}_p\left(\mathrm{Coh}(U)_{(l)}\right) \to \mathrm{K}_p\left(\frac{\mathrm{Coh}(U)_{(l)}}{\mathrm{Coh}(U)_{(l-1)}}\right) \to \mathrm{K}_{p-1}\left(\mathrm{Coh}(U)_{(l-1)}\right) \to \cdots$$

from Quillen K-theory for all $l \in \mathbb{Z}$, where $\mathrm{Coh}(U)_{(l)}$ denotes the abelian category of coherent sheaves on the open subscheme $U \subset X$, with codimension of support $\geq -l$.

Therefore the stalks of the exact sequence (17) are exact sequences isomorphic to the usual ones in the Gersten conjecture, which is satisfied for regular local rings of finite type over a field (see [Qui73, Theorem 5.11]). This implies the statement as we can check the vanishing of a map of sheaves on the stalks. $\square$

6.7.6. LEMMA. *There are isomorphisms*

$${}_{\cap}\mathrm{CH}^{\Delta}_{-p}(\mathcal{T}) \cong \mathrm{CH}^p(X)$$

for all $p \in \mathbb{Z}$.

PROOF. Under our assumptions, Theorem 2.3.5 shows that $\mathrm{CH}^{\Delta}_{-p}(\mathcal{T}) \cong \mathrm{CH}^p(X)$ for all $p \in \mathbb{Z}$. The isomorphisms ${}_{\cap}\mathrm{CH}^{\Delta}_{-p}(\mathcal{T}) \cong \mathrm{CH}^{\Delta}_{-p}(\mathcal{T})$ are a consequence of the fact that the categories $\mathcal{T}^{\natural}_{(-p)}/\mathcal{T}^{\natural}_{(-p-1)}$ can be expressed as derived categories of abelian categories (as we assumed that X is non-singular) and are therefore idempotent complete already (see Section 2.3). Thus there is an equivalence

$$\mathcal{T}^{\natural}_{(-p)}/\mathcal{T}^{\natural}_{(-p-1)} \to \left(\mathcal{T}^{\natural}_{(-p)}/\mathcal{T}^{\natural}_{(-p-1)}\right)^{\natural} \cong \left(\mathcal{T}_{(-p)}/\mathcal{T}_{(-p-1)}\right)^{\natural}$$

induced by the inclusion functor, which gives the isomorphism by Remark 6.5.2. □

We now want to compare the usual intersection product on X and the product from Definition 6.6.3 on the tensor triangular Chow groups of $\mathrm{D}^{\mathrm{perf}}(X)$, coming from the tensor Frobenius pair **sPerf**. In order to do this, consider the isomorphisms

$$\mathbb{K}_i(\boldsymbol{\mathcal{T}}_U) \to \mathrm{K}_i(\mathrm{Coh}(U))$$

that were constructed in the proof of Lemma 6.7.5. If we denote them by s_i^U, then for all $i, j \geq 0$ and $U \subset X$ open, they fit into a diagram

$$\begin{tikzcd}
\mathbb{K}_i(\boldsymbol{\mathcal{T}}_U) \otimes \mathbb{K}_j(\boldsymbol{\mathcal{T}}_U) \arrow[r] \arrow[d, "s_i^U \otimes s_j^U"] & \mathbb{K}_{i+j}(\boldsymbol{\mathcal{T}}_U) \arrow[d, "s_{i+j}^U"] \\
\mathrm{K}_i(\mathrm{Coh}(U)) \otimes \mathrm{K}_j(\mathrm{Coh}(U)) \arrow[r] & \mathrm{K}_{i+j}(\mathrm{Coh}(U))
\end{tikzcd} \tag{26}$$

where the horizontal arrows are given by the products in the Waldhausen K-theory of $\boldsymbol{\mathcal{T}}_U$ and in the Quillen K-theory of $\mathrm{Coh}(U)$, respectively.

6.7.7. THEOREM. *Let α denote the intersection product from Definition 6.6.3 coming from the tensor Frobenius pair* **sPerf** *and let α' be the usual intersection product on X. Assume that diagram (26) commutes for all $i, j \geq 0$ and all opens $U \subset X$. Then the diagram*

$$\begin{tikzcd}
{}_{\cap}\mathrm{CH}^{\Delta}_{-p}(\mathcal{T}) \otimes {}_{\cap}\mathrm{CH}^{\Delta}_{-q}(\mathcal{T}) \arrow[r, "\alpha"] \arrow[d, "\cong"] & {}_{\cap}\mathrm{CH}^{\Delta}_{-p-q}(\mathcal{T}) \arrow[d, "\cong"] \\
\mathrm{CH}^p(X) \otimes \mathrm{CH}^q(X) \arrow[r, "\alpha'"] & \mathrm{CH}^{p+q}(X)
\end{tikzcd}$$

commutes up to a sign $(-1)^{pq}$ for all $p, q \geq 0$.

6.7.8. REMARK. The construction of the products in Quillen and Waldhausen K-theory is so natural that it seems very plausible that diagram (26) always commutes for all $i, j \geq 0$ and all opens $U \subset X$. However, we could not find the statement in the literature and we were unable to prove it.

PROOF OF THEOREM 6.7.7. As we have $\mathrm{Spc}(\mathcal{T}) \cong X$ and X is regular, the sheaves $\mathcal{K}_p^{(0)}$ will be isomorphic to the sheaves $\mathcal{F}_p$ associated to the presheaf $U \mapsto \mathrm{K}_p(\mathrm{Coh}(U))$ on X via the isomorphisms s_p. By the Bloch formula, $\mathrm{H}^p(X, \mathcal{F}_p) \cong \mathrm{CH}^p(X)$. The statement now follows by the commutativity of diagram (26) and the main result of [Gra78], where it is shown that the product

$$\mathrm{H}^p(X, \mathcal{F}_p) \otimes \mathrm{H}^q(X, \mathcal{F}_q) \to H^{p+q}(X, \mathcal{F}_p \otimes \mathcal{F}_q) \to \mathrm{H}^{p+q}(X, \mathcal{F}_{p+q})$$

agrees with the usual intersection product up to a sign $(-1)^{pq}$, where the second map comes from the product on Quillen K-theory induced by the tensor product. □

Glossary

Category theory

Abelian category A category is abelian if it is additive, every morphism has a kernel and a cokernel and every monomorphism is a kernel and every epimorphism is a cokernel.
Additive category A category is additive if it has a zero object, all finite biproducts exist and all Hom-sets are endowed with the structure of an abelian group, such that composition is bilinear.
Biexact functor An additive functor $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ for $\mathcal{A}, \mathcal{B}, \mathcal{C}$ exact (resp. triangulated) categories is biexact if for all objects $A \in \mathcal{A}$ and $B \in \mathcal{B}$, the functors $F(A, -)$ and $F(-, B)$ are exact, i.e. they send conflations to conflations (resp. distinguished triangles to distinguished triangles and commute with the corresponding suspension functors).
Cofibration See *Waldhausen category*.
Cofinal functor A fully faithful functor $\varphi : \mathcal{J} \to \mathcal{I}$ into a filtered category $\mathcal{I}$ is cofinal if for any object $I \in \mathcal{I}$ there exists an object $J \in \mathcal{J}$ and a morphism $I \to \varphi(J)$. If φ is cofinal, then for any functor $\alpha : \mathcal{I} \to \mathcal{C}$, we have an isomorphism $\varinjlim \alpha \circ \varphi \cong \varinjlim \alpha$ (see [KS06]).
Cofinally small category A category $\mathcal{C}$ is cofinally small if there exists a small category $\mathcal{C}_0$ and a cofinal functor $\mathcal{C}_0 \to \mathcal{C}$.
Conflation See *Exact category*.
Deflation See *Exact category*.
Dense subcategory A triangulated subcategory $\mathcal{S} \subset \mathcal{T}$ of a triangulated category $\mathcal{T}$ is dense if each object of $\mathcal{T}$ is a direct summand of an object isomorphic to an object of $\mathcal{S}$.
Enough injective/projective objects An exact category $\mathcal{E}$ has enough injective objects if for every object A there exists an inflation $A \hookrightarrow I$ to an injective object I of $\mathcal{E}$. Dually, $\mathcal{E}$ has enough projective objects if for every object A there exists a deflation $P \twoheadrightarrow A$ from a projective object P of $\mathcal{E}$.
Essential image The essential image of a functor $F : \mathcal{S} \to \mathcal{T}$ is the full subcategory of $\mathcal{T}$ consisting of those objects B such that $B \cong F(A)$ for some object $A \in \mathcal{S}$.
Essentially small category A category is essentially small if it is equivalent to a small category.
Exact category (in the sense of Quillen) An exact category is an additive category $\mathcal{E}$ equipped with a class E of pairs of composable morphisms (f, g)

$$A \xrightarrow{f} B \xrightarrow{g} C$$

such that f is a kernel for g and g is a cokernel for f. The morphisms f, g that appear in such a pair have to satisfy a list of axioms that mimick the behavior of monomorphisms and epimorphisms in an abelian category (see e.g. [Büh10]). A pair (f, g) in E is called a *conflation* (or *admissible exact sequence*), f is called an *inflation* (or *admissible monomorphism*) and g is called a *deflation* (or *admissible epimorphism*). It can be shown that any small exact category $\mathcal{E}$ can be embedded into an abelian category $\mathcal{A}$ such that the embedding sends conflations in $\mathcal{E}$ to short exact sequences in $\mathcal{A}$ and $\mathcal{E}$ is closed under extensions in $\mathcal{A}$. One associates to any exact category $\mathcal{E}$ a based topological space $\mathrm{BQ}\mathcal{E}$ whose higher homotopy groups are the *Quillen* K-*groups* of $\mathcal{E}$ (see [Qui73]).

Filtered category A category $\mathcal{I}$ is filtered if it is non-empty and satisfies the following conditions:

(1) For all objects $I, J \in \mathcal{I}$ there exists an object $K \in \mathcal{I}$ and morphisms $I \to K$ and $J \to K$.
(2) For all parallel morphisms $f, g : I \rightrightarrows J$ in $\mathcal{I}$, there exists a morphism $h : J \to K$ such that $h \circ f = h \circ g$.

Frobenius category A Frobenius category is an exact category that has enough injective and enough projective objects and is such that the classes of injective objects and projective objects coincide.

Full subcategory A subcategory $\mathcal{D} \subset \mathcal{C}$ is full if $\mathrm{Hom}_{\mathcal{D}}(A, B) = \mathrm{Hom}_{\mathcal{C}}(A, B)$ for all objects A, B of $\mathcal{D}$.

Inflation See *Exact category*.

Injective object An object I of an exact category $\mathcal{E}$ is injective if the functor

$$\mathrm{Hom}_{\mathcal{E}}(-, I) : \mathcal{E}^{\mathrm{op}} \to \mathbf{Ab}$$

is exact.

Localizing subcategory A triangulated subcategory $\mathcal{S} \subset \mathcal{T}$ of a triangulated category $\mathcal{T}$ is localizing if $\mathcal{S}$ is closed under taking set-indexed coproducts.

Projective object An object P of an exact category $\mathcal{E}$ is projective if the functor

$$\mathrm{Hom}_{\mathcal{E}}(P, -) : \mathcal{E} \to \mathbf{Ab}$$

is an exact functor.

Quasi-inverse A quasi-inverse to a functor $F : \mathcal{A} \to \mathcal{B}$ is a functor $G : \mathcal{B} \to \mathcal{A}$ such that $G \circ F$ is naturally isomorphic to $\mathrm{id}_{\mathcal{A}}$ and $F \circ G$ is naturally isomorphic to $\mathrm{id}_{\mathcal{B}}$. If F has a quasi-inverse, then F is an equivalence of categories.

Quillen K-**groups** see *Exact category*.

Serre subcategory A subcategory $\mathcal{B}$ of an Abelian category $\mathcal{A}$ is a Serre subcategory if it is non-empty, full and for all short exact sequences

$$0 \to M' \to M \to M'' \to 0$$

in $\mathcal{A}$, we have that M is in $\mathcal{B}$ if and only if both M' and M'' are in $\mathcal{B}$.

Small category A category is small if it has a set (as opposed to a proper class) of objects and a set of morphisms.

Symmetric monoidal category A symmetric monoidal category $\mathcal{S}$ is a category $\mathcal{S}$ which is equipped with a commutative "tensor product". That is, we are given a bifunctor $\otimes : \mathcal{S} \times \mathcal{S} \to \mathcal{S}$, a unit object $\mathbb{I} \in \mathcal{S}$ and natural isomorphisms $A \otimes \mathbb{I} \cong A \cong \mathbb{I} \otimes A$, $A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$, $A \otimes B \cong B \otimes A$ for all objects $A, B, C \in \mathcal{S}$ which have to satisfy some coherence conditions (see [ML98]).

Waldhausen category A Waldhausen category $\mathcal{W}$ is a category with a zero object that

is equipped with two classes of morphisms, the *cofibrations* $\mathrm{co}(\mathcal{W})$ and the *weak equivalences* $\mathrm{we}(\mathcal{W})$, which have to satisfy a number of axioms (see [Wal85]). One associates to any Waldhausen category $\mathcal{W}$ a based topological space $\Omega|wS_\bullet\mathcal{W}|$ whose higher homotopy groups are the *Waldhausen* K*-groups* of $\mathcal{W}$. If we are given an exact category $\mathcal{E}$ and let $\mathrm{co}(E)$ be the class of inflations in $\mathcal{E}$ and $\mathrm{we}(\mathcal{E})$ the class of isomorphisms, then $\mathcal{E}$ becomes a Waldhausen category in this way.
Waldhausen K**-groups** see *Waldhausen category*.
Weak equivalence See *Waldhausen category*.

Algebraic geometry

Homological support The homological support of a chain complex of modules/sheaves is the support of the coproduct over all its homology modules/sheaves.
Perfect complex A perfect complex on a scheme $(X, \mathcal{O}_X)$ is a chain complex of $\mathcal{O}_X$-modules that is locally quasi-isomorphic to a bounded chain complex of locally free sheaves of finite rank.
Quasi-separated scheme A scheme is quasi-separated if the intersection of any two quasi-compact open subsets is quasi-compact.
Regular scheme A scheme $(X, \mathcal{O}_X)$ is regular if for any point $x \in X$ the local ring $\mathcal{O}_{X,x}$ is a regular local ring.
Separated scheme A scheme X over a base scheme S is separated if the diagonal map $X \to X \times_S X$ is a closed immersion.
Specialization closed A subset Y of a topological space X is specialization closed if $P \in Y$ implies that $\overline{\{P\}} \subset Y$.
Topologically noetherian scheme A scheme $(X, \mathcal{O}_X)$ is topologically noetherian if X is a noetherian topological space.

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Index

Samenvatting

In dit hoofdstuk zal de inhoud van dit proefschrift worden samengevat voor lezers met enige achtergrondkennis op het gebied van algebraïsche meetkunde en/of getrianguleerde categorieën.

Algebraïsche variëteiten en Chow-groepen. De objecten die in in de algebraïsche meetkunde worden bestudeerd zijn schema's en morfismes van schema's. In deze samenvatting beperken wij ons tot makkelijkere (maar zeker niet-triviale!) varianten van een schema, namelijk *algebraïsche variëteiten (over* $\mathbb{C}$*)*. Voor een geheel getal $n \geq 0$ wordt $\mathbb{A}^n_{\mathbb{C}}$, de *affiene n-ruimte over* $\mathbb{C}$, gegeven als de verzameling $\mathbb{C}^n$ voorzien van de *Zariski-topologie*: een verzameling $V \subset \mathbb{A}^n_{\mathbb{C}}$ is gesloten als er een verzameling $T \subset \mathbb{C}[x_1, \ldots, x_n]$ bestaat zodat V de gezamenlijke nulpuntenverzamelingen van de polynomen in T is. Een *affiene algebraïsche variëteit (over* $\mathbb{C}$*)* is een gesloten deelverzameling $V \subset \mathbb{A}^n_{\mathbb{C}}$, samen met de *coördinatenring*

$$\mathrm{A}(V) := \mathbb{C}[x_1, \ldots, x_n]/\mathrm{I}(V) .$$

Hierbij is $\mathrm{I}(V) \subset \mathbb{C}[x_1, \ldots, x_n]$ het ideaal van alle polynomen die op V verdwijnen.

VOORBEELD. Zij $f \in \mathbb{C}[x_1, \ldots, x_n]$. Het polynoom f definieert een affiene algebraïsche variëteit $V(f) \subset \mathbb{A}^n_{\mathbb{C}}$ waarbij $T = \{f\}$. Men kan laten zien dat

$$\mathrm{A}(V(f)) = \mathbb{C}[x_1, \ldots, x_n]/\sqrt{(f)} ,$$

waarbij $\sqrt{(f)}$ het ideaal

$$\{g : \exists\, r > 0 \text{ zo dat } g^r \in (f)\} \subset \mathbb{C}[x_1, \ldots, x_n]$$

is.

VOORBEELD. Zij $V \subset \mathbb{A}^n_{\mathbb{C}}$ een affiene algebraïsche variëteit en $f \in \mathbb{C}[x_1, \ldots, x_n]$. Men kan laten zien dat de open deelverzameling

$$U^V_f := V \setminus V(f) \subset V$$

weer een affiene algebraïsche variëteit is met coördinatenring $\mathrm{A}(V)_f$, de lokalisering van $\mathrm{A}(V)$ bij de multiplicatieve vezameling $\{f^n : n \in \mathbb{Z}_{\geq 0}\}$.

Een algebraïsche variëteit is grofweg een object, dat lokaal "eruit ziet" als een affiene algebraïsche variëteit. Voor het topologische gedeelte is dat eenvoudig te realiseren: de onderliggende ruimte van een algebraïsche variëteit is een topologische ruimte X met een eindige open overdekking

$$X = \bigcup V_i$$

zodat V_i homeomorf is met een affiene algebraïsche variëteit. Om lokaal over coördinatenringen te kunnen spreken eisen wij dat X voorzien is van een *schoof van ringen* $\mathcal{O}_X$: voor elke open deelverzameling $U \subset X$ zijn ringen $\mathcal{O}_X(U)$ gegeven, en voor een inclusie $U_1 \subset U_2$ zijn er ringhomomorfismes $\mathcal{O}_X(U_2) \to \mathcal{O}_X(U_1)$, de *restrictieafbeeldingen*, welke aan een aantal axioma's moeten voldoen. In het geval van onze algebraïsche variëteit X eisen wij dat $\mathcal{O}_X(U_f^{V_i}) \cong \mathrm{A}(V_i)_f$ voor alle f, en dat de restrictieafbeeldingen overeenkomen met de lokaliseringsafbeeldingen via dit isomorfisme. Omdat de verzamelingen $U_f^{V_i}$ een basis voor de topologie op X vormen, zijn de ringen $\mathcal{O}_X(U)$ op deze manier volledig vastgelegd voor alle open verzamelingen $U \subset X$.

VOORBEELD. Zij V een affiene algebraïsche variëteit. Dan is V een algebraïsche variëteit: de schoof $\mathcal{O}_V$ wordt vastgelegd door te eisen dat $\mathcal{O}(U_f^V) = \mathrm{A}(V)_f$ en dat de restrictieafbeeldingen gegeven worden door de lokaliseringsafbeeldingen. We noemen $\mathcal{O}_V$ de *schoof van reguliere rationale functies op* V.

VOORBEELD. De Riemannsfeer $\mathbb{P}^1_{\mathbb{C}}$ is een voorbeeld van een algebraïsche variëteit die niet affien is. We kunnen $\mathbb{P}^1_{\mathbb{C}}$ overdekken met de open deelverzamelingen

$$U_1 := \mathbb{P}^1_{\mathbb{C}} \setminus \{\infty\} \quad \text{en} \quad U_2 := \mathbb{P}^1_{\mathbb{C}} \setminus \{0\}\,.$$

Een toepassing van de stereografische projectie laat zien dat zowel U_1 als U_2 homeomorf zijn met $\mathbb{C}$, wat men kan schrijven als de nulpuntenverzameling van $0 \in \mathbb{C}[x]$. Wij definïeren nu een schoof van ringen op $\mathbb{P}^1_{\mathbb{C}}$ door de schoven van reguliere rationale functies op $U_i \cong \mathbb{C}$ aan elkaar te plakken op $U_1 \cap U_2 = \mathbb{C} \setminus \{0\}$ door middel van de functie $x \mapsto 1/x$.

De *Chow-groepen* van een algebraïsche variëteit X zijn invarianten van X in de vorm van abelse groepen $\mathrm{CH}^p(X)$, $p \geq 0$, die voortkomen uit de systematische studie van irreducibele gesloten deelvariëteiten van X "modulo rationale equivalentie". Een deelvariëteit Y van een algebraïsche variëteit X is grofweg een gesloten deelverzameling van X die zelf weer een algebraïsche variëteit is. Een deelvariëteit Y is irreducibel als Y niet kan worden geschreven als een eindige vereniging van gesloten deelvariëteiten $\neq Y$. Voor een geheel getal $p \geq 0$ beschouwt men alle irreducibele deelvariëteiten van X van codimensie p en maakt dit tot een abelse groep $\mathrm{Z}^p(X)$ door formele eindige sommen van codimensie p deelvariëteiten van X (zogenaamde *cykels*) met geheeltallige coëfficiënten toe te laten. Nu definiëren wij op deze groep een equivalentierelatie waarbij twee codimensie p cykels *rationeel equivalent* zijn als zij door middel van een "algebraïsche homotopie in X" in elkaar over kunnen gaan. Dit kan men zich voorstellen als volgt: de twee cykels zijn elementen van een continue familie van cykels $C \subset X \times \mathbb{P}^1$, waarbij C zelfs weer een deelvariëteit van $X \times \mathbb{P}^1$ is. Wij formaliseren dit idee door te zeggen dat twee cykels rationeel equivalent zijn als hun verschil gelijk is aan de divisor (d.w.z. de formele som van nulpunten en polen, geteld met multipliciteiten) van een rationele functie op een irreducibele deelvariëteit van X van codimensie $p-1$. Het quotiënt van $\mathrm{Z}^p(X)$ modulo deze equivalentierelatie is de codimensie p Chow-groep $\mathrm{CH}^p(X)$.

VOORBEELD. Voor een algebraïsche variëteit X is $\mathrm{Z}^0(X) = \mathrm{CH}^0(X)$ de vrije abelse groep op de irreducibele componenten van X. Voor een grote klasse van algebraïsche variëteiten X is $\mathrm{CH}^1(X)$ gelijk aan $\mathrm{Pic}(X)$, de Picardgroep van X.

Op een gladde algebraïsche variëteit kan van

$$\mathrm{CH}(X) = \bigoplus_{p \geq 0} \mathrm{CH}^p(X)$$

een gegradueerde ring worden gemaakt door een *snijproduct*

$$\mathrm{CH}^p(X) \times \mathrm{CH}^q(X) \to \mathrm{CH}^{p+q}(X)$$

voor alle $p, q \geq 0$ te definiëren. Het idee achter deze constructie is om deelvariëteiten $V, W \subset X$ van codimensie p en q te snijden en een deelvariëteit $V \cap W \subset X$ van codimensie $p+q$ te verkrijgen. De deelvariëteit $V \cap W$ is niet noodzakelijk irreducibel maar heeft een eindig aantal irreducibele componenten, die een cykel op X definiëren. Om rekening te houden met snijmultipliciteiten voorzien we elk irreducibel component van $V \cap W$ van de bijbehorende multipliciteit als coëfficiënt. Dit kunnen wij vervolgens lineair uitbreiden tot cykels. Aan dit idee zijn echter een aantal problemen verbonden: zoals men snel ziet kan het gebeuren dat de codimensie van $V \cap W$ niet gelijk is aan $p+q$ (neem bijvoorbeeld $V = W$, met $\mathrm{codim}(V) > 0$) en het blijkt ook niet makkelijk te zijn om een goede definitie van snijmultipliciteiten te geven. Voor gladde algebraïsche variëteiten is er echter een oplossing: *Chow's moving lemma* (1956) zegt dat er altijd V' en W' in de respectievelijke equivalentieklassen van V en W in $\mathrm{CH}(X)$ zijn, zodat de codimensie van $V' \cap W'$ gelijk is aan $p+q$. Verder is het mogelijk om in dit geval een goede definitie voor intersectiemultipliciteiten te geven, bijvoorbeeld door de *Tor-formule van Serre* (1965). Grayson (1978) geeft een andere methode om het snijproduct door middel van de hogere algebraïsche K-theorie van X te definiëren.

Afgeleide en getrianguleerde categorieën. Een categorie is een wiskundig object dat kan worden beschouwd als een formele abstractie van de volgende algemene observatie: in alle takken van de wiskunde bestudeert men een bepaalde klasse van objecten (zoals verzamelingen, topologische ruimtes, groepen etc.) en een bepaalde soort van afbeeldingen tussen deze objecten (zoals functies, continue functies, groepshomomorfismes etc.). Een categorie $\mathcal{C}$ bestaat uit de data van een verzameling van *objecten* $\mathrm{Ob}(\mathcal{C})$ en voor elk tweetal van objecten $A, B \in \mathrm{Ob}(\mathcal{C})$ een verzameling van *morfismes* $\mathrm{Hom}_{\mathcal{C}}(A, B)$. Verder is er voor elk drietal van objecten $A, B, C \in \mathrm{Ob}(\mathcal{C})$ een associatieve samenstellingsafbeelding

$$\circ : \mathrm{Hom}_{\mathcal{C}}(A, B) \times \mathrm{Hom}_{\mathcal{C}}(B, C) \to \mathrm{Hom}_{\mathcal{C}}(A, B) \times \mathrm{Hom}_{\mathcal{C}}(B, C)$$

en een identiteitselement $\mathrm{id}_A \in \mathrm{Hom}_{\mathcal{C}}(A, A)$ voor elk object A, dat als links- en rechtseenheid voor de samenstellingsafbeelding fungeert.

Veel categorieën hebben meer structuur dan de boven genoemde: getrianguleerde categorieën zijn een soort categorie die men vooral tegenkomt in de context van homotopie, zowel in de algebraïsche zin (homotopie van ketencomplexen) als in de topologische zin (homotopietheorie van topologische ruimtes). Het meest basale voorbeeld aan de algebraïsche kant is de *afgeleide categorie* van een abelse categorie, voor het eerst bestudeert door Grothendieck en Verdier (1967). Dit leggen wij nu verder uit.

Een abelse categorie is grofweg een categorie waarin men morfismes kan optellen, en waarin elk morfisme een kern en een cokern heeft. Het naamgevende voorbeeld is de categorie van abelse groepen **Ab**. Een observatie die leidt tot de constructie van de bijbehorende afgeleide categorie is dat in veel wiskundige contexten ketencomplexen van abelse groepen een belangrijke rol spelen, zoals in de volgende twee voorbeelden: bij het uitrekenen van de homologiegroepen van een topologische ruimte, of bij het uitrekenen van groepencohomologie kunnen verschillende methodes worden toegepast, die allemaal het doel hebben om een ketencomplex van abelse groepen te produceren, waarvan men vervolgens homologiegroepen berekent. Een ketencomplex van abelse groepen is een diagram

$$A_\bullet : \cdots \to A_{i+1} \xrightarrow{\partial_{i+1}} A_i \xrightarrow{\partial_i} A_{i-1} \to \cdots$$

waarbij A_i abelse groepen zijn en ∂_i morfismes ervan (de *differentialen*), en $\partial_i \partial_{i+1} = 0$ voor alle i geldt. De eerste stap in de constructie van de afgeleide categorie is de constructie van de categorie van ketencomplexex C(**Ab**). De objecten van C(**Ab**) zijn ketencomplexen van abelse groepen en een afbeelding $A_\bullet \to B_\bullet$ is een verzameling groepshomomorfismes $A_i \to B_i$ die moeten commuteren met de differentialen. We merken op dat C(**Ab**) weer een abelse categorie is.

VOORBEELD. Voor een topologische ruimte X produceert men het *singuliere ketencomplex* $C_\bullet(X)$, waarbij $C_n(X)$ wordt gegeven als de vrije abelse groep op de continue afbeeldingen van de standaard n-simplex naar X. De differentialen worden gegeven door restrictie tot de zijvlakken.

Zij G een groep. Om de cohomologie van een G-moduul M uit te rekenen produceert men een *injectieve resolutie van* M en bekijkt de restrictie van de resolutie tot het G-invariante deel. Het resultaat is ook hier een ketencomplex van abelse groepen.

In de boven genoemde voorbeelden is men eigenlijk niet geïnteresseerd in het ketencomplex $A_\bullet$ zelf, maar in de homologiegroepen

$$H_i(A_\bullet) := \ker \partial_i / \operatorname{im}(\partial_{i+1}) \ .$$

In het geval van de groepencohomologie is het zelfs zo, dat een andere keuze van injectieve resolutie kan leiden tot een niet-isomorf ketencomplex. Er is echter één belangrijk verband: een keuze van twee injectieve resoluties geeft altijd een afbeelding tussen de twee geassocieerde ketencomplexen, die isomorfismes op de homologiegroepen induceren. Wij noemen afbeeldingen van ketencomplexen, die isomorfismes op de homologiegroepen induceren *quasi-isomorfismes*.

Met deze observaties in het achterhoofd zien wij dat de categorie C(**Ab**) "te groot" is: wij zijn op zoek naar een categorie waarin quasi-isomorfe ketencomplexen kunnen worden geïdentificeerd. Het idee om dit probleem op te lossen is makkelijk: De afgeleide categorie D(**Ab**) wordt geconstrueerd uit C(**Ab**) door de quasi-isomorfismes formeel te inverteren. Een morfisme $A_\bullet \to B_\bullet$ in D(**Ab**) wordt gerepresenteerd door een "breuk"

$$A_\bullet \xleftarrow{\alpha} C_\bullet \to B_\bullet$$

waarbij α een quasi-isomorfisme is (die wij als "noemer" beschouwen). Op deze manier forceren wij dat quasi-isomorfe ketencomplexen isomorf worden in D(**Ab**).

De prijs die men hiervoor moet betalen is dat D(**Ab**) geen abelse categorie meer is. De categorie D(**Ab**) heeft nog wél een additieve structuur: wij kunnen morfismes optellen. Een kenmerkende eigenschap van abelse categoriën, het bestaan van kernen en cokernen, is echter niet meer gegeven. Daardoor kan men in D(**Ab**) niet meer praten over exacte rijen. Er bestaat wél een vervanger voor dit belangrijke concept: een *exact driehoek* in

D(**Ab**) is een diagram van de vorm

$$A_\bullet \to B_\bullet \to C_\bullet \to \Sigma(A_\bullet)$$

dat afkomstig is van een korte exacte rij

$$0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$$

in de (abelse!) categorie C(**Ab**) die in elke graad splijt. De uitdrukking $\Sigma(A_\bullet)$ staat hier voor het ketencomplex $A'_\bullet$ met $A'_i = A_{i+1}$ en $\partial_i^{A'_\bullet} = -\partial_{i+1}^{A_\bullet}$. Wij merken op dat de constructie van $\mathrm{D}(\mathcal{A})$ voor een willekeurige abelse categorie $\mathcal{A}$ in volledige analogie is met de constructie van D(**Ab**).

VOORBEELD. Op een algebraïsche variëteit X beschouwen wij de categorie van coherente schoven van $\mathcal{O}_X$-modulen $\mathrm{Coh}(X)$. De objecten van deze categorie zijn schoven van abelse groepen $\mathcal{F}$ op X zodanig dat $\mathcal{F}(U)$ een eindig voortgebracht $\mathcal{O}_X(U)$-moduul is en zodanig dat deze moduulstructuur compatibel is met de restrictieafbeeldingen van $\mathcal{F}$ en $\mathcal{O}_X$. De categorie $\mathrm{Coh}(X)$ is abels en $\mathrm{D^b}(\mathrm{Coh}(X))$ is de volle deelcategorie van $\mathrm{D}(\mathrm{Coh}(X))$ van begrensde ketencomplexen in $\mathrm{Coh}(X)$, d.w.z. complexen $\mathcal{F}_\bullet$ met $\mathcal{F}_i \neq 0$ voor eindig veel i. Als X een affiene algebraïsche variëteit is, dan is de categorie $\mathrm{Coh}(X)$ equivalent met $\mathrm{A}(X)$-mod, de categorie van eindig voortgebrachte $\mathrm{A}(X)$-modulen en geldt $\mathrm{D^b}(\mathrm{Coh}(X)) \cong \mathrm{D^b}(\mathrm{A}(X)\text{-mod})$.

Een getrianguleerde categorie kan men beschouwen als de axiomatisering van een afgeleide categorie: het is een additieve categorie $\mathcal{T}$, samen met een additieve equivalentie van categoriën $\Sigma_{\mathcal{T}} : \mathcal{T} \to \mathcal{T}$ (de *shift* of *suspension*) en een klasse van exacte driehoeken

$$A \to B \to C \to \Sigma(A)$$

die aan een aantal axioma's moeten voldoen. Een functor $F : \mathcal{T} \to \mathcal{S}$ tussen twee getrianguleerde categorieën heet *exact* als hij met de shifts commuteert en exacte driehoeken in $\mathcal{T}$ naar exacte driehoeken in $\mathcal{S}$ stuurt.

In dit proefschrift bestuderen wij een klasse van getrianguleerde categorieën die nog meer structuur hebben: een tensor-getrianguleerde categorie is een getrianguleerde categorie $\mathcal{T}$ samen met een compatibele symmetrisch-monoïdale structuur, dat wil zeggen er is een bi-functor

$$\otimes : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$$

die, op natuurlijk isomorfisme na, een associatieve en commutatieve operatie op $\mathcal{T}$ met eenheidsobject $\mathbb{I}$ definieert. Verder eisen wij dat voor elk object $A \in \mathcal{T}$ de functor

$$A \otimes - : \mathcal{T} \to \mathcal{T}$$

exact is.

VOORBEELD. Zij X een niet-singuliere algebraïsche variëteit. Dan induceert het tensorproduct $\otimes_{\mathcal{O}_X}$ van coherente schoven een bi-functor

$$\otimes^{\mathrm{L}} : \mathrm{D^b}(\mathrm{Coh}(X)) \times \mathrm{D^b}(\mathrm{Coh}(X)) \to \mathrm{D^b}(\mathrm{Coh}(X))$$

die $\mathrm{D^b}(\mathrm{Coh}(X))$ de structuur van een tensor-getrianguleerde categorie geeft. Wij merken op dat de constructie van $\otimes^{\mathrm{L}}$ op $\mathrm{D^b}(\mathrm{Coh}(X))$ niet altijd mogelijk is als X niet glad is: voor twee begrensde ketencomplexen $\mathcal{F}_\bullet, \mathcal{G}_\bullet \in \mathrm{D^b}(\mathrm{Coh}(X))$ is het ketencomplex $\mathcal{F}_\bullet \otimes^{\mathrm{L}} \mathcal{G}_\bullet$ dan mogelijk niet begrensd.

Chow-groepen van tensor-getrianguleerde categorieën. Een bekende stelling van P. Gabriel (1962) zegt dat alle informatie over een algebraïsche variëteit X bevat is in de abelse categorie $\mathrm{Coh}(X)$: men kan voor elke abelse categorie $\mathcal{A}$ een topologische ruimte $\mathrm{Sp}(\mathcal{A})$ en een schoof van ringen $\mathcal{O}_{\mathcal{A}}$ op $\mathrm{Sp}(\mathcal{A})$ definieren, zodanig dat het paar $(\mathrm{Sp}(\mathrm{Coh}(X)), \mathcal{O}_{\mathrm{Coh}(X)})$ isomorf is met de algebraïsche variëteit X. Een natuurlijke vraag die hieruit voorkomt is of de getrianguleerde categorie $\mathrm{D}^{\mathrm{b}}(\mathrm{Coh}(X))$ ook alle informatie over X bevat. Het antwoord hierop is negatief: voor een abelse variëteit X en de duale variëteit $\hat{X}$ zijn de categorieën $\mathrm{D}^{\mathrm{b}}(\mathrm{Coh}(X))$ en $\mathrm{D}^{\mathrm{b}}(\mathrm{Coh}(\hat{X}))$ altijd equivalente getrianguleerde categorieën, maar X en $\hat{X}$ zijn niet noodzakelijk isomorf.

Een mogelijkheid om dit te “repareren” is om $\mathrm{D}^{\mathrm{b}}(\mathrm{Coh}(X))$ in het niet-singuliere geval als tensor-getrianguleerde categorie te beschouwen. Een stelling van P. Balmer (2005) zegt, dat men voor elke tensor-getrianguleerde categorie $\mathcal{T}$ een topologische ruimte $\mathrm{Spc}(\mathcal{T})$ en een schoof van ringen $\mathcal{O}_{\mathcal{T}}$ op $\mathrm{Spc}(\mathcal{T})$ kan definiëren zodat

$$(\mathrm{Spc}(\mathrm{D}^{\mathrm{b}}(\mathrm{Coh}(X))), \mathcal{O}_{\mathrm{D}^{\mathrm{b}}(\mathrm{Coh}(X))})$$

isomorf is met X. De stelling is nog algemener: voor een willekeurige (mogelijk singuliere) algebraïsche variëteit X beschouwen wij de getrianguleerde deelcategorie van *perfecte complexen* $\mathrm{D}^{\mathrm{perf}}(X) \subset \mathrm{D}^{\mathrm{b}}(\mathrm{Coh}(X))$. De categorie $\mathrm{D}^{\mathrm{perf}}(X)$ is altijd een *tensor*-getrianguleerde categorie en $\mathrm{D}^{\mathrm{perf}}(X)$ is equivalent met $\mathrm{D}^{\mathrm{b}}(\mathrm{Coh}(X))$ als X glad is. Volgens de stelling geldt altijd dat

$$(\mathrm{Spc}(\mathrm{D}^{\mathrm{perf}}(X)), \mathcal{O}_{\mathrm{D}^{\mathrm{perf}}(X)})$$

isomorf is met X.

Uit het oogpunt van deze stelling zou het daarom in theorie mogelijk moeten zijn om de studie van een algebraïsche variëteit X te vervangen door de studie van $\mathrm{D}^{\mathrm{perf}}(X)$. Andersom kan men de studie van tensor-getrianguleerde categorieën opvatten als een uitbreiding van de studie van algebraïsche variëteiten. In dit proefschrift wordt de vraag bestudeert in hoeverre het mogelijk is om de invarianten “Chow groepen van een algebraïsche variëteit X” uit te breiden naar tensor-getrianguleerde categorieën. Wij verkrijgen onder andere de volgende resultaten:

- Een definitie van cykelgroepen en Chow-groepen voor tensor-getrianguleerde categorieën $\mathcal{T}$ van P. Balmer (2013) geeft invarianten die de cykelgroepen en Chow-groepen van een niet-singuliere algebraïsche variëteit X reconstrueren als $\mathcal{T} = \mathrm{D}^{\mathrm{perf}}(X)$.
- Deze invarianten hebben goede functorialiteitseigenschappen. Hiermee bedoelen wij dat een grote klasse van functoren tussen tensor-getrianguleerde categorieën groepshomomorfismes induceert tussen de bijbehorende cykelgroepen en Chow-groepen.
- De theorie is toepasbaar in nieuwe situaties, bijvoorbeeld in de modulaire representatietheorie.
- Onder speciale voorwaarden kan men de methode van Grayson generaliseren om een snijproduct op de Chow-groepen van tensor-getrianguleerde categorieën te verkrijgen.

Wij verwijzen de lezer naar de introductie voor de exacte formuleringen van bovenstaande uitspraken.

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Curriculum Vitæ

Sebastian Arne Klein was born on May 8th, 1984 in Offenbach am Main, Germany. He attended the secondary school “Hanns-Seidel-Gymnasium” in Hösbach from 1994 to 2003.

After a 10-months alternative civilian service at the local hospital in Aschaffenburg, he enrolled for a degree in mathematics at the Albert-Ludwigs-Universität Freiburg in 2004, where he obtained his “Vordiplom” in 2006.

From 2007 to 2008 he stayed as an exchange student at Universiteit Utrecht and enrolled for the local Master’s programme in 2008. He completed his Master’s degree “cum laude” in 2010, with a thesis that was supervised by Prof. dr. Gunther Cornelissen and won the 2010 student prize of the NWO research cluster Geometry and Quantum Theory.

From 2010 to 2014 he was a Ph.D. student at Universiteit Utrecht, under the joint supervision of Prof. dr. Gunther Cornelissen and Prof. dr. Paul Balmer. During this period he made research visits to the University of California, Los Angeles, and Universität Bielefeld and gave talks at research conferences in Leiden and Warsaw and at the Max-Planck-Institut für Mathematik in Bonn.

In November 2014, he will be joining Prof. dr. Wendy Lowen’s research group on non-commutative algebra and geometry at Universiteit Antwerpen as a postdoctoral researcher.