Convexity theorems for symmetric spaces and representations of n-Lie algebras
Two studies in Lie theory

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Convexity theorems for symmetric spaces and representations of n-Lie algebras
Two studies in Lie theory

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Twee studies in Lie theorie

(met een samenvatting in het Nederlands)

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Pentru Buni şi Bunu, pentru răbdarea lor

Contents

Introduction

This thesis presents the outcome of two research projects. The first of these dealt with the classification of the irreducible highest weight representations of the simple complex n-Lie algebra. The results of this project have been published in [3].

In the second project, it was investigated whether Kostant's non-linear convexity theorem for real semisimple groups can be generalized to a new setting arising from the theory of semisimple symmetric spaces. As a result of our research, such a generalization was indeed found, and is presented as Theorem 4.10.1, the main result of this thesis. The results of the second project were published earlier in [4].

The present thesis is organized in two parts which essentially contain the mentioned papers [3] and [4], respectively, together with self-contained introductions to them. The first part is contained in Chapter 1, whereas the second part is contained in Chapters 2, 3 and 4.

These two parts can be read independently. We will now outline their contents in more detail.

Representations of the simple n-Lie algebra

The theory of n-Lie algebras was first developed in 1985 by Filippov in [16]. Subsequently, the interest in this type of algebras grew due to their importance in physics. For instance, a metric 3-Lie algebra is used in the Lagrangian description of a certain $2+1$ dimensional field theory, called the Bagger-Lambert-Gustavsson theory.

Let $\mathbb{K}$ be a field of characteristic zero. An n-Lie algebra over $\mathbb{K}$ is a natural generalization of a Lie algebra, for which the Lie bracket is not a binary operation, but an n-ary one ($n \geq 3$). The n-bracket is n-linear, antisymmetric and satisfies a generalized Jacobi identity.

Many of the classical concepts and theorems in the theory of Lie algebras have very natural counterparts in the setting of n-Lie algebras. For example, Ling proved in his PhD-thesis [43], that every semisimple n-Lie algebra is the direct sum of its simple ideals. Moreover, he classified all simple n-Lie algebras, under the assumption that the field $\mathbb{K}$ is algebraically closed. It turns out that, up to isomorphism, there exists a unique simple n-Lie algebra, which has to be of dimension $n+1$. For $\mathbb{K} = \mathbb{C}$, a realization of the simple n-Lie algebra is given by $\mathbb{C}^{n+1}$ with the bracket given by

$$[e_1, \ldots, \hat{e}_i, \ldots, e_{n+1}] = (-1)^{n+i+1} e_i.$$

Here $\{e_1, \ldots, e_{n+1}\}$ denotes the canonical basis of $\mathbb{C}^{n+1}$ and $\hat{e}_i$ means that the elements e_i has been omitted from the n-bracket. We denote the given complex simple n-Lie algebra by A and reserve the notation V for a generic n-Lie algebra.

To every n-Lie algebra V we can associate a Lie algebra ($\wedge^{n-1}V$), called the basic Lie algebra. For the simple n-Lie algebra A the basic Lie algebra is isomorphic to the complex Lie algebra $so(n+1)$. The importance of the basic Lie algebra becomes obvious when studying representation theory of n-Lie algebras. Representations of the n-Lie algebra V are in 1-1 correspondence with representations of its basic Lie algebra $\wedge^{n-1}V$ on which a certain ideal $Q(V)$ in the universal enveloping algebra of the basic Lie algebra acts trivially. Furthermore, the concepts of irreducibility and complete reducibility of modules stay true under this correspondence.

In [14] A. Dzhumadil'daev classified the finite-dimensional irreducible highest weight representations of the simple complex n-Lie algebra A. In the first part of this thesis we classify all irreducible highest-weight representations of A, finite- and infinite-dimensional alike.

Let $\mathfrak{b}$ be a Borel subalgebra of the Lie algebra $so(n+1)$ and let $\lambda \in (\mathfrak{b}/\mathrm{Rad}(\mathfrak{b}))^*$. Denote by $V(\lambda)$ the associated Verma module of the highest weight λ and by $Z(\lambda)$ its unique irreducible quotient, with highest weight λ. We give conditions on λ, such that $Z(\lambda)$ is an n-Lie algebra module for the simple n-Lie algebra A.

Theorem (1.3.2) *Let $n \geq 3$, $n+1 = 2N$ and $t \in \{1, \ldots, N\}$. Denote by $\pi_1, \ldots, \pi_N$ the fundamental weights of $so(2N)$. Then, $Z(\lambda)$ is an irreducible representation of the simple n-Lie algebra A if and only if λ has one of the following values*

$$\begin{cases} x\pi_t & t = 1, \\ (-1-x)\pi_{t-1} + x\pi_t & 1 < t < N-1, \\ (-1-x)\pi_{t-1} + x\pi_t + x\pi_{t+1} & t = N-1, \\ (-1-x)\pi_{t-1} + (-1+x)\pi_t & t = N, \end{cases}$$

where $x \in \mathbb{C}$.

Theorem (1.3.3) *Let $n \geq 3$, $n+1 = 2N+1$ and $t \in \{1, \ldots, N\}$. Denote the fundamental weights of $so(2N+1)$ by $\pi_1, \ldots, \pi_N$. Then, $Z(\lambda)$ is an irreducible highest weight representation of the simple n-Lie algebra A if and only if λ has one of the following values*

$$\begin{cases} x\pi_t & t = 1, \\ (-1-x)\pi_{t-1} + x\pi_t & 1 < t \leq N, \end{cases}$$

where $x \in \mathbb{C}$.

It is clear that for $t = 1$ and x a positive integer, $Z(\lambda)$ is a finite-dimensional module.

The proof of the theorem relies on the above mentioned correspondence between representations of an n-Lie algebra and representations of its basic Lie algebra. The techniques used are classical techniques in representation theory of semisimple Lie algebras.

Chapter 1 is organized as follows. In Section 1.1 we begin with a short introduction to the theory of n-Lie algebras. We introduce here the standard definitions and results, define the basic Lie algebra associated to an n-Lie algebra and give its construction in Subsection 1.1.2. In Subsection 1.1.3 we sketch the proof of the classification theorem of simple n-Lie algebras obtained in [43].

Section 1.2 is entirely devoted to the structure of the simple n-Lie algebra A. We begin by presenting a different construction of this n-Lie algebra and we prove that its basic Lie algebra is isomorphic to $so(n+1)$. In Subsection 1.2.1 we describe a pictorial representation of the elements of the basic Lie algebra of the simple n-Lie algebra.

In Section 1.3 we give the statements of our main theorem. Depending on the parity of $n+1$ we obtain Theorem 1.3.2 or Theorem 1.3.3.

The strategy used to prove these theorems is explained in detail in Subsection 1.4.2, while the actual computations are done in Subsections 1.5.3 and 1.5.4. In Subsection 1.5.5 we give a second method for obtaining the 'only if'-implication of these theorems, by means of the pictorial representation mentioned above.

Finally, in Section 1.6, we give an application of our theorem. Namely, we classify the primitive ideals of the universal enveloping algebra of the simple n-Lie algebra A. Moreover, we show that the Joseph ideal constructed in [33] is such a primitive ideal.

The convexity theorem

A well-known result of Schur, Horn and Thompson says that for the space of all n-by-n Hermitian matrices with a given sequence $(\lambda_1, \ldots, \lambda_n)$ of real eigenvalues, a vector in $\mathbb{R}^n$ is the diagonal part of such a matrix if and only if this vector belongs to the convex polytope with vertices given by the set $\{(\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(n)}) \,|\, \sigma \in S_n\}$ of all permutations of the eigenvalues. This result was generalized by Kostant in [37] to the setting of connected real semisimple Lie groups, with finite center. Kostant's generalization is known in the literature as the linear convexity theorem of Kostant.

Let G be a connected real semisimple Lie group with finite center (or more generally a reductive Lie group of the Harish-Chandra class), K a maximal compact subgroup and $G = KAN_P$ an associated Iwasawa decomposition. Let $E_{\mathfrak{a}} : \mathfrak{g} \to \mathfrak{a}$ denote the projection associated with the corresponding infinitesimal Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}_P$ and let $X \in \mathfrak{a}$. The linear convexity theorem of Kostant says that the image of the adjoint orbit $\mathrm{Ad}(K)X$ under $E_{\mathfrak{a}}$ is a convex polytope in $\mathfrak{a}$. More precisely, this convex polytope has vertices given by the orbit of X under the action of the Weyl group $W(\mathfrak{a})$ of the root system $\Sigma(\mathfrak{g}, \mathfrak{a})$. In formula,

$$E_{\mathfrak{a}}(\mathrm{Ad}(K)X) = \mathrm{conv}(W(\mathfrak{a}) \cdot X),$$

where 'conv' indicates that the convex hull is taken.

The term 'linear' is explained by the fact that it is the linear version of another convexity theorem, called Kostant's non-linear convexity theorem, in which the linear projection is replaced by the Iwasawa projection $\mathfrak{H}_P : G \to \mathfrak{a}$, defined by $\mathfrak{H}_P(kan) = \log a$. Furthermore, the action is replaced by the action of K on G by conjugation. Namely, for $a = \exp X \in A$, we have that

$$\mathfrak{H}_P(\mathrm{Ad}(K)a) = \mathfrak{H}_P(aK) = \mathrm{conv}(W(\mathfrak{a}) \cdot X).$$

Both convexity theorems of Kostant have been generalized to the framework of symplectic geometry: see [2], [9], [23], [24], [35] for the linear convexity theorem and [30], [44] for the non-linear one. The non-linear convexity theorem has also been generalized to the setting of semisimple symmetric spaces in [5] and [48]. In the second part of this thesis we present a remarkable further generalization of van den Ban's convexity theorem of [5].

Let $\theta : G \to G$ be a Cartan involution on G associated to K. Furthermore, let $\sigma : G \to G$ be an involution on G commuting θ and let H be an open subgroup of the fixed point group G^σ. We may select A such that A is σ-stable and $A \cap H$ is of smallest possible dimension. Our goal is to determine the image of aH under the Iwasawa projection $\mathfrak{H}_P$, where $a \in A$. This problem was studied in [5] under the extra assumption that the minimal parabolic subgroup $P := Z_K(\mathfrak{a})AN_P$ is $\mathfrak{q}$-extreme, i.e. $N_P \cap H$ is of smallest possible dimension. We provide an answer without imposing this extra assumption.

Thus, van den Ban's convexity theorem is a particular case of our main theorem. Kostant's non-linear convexity theorem in turn arises as a particular case of van den Ban's result by taking $\sigma = \theta$. Our result also implies Kostant's result in a different way, by viewing the group G as a symmetric space. This will be explained at a later stage in this introduction.

We will now present the formulation of our main result in some detail. First of all, the image $\mathfrak{H}_P(aH)$ is readily seen to be $\mathfrak{a} \cap \mathfrak{h}$-invariant and therefore completely determined by its projection $\mathrm{pr}_{\mathrm{q}}\mathfrak{H}_P(aH)$ onto $\mathfrak{a} \cap \mathfrak{q}$ along $\mathfrak{a} \cap \mathfrak{h}$ (here $\mathfrak{q}$ denotes the -1-eigenspace of $\mathfrak{g}$ for the infinitesimal involution σ). Moreover, it suffices to describe the image for $a \in \exp(\mathfrak{a} \cap \mathfrak{q})$.

Theorem (4.10.1) *Let G be a reductive Lie group of the Harish-Chandra class, σ an involution on G and H an essentially connected open subgroup of G^σ. Let P be any minimal parabolic subgroup containing A and $a \in \exp(\mathfrak{a} \cap \mathfrak{q})$. Then*

$$\mathrm{pr}_{\mathrm{q}}\mathfrak{H}_P(aH) = \mathrm{conv}(W_{K\cap H} \cdot \log a) + \Gamma(P),$$

where 'conv' *denotes the convex hull, $\Gamma(P)$ is a closed convex polyhedral cone and $W_{K\cap H}$ denotes the Weyl group $N_{K\cap H}(\mathfrak{a} \cap \mathfrak{q})/Z_{K\cap H}(\mathfrak{a} \cap \mathfrak{q})$.*

Every semisimple Lie group G can be realized as the semisimple symmetric space $G \times G/\mathrm{diag}(G \times G)$. In this setting the considered minimal parabolic subgroups of $G \times G$ are of the form $P \times Q$, with P and Q minimal parabolic subgroups of G containing A. Van den Ban's condition on the parabolic subgroup amounts (in this setting) to $Q = \bar{P} = \theta(P)$. On the other hand, applying our theorem to the group, seen as a semisimple symmetric space, with $P = Q$, we can recover the non-linear convexity theorem of Kostant in a second way, as mentioned above.

The techniques used to prove the theorem are inspired by [26] and [13]. In [26] Heckman uses these techniques to prove the linear convexity theorem of Kostant. However, he obtains the non-linear convexity theorem from the linear one by a homotopy argument.

The second part of the thesis is organized into three chapters.

The first of these chapters is Chapter 2 in which we give a brief introduction to the subject. In Section 2.1 we discuss some structure theory of semisimple Lie groups. We explain in this section both convexity theorems of Kostant and give a few examples. In Section 2.2 we introduce parabolic subalgebras and parabolic subgroups, while in Section 2.3 we define semisimple symmetric spaces. Chapter 2 ends with the definition of a reductive Lie group of the Harish-Chandra class and a short motivation for using such Lie groups, see Section 2.4.

We begin Chapter 3 with the precise statement of Theorem 4.10.1 in Section 3.1. The rest of this chapter is dedicated to the case of the semisimple symmetric space $G \times G/\mathrm{diag}(G \times G)$ (the group case). Our detailed exposition of the group case starts in Section 3.2 where we illustrate on a particular example how the polyhedral convex cone $\Gamma(P)$ depends on the parabolic subgroup P. Namely, by choosing different minimal parabolic subgroups of this semisimple symmetric space with the same split component, we obtain different convex cones. Inspired by the independent proof of van den Ban's convexity result for the group case, see [5, Theorem A.1], we present in Subsection 3.2.2 a computational proof for the case of the group. This proof amounts to the use of the classical non-linear convexity theorem of Kostant and a well-known result about the Iwasawa projection of unipotent radicals (Lemma 3.2.4). Conversely, both these results can be obtained from our convexity theorem applied to the case of the group. In Subsection 3.2.3 we show how to obtain Lemma 3.2.4 as a consequence of Theorem 4.10.1.

The last chapter, Chapter 4, is entirely devoted to the proof of the convexity resulted stated above. The proof is divided into a series of steps, each contained in a section of this chapter. Each step proves a smaller result necessary for our argumentation in the proof of the main theorem. A summary of the proof and the multiple steps it involves is presented in Section 4.1. Finally, in Section 4.10, by induction on the real rank of the Lie group G and some topological arguments, all these steps contribute to our final reasoning and as such prove Theorem 4.10.1. We conclude the thesis with an appendix, A, in which we prove a lemma (Lemma 4.2.10) concerning the decomposition for nilpotent groups in terms of subgroups generated by roots.

In the end, we wish to mention that our choice of notation differs from the first part of the thesis (Chapter 1) to the second (Chapters 2, 3 and 4).

Chapter 1

Highest weight representations of the simple n-Lie algebra

This chapter contains the first part of this thesis. As mentioned in the introduction, we classify here irreducible highest-weight representations of the simple complex n-Lie algebra.

We start in Section 1.1 with a brief introduction to the theory of n-Lie algebras. In the beginning of this section we recall the standard definitions and results. Next we define the basic Lie algebra associated to an n-Lie algebra and give its construction in Subsection 1.1.2. For the simple n-Lie algebra A the basic Lie algebra is isomorphic to the complex algebra $so(n+1)$.

In Section 1.3 we give the statements of our main theorem. Depending on the parity of $n+1$ we obtain Theorem 1.3.2 or Theorem 1.3.3. The strategy used to prove these theorems is presented in Subsection 1.4.2, while the actual computations are done in Subsections 1.5.3 and 1.5.4. In Subsection 1.5.5 we give a second method of obtaining the 'only if'-implication of these theorems, by means of a pictorial representation for the elements of the basic Lie algebra of the simple n-Lie algebra. This pictorial representation is introduced in Subsection 1.2.1.

In the end of this chapter, Section 1.6, we give an application of our theorem: we classify the primitive ideals of the universal enveloping algebra of the simple n-Lie algebra A.

Throughout this chapter V denotes a finite-dimensional vector space over a field $\mathbb{K}$ of characteristic zero.

1.1 Introduction to n-Lie algebras

This section presents the standard definitions and results in the theory of n-Lie algebras. Many of the classical notions and well-know results of the theory of Lie algebras have a natural counterpart in the setting of n-Lie algebras. The theory presented here can be found in [16] and [43].

1.1.1 Definitions and notations

We start our exposition by a succinct recollection of the main definitions and results for the theory of n-Lie algebras.

Definition 1.1.1. The vector space V together with a multi-linear, antisymmetric n-ary operation $[\cdot, \ldots, \cdot] : \times^n V \to V$ is called an *n-Lie algebra*, $n \geq 2$, if the n-ary bracket satisfies the equation

$$[[v_1, \ldots, v_n], v_{n+1}, \ldots, v_{2n-1}] = \sum_{i=1}^{n} [v_1, \ldots, [v_i, v_{n+1}, \ldots, v_{2n-1}], \ldots, v_n], \quad (1.1)$$

where $v_1, \ldots, v_{2n-1} \in V$.

For $n = 2$ this equation is the classical Jacobi identity, thus 2-Lie algebras are the same as Lie algebras. Henceforth we will always assume that $n \geq 3$. However, we advise the reader to keep in mind that in the present section we develop the theory of n-Lie algebras in analogy with the classical theory of Lie algebras.

Example 1.1.2. The vector space $\mathbb{K}^{n+1}$ endowed with the n-bracket

$$[e_1, \ldots, \hat{e_i}, \ldots, e_{n+1}] = (-1)^{n+i+1} e_i,$$

is an n-Lie algebra. Here $\{e_1, \ldots, e_{n+1}\}$ denotes the canonical basis of $\mathbb{K}^{n+1}$ and $\hat{e_i}$ means that e_i is omitted from the n-bracket. The importance of this example comes from the fact that for $\mathbb{K} = \mathbb{C}$ and $n \geq 3$ every simple n-Lie algebra is isomorphic to this one. Later on we will give an alternative construction of this n-Lie algebra.

Example 1.1.3. Let $V = \{f : \mathbb{R}^n \to \mathbb{R} | \, f \text{ of class } C^\infty\}$, $n \geq 3$, and define the n-ary bracket on V as

$$[f_1, \ldots, f_n] = \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_1}{\partial x_n} & \cdots & \frac{\partial f_n}{\partial x_n} \end{vmatrix}.$$

The vector space V together with this operation forms an n-Lie algebra. We will call this operation *the Jacobian*.

Definition 1.1.4. A linear map $D : V \to V$ is called a *derivation* of the n-Lie algebra V if for all $v_1, \ldots, v_n \in V$

$$D[v_1, \ldots, v_n] = \sum_{i=1}^{n} [v_1, \ldots, D(v_i), \ldots, v_n].$$

Consider the map $\mathrm{ad} : \wedge^{n-1} V \to \mathrm{End}(V)$ defined on monomials $v_1 \wedge \ldots \wedge v_{n-1} \in \wedge^{n-1} V$ by

$$\mathrm{ad}(v_1 \wedge \ldots \wedge v_{n-1})(w) = [v_1, \ldots, v_{n-1}, w] \tag{1.2}$$

and extended linearly to sums. Equation (1.1) is equivalent to the map $\mathrm{ad}(v_1 \wedge \ldots \wedge v_{n-1})$ being a derivation of V, and hence we call it *the generalized Jacobi identity*.

Definition 1.1.5. A derivation D of the n-Lie algebra V is called an *inner derivation* if it is in the image of $\mathrm{ad} : \wedge^{n-1} V \to \mathrm{End}(V)$.

The space of all derivations of V, which we denote by $\mathrm{Der}(V)$, is a Lie algebra relative to the Lie bracket

$$[D_1, D_2] = D_1 D_2 - D_2 D_1.$$

Moreover, the space $\mathrm{Inder}(V)$ of inner derivations of V is an ideal of $\mathrm{Der}(V)$, see Proposition 1.1.2 in [43].

Definition 1.1.6. Let V together with $[\cdot, \ldots, \cdot] : \times^n V \to V$ be an n-Lie algebra. A subspace $V' \subseteq V$ together with the inherited operation is called a *subalgebra* of the n-Lie algebra V if

$$[V', \ldots, V'] \subseteq V'.$$

Example 1.1.7. Let us return to the Example 1.1.3 above, and denote by V' the subset of V consisting of the polynomial functions in V. Then V' together with the inherited operation forms a subalgebra of the n-Lie algebra V.

Definition 1.1.8. A subalgebra $I \subseteq V$ of an n-Lie algebra V is called an *ideal* if

$$[V, \ldots, V, I] \subseteq I.$$

An n-Lie algebra V is called *simple* if it is non-abelian, i.e. $[V, \ldots, V] \neq \{0\}$, and it has no other ideals besides 0 and itself.

Example 1.1.9. As in Example 1.1.3 above, we define

$$V = \{f : \mathbb{R}^n \to \mathbb{R} | \, f \text{ of class } C^\infty\},$$

for $n \geq 3$, and put the n-ary bracket on V to be the Jacobian. Denote by I the set of functions in V which are flat at the origin, that is all partial derivatives at the origin are zero. Then I is an ideal of V.

Definition 1.1.10. Let V_1 and V_2 be two n-Lie algebras over $\mathbb{K}$, $n \geq 3$, and $\tau : V_1 \to V_2$ a linear map. The map τ is an *n-Lie algebra homomorphism* if

$$\tau[v_1, \ldots, v_n]_1 = [\tau v_1, \ldots, \tau v_n]_2$$

and it is an *isomorphism* if in addition τ is a bijection.

Definition 1.1.11. Let V be an n-Lie algebra and M a vector space over $\mathbb{K}$. A *representation* of V on M is given by a map

$$\rho : \wedge^{n-1} V \to \mathrm{End}(M)$$

such that for all $v_1, \ldots, v_{2n-2} \in V$ the following equality holds.

$$\begin{aligned} &\rho([v_1, \ldots, v_n] \wedge v_{n+1} \wedge \ldots \wedge v_{2n-2}) = \\ =&\sum_{i=1}^{n} (-1)^{i+n} \rho(v_1 \wedge \ldots \wedge \hat{v}_i \wedge \ldots \wedge v_n) \rho(v_i \wedge v_{n+1} \wedge \ldots \wedge v_{2n-2}) \end{aligned} \tag{1.3}$$

Remark 1.1.12 (Alternative definition)**.** The definition given above is equivalent to the following definition of an n-Lie module, which is more commonly used in the existing literature.

Let V be an n-Lie algebra with n-ary bracket denoted by

$$[\cdot, \ldots, \cdot]_V : \times^n V \to V.$$

An n-Lie module structure on a vector space M for the n-Lie algebra V is defined by an n-Lie algebra structure on the direct sum $V \oplus M$

$$[\cdot, \ldots, \cdot] : \times^n (V \oplus M) \to V \oplus M,$$

such that the following conditions are satisfied:

i) V is a subalgebra of $V \oplus M$, i.e. for $v_1, \ldots, v_n \in V$

$$[v_1, \ldots, v_n] = [v_1, \ldots, v_n]_V,$$

ii) M is an abelian ideal, i.e. for $m_1, m_2 \in M$ and $x_1, \ldots, x_{n-1} \in V \oplus M$

$$[x_1, \ldots, x_{n-1}, m_1] \in M \quad \text{and} \quad [x_1, \ldots, x_{n-2}, m_1, m_2] = 0.$$

The equivalence between this definition and Definition 1.1.11 above is obtained by defining the representation $\rho : \wedge^{n-1} V \to \mathrm{End}(M)$ via the equality

$$\rho(v_1 \wedge \ldots \wedge v_{n-1})(m) = [v_1, \ldots, v_{n-1}, m].$$

Example 1.1.13. Any ideal I of the n-Lie algebra V is an n-Lie module of V, where $\rho(v_1 \wedge \ldots \wedge v_{n-1})(i) = [v_1, \ldots, v_{n-1}, i]$.

Definition 1.1.14. A linear subspace N of an n-Lie module M is called a *submodule* if for all $l \in N$ and $v_1, \ldots, v_{n-1} \in V$

$$\rho(v_1 \wedge \ldots \wedge v_{n-1})(l) \in N.$$

Any module M of V has two trivial submodules: 0 and M. If these are the only submodules that M possesses, we call M *irreducible*. If, on the other hand, M is decomposable as a direct sum of irreducible submodules, then M is called *completely reducible*.

1.1.2 The basic Lie algebra

To an n-Lie algebra V we can associate a Lie algebra, called the *basic Lie algebra* of V. This construction goes as follows.

The map ad, defined by Equation (1.2), extends to a map

$$\nabla : \wedge^{n-1}V \to \text{End}(\wedge^m V) \quad \text{for} \quad 1 \leq m \leq \dim V$$

defined on monomials as

$$\nabla_{(a_1 \wedge \ldots \wedge a_{n-1})}(b_1 \wedge \ldots \wedge b_m) = \sum_{i=1}^{m} b_1 \wedge \ldots \wedge [a_1, \ldots, a_{n-1}, b_i] \wedge \ldots \wedge b_m$$

and extended linearly to sums of monomials.

We define a bilinear operation on $\wedge^{n-1}V$. On monomials $a = a_1 \wedge \ldots \wedge a_{n-1}$ and $b = b_1 \wedge \ldots \wedge b_{n-1}$ we set

$$[a, b] = \frac{1}{2}(\nabla_a b - \nabla_b a)$$

and extend it bilinearly to sums.

Proposition 1.1.15. *The bracket $[\cdot\,,\cdot]$, defined above, endows $\wedge^{n-1}V$ with a Lie algebra structure. Moreover,* $\mathrm{ad} : \wedge^{n-1}V \to$ *Inder(V) is a surjective Lie algebra homomorphism.*

Proof. The skew-symmetry of the bracket is obvious, and so is the surjectivity of ad, thus we need to prove that the Jacobi identity holds and that ∇ is a Lie algebra homomorphism. In order to do this, we first show that

$$\nabla_{\nabla_a b} = \nabla_a \nabla_b - \nabla_b \nabla_a = [\nabla_a, \nabla_b].$$

Since both the left-hand-side and the right-hand-side of the equation are derivations of the exterior algebra $(\wedge^\bullet V, \wedge)$ it suffices to show the equality for some arbitrary $c \in V$.

$$\nabla_{\nabla_a b} c = \nabla_{\nabla_{a_1 \wedge \ldots \wedge a_{n-1}} b_1 \wedge \ldots \wedge b_{n-1}}(c)$$

$$
\begin{aligned}
&= \nabla_{\sum_{i=1}^{n-1} b_1\wedge\ldots\wedge[a_1,\ldots,a_{n-1},b_i]\wedge\ldots\wedge b_{n-1}}(c)\\
&= \sum_{i=1}^{n-1}[b_1,\ldots,[a_1,\ldots,a_{n-1},b_i],\ldots,b_{n-1},c]\\
&= [a_1,\ldots,a_{n-1},[b_1,\ldots,b_{n-1},c]] - [b_1,\ldots,b_{n-1},[a_1,\ldots,a_{n-1},c]]\\
&= (\nabla_a\nabla_b - \nabla_b\nabla_a)(c).
\end{aligned}
$$

Hence, as a consequence of the above, we obtain

$$
\begin{aligned}
\nabla_{[a,b]} &= \nabla_{\frac{1}{2}(\nabla_a b - \nabla_b a)}\\
&= \frac{1}{2}(\nabla_{\nabla_a b} - \nabla_{\nabla_b a})\\
&= \nabla_a\nabla_b - \nabla_b\nabla_a\\
&= [\nabla_a, \nabla_b]. \qquad (1.4)
\end{aligned}
$$

The Jacobi identity now follows easily. Namely, for $a, b, c \in \wedge^{n-1}V$ we have

$$
\begin{aligned}
[a,[b,c]] &= \frac{1}{2}(\nabla_a\nabla_{[b,c]} - \nabla_{[b,c]}\nabla_a)\\
&= \frac{1}{2}(\nabla_a(\nabla_b\nabla_c - \nabla_c\nabla_b) - (\nabla_b\nabla_c - \nabla_c\nabla_b)\nabla_a)\\
&= \frac{1}{2}(\nabla_a\nabla_b\nabla_c - \nabla_a\nabla_c\nabla_b - \nabla_b\nabla_c\nabla_a - \nabla_c\nabla_b\nabla_a)+\\
&+ \frac{1}{2}(\nabla_b\nabla_a\nabla_c - \nabla_b\nabla_a\nabla_c) + \frac{1}{2}(\nabla_c\nabla_a\nabla_b - \nabla_c\nabla_a\nabla_b)\\
&= \frac{1}{2}(\nabla_{[a,b]}\nabla_c - \nabla_c\nabla_{[a,b]}) + \frac{1}{2}(\nabla_b\nabla_{[a,c]} - \nabla_{[a,c]}\nabla_b)\\
&= [[a,b],c] + [b,[a,c]].
\end{aligned}
$$

□

Remark 1.1.16. In general the operation $\nabla : \wedge^{n-1}V \to \mathrm{End}(\wedge^{n-1}V)$ is not skew-symmetric, i.e. for $a, b \in \wedge^{n-1}V$

$$\nabla_a b \neq -\nabla_b a.$$

As an example, consider the subalgebra spanned by $\{1, x, y, z, x^2\}$ inside $C^\infty(\mathbb{R}^3)$, see Example 1.1.3. Then

$$\nabla_{x\wedge y}(z\wedge x^2) = [x,y,z]\wedge x^2 + z\wedge[x,y,x^2] = 1\wedge x^2 + z\wedge\frac{\partial x^2}{\partial z} = 1\wedge x^2$$

while

$$\nabla_{z\wedge x^2}(x\wedge y) = -[x,x^2,z]\wedge y + x\wedge[x^2,y,z] = -\frac{\partial x^2}{\partial y}\wedge y + x\wedge\frac{\partial x^2}{\partial x} = 0.$$

Thus,

$$\nabla_{x\wedge y}(z\wedge x^2) \neq -\nabla_{z\wedge x^2}(x\wedge y).$$

Definition 1.1.17. An n-Lie algebra V, such that $\nabla : \wedge^{n-1}V \to \mathrm{End}(\wedge^{n-1}V)$ is skew-symmetric, is called ∇-*skew-symmetric* (for short ∇-*skew*).

Proposition 1.1.18. *The map $\nabla : \wedge^{n-1}V \to \mathrm{End}(\wedge^{n-1}V)$ is a Lie algebra representation of the basic Lie algebra on itself. Moreover, for each $v \in \wedge^{n-1}V$ the endomorphism ∇_v is a derivation of the basic Lie algebra.*

Proof. For the first assertion of the proposition we have to show that for any $a, b \in \wedge^{n-1}V$

$$\nabla_{[a,b]} = [\nabla_a, \nabla_b].$$

By Equation (1.4) above, this holds.

For the second assertion of the proposition let $v, a, b \in \wedge^{n-1}V$. We want to show that

$$\nabla_v[a, b] = [\nabla_v a, b] + [a, \nabla_v b].$$

By using again some of the computations in the proof of Proposition 1.1.15 above, we obtain:

$$\begin{aligned}
&[\nabla_v a, b] + [a, \nabla_v b] = \\
&\frac{1}{2}(\nabla_{\nabla_v a} b - \nabla_b \nabla_v a + \nabla_a \nabla_v b - \nabla_{\nabla_v b} a) = \\
&\frac{1}{2}(\nabla_v \nabla_a b - \nabla_a \nabla_v b - \nabla_b \nabla_v a + \nabla_a \nabla_v b - \nabla_v \nabla_b a + \nabla_b \nabla_v a) = \\
&\frac{1}{2}(\nabla_v \nabla_a b - \nabla_v \nabla_b a) = \nabla_v[a, b].
\end{aligned}$$

□

Corollary 1.1.19. *Let V be an n-Lie algebra and $\wedge^{n-1}V$ its basic Lie algebra. The operations*

$$\nabla : \wedge^{n-1}V \to \mathrm{End}(\wedge^{n-1}V)$$

and

$$\mathrm{ad} : \wedge^{n-1}V \to \mathrm{End}(\wedge^{n-1}V)$$

give two possibly different representations (by derivations) of $\wedge^{n-1}V$ on itself. Here ad *is defined in the usual way, namely for $a, b \in \wedge^{n-1}V$*

$$\mathrm{ad}(a)(b) = [a, b].$$

These two representations coincide if and only if V is ∇-skew.

The basic Lie algebra $\wedge^{n-1}V$ of the n-Lie algebra V is an important tool in the theory of n-Lie algebras. It allows us to study several concepts in n-Lie theory by relating them to their Lie counterpart. In particular, we will study irreducible representations of the n-Lie algebra V by viewing them as irreducible representations of the basic Lie algebra $\wedge^{n-1}V$ with some special property.

Let M be an n-Lie module of the n-Lie algebra V. Our aim in what follows is to obtain on M a Lie module structure for the basic Lie algebra $\wedge^{n-1}V$.

Proposition 1.1.20. *Let $\rho : \wedge^{n-1}V \to \mathrm{End}(M)$ be a representation of V on M. Then ρ is a Lie algebra representation of the basic Lie algebra $\wedge^{n-1}V$ on M.*

Proof. We have to check that ρ respects the Lie bracket on $\wedge^{n-1}V$. Namely, that for all $v_1, \dots, v_{n-1}, w_1, \dots, w_{n-1} \in V$ and $m \in M$ the following equality holds

$$\begin{aligned}
&\rho([v_1 \wedge \ldots \wedge v_{n-1}, w_1 \wedge \ldots \wedge w_{n-1}])(m) = \\
&\rho(v_1 \wedge \ldots \wedge v_{n-1})\rho(w_1 \wedge \ldots \wedge w_{n-1})(m) - \\
&\rho(w_1 \wedge \ldots \wedge w_{n-1})\rho(v_1 \wedge \ldots \wedge v_{n-1})(m).
\end{aligned}$$

Thus, we compute

$$\begin{aligned}
&\rho([v_1 \wedge \ldots \wedge v_{n-1}, w_1 \wedge \ldots \wedge w_{n-1}])(m) = \\
=&\frac{1}{2}\rho(\sum_{i=1}^{n-1} w_1 \wedge \ldots \wedge [v_1, \ldots, v_{n-1}, w_i] \wedge \ldots \wedge w_{n-1} \\
&- \sum_{i=1}^{n-1} v_1 \wedge \ldots \wedge [w_1, \ldots, w_{n-1}, v_i] \wedge \ldots \wedge v_{n-1})(m) \\
=&\frac{1}{2}(\sum_{i=1}^{n-1} \rho(w_1 \wedge \ldots \wedge [v_1, \ldots, v_{n-1}, w_i] \wedge \ldots \wedge w_{n-1})(m) \\
&- \sum_{i=1}^{n-1} \rho(v_1 \wedge \ldots \wedge [w_1, \ldots, w_{n-1}, v_i] \wedge \ldots \wedge v_{n-1})(m)) \\
=&\frac{1}{2}(\rho(v_1 \wedge \ldots \wedge v_{n-1})\rho(w_1 \wedge \ldots \wedge w_{n-1})(m) \\
&- \rho(w_1 \wedge \ldots \wedge w_{n-1})\rho(v_1 \wedge \ldots \wedge v_{n-1})(m)) \\
&- \frac{1}{2}(\rho(w_1 \wedge \ldots \wedge w_{n-1})\rho(v_1 \wedge \ldots \wedge v_{n-1})(m) \\
&- \rho(v_1 \wedge \ldots \wedge v_{n-1})\rho(w_1 \wedge \ldots \wedge w_{n-1})(m)) \\
=&\rho(v_1 \wedge \ldots \wedge v_{n-1})\rho(w_1 \wedge \ldots \wedge w_{n-1})(m) \\
&- \rho(w_1 \wedge \ldots \wedge w_{n-1})\rho(v_1 \wedge \ldots \wedge v_{n-1})(m). \square
\end{aligned}$$

The converse of Proposition 1.1.20 above does not hold in general, as a Lie module M of the basic Lie algebra $\wedge^{n-1}V$ is not necessarily an n-Lie module of the n-Lie algebra V; for this we need Equation (1.3) to hold. We state this as a corollary.

Corollary 1.1.21. *There is a 1-1 correspondence between representations of the n-Lie algebra V and representations of its basic Lie algebra $\wedge^{n-1}V$ for which the condition given by Equation (1.3) is satisfied.*

A consequence of this corollary is given in the proposition below. According to this proposition we may study irreducibility/complete reducibility of n-Lie modules of V by viewing them as Lie modules of the basic Lie algebra $\wedge^{n-1}V$.

Proposition 1.1.22 ([14, Proposition 2.1]). *Let M be an n-Lie module of the n-Lie algebra V. Then M is irreducible if and only if M is irreducible as a Lie module of $\wedge^{n-1}V$. Similarly, the module M is completely reducible if and only if M is completely reducible as a Lie module of the basic Lie algebra $\wedge^{n-1}V$.*

Representations of the Lie algebra $\wedge^{n-1}V$ are in 1-1 correspondence with representations of the universal enveloping algebra $U(\wedge^{n-1}V)$. In the associative algebra $U(\wedge^{n-1}V)$ consider the elements

$$\begin{aligned} x_{v_1,\ldots,v_{2n-2}} =& [v_1,\ldots,v_n] \wedge v_{n+1} \wedge \ldots \wedge v_{2n-2} - \\ & \sum_{i=1}^{n} (-1)^{i+n} (v_1 \wedge \ldots \wedge \hat{v}_i \wedge \ldots \wedge v_n)(v_i \wedge v_{n+1} \wedge \ldots \wedge v_{2n-2}), \end{aligned} \tag{1.5}$$

where $\hat{v}_i$ means that the element v_i has been omitted. Define

$$R := \operatorname{Span}\{x_{v_1,\ldots,v_{2n-2}} | v_1,\ldots,v_{2n-2} \in V\}, \tag{1.6}$$

and consider the two sided ideal generated by R in $U(\wedge^{n-1}V)$

$$Q(V) := U(\wedge^{n-1}V) R U(\wedge^{n-1}V).$$

Let $\rho : \wedge^{n-1}V \to \operatorname{End}(M)$ be a representation of the n-Lie algebra V. The induced representation of the universal enveloping algebra will be denoted by the same symbol:

$$\rho : U(\wedge^{n-1}V) \to \operatorname{End}(M).$$

Then, by Equation (1.3)

$$Q(V) \subseteq \ker \rho.$$

Moreover, representations of $U(\wedge^{n-1}V)$ for which the ideal $Q(V)$ acts trivially are representations of the n-Lie algebra V. Thus, we have a 1-1 correspondence between representations of V and representations of $U(\wedge^{n-1}V)$ which contain $Q(V)$ in their kernel. Therefore, we define the universal enveloping algebra of the n-Lie algebra V as follows.

Definition 1.1.23. The *universal enveloping algebra* of the n-Lie algebra V is defined as

$$U(V) := U(\wedge^{n-1}V)/Q(V).$$

The argument above leads us to conclude:

Theorem 1.1.24. *Representations of the n-Lie algebra V are the same as representations of the associative algebra $U(V)$.*

The representation of the basic Lie algebra $\wedge^{n-1}V$ on itself

$$\nabla : \wedge^{n-1}V \to \mathrm{End}(\wedge^{n-1}V)$$

acts by derivations. We can extend this action to an action

$$\nabla : \wedge^{n-1}V \to \mathrm{End}(\otimes^k \wedge^{n-1} V)$$

by defining it to be zero for $k = 0$ and

$$\nabla_v(v_1 \otimes \ldots \otimes v_k) = \sum_{i=1}^{k} v_1 \otimes \ldots \otimes \nabla_v(v_i) \otimes \ldots \otimes v_k$$

for $k > 0$. Here $v, v_1, \ldots, v_k \in \wedge^{n-1}V$.

Since ∇ acts by derivations on the Lie algebra $\wedge^{n-1}V$, a simple computation shows that the ideal in the tensor algebra $T(\wedge^{n-1}V)$, generated by elements of the form $v_1 \otimes v_2 - v_2 \otimes v_1 - [v_1, v_2]$, will be invariant under the action of ∇. This means that ∇ can be extended by derivations to a representation of the basic Lie algebra on its universal enveloping algebra $U(\wedge^{n-1}V)$. This extension will be denoted by the same symbol

$$\nabla : \wedge^{n-1}V \to \mathrm{End}(U(\wedge^{n-1}V)).$$

Lemma 1.1.25. *The set R defined in (1.6) is invariant under the ∇-action of $\wedge^{n-1}V$ on $U(\wedge^{n-1}V)$.*

Proof. Since $\nabla : \wedge^{n-1}V \to \mathrm{End}(U(\wedge^{n-1}V))$ acts by derivations on the universal enveloping algebra $U(\wedge^{n-1}V)$, we have that for $a, b \in U(\wedge^{n-1}V)$ and $v \in \wedge^{n-1}V$

$$\nabla_v(a \cdot b) = \nabla_v a \cdot b + a \cdot \nabla_v b.$$

Let $x_{v_1,\ldots,v_{2n-2}}$ be an element of R and $w = w_1 \wedge \ldots \wedge w_{n-1} \in \wedge^{n-1}V$. Then a straightforward computation shows that

$$\nabla_w x_{v_1,\ldots,v_{2n-2}} = \sum_{i=1}^{2n-2} x_{v_1,\ldots,[w_1,\ldots,w_{n-1},v_i],\ldots,v_{2n-2}}.$$

□

As a corollary of this lemma we obtain:

Proposition 1.1.26. *If the n-Lie algebra V is ∇-skew, then the left ideal generated by R equals the right ideal generated by R and equals the two sided ideal $Q(V)$, i.e.*

$$U(\wedge^{n-1}V) \cdot R = R \cdot U(\wedge^{n-1}V) = Q(V).$$

Proof. Let $v, w \in \wedge^{n-1}V$. Then, by the ∇-skew-symmetry of the n-Lie algebra V

$$\nabla_v w = \mathrm{ad}(v)(w) = [v, w].$$

Moreover, the map $\mathrm{ad} : \wedge^{n-1}V \to \mathrm{End}(\wedge^{n-1}V)$ extends by derivations to a map $\mathrm{ad} : \wedge^{n-1}V \to \mathrm{End}(U(\wedge^{n-1}V))$ given by the commutator. Namely, for $v \in \wedge^{n-1}V$ and $a \in U(\wedge^{n-1}V)$

$$[v, a] = va - av.$$

By Lemma 1.1.25 above, we know that R is invariant under the ∇-action of the basic Lie algebra $\wedge^{n-1}V$ on the universal enveloping algebra $U(\wedge^{n-1}V)$. Hence, for all $v \in \wedge^{n-1}V$ and $r \in R$ the following holds

$$vr - rv \in R.$$

Since a generic element in $U(\wedge^{n-1}V)$ is a sum of elements of the form $v_1 \ldots v_k$, by induction on k we can easily show that $U(\wedge^{n-1}V) \cdot R \subseteq R \cdot U(\wedge^{n-1}V)$ and that the converse inclusion also holds. Thus, we obtain equality. $\square$

Let I be a left ideal of the universal enveloping algebra of the basic Lie algebra $U(\wedge^{n-1}V)$. Then $U(\wedge^{n-1}V)$ acts on $U(\wedge^{n-1}V)/I$. Our aim is to determine conditions on I such that $U(\wedge^{n-1}V)/I$ is an n-Lie module of V. By Theorem 1.1.24 above, we know that a representation of the n-Lie algebra V is a representation of the universal enveloping algebra $U(V)$ and that the converse also holds.

Lemma 1.1.27. *$Q(V) \subseteq I$ if and only if the action of $U(\wedge^{n-1}V)$ on $U(\wedge^{n-1}V)/I$ factors through an action of $U(V)$ on $U(\wedge^{n-1}V)/I$.*

Proof. Let $\rho : U(\wedge^{n-1}V) \to \mathrm{End}(U(\wedge^{n-1}V)/I)$ be the representation, given by left multiplication

$$\rho(u)(x) = u \cdot x.$$

Then, obviously, $\ker\rho \subseteq I$.

Assume that $U(V)$ acts on $U(\wedge^{n-1}V)/I$. Then, $Q(V) \subseteq \ker\rho \subseteq I$.

Conversely, assume that $Q(V) \subseteq I$. We want to show that $Q(V)$ is contained in $\ker\rho$. Let $q \in Q(V)$ and $[u] \in U(\wedge^{n-1}V)/I$, where $u \in U(\wedge^{n-1}V)$. Then

$$\rho(q)([u]) = \rho(q \cdot u)(1) \in \rho(Q(V))(1) \subseteq \rho(I)(1) = I.$$

Thus $\rho(q)([u]) = 0$ and we are done. $\square$

If V is ∇-skew, then by Proposition 1.1.26 and Lemma 1.1.27 above we obtain the following condition on I which insures that $U(\wedge^{n-1}V)/I$ is an n-Lie module of V.

Proposition 1.1.28. *Let V be ∇-skew-symmetric. The representation of $U(\wedge^{n-1}V)$ on $U(\wedge^{n-1}V)/I$ factors through a representation of $U(V)$ on $U(\wedge^{n-1}V)/I$ if and only if $R \subseteq I$.*

1.1.3 Classification of semisimple n-Lie algebras

In this subsection we give the classification of the simple, and hence the semisimple n-Lie algebras. The theory presented in this subsection is contained in [43].

We begin with the definition of a semisimple n-Lie algebra.

Let I be an ideal of the n-Lie algebra V and write

$$I^{(0)} := I \text{ and } I^{(k+1)} := [I^{(k)}, \ldots, I^{(k)}]. \tag{1.7}$$

Definition 1.1.29. An ideal $I \subseteq V$ is called *solvable* if there exists $k \geq 0$ such that

$$I^{(k+1)} = \{0\}.$$

Remark 1.1.30. Solvability of ideals as defined above, is also know in the literature as *solvability in the sense of Filippov*. According to how many slots of the n-ary operation $[\cdot, \ldots, \cdot] : \times^n V \to V$ are occupied by elements of the derived series (Equation (1.7)), one can define the notion of k-*solvability* of an ideal I (see for instance [34]). However, for simplicity, we will only be concerned with ideals which are solvable in the sense of Filippov.

Of course, the sum of two ideals is again an ideal, and moreover, the sum of two solvable ideals is again a solvable ideal (see [43, Proposition 2.2]). Hence, by finite-dimensionality of V, there exists a maximal solvable ideal $I \subseteq V$, also known as the *radical* of the n-Lie algebra V.

Definition 1.1.31. An n-Lie algebra V is said to be *semisimple* if it has no nonzero solvable ideals.

In the case of Lie algebras, we know that a semisimple Lie algebra is the direct sum of its simple ideals. The same holds for n-Lie algebras, as has been shown in [43].

Theorem 1.1.32. *An n-Lie algebra V is semisimple if and only if V is a direct sum of simple ideals.*

Proof. We present a sketch of the proof. For details we refer the reader to [43], Theorem 2.7.

Assume that V is a direct sum of simple ideals

$$V = V_1 \oplus \ldots \oplus V_k.$$

Assume there exists I a nonzero solvable ideal of V. We may as well assume that I is the radical of V. Then, by Theorem 2.5 in [43], it follows that

$$I = I_1 \oplus \ldots \oplus I_k,$$

where $I_i = I \cap V_i$ is the radical of the simple ideal V_i. We obtain that $I_i = \{0\}$ for every $1 \geq i \geq k$ and thus, $I = \{0\}$.

For the converse implication, assume that the n-Lie algebra V is semisimple. Denote by Rad the radical of the Lie algebra $\mathrm{Der}(V)$ and let $M := [\mathrm{Rad}, \mathrm{Der}(V)] \subseteq \mathrm{Rad}$. Since Rad is a solvable ideal, M is solvable as well.

Consider the ideal $M(V)$ of the n-Lie algebra V:

$$M(V) := \{D_1 \dots D_k(v) | \, D_1, \dots, D_k \in M \text{ and } v \in V\}.$$

It is proven in [43] that $M(V)$ is a proper solvable ideal of V. Since V is semisimple, we obtain that $M(V) = \{0\}$ and therefore $M = \{0\}$. Thus,

$$\mathrm{Rad} = Z(\mathrm{Der}(V)).$$

Here $Z(\mathrm{Der}(V))$ denotes the center of the Lie algebra $\mathrm{Der}(V)$. It follows that the Lie algebra $\mathrm{Der}(V)$ is reductive. Lemma 1.1.4 in [43] tells us that if a derivation D of V commutes with every inner derivation of V, then $D = 0$ and it commutes with every derivation of V. By application of this lemma, we obtain that $Z(\mathrm{Der}(V)) = 0$ and therefore $\mathrm{Der}(V)$ is in fact semisimple.

Let I be an ideal in $\mathrm{Der}(V)$ such that

$$\mathrm{Der}(V) = I \oplus \mathrm{Inder}(V).$$

We apply again Lemma 1.1.4 in [43] and obtain that $\mathrm{Der}(V) = \mathrm{Inder}(V)$.

Now the n-Lie algebra V is a completely reducible $\mathrm{Der}(V)$-module. Hence, it is a direct sum of irreducible submodules. These submodules are exactly the simple ideals we are looking for. □

Theorem 1.1.33. *Assume the field $\mathbb{K}$ is algebraically closed. Let V be a simple n-Lie algebra over $\mathbb{K}$. Then V is of dimension $n + 1$. Moreover, up to isomorphism there exists a unique simple n-Lie algebra.*

Remark 1.1.34. For $\mathbb{K} = \mathbb{C}$ a realization of the simple n-Lie algebra is given by the vector space $\mathbb{C}^{n+1}$ with the product as explained in Example 1.1.2.

Proof. For completeness of our presentation, we sketch here the proof given in [43]. For details, the reader is advised to check [43].

V is a simple n-Lie algebra. Hence, by the proof of Theorem 1.1.32, we know that $\mathrm{Inder}(V) = \mathrm{Der}(V)$ is a semisimple Lie algebra, that acts faithfully and irreducibly on V. We denote the Lie algebra $\mathrm{Inder}(V)$ by L. Moreover, by Proposition 1.1.15 and its proof, the map $\mathrm{ad} : \wedge^{n-1}V \to L$ is a surjective Lie algebra homomorphism, such that the map

$$(v_1, \dots, v_n) \mapsto \mathrm{ad}(v_1 \wedge \dots \wedge v_{n-1}).v_n$$

is alternating.

Such a triple (L, V, ad), with the properties mentioned above is called a *good triple*. It is shown in [43] that there is a 1-1 correspondence between the set of simple

n-Lie algebras and the set of good triples (L, V, ad). Thus, determining the simple n-Lie algebras, now translates into determining the set of good triples.

This can be done by showing that each simple component of the semisimple Lie algebra L has a simple root α with a the property that $\lambda_+ - \lambda_- - \alpha$ is a root of the Lie algebra L. Here, λ_+ is a maximal weight of the irreducible representation V and λ_- is a minimal weight of V obtained by applying the longest Weyl group element to λ_+. The number of irreducible representations V of L, with the property mentioned above, is shown in [43] to be finite. All good triples (L, V, ad) will be found among them. Assuming that L is semisimple, but not simple, with the property above, leads to the conclusion that $L \cong so(4, \mathbb{K})$ and that V is 4 dimensional, with 3-Lie bracket given by the vector product.

It remains to investigate the case where L is a simple Lie algebra with the property that for some simple root α

$$\lambda_+ - \lambda_- - \alpha$$

is again a root. This is done in [43] by first determining all the possible pairs (L, λ_+). Then, a case by case study completes the proof of the theorem. $\square$

Remark 1.1.35. From now on we will restrict our attention to complex n-Lie algebras, i.e. we assume that $\mathbb{K} = \mathbb{C}$. The simple n-Lie algebra will be denoted by A, while a generic n-Lie algebra will be denoted as before by V.

1.2 The simple n-Lie algebra

In this section we give a different description of the simple complex n-Lie algebra $\mathbb{C}^{n+1}$ and compute its basic Lie algebra.

Let A denote an $(n+1)$-dimensional complex vector space and $\langle \cdot, \cdot \rangle : A \times A \to \mathbb{C}$ a non-degenerate symmetric bilinear form on A. We extend this bilinear form to a bilinear form on $\wedge^k A$ ($1 \leq k \leq n+1$) which we denote the same

$$\langle v, w \rangle := \det(\langle v_i, w_j \rangle_{i,j}),$$

where $v = v_1 \wedge \ldots \wedge v_k$, $w = w_1 \wedge \ldots \wedge w_k \in \wedge^k A$. Fix $\omega \in \wedge^{n+1} A$ such that

$$\langle \omega, \omega \rangle = 1.$$

On the exterior algebra of A we define *the Hodge star operator*

$$* : \wedge^k A \to \wedge^{n+1-k} A$$

by

$$v \wedge *w := \langle v, w \rangle \omega.$$

Here v and w are as before k-vectors in $\wedge^k A$.

Proposition 1.2.1. *The vector space A together with the operation $[\cdot, \ldots, \cdot] : \times^n A \to A$ given by*

$$[v_1, \ldots, v_n] = *(v_1 \wedge \ldots \wedge v_n)$$

is an n-Lie algebra.

Proof. We will show that on a well-chosen basis of A this operation coincides with the one of Example 1.1.2.

Let $\{e_1, \ldots, e_{n+1}\} \subset A$ be a basis of A such that for any $1 \leq i, j \leq n+1$

$$\langle e_i, e_j \rangle = \delta_{i,j}.$$

Then $e_1 \wedge \ldots \wedge e_{n+1} = \pm\omega \in \wedge^{n+1} A$. In case $e_1 \wedge \ldots \wedge e_{n+1} = -\omega$ we interchange e_1 and e_2. Hence, $\{e_1, \ldots, e_{n+1}\}$ is an orthonormal oriented basis of A. Now the conclusion follows directly. $\square$

On A, fix the basis used in the proof of the proposition above: $\{e_1, \ldots, e_{n+1}\} \subset A$ is orthonormal and oriented. Then, $\{e_i \wedge e_j | \, 1 \leq i < j \leq n+1\}$ is a basis for $\wedge^2 A$. For a basis element $e_1 \wedge \ldots \wedge \hat{e}_i \wedge \ldots \wedge \hat{e}_j \wedge \ldots \wedge e_{n+1}$ ($1 \leq i < j \leq n+1$) of the basic Lie algebra $\wedge^{n-1} A$ we have

$$*(e_1 \wedge \ldots \wedge \hat{e}_i \wedge \ldots \wedge \hat{e}_j \wedge \ldots \wedge e_{n+1}) = (-1)^{i+j+1} e_i \wedge e_j.$$

Observe that $*^2 = \mathrm{id}$.

On $\wedge^2 A$ we define:

$$\nabla_{v_1 \wedge v_2}(w_1 \wedge w_2) = *\nabla_{*(v_1 \wedge v_2)} * (w_1 \wedge w_2).$$

A straightforward computation shows that

$$\begin{aligned} \nabla_{v_1 \wedge v_2}(w_1 \wedge w_2) = {} & \langle v_1, w_1 \rangle v_2 \wedge w_2 + \langle v_1, w_2 \rangle w_1 \wedge v_2 \\ & + \langle v_2, w_1 \rangle w_2 \wedge v_1 + \langle v_2, w_2 \rangle v_1 \wedge w_1. \end{aligned} \tag{1.8}$$

As a consequence of Equation (1.8) we obtain:

Corollary 1.2.2. *The simple n-Lie algebra A is ∇-skew.*

We define the Lie bracket on $\wedge^2 A$ by $[v, w] = *[*v, *w]$. Then, on basis elements of $\wedge^2 A$ the Lie bracket is given by

$$[e_i \wedge e_j, e_k \wedge e_l] = \delta_{i,k} e_j \wedge e_l + \delta_{i,l} e_k \wedge e_j + \delta_{j,k} e_l \wedge e_i + \delta_{j,l} e_i \wedge e_k.$$

Let $\{e^{i,j} := E_{ij} - E_{ji} | 1 \leq i < j \leq n+1\}$ be a basis of the complex Lie algebra $so(n+1)$.

Proposition 1.2.3. *The map $\varphi : \wedge^2 A \to so(n+1)$ given by*

$$e_i \wedge e_j \mapsto (-1)^{i+j+1} e^{i,j}$$

is an isomorphism of Lie algebras.

Hence, we conclude

Corollary 1.2.4. *The map $\varphi \circ \star : \wedge^{n-1} A \to so(n+1)$ given by*

$$e_1 \wedge \ldots \wedge \hat{e}_i \wedge \ldots \wedge \hat{e}_j \wedge \ldots \wedge e_{n+1} \mapsto e^{i,j}$$

is an isomorphism of Lie algebras.

We use Corollary 1.2.4 above to express the generators of the ideal $Q(A) \subseteq U(\wedge^{n-1} A)$ in terms of the basis for $so(n+1)$

$$\{e^{i,j} | 1 \leq i < j \leq n+1\} \subset so(n+1).$$

Let $v_1, \ldots, v_{2n-2}$ be elements of the orthonormal basis of A, namely

$$\{v_1, \ldots, v_{2n-2}\} \subset \{e_1, \ldots, e_{n+1}\},$$

and use the map $\varphi \circ \star$ to identify $e_1 \wedge \ldots \wedge \hat{e_i} \wedge \ldots \wedge \hat{e_j} \wedge \ldots \wedge e_{n+1}$ and $e^{i,j}$. Equation (1.5) now becomes:

$$x_{i,k,l,m} = \begin{cases} e^{i,k} e^{l,m} - e^{i,l} e^{k,m} + e^{i,m} e^{k,l} & \text{if } i, k, l, m \text{ are all distinct,} \\ 0 & \text{otherwise,} \end{cases}$$

where $i < k < l < m \in \{1, \ldots, n+1\}$. Here, $x_{i,k,l,m}$ is just a short-hand notation of the generator denoted before by $x_{a_1, \ldots, a_{2n-2}}$. For the detailed computation, we refer the reader to [14]. Later on we will need these elements expressed also in terms of the basis of $\wedge^2 A$ given by $\{e_i \wedge e_j | \, 1 \leq i < j \leq n+1\}$.

Remark 1.2.5. We will see later on, that the ordering on the indices can be dropped. Although it is not necessary, we will still assume them to be ordered.

A simple computation shows that for distinct indices i, k, l, m

$$x_{i,k,l,m} = (-1)^{i+k+l+m}((e_i \wedge e_k)(e_l \wedge e_m) - (e_i \wedge e_l)(e_k \wedge e_m) + (e_i \wedge e_m)(e_k \wedge e_l)).$$

1.2.1 A graphical interpretation of the generators of R

In order to understand the relations which generate $Q(A)$ better, we want to represent them graphically. On the basis $\{e^{i,j} | 1 \leq i < j \leq n+1\}$ of $so(n+1)$ we define the lexicographical order:

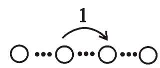
$$e^{i_1,j_1} \leq e^{i_2,j_2} \iff i_1 < i_2 \text{ or } (i_1 = i_2 \text{ and } j_1 \leq j_2).$$

If $e^{i_1,j_1} \leq e^{i_2,j_2}$, we say that $(i_1, j_1) \leq (i_2, j_2)$. We always assume that $i < j$, unless otherwise mentioned. If $i > j$, we can interchange them by the following rule: $e^{i,j} = -e^{j,i}$. Taking into account that two basis elements of $so(n+1)$, which are not in lexicographical order, can be reordered using the Lie bracket at the expense of some term of degree one less, it becomes easy to give a PBW-basis of the algebra $U(\wedge^{n-1}A)$, the universal enveloping algebra of the basic Lie algebra $\wedge^{n-1}A \simeq so(n+1)$. Let

$$U_k = \{e \in U | e = e^{i_1,j_1}e^{i_2,j_2}\dots e^{i_k,j_k}, \text{ where } (i_1,j_1) \leq (i_2,j_2) \leq \dots \leq (i_k,j_k)\},$$

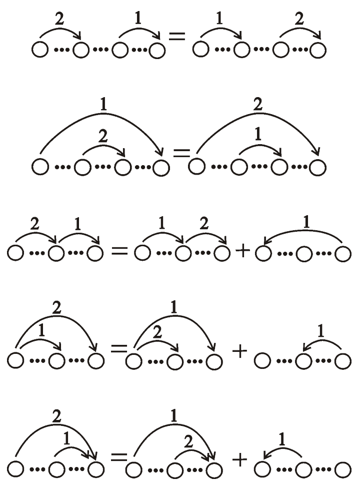
i.e. all simple elements of degree k. Then a Poincare-Birkhoff-Witt basis is given by $\bigcup_k U_k$.

Any simple element of degree 1, i.e. some $e^{i,j}$, can be represented as $n+1$ ordered points with an oriented arrow going from the i'th point to the j'th. (Recall that both i and j range from 1 to $n+1$.)



Changing the orientation of the arrow is the same as changing the order of i and j, thus it results in a minus sign in front of the diagram. Some arbitrary product in $U(\wedge^{n-1}A)$ can be represented similarly as $n+1$ ordered points with arcs connecting them. Multiple arcs between the same two points are allowed, each of these arcs having its own number above it. This number stands for the position the basis element occupies in the product. Multiplication of such elements can be translated, in the language of diagrams, as overlapping, where the numbers above the arrows in the second diagram have to be shifted by a number equal to the number of arrows in the first diagram.

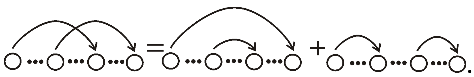
In $U(\wedge^{n-1}A)$ the following commutation relations hold.



For instance, the last diagram represents the commutation relation $e^{j,k}e^{i,k} = e^{i,k}e^{j,k} + e^{j,i}$ for $i<j<k$.

In view of these relations, every diagram can be rewritten in a way such that all products are in lexicographic order. Hence, we can drop the numbers above the arrows in a diagram.

The generators $x_{i,j,k,l}$ of $Q(A)$ tell us that in any diagram intersections can be resolved. Hence, we represent $x_{i,j,k,l}$ graphically as:



Although this graphical interpretation does not supply much insight just yet, it will become very useful further on.

1.3 Infinite-dimensional irreducible highest weight representations of A

Theorem 1.1 in [14], classifies the finite-dimensional irreducible highest weight representations of the simple n-Lie algebra A. We recover this result by classifying the infinite-dimensional highest weight irreducible representations of A.

Definition 1.3.1. A module of the n-Lie algebra V is called a *highest weight module*, if it is a highest weight module of the basic Lie algebra $\wedge^{n-1}V$.

Let $\mathfrak{b}$ be a Borel subalgebra of the Lie algebra $so(n+1)$ and let $\lambda \in (\mathfrak{b}/\mathrm{Rad}(\mathfrak{b}))^*$. Denote by $V(\lambda)$ the associated Verma module of the highest weight λ. Denote by $Z(\lambda)$ the unique irreducible quotient of $V(\lambda)$, with highest weight λ. The main result of this chapter, contained in the two theorems below, gives conditions on λ, such that $Z(\lambda)$ becomes an n-Lie algebra module for the simple n-Lie algebra A.

Theorem 1.3.2. *Let $n \geq 3$, $n+1 = 2N$ and $t \in \{1, \dots, N\}$. Denote by $\pi_1, \dots, \pi_N$ the fundamental weights of $so(2N)$. Then, $Z(\lambda)$ is an irreducible representation of the simple n-Lie algebra A if and only if λ has one of the following values*

$$\begin{cases} x\pi_t & t = 1, \\ (-1-x)\pi_{t-1} + x\pi_t & 1 < t < N-1, \\ (-1-x)\pi_{t-1} + x\pi_t + x\pi_{t+1} & t = N-1, \\ (-1-x)\pi_{t-1} + (-1+x)\pi_t & t = N, \end{cases}$$

where $x \in \mathbb{C}$.

Theorem 1.3.3. *Let $n \geq 3$, $n+1 = 2N+1$ and $t \in \{1, \dots, N\}$. Denote the fundamental weights of $so(2N+1)$ by $\pi_1, \dots, \pi_N$. Then, $Z(\lambda)$ is an irreducible highest weight representation of the simple n-Lie algebra A if and only if λ has one of the following values*

$$\begin{cases} x\pi_t & t = 1, \\ (-1-x)\pi_{t-1} + x\pi_t & 1 < t \leq N, \end{cases}$$

where $x \in \mathbb{C}$.

The strategy used to prove these theorems is explained in Subsection 1.4.2 below, while the actual computations can be found in Section 1.5.

1.4 Main idea of the proof

Our main goal for this section is to describe the strategy we will use to prove Theorems 1.3.2 and 1.3.3 above. We will find irreducible, highest weight representations of the simple n-Lie algebra A. Once this is done, it will be easy, by looking at the highest weight, to figure out which of these representations are finite dimensional and which are not.

The main algebraic object we will work with is the simple Lie algebra $so(n+1)$ (respectively the semi-simple one, in the case of $so(4)$). Its universal enveloping algebra will be denoted as before by $U(so(n+1))$. Let $\mathfrak{h}$ be a Cartan subalgebra of $so(n+1)$ and Φ the corresponding root system. We denote by Φ^+ a choice of positive roots of Φ and by Φ^- the corresponding choice of negative ones. Fix λ in $\mathfrak{h}^*$. Consider the left ideal of $U(so(n+1))$, $I(\lambda)$, generated by all $x_\alpha,$ with $\alpha \in \Phi^+$ and all $h - \lambda(h)1$, where $h \in H$. Then

$$V(\lambda) := U(so(n+1))/I(\lambda)$$

is a highest weight module of $so(n+1)$ with highest weight λ, called the *Verma module* of weight λ. The module $V(\lambda)$ need not be irreducible but it has a unique irreducible quotient. Define $Z(\lambda)$ to be

$$Z(\lambda) := U(so(n+1))/J(\lambda),$$

where $J(\lambda)$ is the unique maximal left ideal of $U(so(n+1))$ containing the left ideal $I(\lambda)$. Then $Z(\lambda)$ is an irreducible highest weight module with highest weight λ. Our goal is to determine for which $\lambda \in \mathfrak{h}^*$, $Z(\lambda)$ is an irreducible module of the n-Lie algebra A, i.e. for which $\lambda \in \mathfrak{h}^*$, the two-sided ideal $Q(A)$ acts trivially on $Z(\lambda)$.

1.4.1 Independence of the choice of Borel subalgebra

In this subsection we show that our result will be independent of the choice of Cartan subalgebra and the choice of positive system. We show that for any two Borel subalgebras $\mathfrak{b}$ and $\mathfrak{b}'$ of $so(n+1)$, the set of weights corresponding to these subalgebras, such that $Z(\lambda)$ is an irreducible representation of the n-Lie algebra A, are related by an isomorphism. Note that we want $Z(\lambda)$ to be an irreducible A-module and not only an irreducible $so(n+1)$-module; as an $so(n+1)$-module, this is a well-known result.

Let $\mathfrak{h}$ and $\mathfrak{h}'$ be two Cartan subalgebras of $so(n+1)$, $\lambda \in \mathfrak{h}^*$, $\lambda' \in (\mathfrak{h}')^*$ and Φ^+ and Φ'^+ two choices of positive systems. Denote by

$$\mathfrak{b} := \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}^\alpha$$

and by

$$\mathfrak{b}' := \mathfrak{h}' \oplus \bigoplus_{\alpha \in \Phi'^+} \mathfrak{g}^\alpha$$

the two corresponding Borel subalgebras. Let $I(\mathfrak{b}, \lambda)$, $I(\mathfrak{b}', \lambda')$ the two left ideals corresponding to each of these Borel subalgebras (the notation now keeps track of the Borel subalgebra as well). Denote by $J(\mathfrak{b}, \lambda)$, $J(\mathfrak{b}', \lambda')$ the maximal left ideals of $U(so(n+1))$ which include the two ideals above. By Lemma 1.1.27 above, the problem of $Z(\lambda)$, respectively $Z(\lambda')$, being an irreducible A-module translates as $Q(A) \subseteq J(\mathfrak{b}, \lambda)$, respectively $Q(A) \subseteq J(\mathfrak{b}', \lambda')$.

Let

$$\Lambda := \{\alpha \in \mathfrak{h}^* | Q(A) \subseteq J(\mathfrak{b}, \alpha)\},$$
$$\Lambda' := \{\alpha' \in (\mathfrak{h}')^* | Q(A) \subseteq J(\mathfrak{b}', \alpha')\}.$$

By Theorems 16.2 and 16.4 in [32] there exists an inner automorphism $\varphi : so(n+1) \to so(n+1)$ for which $\varphi(\mathfrak{b}) = \mathfrak{b}'$ and $\varphi(\mathfrak{h}) = \mathfrak{h}'$. Then, φ induces an automorphism $\tilde{\varphi} : U(so(n+1)) \to U(so(n+1))$. We want the following equality to hold:

$$(\varphi_{|\mathfrak{h}})^*(\Lambda') = \Lambda,$$

which is the same thing as

$$Q(A) \subseteq J(\mathfrak{b}, \lambda) \text{ if and only if } Q(A) \subseteq J(\mathfrak{b}', \lambda'),$$

for $\lambda' = (\varphi_{|\mathfrak{h}})^*(\lambda)$. Since φ is an isomorphism, it is enough to show just one implication of the above equivalence. Assume that $Q(A) \subseteq J(\mathfrak{b}, \lambda)$. Then,

$$Q(A) \subseteq J(\mathfrak{b}, \lambda) \Rightarrow \tilde{\varphi}(Q(A)) \subseteq \tilde{\varphi}(J(\mathfrak{b}, \lambda)) = J(\mathfrak{b}', \lambda').$$

Hence, if we can show that $\tilde{\varphi}(Q(A)) = Q(A)$ we are done. This follows automatically from the fact that $Q(A)$ is a two-sided ideal in $U(so(n+1))$ and φ is an inner automorphism of $so(n+1)$.

1.4.2 The strategy for the proof

From the above subsection we see that keeping track of the Borel subalgebra in the notation of the two ideals $I(\mathfrak{b}, \lambda)$ and $J(\mathfrak{b}, \lambda)$ is superfluous. Hence we revert to our original notation, used in the introduction to this section: $I(\lambda)$ and $J(\lambda)$.

Lemma 1.1.27 tells us that $Z(\lambda)$ has an induced structure of an A-module if and only if $Q(A) \subseteq J(\lambda)$. Thus, we want to see for which $\lambda \in \mathfrak{h}^*$ the inclusion above holds. The next lemma gives us a useful method for checking this inclusion.

Lemma 1.4.1. $Q(A) \not\subseteq J(\lambda)$ *if and only if* $Q(A) + I(\lambda) = U(so(n+1))$.

Proof. Denote $J' := Q(A) + I(\lambda)$. Then J' is an ideal of $U(so(n+1))$ containing $I(\lambda)$.

We prove the implication from right to left. Assume $Q(A) + I(\lambda) \neq U(so(n+1))$; then J' is a proper ideal containing $I(\lambda)$, and thus, by maximality of $J(\lambda)$, we have the inclusion $J' \subseteq J(\lambda)$. We conclude that $Q(A) \subseteq J(\lambda)$.

Conversely, if $Q(A) \subseteq J(\lambda)$ then $J' \subseteq J(\lambda) \subsetneq U(so(n+1))$ and we see that J' is a proper ideal. □

The equality $Q(A) + I(\lambda) = U(so(n+1))$ is equivalent to the assertion that $1 \in Q(A) + I(\lambda)$, where 1 is the unit in $U(so(n+1))$. Denote

$$\hat{Q}(A) := (Q(A) + I(\lambda))/I(\lambda) \subseteq V(\lambda) = U(so(n+1))/I(\lambda)$$

and

$$\mathbb{1} := 1 + I(\lambda).$$

The equality $Q(A) + I(\lambda) = U(so(n+1))$ is equivalent to $\mathbb{1} \in \hat{Q}(A)$. Observe that if $Q(A) + I(\lambda) = U(so(n+1))$ then $\hat{Q}(A) = V(\lambda)$ and the converse also holds.

Consider the projection along lower weight spaces on the highest weight space:

$$\mathrm{pr}_\lambda : V(\lambda) \to \mathrm{Span}\,\mathbb{1} = V(\lambda)_\lambda,$$

where $V(\lambda)_\lambda$ denotes the 1-dimensional subspace of $V(\lambda)$ with weight λ. With this notation we obtain the following equivalence:

$$\mathbb{1} \in \hat{Q}(A) \text{ if and only if there exists } x \in \hat{Q}(A) \text{ s.t. } \mathrm{pr}_\lambda(x) \neq 0. \tag{1.9}$$

The implication from left to right is trivial, hence in order to convince ourselves that the above equivalence holds we only need to check the converse implication.

Let $x \in \hat{Q}(A)$ with $\mathrm{pr}_\lambda(x) \neq 0$, then $\hat{Q}(A) \nsubseteq \ker(\mathrm{pr}_\lambda)$. On the other hand $\hat{J}(\lambda) = J(\lambda)/I(\lambda) \subseteq \ker(\mathrm{pr}_\lambda)$. Thus $\hat{Q}(A) \nsubseteq \hat{J}(\lambda)$ and therefore $\mathbb{1} \in \hat{Q}(A)$.

The next lemma is the most important one in this section.

Lemma 1.4.2. *The $so(n+1)$-module structure on $Z(\lambda)$ factors through an A-module structure if and only if $\mathrm{pr}_\lambda(\hat{R}) = 0$, where $\hat{R} = (R + I(\lambda))/I(\lambda)$.*

Proof. By the Equivalence (1.9) above and the Lemmas 1.4.1 and 1.1.27, it follows that

$$Z(\lambda) \text{ has an induced } A\text{-module structure if and only if } \hat{Q}(A) \subseteq \ker(\mathrm{pr}_\lambda).$$

We will prove that the equality

$$\mathrm{pr}_\lambda(\hat{Q}(A)) = \mathrm{pr}_\lambda(\hat{R})$$

holds, which implies the conclusion.

Since $\hat{R} \subseteq \hat{Q}(A)$, we have that $\mathrm{pr}_\lambda(\hat{R}) \subseteq \mathrm{pr}_\lambda(\hat{Q}(A))$.

We prove the converse inclusion. By Lemma 1.1.26, the ideal $Q(A)$ is spanned by elements of the form $q = r \cdot u$, where $r \in R$ and $u \in U(so(n+1))$. Since the elements $x_{\alpha_1} \dots x_{\alpha_k} \cdot \mathbb{1}$ ($\alpha_i \in \Phi^-$ for all $1 \leq i \leq k$) span $V(\lambda) = U(so(n+1)) \cdot \mathbb{1}$, we conclude that every element in $\hat{Q}(A)$ can be written as a sum of elements of the form

$$r \cdot x_{\alpha_1} x_{\alpha_2} \dots x_{\alpha_k} \cdot \mathbb{1},$$

where $r \in R$ and $\alpha_i \in \Phi^-$.

We prove by induction on $k \geq 0$ that $\mathrm{pr}_\lambda(r \cdot x_{\alpha_1} x_{\alpha_2} \dots x_{\alpha_k} \cdot \mathbb{1}) \in \mathrm{pr}_\lambda(\hat{R})$, for all r and α_i as above. For $k = 0$ this is clear. Assume the claim holds for $k - 1 \geq 0$. We write

$$\mathrm{pr}_\lambda(r \cdot x_{\alpha_1} x_{\alpha_2} \dots x_{\alpha_k} \cdot \mathbb{1}) = \mathrm{pr}_\lambda([r, x_{\alpha_1}] x_{\alpha_2} \dots x_{\alpha_k} \cdot \mathbb{1}) + \mathrm{pr}_\lambda(x_{\alpha_1} \cdot r \cdot x_{\alpha_2} \dots x_{\alpha_k} \cdot \mathbb{1}).$$

Since R is a $\wedge^{n-1} A \cong so(n+1)$-module (see Lemma 1.1.25), $[r, x_{\alpha_1}] \in R$ and therefore by the inductive hypothesis the first term belongs to $\mathrm{pr}_\lambda(\hat{R})$. Since λ is the highest weight of $V(\lambda)$, we have that $\mathrm{weight}(r \cdot x_{\alpha_2} \dots x_{\alpha_k} \cdot \mathbb{1}) \preceq \lambda$ and thus, $\mathrm{weight}(x_{\alpha_1} \cdot r \cdot x_{\alpha_2} \dots x_{\alpha_k} \cdot \mathbb{1}) \prec \lambda$. Hence, the second term is zero and this finishes the proof. $\square$

R is a finite dimensional $so(n+1)$-module and $so(n+1)$ is (semi)simple. Thus, R is decomposable into weight spaces. If $r \in R$ is such that $\mathrm{weight}(r) = \mu \neq 0$, then clearly $\mathrm{pr}_\lambda(r \cdot \mathbb{1}) = 0$. Therefore:

Corollary 1.4.3. *The $so(n+1)$-module structure on $Z(\lambda)$ factors through an A-module structure if and only if $\mathrm{pr}_\lambda(r \cdot \mathbb{1}) = 0$, for all $r \in R$ of weight 0.*

1.5 A unified statement for the main theorems and the proof

In this section we will reformulate the main theorems in a single statement, not dependent on the parity of $n+1$, and present the proof of this new statement.

The basic Lie algebra of the simple n-Lie algebra A is isomorphic to $so(n+1)$ and we denote by $\mathfrak{h}$ a Cartan subalgebra of this Lie algebra. Recall that our goal is to determine for which $\lambda \in \mathfrak{h}^*$, the two-sided ideal $Q(A)$ acts trivially on the irreducible, highest weight module $Z(\lambda)$. This will ensure that $Z(\lambda)$ is an n-Lie module of A. We will first introduce some new notation.

1.5.1 The Lie algebra so(n+1)

Until now, on the $(n+1)$-dimensional, complex vector space A we considered the orthonormal basis $e_1, \dots, e_{n+1}$, the inner product $\langle e_i, e_j \rangle = \delta_{i,j}$ and the orientation form $e_1 \wedge \dots \wedge e_{n+1}$. This was useful for us because of several reasons: the easy expression of the n-ary bracket, the simple form of the generators of $Q(A)$. It will be useful for us, further on, to have a complex bilinear form instead of an inner product on A and a new basis of the basic Lie algebra $\wedge^2 A \cong so(n+1)$. The theory presented here has been inspired by [38].

Denote by $(\cdot, \cdot)$ the bilinear form on A defined by

$$(e_i, e_j) = \delta_{i,j}, \quad 1 \leq i, j \leq n+1. \tag{1.10}$$

To give the root basis of $so(n+1)$ it will be convenient to have another basis of A. For this we need to take into account the parity of $n+1$.

If $n+1=2N$, we define

$$v_{\pm j}=\frac{e_{2j-1}\mp ie_{2j}}{\sqrt{2}},\quad j=1,2,\ldots,N, \tag{1.11}$$

then $\{v_{-N},\ldots,v_{-1},v_1,\ldots,v_N\}$ forms a basis of A and

$$(v_i,v_j)=\delta_{i+j,0}.$$

If, on the other hand, $n+1=2N+1$ then we need one more element, viz. $v_0:=e_{2N+1}$. Then the basis of A is given by $\{v_{-N},\ldots,v_{-1},v_0,v_1,\ldots,v_N\}$, and again $(v_i,v_j)=\delta_{i+j,0}$.

Hence, a basis of $so(n+1)\simeq\wedge^2A$ is given by elements of the form $v_j\wedge v_k$, where $-N\leq j<k\leq N$ and $j,k\neq 0$ if $n+1$ is even. The relation between the two bases used is given by:

$$v_{\nu j}\wedge v_{\mu k}=\frac{1}{2}(e_{2j-1}\wedge e_{2k-1}-\nu\mu e_{2j}\wedge e_{2k}-i(\nu e_{2j}\wedge e_{2k-1}+\mu e_{2j-1}\wedge e_{2k})),$$

$$v_0\wedge v_{\nu j}=\frac{1}{\sqrt{2}}(e_{2j-1}\wedge_{2N+1}-i\nu e_{2j}\wedge e_{2N+1}).$$

Here ν,μ are +1 or -1. A non-degenerate, invariant, symmetric bilinear form on $\wedge^2A\simeq so(n+1)$ is given by

$$(u_1\wedge u_2,v_1\wedge v_2)=(u_1,v_2)(u_2,v_1)-(u_1,v_1)(u_2,v_2).$$

Define

$$\epsilon_j:=ie_{2j-1}\wedge e_{2j},\quad \text{where } 1\leq j\leq N. \tag{1.12}$$

Here i denotes the imaginary unit. Then $\epsilon_j=v_j\wedge v_{-j}$ and

$$\mathfrak{h}:=\bigoplus_{i=1}^{N}\mathbb{C}\epsilon_i$$

is a Cartan subalgebra of the Lie algebra $so(n+1)$. Let $h\in\mathfrak{h}$, then the commutator of h and $v_{\nu j}\wedge v_{\mu k}$ is:

$$[h,v_{\nu j}\wedge v_{\mu k}]=(\nu\epsilon_j+\mu\epsilon_k,h)v_{\nu j}\wedge v_{\mu k},$$

$$[h,v_0\wedge v_{\nu j}]=(\nu\epsilon_j,h)v_0\wedge v_{\nu j},$$

while the commutator of $v_{\nu j}\wedge v_{\mu k}$ and $v_{-\nu j}\wedge v_{-\mu k}$ is:

$$[v_{\nu j}\wedge v_{\mu k},v_{-\nu j}\wedge v_{-\mu k}]=-\nu\epsilon_j-\mu\epsilon_k. \tag{1.13}$$

Remark 1.5.1. In order to express the roots of the Lie algebras $so(2N)$ and $so(2N+1)$ in a simpler fashion we identify $\mathfrak{h}^*$ with $\mathfrak{h}$ via the map

$$(\epsilon_j,\cdot)\longmapsto\epsilon_j.$$

Thus, if $n+1=2N$, the set of roots is given by

$$\Phi = \{\pm(\epsilon_i \pm \epsilon_j) | 1 \leq i \neq j \leq N\}$$

and a base for this root system is given by

$$\Delta = \{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \dots, \epsilon_{N-1} - \epsilon_N, \epsilon_{N-1} + \epsilon_N\}. \tag{1.14}$$

Hence, we obtain the following root space decomposition for $so(2N)$:

$$so(2N) = \bigoplus_{\substack{1\leq j<k\leq N \\ \mu\in\{+1,-1\}}} \mathbb{C}(v_{-j} \wedge v_{\mu k}) \oplus \bigoplus_{1\leq j\leq N} \mathbb{C}\epsilon_j \oplus \bigoplus_{\substack{1\leq j<k\leq N \\ \mu\in\{+1,-1\}}} \mathbb{C}(v_j \wedge v_{\mu k}).$$

If $n+1 = 2N+1$, then the set of roots is

$$\Phi = \{\pm(\epsilon_i \pm \epsilon_j) | 1 \leq i \neq j \leq N\} \cup \{\pm\epsilon_i | 1 \leq i \leq N\},$$

while a base for this root system is given by

$$\Delta = \{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \dots, \epsilon_{N-1} - \epsilon_N, \epsilon_N\}. \tag{1.15}$$

This allows us to give the following root space decomposition of the Lie algebra $so(2N+1)$:

$$\begin{aligned} so(2N+1) = &\bigoplus_{\substack{1\leq j<k\leq N \\ \mu\in\{+1,-1\}}} \mathbb{C}(v_{-j} \wedge v_{\mu k}) \oplus \bigoplus_{1\leq j\leq N} \mathbb{C}(v_{-j} \wedge v_0) \oplus \bigoplus_{1\leq j\leq N} \mathbb{C}\epsilon_j \\ &\oplus \bigoplus_{1\leq j\leq N} \mathbb{C}(v_0 \wedge v_j) \oplus \bigoplus_{\substack{1\leq j<k\leq N \\ \mu\in\{+1,-1\}}} \mathbb{C}(v_j \wedge v_{\mu k}). \end{aligned}$$

The first line of the formula above contains the negative side of the root space decomposition and the Cartan subalgebra, while on the second line just the positive side is listed.

1.5.2 A unified statement for the main theorems

Let $\mathfrak{h} := \oplus_{i=1}^N \mathbb{C}\epsilon_i$ be the Cartan subalgebra of the Lie algebra $so(n+1)$ constructed above. Consider the base of the root system of $so(n+1)$ given by Equation (1.14) or (1.15), depending on the parity of $n+1$. Let λ be the highest weight of the highest weight module $Z(\lambda)$. Denote

$$\lambda_i := \lambda(\epsilon_i).$$

Theorem 1.5.2. *The highest weight, irreducible representation $Z(\lambda)$ of $so(n+1)$ factors through a representation of the simple n-Lie algebra A if and only if $\lambda \in \mathfrak{h}^*$ is such that $\lambda_1 = \lambda_2 = \dots = \lambda_{t-1} = -1$, $\lambda_t = x \in \mathbb{C}$ and $\lambda_{t+1} = \dots = \lambda_{\lfloor \frac{n+1}{2} \rfloor} = 0$, for some $1 \leq t \leq \lfloor \frac{n+1}{2} \rfloor$.*

From general theory of irreducible Lie algebra representations (see for instance Theorem 21.2 in [32]) we know that for $Z(\lambda)$ to be finite-dimensional we need the highest weight to be integral dominant. It follows that for $x \in \mathbb{Z}_+$ and $t = 1$ we obtain a finite dimensional irreducible representation of A and otherwise an infinite dimensional irreducible one. Thus, we have recovered the result in [14]. Our proof however, will be different from the one presented there.

When stating the above theorem in terms of the fundamental weights of $so(n+1)$, one needs to be careful about the distinction between the two cases: $n+1$ is even or $n+1$ is odd. This gives Theorems 1.3.2 and 1.3.3.

The rest of this section will be devoted to the proof of Theorem 1.5.2. We will follow the strategy described by Corollary 1.4.3. Namely, with the notation as in Subsection 1.4.2, we will determine the elements $r \in R$ of $\mathfrak{h}$-weight zero and impose the condition that for such elements $\mathrm{pr}_\lambda(r.\mathbb{1}) = 0$. This will lead to the necessary and sufficient conditions on the highest weight λ such that the $so(n+1)$-module structure on $Z(\lambda)$ factors through an A-module structure. These conditions can be expressed as the zero set of a set of polynomials in $\lambda_1, \ldots, \lambda_{\lfloor \frac{n+1}{2} \rfloor}$. The solutions can be read in the statement of the theorem. The proof presented below is a computational proof which will differentiate between the two cases: $n+1$ is even or odd. In Subsection 1.5.5 we will show a second method of obtaining the same polynomials which will make use of the graphic representation of the generators of R introduced in Subsection 1.2.1. This second method only proves the implication from left to right in Theorem 1.5.2 above. It has the advantage of working in both cases while no additional distinction needs to be made.

1.5.3 Proof for the case: n+1 is even

First, we treat the case: $n+1 = 2N$. This means that the Lie algebra $\wedge^{n-1}A$ is $so(2N)$.

We want to compute the elements of R in terms of the elements $v_{\nu j} \wedge v_{\mu k}$. By inverting the matrix which defines the $v_{\nu j} \wedge v_{\mu k}$'s in terms of the $e_j \wedge e_k$'s we obtain the following equality.

$$\begin{pmatrix} e_{2j-1} \wedge e_{2k-1} \\ e_{2j} \wedge e_{2k} \\ e_{2j} \wedge e_{2k-1} \\ e_{2j-1} \wedge e_{2k} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ i & -i & i & -i \\ i & i & -i & -i \end{pmatrix} \begin{pmatrix} v_j \wedge v_k \\ v_{-j} \wedge v_k \\ v_j \wedge v_{-k} \\ v_{-j} \wedge v_{-k} \end{pmatrix} \tag{1.16}$$

We will avoid to write long, tedious computations, and jump ahead to the final result. The most important piece of information is Equation (1.16) above.

Recall that $x_{i_1,i_2,i_3,i_4} = e^{i_1,i_2}e^{i_3,i_4} - e^{i_1,i_3}e^{i_2,i_4} + e^{i_1,i_4}e^{i_2,i_3}$. This formula can also be rewritten as:

$$x_{i_1,i_2,i_3,i_4} = \frac{1}{8} \sum_{\sigma \in S_4} sgn(\sigma) e^{i_{\sigma(1)},i_{\sigma(2)}} e^{i_{\sigma(3)},i_{\sigma(4)}}.$$

It follows that for any τ in S_4,

$$x_{i_{\tau(1)},i_{\tau(2)},i_{\tau(3)},i_{\tau(4)}} = sgn(\tau)x_{i_1,i_2,i_3,i_4}.$$ [1]

Therefore, if two indices are equal, the formula above tells us that $x_{i_1,i_2,i_3,i_4} = 0$. Hence

$$\begin{aligned} R &= \text{Span}\{x_{i_1,i_2,i_3,i_4}\}_{1\le i_1,i_2,i_3,i_4\le 2N} = \\ &= \text{Span}\{1/8 \sum_{\sigma\in S_4} sgn(\sigma)e^{i_{\sigma(1)},i_{\sigma(2)}}e^{i_{\sigma(3)},i_{\sigma(4)}}\}_{1\le i_1,i_2,i_3,i_4\le 2N} = \\ &= \text{Span}\{1/8 \sum_{\sigma\in S_4} sgn(\sigma)(e_{i_{\sigma(1)}}\wedge e_{i_{\sigma(2)}})(e_{i_{\sigma(3)}}\wedge e_{i_{\sigma(4)}})\}_{1\le i_1,i_2,i_3,i_4\le 2N}. \end{aligned} \tag{1.17}$$

By using Equation (1.16) and Equation (1.17) we can now express the elements x_{i_1,i_2,i_3,i_4} in the new basis of $so(2N)$ introduced in Subsection 1.5.1 above.

$$\text{Span}\{x_{i_1,i_2,i_3,i_4}\}_{1\le i_1,i_2,i_3,i_4\le 2N} =$$

$$\begin{aligned} = \text{Span}\{1/2((v_i\wedge v_j)(v_k\wedge v_l) - (v_i\wedge v_k)(v_j\wedge v_l) + (v_i\wedge v_l)(v_j\wedge v_k) + \\ +(v_k\wedge v_l)(v_i\wedge v_j) - (v_j\wedge v_l)(v_i\wedge v_k) + (v_j\wedge v_k)(v_i\wedge v_l))\}, \end{aligned}$$

where $i \le j \le k \le l$ range from $-N$ to N excluding 0 and the product in the expression above is the product in $U(so(2N))$.

We denote an element expressed in this notation by $v_{i,j,k,l}(\alpha,\beta,\gamma,\delta)$, where α,β,γ and δ represent the signs of the indices. Hence, we may assume $1 \le i \le j \le k \le l \le N$.

Recall that $x_{a,b,c,d}$ is zero as soon as any two indices are equal. Without loss of generality we may assume that $1 \le a \le b \le c \le d \le 2N$. For $v_{a,b,c,d}(\alpha,\beta,\gamma,\delta)$ this fact does not hold anymore, i.e. if any two indices are equal then $v_{a,b,c,d}(\alpha,\beta,\gamma,\delta)$ is zero if and only if the corresponding signs are also equal.

Under the action of $so(2N)$ on R all the 6 terms in the expression of the element $v_{a,b,c,d}(\alpha,\beta,\gamma,\delta)$ have the same $\mathfrak{h}$-weight. We compute this weight for $(v_{\alpha a}\wedge v_{\beta b})(v_{\gamma c}\wedge v_{\delta d})$. Let $\epsilon_i \in \mathfrak{h}$. Then

$$\begin{aligned} &[\epsilon_i, (v_{\alpha a}\wedge v_{\beta b})(v_{\gamma c}\wedge v_{\delta d})] = \\ &[\epsilon_i, v_{\alpha a}\wedge v_{\beta b}](v_{\gamma c}\wedge v_{\delta d}) + (v_{\alpha a}\wedge v_{\beta b})[\epsilon_i, v_{\gamma c}\wedge v_{\delta d}] = \\ &(\alpha(\epsilon_a,\epsilon_i) + \beta(\epsilon_b,\epsilon_i))(v_{\alpha a}\wedge v_{\beta b})(v_{\gamma c}\wedge v_{\delta d}) + \\ &(\gamma(\epsilon_c,\epsilon_i) + \delta(\epsilon_d,\epsilon_i))(v_{\alpha a}\wedge v_{\beta b})(v_{\gamma c}\wedge v_{\delta d}) = \\ &(\alpha(\epsilon_a,\epsilon_i) + \beta(\epsilon_b,\epsilon_i) + \gamma(\epsilon_c,\epsilon_i) + \delta(\epsilon_d,\epsilon_i))(v_{\alpha a}\wedge v_{\beta b})(v_{\gamma c}\wedge v_{\delta d}). \end{aligned}$$

Thus, we obtain the weight

$$\text{weight}(v_{a,b,c,d}(\alpha,\beta,\gamma,\delta)) = \alpha\epsilon_a + \beta\epsilon_b + \gamma\epsilon_c + \delta\epsilon_d,$$

This proves the following lemma.

[1] This proves Remark 1.2.5.

Lemma 1.5.3. *The subspace $R_0 \subset R$ of elements of $\mathfrak{h}$-weight zero for the action of $so(2N)$ on R is spanned by the elements $v_{a,b,c,d}(\alpha, \beta, \gamma, \delta)$ with $a = b$, $c = d$ and $\alpha = \gamma = 1$, $\beta = \delta = -1$.*[2]

Thus, R_0 is spanned by the elements of the form:

$$v_{a,a,c,c}(1,-1,1,-1) =$$
$$\frac{1}{2}((v_a \wedge v_{-a})(v_c \wedge v_{-c}) - (v_a \wedge v_c)(v_{-a} \wedge v_{-c}) + (v_a \wedge v_{-c})(v_{-a} \wedge v_c)+$$
$$+ (v_c \wedge v_{-c})(v_a \wedge v_{-a}) - (v_{-a} \wedge v_{-c})(v_a \wedge v_c) + (v_{-a} \wedge v_c)(v_a \wedge v_{-c})).$$

By using that $v_j \wedge v_{\nu k} \cdot \mathbb{1} = 0$ (for all $1 \leq j < k \leq N$ and $\nu \in \{+1, -1\}$) and Equation (1.13), we can compute $\mathrm{pr}_\lambda(v_{a,a,c,c}(1,-1,1,-1) \cdot \mathbb{1})$.

$$v_{a,a,c,c}(1,-1,1,-1) \cdot \mathbb{1} =$$
$$\frac{1}{2}(2\epsilon_a\epsilon_c + (-\epsilon_a + \epsilon_c) - (-\epsilon_a - \epsilon_c)) \cdot \mathbb{1} =$$
$$(\epsilon_a\epsilon_c + \epsilon_c) \cdot \mathbb{1}$$

Since $h \cdot \mathbb{1} = \lambda(h)\mathbb{1}$ for all $h \in \mathfrak{h}$, we obtain that

$$\mathrm{pr}_\lambda(v_{a,a,c,c}(1,-1,1,-1) \cdot \mathbb{1}) = \lambda(\epsilon_c)(\lambda(\epsilon_a) + 1) = \lambda_c(\lambda_a + 1).$$

Corollary 1.5.4. *The space $R_0 \subset R$, of elements of $\mathfrak{h}$-weight zero, has the property that $\mathrm{pr}_\lambda(R_0.\mathbb{1}) = 0$ if and only if for all $1 \leq a < c \leq 2N$ the equality $\lambda_c(\lambda_a+1) = 0$ holds.*

By combining this corollary with Corollary 1.4.3 we obtain:

Corollary 1.5.5. *The $so(n + 1)$-module structure on $Z(\lambda)$ factors through an A-module structure if and only if for all $1 \leq a < c \leq 2N$ we have that the following equality holds*

$$\lambda_c(\lambda_a + 1) = 0.$$

The Corollary 1.5.5 above also implies Theorem 1.5.2 for the case when $n + 1$ is even.

1.5.4 Proof for the case: n+1 is odd

Next, we treat the case: $n + 1 = 2N + 1$. Our basic Lie algebra $\wedge^{n-1}A$ is now the Lie algebra $so(2N+1)$ with basis given by elements of the form $v_i \wedge v_j$, where $i < j$ are contained in the set $\{-N, \dots, 0, \dots, N\}$.

[2]Recall that $a \leq b \leq c \leq d$ and that the equality of 3 consecutive indices is not possible. Observe that for different choices of signs, the resulting generator gets multiplied by ± 1.

We will apply the same strategy as in the "$n+1$-even" case, namely: rewrite the elements of R in terms of the elements above, and afterwards find those which have weight zero under the action of $so(2N+1)$.

Observe first that the elements $v_{\nu j} \wedge v_{\mu k}$ have the same definition as in the previous case. This proves that:

$$\begin{aligned}
&\mathrm{Span}\{x_{i_1,i_2,i_3,i_4}\}_{1\le i_1,i_2,i_3,i_4\le 2N} = \\
&\mathrm{Span}\{(v_{\alpha a} \wedge v_{\beta b})(v_{\gamma c} \wedge v_{\delta d}) + (v_{\alpha a} \wedge v_{\delta d})(v_{\beta b} \wedge v_{\gamma c}) - (v_{\alpha a} \wedge v_{\gamma c})(v_{\beta b} \wedge v_{\delta d}) + \\
&+ (v_{\gamma c} \wedge v_{\delta d})(v_{\alpha a} \wedge v_{\beta b}) + (v_{\beta b} \wedge v_{\gamma c})(v_{\alpha a} \wedge v_{\delta d}) - (v_{\beta b} \wedge v_{\delta d})(v_{\alpha a} \wedge v_{\gamma c})\},
\end{aligned}$$

where $1 \le a,b,c,d \le N$ and $\alpha,\beta,\gamma,\delta \in \{\pm 1\}$. As before, the elements in the set on the right-hand-side will be denoted by $v_{a,b,c,d}(\alpha,\beta\gamma,\delta)$.

This shows that if all indices involved in $x_{a,b,c,d}$ are strictly less than $2N+1$, then the "weight-zero" relations are those obtained in the "$n+1$-even" case. Moreover, the following lemma proves that these are indeed all "weight-zero" relations in R.

Lemma 1.5.6. *The set R_0 of elements of $\mathfrak{h}$-weight zero in R is spanned by the elements $v_{a,b,c,d}(\alpha,\beta,\gamma,\delta)$ with $a=b<c=d<2N+1$ and $\alpha=\gamma=1$, $\beta=\delta=-1$.*

Proof. It follows from the results in Subsection 1.5.3 that R_0 contains the given span, which is also the span of the elements $x_{a,b,c,d}$ with $1 \le a \le b \le c \le d < 2N+1$.

Suppose that R_0 strictly contains the mentioned span. Then there would exist a sequence $1 \le a \le b \le c \le d = 2N+1$ such that $x_{a,b,c,d}$ has a "weight-zero" component.

By inverting the matrix which defines the elements $v_0 \wedge v_a$ and $v_{-a} \wedge v_0$ we obtain that:

$$e_{2a-1+\alpha} \wedge e_{2N+1} = \frac{1}{\sqrt{2}} \sum_{\nu\in\{-1,1\}} (i\nu)^{\alpha} v_0 \wedge v_{\nu a},$$

where $\alpha \in \{0,1\}$. This gives us the following equality:

$$\begin{aligned}
&x_{2a-1+\alpha,2b-1+\beta,2c-1+\gamma,2N+1} = \\
&\frac{1}{4\sqrt{2}} \sum_{\nu,\mu,\omega\in\{1,-1\}} (i\nu)^{\alpha}(i\mu)^{\beta}(i\omega)^{\gamma}\cdot \\
&\Big((v_{\nu a} \wedge v_{\mu b})(v_0 \wedge v_{\omega c}) + (v_0 \wedge v_{\nu a})(v_{\mu b} \wedge v_{\omega c}) - (v_{\nu a} \wedge v_{\omega c})(v_0 \wedge v_{\mu b}) \\
&+ (v_0 \wedge v_{\omega c})(v_{\nu a} \wedge v_{\mu b}) + (v_{\mu b} \wedge v_{\omega c})(v_0 \wedge v_{\nu a}) - (v_0 \wedge v_{\mu b})(v_{\nu a} \wedge v_{\omega c})\Big).
\end{aligned}$$

In this sum the term with indices $0, \nu a, \mu b, \omega c$ has weight $\nu\epsilon_a + \mu\epsilon_b + \omega\epsilon_c$. For a particular choice of ν,μ,ω this weight must be zero. This in turn implies that $a=b=c$ and $\nu+\mu+\omega=0$, which is clearly impossible. □

As in Subsection 1.5.3 we now conclude that Corollary 1.5.5 holds also in this case. This proves Theorem 1.5.2 for $n+1$ is odd.

1.5.5 The graphical method

In the following we will demonstrate another way of obtaining the same polynomials, by making use of the graphical language developed in Subsection 1.2.1.

By the definition of the ideal $I(\lambda)$, it follows that in $V(\lambda)$ the elements $v_j \wedge v_k.\mathbb{1}$ and $v_j \wedge v_{-k}.\mathbb{1}$ ($j < k$) are zero. We use the graphical representation of these elements to obtain the following equalities:

$$v_j \wedge v_k.\mathbb{1} = \frac{1}{2}(-\circ\ \circ\ \circ\ \circ + \circ\ \circ\ \circ\ \circ - i\circ\ \circ\ \circ\ \circ - i\circ\ \circ\ \circ\ \circ).\mathbb{1} = 0,$$

$$v_j \wedge v_{-k}.\mathbb{1} = \frac{1}{2}(-\circ\ \circ\ \circ\ \circ - \circ\ \circ\ \circ\ \circ + i\circ\ \circ\ \circ\ \circ - i\circ\ \circ\ \circ\ \circ).\mathbb{1} = 0.$$

Remember that we chose $j < k$ and observe that in these equalities the four points are labeled $2j-1, 2j, 2k-1, 2k$.

The computations presented below, should be seen as taking place in $U(so(n+1))/(Q(A)+I(\lambda))$.

Remark 1.5.7. Note that we assume here that $Q(A)+I(\lambda) \subsetneq U(so(n+1))$, i.e. the $so(n+1)$-module structure on $Z(\lambda)$ factors through an A-module structure.

By adding and subtracting the two equalities above we obtain the following two relations:

$$\circ\ \circ\ \circ\ \circ = -i\circ\ \circ\ \circ\ \circ,$$

$$\circ\ \circ\ \circ\ \circ = i\circ\ \circ\ \circ\ \circ.$$

They tell us that if an arc connects two points in the diagram which have different parities, then we can shift the left leg of the arc, such that the result is an arc between two points of the same parity. By doing this we acquire either $+i$ or $-i$ in front of the diagram.

The ideal $I(\lambda)$ is a left ideal, hence:

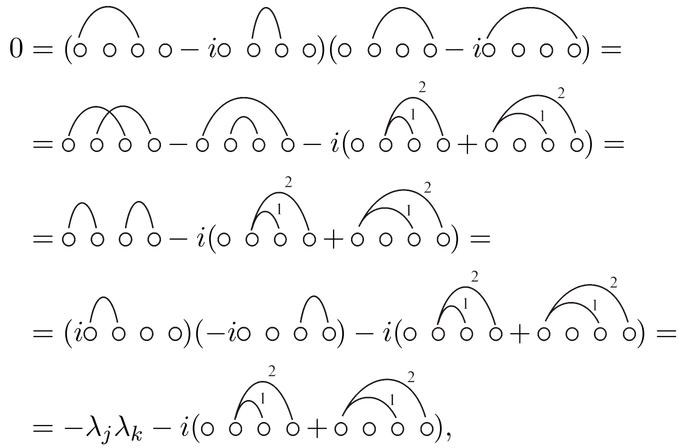
$$(\circ\ \circ\ \circ\ \circ - i\circ\ \circ\ \circ\ \circ)(\circ\ \circ\ \circ\ \circ - i\circ\ \circ\ \circ\ \circ) = 0$$

and

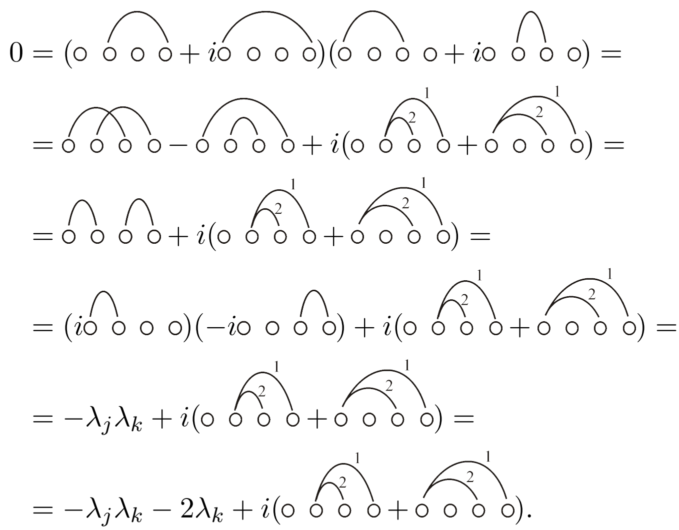
$$(\circ\ \circ\ \circ\ \circ + i\circ\ \circ\ \circ\ \circ)(\circ\ \circ\ \circ\ \circ + i\circ\ \circ\ \circ\ \circ) = 0.$$

Recall that the Cartan subalgebra $\mathfrak{h}$ of $so(n+1)$ was defined as $\bigoplus_{i=1}^{N} \mathbb{C}\epsilon_i$, where $\epsilon_j = ie^{2j-1,2j} = i\circ\ \circ$ (a simple diagram with one arc connecting the points $2j-1$ and $2j$). The element ϵ_j acts on the highest weight vector by multiplication with $\lambda(\epsilon_j)$, which we agreed to denote by λ_j. Using this fact we compute the action of the products above on the highest weight vector $\mathbb{1}$.

The first equality becomes:



while the second can be rewritten as:



Summing up, we obtain the following equation:

$$-2\lambda_j\lambda_k - 2\lambda_k = 0,$$

hence the same polynomials in λ:

$$\lambda_k(\lambda_j + 1) = 0.$$

1.5.6 A third method for the smallest case

In this subsection we consider the case $n + 1 = 4$. Then A is the simple 3-Lie algebra and its basic Lie algebra is $so(4)$. We will present a third method for obtaining the possible highest weights such that the representation of $so(4)$ on $Z(\lambda) = U(so(4))/J(\lambda)$ factors through a representation of A on $Z(\lambda)$.

Unlike before, $so(4)$ is not a simple Lie algebra, but it is isomorphic to $sl(2) \oplus sl(2)$. Our approach will be to find the two $sl(2)$'s sitting inside $so(4)$ and compute the set R, generating $Q(A)$, in terms of the elements of these two simple Lie algebras. We will then impose the condition that R acts trivially on the highest weight module $Z(\lambda)$.

In the notation of Subsection 1.5.1 define

$$x_1 = iv_1 \wedge v_{-2} \quad y_1 = iv_{-1} \wedge v_2 \quad x_2 = iv_1 \wedge v_2 \quad y_2 = iv_{-1} \wedge v_{-2}.$$

Then

$$[x_1, y_1] = \epsilon_1 - \epsilon_2 =: h_1 \text{ and } [x_2, y_2] = \epsilon_1 + \epsilon_2 =: h_2,$$

while $[x_1, y_2] = [x_2, y_1] = 0$. We obtain in this manner the two $sl(2)$'s sitting inside $so(4)$, namely $\mathrm{Span}\{x_1, h_1, y_1\}$ and $\mathrm{Span}\{x_2, h_2, y_2\}$.

In the notation of Section 1.2, the generating set R of the two-sided ideal $Q(A) \subset U(so(4))$ is spanned by the element:

$$\begin{aligned} X := x_{1,2,3,4} &= e^{12}e^{34} + e^{14}e^{23} - e^{13}e^{24} = \\ &= (e_1 \wedge e_2)(e_3 \wedge e_4) + (e_1 \wedge e_4)(e_2 \wedge e_3) - (e_1 \wedge e_3)(e_2 \wedge e_4). \end{aligned}$$

We express this generator in terms of the elements $x_1, h_1, y_1, x_2, h_2, y_2$. Since $h_1 = \epsilon_1 - \epsilon_2$ and $h_2 = \epsilon_1 + \epsilon_2$ it follows that

$$e^{12} = \frac{h_1 + h_2}{2i} \text{ and } e^{34} = \frac{h_2 - h_1}{2i}.$$

By using Equation (1.16) we obtain

$$e^{23} = \frac{1}{2}(x_1 + x_2 - y_1 - y_2),$$

$$\begin{aligned} e^{14} &= \frac{1}{2}(-x_1 + x_2 + y_1 - y_2), \\ e^{13} &= \frac{-1}{2i}(x_1 + x_2 + y_1 + y_2), \\ e^{24} &= \frac{1}{2i}(-x_1 + x_2 - y_1 + y_2), \end{aligned}$$

which in turn gives us:

$$e^{14}e^{23} - e^{13}e^{24} = \frac{h_1 + 2y_1x_1 - h_2 - 2y_2x_2}{2}.$$

The element X is thus given by:

$$X = \frac{h_1^2 - h_2^2}{4} + \frac{h_1 - h_2}{2} + y_1x_1 - y_2x_2.$$

Let $\mu_1, \mu_2 \in \mathbb{C}$ and let $Z(\mu_1, \mu_2)$ be a highest weight irreducible module for the Lie algebra $so(4)$, of highest weight μ determined by $\mu(h_i) = \mu_i$, $(i = 1, 2)$. Let $\mathbb{1}$ be the highest weight vector of $Z(\mu_1, \mu_2)$. For this module to be a module for the simple 3-Lie algebra A we want the element X to act trivially on $\mathbb{1}$. Since $h_j \cdot \mathbb{1} = \mu_j \mathbb{1}$ for all $j \in \{1, 2\}$ and $x_j \cdot \mathbb{1} = 0$, we can conclude that X acts on $\mathbb{1}$ as:

$$\frac{\mu_1^2 - \mu_2^2}{4} + \frac{\mu_1 - \mu_2}{2}.$$

We observe that

$$X.\mathbb{1} = 0 \Leftrightarrow \frac{\mu_1^2 - \mu_2^2}{4} + \frac{\mu_1 - \mu_2}{2} = 0 \Leftrightarrow (\mu_1 + 1)^2 = (\mu_2 + 1)^2.$$

Hence, we have obtained two solutions, namely

$$\mu_1 = \mu_2 \text{ and } \mu_1 + \mu_2 = -2.$$

Thus, $Z(\mu_1, \mu_2) = Z(\mu_1) \otimes Z(\mu_2)$ is a 3-Lie algebra module if either both weights coincide, or their sum is -2.

There still remains the matter of determining λ_1 and λ_2 (in the notation previously used). The formulas for h_1 and h_2 tell us that $\mu_1 = \lambda_1 - \lambda_2$ while $\mu_2 = \lambda_1 + \lambda_2$. If $\mu_1 = \mu_2$ we obtain that $\lambda_2 = 0$ and if $\mu_1 + \mu_2 = -2$ we obtain the solution $\lambda_1 = -1$. These are precisely the solutions of the polynomial $\lambda_2(\lambda_1 + 1) = 0$.

1.6 Primitive ideals

In this section we present a corollary to Theorem 1.5.2. Namely, we determine which primitive ideals of the universal enveloping algebra $U(\wedge^{n-1} A)$ are also primitive ideals of $U(A)$.

For Lie algebras, primitive ideals are defined to be two sided ideals of the universal enveloping algebra, which are annihilators of irreducible representations. It makes sense to define them in an analogous way in the case of n-Lie algebras.

Definition 1.6.1. Let V be an n-Lie algebra and $U(V)$ its universal enveloping algebra. A two-sided ideal of $U(V)$ is called *primitive* if it is the annihilator of some irreducible module of V.

Let M be an irreducible module of the n-Lie algebra V. The annihilator of this module, $\mathrm{Ann}(M)$, is by definition a primitive ideal of $U(V)$. By Corollary 1.1.21 and Proposition 1.1.22 M is an irreducible module of the basic Lie algebra $\wedge^{n-1} V$ annihilated by the action of $Q(V) \subset U(\wedge^{n-1} V)$. Since $U(V)$ was defined as $U(\wedge^{n-1} V)/Q(V)$, we may conclude that $\mathrm{Ann}(M)$ is a primitive ideal of $U(\wedge^{n-1} V)$ which contains $Q(V)$.

On the other hand, let I be a primitive ideal of $U(\wedge^{n-1} V)$, such that $Q(V) \subseteq I$. Then, there exists an irreducible module of $\wedge^{n-1} V$, call it M, which is annihilated by I and thus by $Q(V)$. This transforms M into an irreducible module of V and I into a primitive ideal of $U(V)$. We conclude that:

Lemma 1.6.2. *Primitive ideals of $U(V)$ are in $1-1$ correspondence with the primitive ideals of $U(\wedge^{n-1}V)$ which contain $Q(V)$.*

Let $\mathfrak{b}$ be a Borel subalgebra of $\wedge^{n-1}V$ and $\mathfrak{h} \subset \mathfrak{b}$ a Cartan subalgebra. Denote, as before, by $Z(\lambda)$ the irreducible highest weight module of highest weight $\lambda \in \mathfrak{h}^*$. It was proven in [8], by M. Duflo, that a primitive ideal I of the universal enveloping algebra $U(\wedge^{n-1}V)$ is the annihilator of the module $Z(\lambda)$, for some highest weight λ.

Hence, the problem of determining the primitive ideals of $U(A)$, can be reformulated as determining the highest weight, irreducible representations of A.

Let $Z(\lambda)$ be such a module, and denote by $\mathrm{Ann}(Z(\lambda))$ its annihilator. For $\mathrm{Ann}(Z(\lambda))$ to be a primitive ideal of $U(A)$, we would need that $Q(A) \subseteq \mathrm{Ann}(Z(\lambda))$, i.e. $Q(A)$ acts trivially on $Z(\lambda)$. Hence, in order to find the primitive ideals of $U(A)$, we want to find those weights λ, such that $Q(A) \subseteq \mathrm{Ann}(Z(\lambda))$. These are precisely the weights we have determined in the previous sections. Hence, we have already proved the following theorem.

Theorem 1.6.3. *Let I be a primitive ideal of $U(A)$. Then $I = \mathrm{Ann}(Z(\lambda))$ for some λ as in Theorem 1.3.2 or Theorem 1.3.3, depending on the parity of $n+1$.*

As an example, we will show that the Joseph ideal is a primitive ideal of $U(A)$. We fix $n > 3$.

Definition 1.6.4. Let $\mathfrak{g}$ be a semisimple Lie algebra and denote by $U(\mathfrak{g})$ its universal enveloping algebra. A two-sided ideal I of $U(\mathfrak{g})$ is said to be *completely prime* if for all $a, b \in U(\mathfrak{g})$:

$$ab \in I \ \Rightarrow a \in I \text{ or } b \in I.$$

In [33], Joseph constructed a primitive, completely prime ideal in $U(so(n+1))$ (or more generally in the universal enveloping algebra of a simple complex Lie algebra) corresponding to the closure of the minimal nilpotent orbit of the coadjoint action; and computed its infinitesimal character. This ideal, called the *Joseph ideal*, will be denoted by J.

Proposition 1.6.5. *J is a primitive ideal of $U(A)$.*

The rest of this chapter contains two proofs for the proposition above. The first proof makes use of the Theorem 1.6.3. The second proof shows through direct calculations that indeed $Q(A) \subseteq J$. In both proofs we will use that $\wedge^{n-1}A \cong so(n+1)$.

First proof. Since J is a primitive ideal of $U(so(n+1))$ there exists an irreducible highest weight module of $so(n+1)$ of highest weight λ, denoted by $Z(\lambda)$, which is annihilated by J. According to Theorem 1.6.3, J is a primitive ideal of the universal enveloping algebra $U(A)$, if λ is as in Theorem 1.3.2 or Theorem 1.3.3, depending on the parity of $n+1$.

Recall that we denoted by $\pi_1, \ldots, \pi_N$ the fundamental weights of $so(n+1)$, for $n+1 = 2N$ or $n+1 = 2N+1$. Denote by ρ the sum of these fundamental weights:

$$\rho = \pi_1 + \ldots + \pi_N.$$

Assume first that $n+1$ is even, i.e. $n+1 = 2N$. Then, by [33, page 15]

$$\lambda + \rho = \sum_{i=1}^{N-3} \pi_i + \pi_{N-1} + \pi_N.$$

Hence,

$$\lambda = -\pi_{N-2},$$

which is of the type described in Theorem 1.3.2 for $x = 0$ and $t = N-1$.

Assume that $n+1$ is odd, i.e. $n+1 = 2N+1$. Then, again by [33, page 15]

$$\lambda + \rho = \sum_{i=1}^{N-3} \pi_i + \frac{1}{2}\pi_{N-2} + \frac{1}{2}\pi_{N-1} + \pi_N.$$

Hence,

$$\lambda = -\frac{1}{2}\pi_{N-2} - \frac{1}{2}\pi_{N-1},$$

which is of the type described in Theorem 1.3.3 for $x = -\frac{1}{2}$ and $t = N-1$.

Thus, we conclude that J is indeed a primitive ideal of $U(A)$. $\square$

Second proof. We begin this proof by realizing $S^2(so(n+1))$ as a submodule of $U(so(n+1))$ for the $so(n+1)$-action.

Let $S(so(n+1)) = \oplus_{k \geq 0} S^k(so(n+1))$ be the symmetric algebra of $so(n+1)$ and denote by $\odot$ its product. Denote by $T^k(so(n+1))$ the k-th tensor power of $so(n+1)$. Then the map $\Sigma : S^k(so(n+1)) \to T^k(so(n+1))$ defined by

$$\Sigma(a_1 \odot \ldots \odot a_k) = \frac{1}{k!} \sum_{\sigma \in S_k} a_{\sigma(1)} \otimes \ldots \otimes a_{\sigma(k)}$$

is a map of $so(n+1)$-modules. Now the composition of the two maps

$$\Sigma : S^k(so(n+1)) \to T^k(so(n+1))$$

and the map of $so(n+1)$-modules given by the natural projection

$$p : T^k(so(n+1)) \to U(so(n+1))$$

is a map

$$i = p \circ \Sigma : S^k(so(n+1)) \to U(so(n+1))$$

and by the Poincare-Birkoff-Witt Theorem we have

$$U(so(n+1)) = \bigoplus_{k\geq 0} i(S^k(so(n+1))). \tag{1.18}$$

We identify $S^k(so(n+1))$ with its image under i.

Denote by $\alpha = \epsilon_1 + \epsilon_2$ the highest root of $so(n+1)$. Then, according to [21, page 590], as a $so(n+1)$-module

$$S^2(so(n+1)) = V(2\alpha) \oplus V(0) \oplus W,$$

where (see [21, page 590])

$$W = \oplus_i V(\alpha + \alpha_i).$$

Here, the α_i's are roots in $so(n+1)$ perpendicular to α. The following remark is obvious by the above decomposition of W, but important further on.

Remark 1.6.6. The complement W has no irreducible subrepresentation in common with $V(2\alpha) \oplus V(0)$.

In [21] it was shown that the Joseph ideal $J \subset U(so(n+1))$ is equal to the ideal generated by W and $C - c_0$, where c_0 is the eigenvalue for the Casimir C for the infinitesimal character that Joseph obtained. Hence, in order to show that $Q(A) \subset J$ it suffices to prove that $R \subset W$. The lemma below completes the proof. $\square$

Lemma 1.6.7. $R \subset W$.

Proof. To prove this Lemma we first want to realize R as a subrepresentation of $S^2(so(n+1))$ for the $so(n+1)$-action.

By Equation (1.18) we can view R as sitting inside $S^2(so(n+1))$. Recall that $so(n+1) \cong \wedge^2 A$. In the notation of Section 1.2, consider the $\wedge^2 A$-module $\wedge^4 A$ and the map

$$\psi : \wedge^4 A \to S^2(\wedge^2 A)$$

defined on monomials as:

$$e_i \wedge e_j \wedge e_k \wedge e_l \mapsto (e_i \wedge e_j) \odot (e_k \wedge e_l) - (e_i \wedge e_k) \odot (e_j \wedge e_l) + (e_i \wedge e_l) \odot (e_j \wedge e_k).$$

Observe that $R \cong \psi(\wedge^4 A)$.

Of course, we can also define the map $\phi : S^2(\wedge^2 A) \to \wedge^4 A$ defined on monomials as:

$$(v_i \wedge v_j) \odot (v_k \wedge v_l) \mapsto v_i \wedge v_j \wedge v_k \wedge v_l.$$

It is easy to see that $\phi \circ \psi = 3Id$. Then, $\ker\phi$ is a subrepresentation of $\wedge^2 A$ in $S^2(\wedge^2 A)$, complementary to R. Hence, we have obtained the following decomposition into submodules:

$$S^2(\wedge^2 A) = \psi(\wedge^4 A) \oplus \mathrm{Ker}\phi.$$

Thus, by Remark 1.6.6, in order to show that $R \subset W$, it suffices to show that $V(2\alpha) \oplus V(0) \subseteq \ker\phi$. Let $w \in \wedge^2 A$ be a highest weight vector for the highest weight α. Then, $V(2\alpha)$ is generated by $w \odot w$ and the conclusion follows directly. $\square$

Chapter 2

Kostant's convexity theorem, parabolic subgroups and groups of the Harish-Chandra class

Let G be a real connected semisimple Lie group with finite center and K the maximal compact subgroup associated to some Cartan involution $\theta : G \to G$. The non-linear convexity theorem of Kostant gives the image of a left translate of K under the Iwasawa projection. In the next chapter we will state a generalization of this result to semisimple symmetric spaces (or more generally, reductive symmetric spaces). Chapter 4 below contains the proof of our convexity theorem.

This chapter serves as preparation for the next one. We start in Section 2.1 with a brief account of the non-linear convexity theorem of Kostant. In Section 2.2 we introduce parabolic subalgebras and parabolic subgroups of a real connected semisimple Lie group G, while in Section 2.3 we give a short introduction to the theory of symmetric spaces. We end this chapter with the definition of a reductive group of the Harish-Chandra class (class $\mathcal{H}$) and a motivation for using this type of Lie groups, see Section 2.4.

The theory in this chapter can be found, among other places, in [36] and [55].

2.1 Basic structure theory of semisimple Lie groups

In this section we discuss some of the basic structure theory of semisimple Lie groups. In particular, it is our aim to explain here the Iwasawa decomposition of a real connected semisimple Lie group G and Kostant's non-linear convexity theorem.

We denote by $\mathfrak{g}$ a real Lie algebra. Recall that the symmetric bilinear form

$$B : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

defined by

$$B(X, Y) = \mathrm{Tr}(\mathrm{ad}X\mathrm{ad}Y), \text{ for all } X, Y \in \mathfrak{g}$$

is called the *Killing form* on $\mathfrak{g}$. The Lie algebra $\mathfrak{g}$ is said to be *semisimple* if it is the sum of its simple ideals. Cartan's criterion for semisimplicity says that the Lie algebra $\mathfrak{g}$ is semisimple if and only if its Killing form is non-degenerate. The Lie algebra $\mathfrak{g}$ is said to be *reductive* if it is the direct sum of two subalgebras, an abelian one and a semisimple one.

Example 2.1.1. Denote by $sl(n, \mathbb{R})$, $n \geq 2$, the set of n-by-n matrices with real entries and trace zero. Then $sl(n, \mathbb{R})$ is a semisimple Lie algebra and its Killing form is given by $B(X, Y) = 2n\mathrm{Tr}(XY)$.

Definition 2.1.2. A real Lie group G is said to be *semisimple* if its Lie algebra $\mathfrak{g}$ is semisimple.

Example 2.1.3. Let $SL(n, \mathbb{R})$, $n \geq 2$, be the set of n-by-n matrices with real entries and determinant 1. Then $SL(n, \mathbb{R})$ is a semisimple Lie group with Lie algebra $sl(n, \mathbb{R})$.

Let G be a real connected semisimple Lie group and denote by $\mathfrak{g}$ its Lie algebra. A map $\theta : \mathfrak{g} \to \mathfrak{g}$ is said to be an *involution* on $\mathfrak{g}$ if θ is a Lie algebra automorphism such that $\theta^2 = \mathrm{id}$. An involution $\theta : \mathfrak{g} \to \mathfrak{g}$ is called a *Cartan involution* if the symmetric bilinear form

$$\langle U, V \rangle := -B(U, \theta V) \quad (U, V \in \mathfrak{g}) \tag{2.1}$$

is positive definite.

Example 2.1.4. Denote by $\mathfrak{g}$ the Lie algebra of Example 2.1.1, $\mathfrak{g} = sl(n, \mathbb{R})$. Let $\theta : \mathfrak{g} \to \mathfrak{g}$ be the map given by

$$\theta(X) = -X^T. \tag{2.2}$$

Then θ is a Cartan involution on $\mathfrak{g}$.

It is known that every semisimple Lie algebra $\mathfrak{g}$ has such a Cartan involution (see Corollary 6.18 in [36]). Moreover, all Cartan involutions on $\mathfrak{g}$ are conjugate under $\mathrm{Int}(\mathfrak{g})$ (see Corollary 6.19 in [36]).

Let $\theta : \mathfrak{g} \to \mathfrak{g}$ be a Cartan involution on the semisimple Lie algebra $\mathfrak{g}$. Then, θ has two eigenvalues, ± 1, and it yields the eigenspace decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},$$

where $\mathfrak{k}$ denotes the $+1$-eigenspace and $\mathfrak{p}$ the -1-eigenspace of $\mathfrak{g}$. This decomposition is known as the *Cartan decomposition* of the Lie algebra $\mathfrak{g}$.

Example 2.1.5. For the semisimple Lie algebra $\mathfrak{g} = sl(n, \mathbb{R})$ and the Cartan involution given by (2.2) the +1-eigenspace is given by

$$\mathfrak{k} := \{X \in sl(n, \mathbb{R}) |\, X = -X^T\} = so(n, \mathbb{R}),$$

while the -1-eigenspace w.r.t. θ equals

$$\mathfrak{p} := \{X \in sl(n, \mathbb{R}) |\, X = X^T\}.$$

The Cartan decomposition tells us that every matrix in the Lie algebra $sl(n, \mathbb{R})$ is the sum of two matrices with trace 0: a symmetric one and a skew-symmetric one.

The Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ has the special property that the Killing form is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$. Furthermore $\mathfrak{k}$ and $\mathfrak{p}$ are orthogonal w.r.t the Killing form.

Let K denote the subgroup of G with Lie algebra $\mathfrak{k}$,

$$K := \langle \exp \mathfrak{k} \rangle.$$

The subgroup K is compact if and only if the center of G is finite, $|Z(G)| < \infty$. Moreover, if K is compact then K is maximally compact and any other maximally compact subgroup of G is conjugate to K (see Theorem 6.31 in [36]).

Remark 2.1.6. From now on we assume that the Lie group G has finite center.

Theorem 2.1.7 (Cartan decomposition)**.** *Let G, K and $\mathfrak{p}$ be as above. The mapping $K \times \mathfrak{p} \to G$ given by*

$$(k, X) \mapsto k \cdot \exp X$$

is a diffeomorphism.

Proof. For the proof see for instance [36, Theorem 6.31]. $\square$

Example 2.1.8. As in Example 2.1.3, let $G = SL(n, \mathbb{R})$. Then its Lie algebra $sl(n, \mathbb{R})$ has the Cartan decomposition

$$sl(n, \mathbb{R}) = so(n, \mathbb{R}) \oplus \mathfrak{p},$$

where $\mathfrak{p}$ denotes the set of symmetric n-by-n matrices of trace zero. The Lie group $SL(n, \mathbb{R})$ has finite center and therefore the subgroup $SO(n, \mathbb{R}) = \langle \exp so(n, \mathbb{R}) \rangle$ is a maximally compact subgroup. According to Theorem 2.1.7 every matrix in $SL(n, \mathbb{R})$ can be written uniquely as the product of two matrices with determinant 1: an orthogonal one and a symmetric one with positive eigenvalues

$$SL(n, \mathbb{R}) = SO(n, \mathbb{R}) \times \exp \mathfrak{p}.$$

We define the *global Cartan involution* on G. Let $g \in G$. Then g can be written uniquely as $k \cdot \exp X$, where $k \in K$ and $X \in \mathfrak{p}$. Define

$$\Theta : G \to G \quad \text{by} \quad \Theta(g) = \Theta(k \cdot \exp X) = k \cdot \exp(-X). \tag{2.3}$$

It is clear that Θ is involutive, i.e. its square is the identity, and that its fixed point set is given by

$$G^{\Theta} = K.$$

Moreover, Θ is a Lie group automorphism and its differential at the identity equals the Cartan involution $\theta : \mathfrak{g} \to \mathfrak{g}$

$$d\Theta(e) = \theta,$$

see [36, Theorem 6.31].

Remark 2.1.9. For simplicity, we will denote both the global Cartan involution and its infinitesimal counterpart by θ. It will be clear from the context which one we are referring to.

Example 2.1.10. Consider the decomposition of $SL(n, \mathbb{R})$ presented in Example 2.1.8

$$SL(n, \mathbb{R}) = SO(n, \mathbb{R}) \times \exp \mathfrak{p}.$$

Define $\theta : SL(n, \mathbb{R}) \to SL(n, \mathbb{R})$ by $\theta(x) = (x^T)^{-1}$. We want to show that θ is precisely the global Cartan involution defined by (2.3). For $k \in SO(n, \mathbb{R})$ and $X \in \mathfrak{p}$ we compute

$$\theta(k \cdot \exp X) = ((k \cdot \exp X)^T)^{-1} = (\exp X^T \cdot k^T)^{-1} = (k^T)^{-1} \cdot \exp(-X^T).$$

Since $k \in SO(n, \mathbb{R})$ we have that $(k^T)^{-1} = k$, while $X^T = X$.

Lemma 2.1.11. *Let $X \in \mathfrak{p}$. Then the map $\mathrm{ad}X : \mathfrak{g} \to \mathfrak{g}$ is diagonalizable with real eigenvalues.*

Proof. We will show that $\mathrm{ad}X$ is self-adjoint, where the adjoint $(\cdot)^*$ is defined relative to the inner product $\langle \cdot, \cdot \rangle$. We note that

$$\begin{aligned} \langle (\mathrm{ad}\theta X)Y, Z \rangle &= -B([\theta X, Y], \theta Z) = B(Y, [\theta X, \theta Z]) \\ &= B(Y, \theta[X, Z]) = -\langle Y, (\mathrm{ad}X)Z \rangle \\ &= -\langle (\mathrm{ad}X)^* Y, Z \rangle. \end{aligned}$$

Hence $-\mathrm{ad}X = \mathrm{ad}\theta X = -(\mathrm{ad}X)^*$ and this implies the desired conclusion. □

Remark 2.1.12. The proof of Lemma 2.1.11 also shows that for $X \in \mathfrak{k}$ the map $(\mathrm{ad}X)_{\mathbb{C}} : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$ is diagonalizable with pure imaginary eigenvalues.

Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$, which exists because of the finite dimensionality of $\mathfrak{p}$. It is unique up to conjugation by an element of K ([36, Theorem 6.51]). By Lemma 2.1.11 above we know that any transformation in the set $\{\mathrm{ad}H |\, H \in \mathfrak{a}\}$ is diagonalizable with real eigenvalues. Since they commute, these transformations are simultaneously diagonalizable with real eigenvalues. Accordingly, a simultaneous eigenvalue or $\mathfrak{a}$-weight in $\mathfrak{g}$ is a linear functional $\alpha \in \mathfrak{a}^*$ such that the space

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g} |\, [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{a}\}$$

is non-trivial. The set of non-zero weights we denote by $\Sigma(\mathfrak{g}, \mathfrak{a})$. The simultaneous eigenspace decomposition is the vectorial direct sum given by

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Sigma(\mathfrak{g},\mathfrak{a})} \mathfrak{g}_\alpha, \tag{2.4}$$

where $\mathfrak{g}_0 = Z_{\mathfrak{g}}(\mathfrak{a})$, the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$. Since $\mathfrak{a}$ is maximal abelian in $\mathfrak{p}$ we have that $\mathfrak{g}_0 \cap \mathfrak{p} = \mathfrak{a}$, and since $\mathfrak{a}$ is θ-stable, so is $\mathfrak{g}_0$. Thus, $\mathfrak{g}_0 = \mathfrak{m} \oplus \mathfrak{a}$, where $\mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{a})$, the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$.

Definition 2.1.13. A weight $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a})$ is called a *root* of $\mathfrak{a}$ in $\mathfrak{g}$, while the corresponding weight space $\mathfrak{g}_\alpha$ is called a *root space*. The decomposition (2.4) is known as the *root space decomposition* of the semisimple Lie algebra $\mathfrak{g}$.

Remark 2.1.14. The set of roots $\Sigma(\mathfrak{g}, \mathfrak{a})$ forms a root system in $\mathfrak{a}^*$ ([36, Lemma 6.53]). This system might be non-reduced, meaning that if $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a})$ is a root, either $\frac{1}{2}\alpha$ or 2α might be a root of $\mathfrak{a}$ in $\mathfrak{g}$ as well. Let

$$\Sigma_\circ(\mathfrak{g}, \mathfrak{a}) := \{\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a}) |\, \frac{1}{2}\alpha \notin \Sigma(\mathfrak{g}, \mathfrak{a})\},$$

i.e. $\Sigma_\circ(\mathfrak{g}, \mathfrak{a})$ is the set of *indivisible* roots. Then

$$\Sigma(\mathfrak{g}, \mathfrak{a}) \subseteq \Sigma_\circ(\mathfrak{g}, \mathfrak{a}) \cup 2\Sigma_\circ(\mathfrak{g}, \mathfrak{a}).$$

Remark 2.1.15. Denote by $W(\mathfrak{a})$ the Weyl group of the root system $\Sigma(\mathfrak{g}, \mathfrak{a})$. Then the map $N_K(\mathfrak{a}) \to GL(\mathfrak{a})$ factors through an isomorphism

$$N_K(\mathfrak{a})/Z_K(\mathfrak{a}) \simeq W(\mathfrak{a}),$$

where $N_K(\mathfrak{a})$ is the normalizer of $\mathfrak{a}$ in K and $Z_K(\mathfrak{a})$ is the centralizer of $\mathfrak{a}$ in K for the adjoint action of G on $\mathfrak{g}$.

Example 2.1.16. Consider the Cartan decomposition of the Lie algebra $sl(n,\mathbb{R})$ given in Example 2.1.5

$$sl(n,\mathbb{R}) = so(n,\mathbb{R}) \oplus \mathfrak{p},$$

where $\mathfrak{p}$ is the set of symmetric traceless matrices in $M_n(\mathbb{R})$. A maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$ is given by

$$\mathfrak{a} = \{H \in \mathfrak{p} |\, H \text{ is a diagonal matrix}\}.$$

Given $H \in \mathfrak{a}$, we denote by H_j the j-th diagonal entry of H. Then the root system of $\mathfrak{a}$ in $sl(n,\mathbb{R})$ is given by

$$\Sigma(sl(n,\mathbb{R}),\mathfrak{a}) = \{\alpha_{i,j} |\, 1 \leq i \neq j \leq n\}, \tag{2.5}$$

where $\alpha_{i,j}(H) = H_i - H_j$. Observe that in this case the root system is reduced. For a root $\alpha_{i,j} \in \Sigma(sl(n,\mathbb{R}),\mathfrak{a})$ the corresponding root space $\mathfrak{g}_{\alpha_{i,j}}$ is spanned by the matrix which has entry 1 in the i-th row and j-th column and 0 everywhere else.

Let $\Sigma(P)$ denote a positive system for the root system $\Sigma(\mathfrak{g},\mathfrak{a})$ and let

$$\mathfrak{n}_P = \bigoplus_{\alpha\in\Sigma(P)} \mathfrak{g}_\alpha.$$

Observe that $\Sigma(\mathfrak{g},\mathfrak{a}) = \Sigma(P) \cup -\Sigma(P)$. For the set of negative roots $-\Sigma(P)$ we will also use the notation $\Sigma(\bar{P})$. The use of the notations $\Sigma(P)$ and $\mathfrak{n}_P$ will be explained in Section 2.2.

Remark 2.1.17. Observe that $\theta(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$, for each $\alpha \in \Sigma(\mathfrak{g},\mathfrak{a})$, since $\theta = -\mathrm{id}$ on $\mathfrak{a}$.

Thus, the root space decomposition (2.4) now implies that

$$\mathfrak{g} = \theta(\mathfrak{n}_P) \oplus \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}_P. \tag{2.6}$$

Proposition 2.1.18 (*Iwasawa decomposition* of Lie algebras)**.** *With the notation as above, $\mathfrak{g}$ decomposes as the vector space direct sum $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}_P$.*

Proof. We begin by showing that the sum is indeed direct. By the decomposition (2.6), the intersection $\mathfrak{a} \cap \mathfrak{n}_P$ is trivial and the sum $\mathfrak{a} + \mathfrak{n}_P$ is direct. Let $Y \in \mathfrak{k} \cap (\mathfrak{a} \oplus \mathfrak{n}_P)$. Then, by the definition of $\mathfrak{k}$ and Remark 2.1.17

$$Y = \theta(Y) \in \mathfrak{a} \oplus \theta(\mathfrak{n}_P).$$

Decomposition (2.6) now implies that $Y \in \mathfrak{k} \cap \mathfrak{a}$ and thus $Y = -Y$. Therefore $Y = 0$.

Let $X \in \mathfrak{g}$. According to Equation (2.6) and Remark 2.1.17, we can write

$$X = Y + H + \sum_{\alpha\in\Sigma(\mathfrak{g},\mathfrak{a})} X_\alpha,$$

where $Y \in \mathfrak{m} \subset \mathfrak{k}$, $H \in \mathfrak{a}$ and $X_\alpha \in \mathfrak{g}_\alpha$. Let X_β be an element appearing in the sum $\sum X_\alpha$. If $\beta \in \Sigma(P)$ then $X_\beta \in \mathfrak{n}_P$. If on the other hand $\beta \in -\Sigma(P)$, we write $X_\beta = X_\beta + \theta X_\beta - \theta X_\beta$. The observations that $X_\beta + \theta X_\beta \in \mathfrak{k}$ and that $\theta X_\beta \in \mathfrak{n}_P$ (β is a negative root) finishes the proof. $\square$

Remark 2.1.19. The subalgebras $\mathfrak{a}$ and $\mathfrak{n}_P$ of $\mathfrak{g}$ have the properties that $\mathfrak{a}$ is abelian, $\mathfrak{n}_P$ is nilpotent, $\mathfrak{a} \oplus \mathfrak{n}_P$ is a solvable Lie subalgebra of $\mathfrak{g}$, and $[\mathfrak{a} \oplus \mathfrak{n}_P, \mathfrak{a} \oplus \mathfrak{n}_P] = \mathfrak{n}_P$.

Example 2.1.20. We continue Example 2.1.16 above. A positive system for the root system $\Sigma(sl(n, \mathbb{R}))$ is given by

$$\Sigma(P) = \{\alpha_{i,j} | \, 1 \leq i < j \leq n\}.$$

The nilpotent subalgebra $\mathfrak{n}_P$ is the subalgebra of strictly upper-triangular matrices in $M_n(\mathbb{R})$. The Iwasawa decomposition tells us that every matrix in $sl(n, \mathbb{R})$ can be written as the sum of three traceless matrices: a skew-symmetric one, a diagonal one and a strictly upper-triangular one.

The Iwasawa decomposition of the Lie algebra $\mathfrak{g}$,

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}_P,$$

defines the projection $E_\mathfrak{a} : \mathfrak{g} \to \mathfrak{a}$ along $\mathfrak{k} \oplus \mathfrak{n}_P$. The *linear convexity theorem of Kostant* says that for $H \in \mathfrak{a}$.

$$E_\mathfrak{a}(\mathrm{Ad}(K)H) = \mathrm{conv}(W(\mathfrak{a}) \cdot H), \tag{2.7}$$

where 'conv' denotes the convex hull.

Example 2.1.21. As an example of the linear convexity theorem, we discuss the easy case of $sl(2, \mathbb{R})$.

A matrix in $sl(2, \mathbb{R})$ is of the form

$$X = \begin{pmatrix} x & a \\ b & -x \end{pmatrix}$$

and the projection $E_\mathfrak{a} : sl(2, \mathbb{R}) \to \mathfrak{a}$ is given by taking the diagonal. Let

$$k = \begin{pmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{pmatrix} \in SO(2, \mathbb{R}) \text{ and } H = \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix} \in \mathfrak{a}.$$

Then

$$\begin{aligned} E_\mathfrak{a}(\mathrm{Ad}(k)H) &= E_\mathfrak{a}(\begin{pmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix} \begin{pmatrix} \cos\gamma & \sin\gamma \\ -\sin\gamma & \cos\gamma \end{pmatrix}) \\ &= \begin{pmatrix} x(\cos^2\gamma - \sin^2\gamma) & 0 \\ 0 & -x(\cos^2\gamma - \sin^2\gamma) \end{pmatrix}. \end{aligned}$$

Since $\cos^2\gamma - \sin^2\gamma \in [-1, 1]$ and $W(\mathfrak{a}) = \{\pm\mathrm{id}\}$ we obtain that for every $H \in \mathfrak{a}$ the equality $\pi(\mathrm{Ad}(K)H) = \mathrm{conv}(W(\mathfrak{a}) \cdot H)$ clearly holds.

Denote by A, respectively N_P, the analytic subgroup of G with Lie algebra $\mathfrak{a}$, respectively $\mathfrak{n}_P$. The groups A and N_P, besides being connected, are simply connected and moreover A normalizes N_P. A proof of these facts and of the theorem below can be found in [36, Theorem 6.46].

Theorem 2.1.22 (*Iwasawa decomposition* of Lie groups)**.** *With the notation as above, the multiplication map $K \times A \times N_P \to G$ given by $(k, a, n) \mapsto kan$ is a diffeomorphism.*

The Iwasawa decomposition of the Lie group G tells us that as a manifold we can decompose this Lie group as the product of three subgroups: a maximal compact subgroup K, and abelian subgroup A and a unipotent subgroup N_P. The following example illustrates this decomposition for the Lie group $SL(n, \mathbb{R})$.

Example 2.1.23. The Lie algebra $sl(n, \mathbb{R})$ of the Lie group $SL(n, \mathbb{R})$ has the Iwasawa decomposition

$$sl(n, \mathbb{R}) = so(n, \mathbb{R}) \oplus \mathfrak{a} \oplus \mathfrak{n}_P,$$

where $\mathfrak{a}$ and $\mathfrak{n}_P$ are as in Examples 2.1.16 and 2.1.20. The subgroup A of $SL(n, \mathbb{R})$ generated by $\mathfrak{a}$ is given by the set of all diagonal matrices with positive entries and determinant 1, while the subgroup N_P consists of upper-triangular matrices with 1's on the diagonal. Thus, the Iwasawa decomposition for $SL(n, \mathbb{R})$ is given by

$$SL(n, \mathbb{R}) \simeq SO(n, \mathbb{R}) \times A \times N_P.$$

We are now ready to give the statement of the non-linear convexity theorem of Kostant.

An element $g \in G$ can be written in a unique way as $g = kan$, where $k \in K$, $a \in A$ and $n \in N_P$. The Iwasawa decomposition $G = KAN_P$ gives rise to the real analytic map

$$\mathfrak{H}_P : G \to \mathfrak{a} \quad \text{given by} \quad g \in K \exp \mathfrak{H}_P(g) N_P. \tag{2.8}$$

This map is known as the *Iwasawa projection*. By its definition we can easily see that the Iwasawa projection is left K-invariant and right N_P-invariant, i.e. for $k \in K$ and $n \in N_P$

$$\mathfrak{H}_P(g) = \mathfrak{H}_P(kgn).$$

Kostant's non-linear convexity theorem ([37, Theorem 4.1]) investigates the Iwasawa projection of a left translate of the maximal compact subgroup K. Namely, for $g \in G$, the non-linear convexity theorem gives us the image $\mathfrak{H}_P(gK)$. By [36, Theorem 7.39], we know that $g = kak'$, where $a \in A$ is unique up to conjugation by elements of $W(\mathfrak{a})$ and $k, k' \in K$. Thus, $\mathfrak{H}_P(gK) = \mathfrak{H}_P(kak'K) = \mathfrak{H}_P(aK)$. This means that it suffices to consider elements of the abelian subgroup A. The *non-linear convexity theorem of Kostant* states that for any $a \in A$

$$\mathfrak{H}_P(aK) = \mathrm{conv}(W(\mathfrak{a}) \cdot \log a). \tag{2.9}$$

Here, 'conv' stands for the convex hull. We have already mentioned that both subgroups A and N_P are connected and simply connected. Since they are also nilpotent, the following standard lemma tells us that the maps $\exp : \mathfrak{a} \to A$ and $\exp : \mathfrak{n}_P \to N_P$ are diffeomorphisms. In (2.9) $\log$ denotes the inverse of the map $\exp : \mathfrak{a} \to A$.

Lemma 2.1.24. *Let N be a connected and simply connected Lie group with nilpotent Lie algebra $\mathfrak{n}$. If $\mathfrak{n}_0$ is a subalgebra of $\mathfrak{n}$, then the exponential map maps $\mathfrak{n}_0$ diffeomorphically onto a closed subgroup of N.*

Remark 2.1.25. In [26, Theorem 1.4.2] the non-linear convexity theorem of Kostant has been reduced to the linear one by a homotopy argument.

We end this section with an example of the non-linear convexity theorem.

Example 2.1.26. Let G be the Lie group $SL(3, \mathbb{R})$ with Lie algebra $\mathfrak{g} = sl(3, \mathbb{R})$. Then, the root system $\Sigma(sl(3, \mathbb{R}), \mathfrak{a})$ is as in the figure below

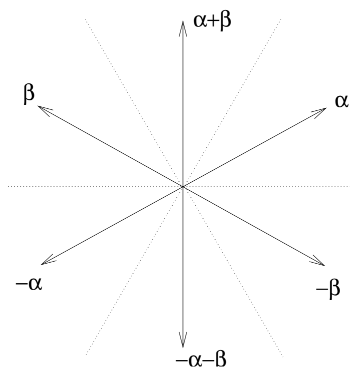


Figure 2.1: The root system $\Sigma(sl(3, \mathbb{R}), \mathfrak{a})$

and the Weyl group $W(\mathfrak{a})$ (which is isomorphic to the permutation group S_3) is generated by reflections in the root hyperplanes.

We identify $\mathfrak{a}$ with $\mathfrak{a}^*$ via the inner product $\langle \cdot, \cdot \rangle$ defined by (2.1). Let $a \in A$. Our aim is to find the image of a left translate of $SO(3, \mathbb{R})$, given by $aSO(3, \mathbb{R})$, under the map $\mathfrak{H}_P$. According to the non-linear convexity theorem this image equals the convex set $\mathrm{conv}(W(\mathfrak{a}) \cdot \log a)$, displayed below.

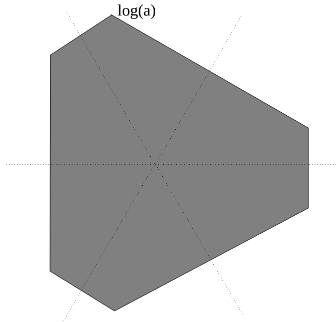


Figure 2.2: The convex hull of $W(\mathfrak{a}) \cdot \log a$

2.2 Parabolic subalgebras and parabolic subgroups

In this section we give a short introduction to parabolic subalgebras and parabolic subgroups. We retain the notation introduced in Section 2.1: G, θ, K, A and N_P and their infinitesimal counterparts $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{a}$ and $\mathfrak{n}_P$, where

$$\mathfrak{n}_P := \bigoplus_{\alpha \in \Sigma(P)} \mathfrak{g}_\alpha$$

for some positive choice of roots $\Sigma(P) \subset \Sigma(\mathfrak{g}, \mathfrak{a})$.

Recall that we used the notation $\mathfrak{m} := Z_{\mathfrak{k}}(\mathfrak{a})$. By Equation (2.6) we have the decomposition

$$\mathfrak{g} = \theta(\mathfrak{n}_P) \oplus \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}_P.$$

Let $X \in \mathfrak{m}$, $Y \in \mathfrak{g}_\alpha$, for some positive root $\alpha \in \Sigma(P)$, and $H \in \mathfrak{a}$. Then

$$[H, [X, Y]] = [[H, X], Y] + [X, [H, Y]] = \alpha(H)[X, Y],$$

which shows that $[\mathfrak{m} \oplus \mathfrak{a}, \mathfrak{n}_P] \subseteq \mathfrak{n}_P$. We conclude that $\underline{\mathfrak{p}} := \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}_P$ is a Lie subalgebra of $\mathfrak{g}$.

Definition 2.2.1. Any subalgebra of $\mathfrak{g}$ conjugate via $\mathrm{Ad}(G)$ to the subalgebra $\underline{\mathfrak{p}}$ is called a *minimal parabolic subalgebra* of $\mathfrak{g}$.

Remark 2.2.2. Because of the Iwasawa decomposition $G = KAN_P$, we may just as well assume that a minimal parabolic subalgebra of $\mathfrak{g}$ is conjugate to the subalgebra $\underline{\mathfrak{p}}$ via $\mathrm{Ad}(K)$.

Example 2.2.3. Consider the semisimple Lie algebra $sl(n, \mathbb{R})$ with Iwasawa decomposition

$$sl(n, \mathbb{R}) = so(n, \mathbb{R}) \oplus \mathfrak{a} \oplus \mathfrak{n}_P,$$

where $\mathfrak{a}$ and $\mathfrak{n}_P$ are as in Examples 2.1.16 and 2.1.20. Let $X \in so(n, \mathbb{R})$ be such that $[X, H] = 0$ for all $H \in \mathfrak{a}$. An easy computation shows that X is the zero matrix, and thus the minimal parabolic subalgebra $\underline{\mathfrak{p}} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}_P$ is given by upper triangular matrices with trace zero.

Definition 2.2.4. Any subalgebra of the Lie algebra $\mathfrak{g}$ which contains a minimal parabolic subalgebra is called a *parabolic subalgebra* of $\mathfrak{g}$.

Remark 2.2.5 (Alternative definition)**.** A parabolic subalgebra of the semisimple Lie algebra $\mathfrak{g}$ can also be defined as any subalgebra $\underline{\mathfrak{q}}$ whose complexification $\underline{\mathfrak{q}}_{\mathbb{C}}$ contains a Borel subalgebra of $\mathfrak{g}_{\mathbb{C}}$, the complexification of $\mathfrak{g}$.

We will present a method of obtaining all parabolic subalgebras $\underline{\mathfrak{q}}$ of $\mathfrak{g}$ which contain the minimal parabolic subalgebra $\underline{\mathfrak{p}}$.

Denote by $\Delta \subset \Sigma(P)$ the set of simple roots of the root system $\Sigma(\mathfrak{g}, \mathfrak{a})$. Fix Δ' a subset of Δ and define

$$\Pi = \Sigma(P) \cup (\mathrm{Span}(\Delta') \cap \Sigma(\mathfrak{g}, \mathfrak{a})).$$

Denote

$$\underline{\mathfrak{q}} := \mathfrak{m} \oplus \mathfrak{a} \oplus \bigoplus_{\alpha \in \Pi} \mathfrak{g}_\alpha. \tag{2.10}$$

Then $\underline{\mathfrak{q}}$ is a parabolic subalgebra of $\mathfrak{g}$ containing the minimal parabolic subalgebra $\underline{\mathfrak{p}}$. Moreover, any parabolic subalgebra of $\mathfrak{g}$ containing the minimal parabolic subalgebra $\underline{\mathfrak{p}}$ can be constructed in this manner (for a proof of this fact see [36, Lemma 7.74]).

Example 2.2.6. Let $\Sigma(sl(n, \mathbb{R}), \mathfrak{a}) = \{\alpha_{i,j} | \, 1 \leq i \neq j \leq n\}$ be the root system of $\mathfrak{a}$ in $sl(n, \mathbb{R})$ (given in Example 2.1.16) with a choice of positive roots $\Sigma(P)$ as in Example 2.1.20. Then, the set of simple roots is given by

$$\Delta = \{\alpha_{i,i+1} | \, 1 \leq i \leq n-1\}.$$

Every subset Δ' of Δ defines a parabolic subalgebra given by block-upper triangular matrices in $sl(n, \mathbb{R})$. For instance, let $\Delta' = \{\alpha_{1,2}, \alpha_{2,3}, \alpha_{n-1,n}\}$. Then

$$\mathrm{Span}(\Delta') \cap \Sigma(sl(n, \mathbb{R}), \mathfrak{a}) = \{\pm\alpha_{1,2}, \pm\alpha_{1,3}, \pm\alpha_{2,3}, \pm\alpha_{n-1,n}\}$$

and we obtain the parabolic subalgebra of $sl(n, \mathbb{R})$ consisting of traceless matrices with 0's under the main diagonal, except for the positions $(2, 1)$, $(3, 1)$, $(3, 2)$ and $(n, n-1)$.

We obtain in this manner, one subalgebra for every arrangement of blocks.

By using the definition of the parabolic subalgebra $\underline{\mathfrak{q}}$ given in (2.10), we can obtain another decomposition of $\underline{\mathfrak{q}}$. Namely, we rewrite the decomposition (2.10) as

$$\underline{\mathfrak{q}} = \mathfrak{m} \oplus \mathfrak{a} \oplus \bigoplus_{\alpha \in \mathrm{Span}(\Delta') \cap \Pi} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha \in \Pi \setminus \mathrm{Span}(\Delta')} \mathfrak{g}_\alpha$$

and denote

$$\mathfrak{n}_{\underline{\mathfrak{q}}} := \sum_{\alpha \in \Pi \setminus \mathrm{Span}(\Delta')} \mathfrak{g}_\alpha, \quad \mathfrak{l}_{\underline{\mathfrak{q}}} := \mathfrak{m} \oplus \mathfrak{a} \oplus \bigoplus_{\alpha \in \mathrm{Span}(\Delta') \cap \Pi} \mathfrak{g}_\alpha.$$

Observe that $\mathfrak{l}_{\underline{\mathfrak{q}}} = \underline{\mathfrak{q}} \cap \theta\underline{\mathfrak{q}}$. The parabolic subalgebra $\underline{\mathfrak{q}}$ is now the vector space direct sum

$$\underline{\mathfrak{q}} = \mathfrak{l}_{\underline{\mathfrak{q}}} \oplus \mathfrak{n}_{\underline{\mathfrak{q}}}.$$

The subalgebra $\mathfrak{l}_{\underline{\mathfrak{q}}}$ is a θ-stable reductive subalgebra of $\mathfrak{g}$. We write

$$\mathfrak{l}_{\underline{\mathfrak{q}}} = [\mathfrak{l}_{\underline{\mathfrak{q}}}, \mathfrak{l}_{\underline{\mathfrak{q}}}] \oplus Z(\mathfrak{l}_{\underline{\mathfrak{q}}}),$$

where $Z(\mathfrak{l}_{\underline{\mathfrak{q}}})$ denotes the center of $\mathfrak{l}_{\underline{\mathfrak{q}}}$. The subspaces $[\mathfrak{l}_{\underline{\mathfrak{q}}}, \mathfrak{l}_{\underline{\mathfrak{q}}}]$ and $Z(\mathfrak{l}_{\underline{\mathfrak{q}}})$ are θ-stable as well and we write

$$\mathfrak{l}_{\underline{\mathfrak{q}}} = (Z(\mathfrak{l}_{\underline{\mathfrak{q}}}) \cap \mathfrak{k}) \oplus (Z(\mathfrak{l}_{\underline{\mathfrak{q}}}) \cap \mathfrak{p}) \oplus ([\mathfrak{l}_{\underline{\mathfrak{q}}}, \mathfrak{l}_{\underline{\mathfrak{q}}}] \cap \mathfrak{k}) \oplus ([\mathfrak{l}_{\underline{\mathfrak{q}}}, \mathfrak{l}_{\underline{\mathfrak{q}}}] \cap \mathfrak{p}).$$

Denote $\mathfrak{a}_{\underline{\mathfrak{q}}} := Z(\mathfrak{l}_{\underline{\mathfrak{q}}}) \cap \mathfrak{p}$ and $\mathfrak{m}_{\underline{\mathfrak{q}}} := (\mathfrak{l}_{\underline{\mathfrak{q}}} \cap \mathfrak{k}) \oplus ([\mathfrak{l}_{\underline{\mathfrak{q}}}, \mathfrak{l}_{\underline{\mathfrak{q}}}] \cap \mathfrak{p})$. We obtain in this manner the decomposition of $\mathfrak{l}_{\underline{\mathfrak{q}}}$ as the direct sum of Lie algebras

$$\mathfrak{l}_{\underline{\mathfrak{q}}} = \mathfrak{m}_{\underline{\mathfrak{q}}} \oplus \mathfrak{a}_{\underline{\mathfrak{q}}}.$$

We conclude that the parabolic subalgebra $\underline{\mathfrak{q}}$ decomposes as

$$\underline{\mathfrak{q}} = \mathfrak{m}_{\underline{\mathfrak{q}}} \oplus \mathfrak{a}_{\underline{\mathfrak{q}}} \oplus \mathfrak{n}_{\underline{\mathfrak{q}}}. \tag{2.11}$$

Definition 2.2.7. The decomposition given in (2.11) is called the *Langlands decomposition* of the parabolic subalgebra $\underline{\mathfrak{q}}$.

Remark 2.2.8. Observe that $\mathfrak{a}_{\underline{\mathfrak{q}}} = \cap_{\alpha \in \Delta'} \ker \alpha$. Moreover, if $\underline{\mathfrak{q}}$ is a minimal parabolic subalgebra of $\mathfrak{g}$, then the two decompositions (2.10) and (2.11) coincide.

Let $M = Z_K(\mathfrak{a})$. Then $MA = Z_G(\mathfrak{a})$ and it is θ-stable (since $\mathfrak{a}$ is θ-stable). Let $x \in MA$, $Y \in \mathfrak{g}_\alpha$, for some positive root α, and $H \in \mathfrak{a}$. Then

$$[H, \mathrm{Ad}(x)Y] = [\mathrm{Ad}(x)H, \mathrm{Ad}(x)Y] = \mathrm{Ad}(x)[H, Y] = \alpha(H)\mathrm{Ad}(x)Y$$

and we see that MA normalizes $\mathfrak{n}_P$. We conclude that MA normalizes N_P and thus

$$P := MAN_P$$

is a subgroup of G. Moreover, the following theorem shows that P is a closed subgroup of G with Lie algebra $\underline{\mathfrak{p}}$.

Theorem 2.2.9. *$P = N_G(\underline{\mathfrak{p}})$ and hence P is a closed subgroup. The subgroup P has Lie algebra $\underline{\mathfrak{p}} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}_P$.*

Proof. For a proof the reader is advised to check [36, Proposition 7.83]. □

By using the Iwasawa decomposition of the group G, it can be shown that the multiplication map $M \times A \times N_P \to P$ is a diffeomorphism. It also follows that $K/M \simeq G/P$.

Definition 2.2.10. Any subgroup of G which is conjugate to P is called a *minimal parabolic subgroup* of G.

Example 2.2.11. Consider the Lie group $SL(n, \mathbb{R})$ with Iwasawa decomposition

$$SL(n, \mathbb{R}) = SO(n, \mathbb{R})AN_P,$$

where A is given by diagonal matrices with positive entries and determinant 1 and N_P is given by the set of upper triangular matrices with 1's on the diagonal, as was shown in Example 2.1.23. A simple computation shows that $M = Z_{SO(n,\mathbb{R})}(\mathfrak{a})$ is given by diagonal matrices with ± 1 on the diagonal and determinant 1. Thus, the minimal parabolic subalgebra P is the set of all upper triangular matrices with determinant 1.

Denote by $\mathcal{P}(A)$ the set of all minimal parabolic subgroups of G containing A and let $Q \in \mathcal{P}(A)$. Then $Q = MAN_Q$, where $N_Q = \exp \mathfrak{n}_Q$ and $\mathfrak{n}_Q$ is the direct sum of the root spaces corresponding to some uniquely determined positive system $\Sigma(Q) \subset \Sigma(\mathfrak{g}, \mathfrak{a})$. The assignment $Q \mapsto \Sigma(Q)$ gives a bijection from $\mathcal{P}(A)$ to the set of positive systems in $\Sigma(\mathfrak{g}, \mathfrak{a})$. As promised, this explains the notation $\Sigma(P)$ and $\mathfrak{n}_P$.

Definition 2.2.12. Let Q be a subgroup of G containing a minimal parabolic subgroup. Then Q is called a *parabolic subgroup* of G.

Let Q be a parabolic subgroup of G with Lie algebra $\underline{\mathfrak{q}}$. Define

$$L_Q = Q \cap \theta Q, \quad \mathfrak{a}_Q = Z(\mathrm{Lie}(L_Q)) \cap \mathfrak{p},$$

where $\mathrm{Lie}(L_Q)$ denotes the Lie algebra of the subgroup L_Q and $Z(\mathrm{Lie}(L_Q))$ denotes the center of $\mathrm{Lie}(L_Q)$. Put $A_Q = \exp \mathfrak{a}_Q$ and $M_Q = Z_K(\mathfrak{a}_Q) \exp(\mathfrak{p} \cap [Z_{\mathfrak{g}}(\mathfrak{a}_Q), Z_{\mathfrak{g}}(\mathfrak{a}_Q)])$. Furthermore, let $\mathfrak{n}_Q$ be the nilpotent radical of the Lie algebra $\mathrm{Lie}(Q)$ and $N_Q = \exp \mathfrak{n}_Q$. Then N_Q is a closed subgroup of G, called the *unipotent radical* of the parabolic subgroup Q.

Theorem 2.2.13 ([55, Theorem II.6.3.10])**.** *The multiplication maps $M_Q \times A_Q \to L_Q$ and $L_Q \times N_Q \to Q$ are diffeomorphisms.*

Definition 2.2.14. The subgroup $L_Q = M_Q A_Q$ is called the *θ-stable Levi component* of the parabolic subgroup Q and A_Q is called the *split component*. We refer to the decomposition

$$Q = M_Q A_Q N_Q$$

as the *Langlands decomposition* of the parabolic subgroup Q.

Remark 2.2.15. The subgroup A_Q is sometimes called *the unique θ-stable split component* of the parabolic subgroup Q.

Let Q be some parabolic subgroup of G. It can be shown that $Q \in \mathcal{P}(A)$ if and only if its split component is given by A. The dimension of A is called the *real rank* of the Lie group G. If Q is a minimal parabolic subgroup of G with split component A, then $M_Q = M$ and $L_Q = MA$.

The following proposition and its proof can be found in [55, Proposition II.6.4.19].

Proposition 2.2.16. *Let $P = M_P A_P N_P$ and $Q = M_Q A_Q N_Q$ be two parabolic subgroups of G. Suppose that $P \subset Q$. Then*

$$M_P \subset M_Q, \quad A_P \supset A_Q, \quad N_P \supset N_Q.$$

We will frequently use the notation $\bar{Q}$ for $\theta(Q)$ and $\bar{N}_Q$ for $\theta(N_Q)$.

2.3 Semisimple symmetric spaces

Let G be a Lie group and $\sigma : G \to G$ and involution on G. Let H be an open subgroup of the fixed point set G^σ, such that $G^{\sigma\circ} \subset H \subset G^\sigma$. Here $G^{\sigma\circ}$ denotes the identity component of G^σ.

Definition 2.3.1. The pair (G, H) is called a *symmetric pair* and the homogeneous space G/H is called a *symmetric space*. If the Lie group G is semisimple then the homogeneous space G/H is called a *semisimple symmetric space*.

Remark 2.3.2. From now on we assume that the real Lie group G is connected, semisimple and has finite center.

Remark 2.3.3. A symmetric space can also be defined via Riemannian geometry, as opposed to the Lie theoretic definition given above. In Riemannian geometry a *Riemannian symmetric space* is a Riemannian manifold such that at every point the geodesic symmetry about that point is an isometry.

The example below shows how one can obtain a Riemannian symmetric space (in the sense of Remark 2.3.3) as a special case of Definition 2.3.1. Remark 2.3.3 above also explains the name 'symmetric space'.

Example 2.3.4. Assume the involution σ is a Cartan involution on the Lie group G. Then the fixed point set G^σ is the maximal compact subgroup K and the symmetric space G/K is a Riemannian symmetric space (as defined in Remark 2.3.3). The Riemannian metric is G-invariant and given by the $\mathrm{Ad}(K)$-invariant inner product $\langle \cdot, \cdot \rangle$ on $T_e(G/K) \simeq \mathfrak{g}/\mathfrak{k} \simeq \mathfrak{p}$. Let $p = hK$ be an arbitrary point in G/K. Define

$$s_p : G/K \to G/K, \quad s_p(gK) = h\sigma(h^{-1}g)K.$$

Then $s_p(p) = p$ and one can check that s_p is the geodesic reflection in p and it is in fact an isometry.

Let $\mathfrak{g}$ be the Lie algebra of G. We denote the infinitesimal correspondent of σ again by $\sigma : \mathfrak{g} \to \mathfrak{g}$. Then $\mathfrak{g}$ has the eigenspace decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q},$$

where $\mathfrak{h}$ denotes the +1 eigenspace and $\mathfrak{q}$ denotes the -1 eigenspace of the involution σ. Observe that $\mathfrak{h}$ is the Lie algebra of the subgroup H.

Let θ be a Cartan involution on G (and its infinitesimal counterpart, denoted the same) which commutes with σ (the existence of which follows from [36, Theorem 6.16]). Then $\sigma\theta := \sigma \circ \theta$ is an involution on $\mathfrak{g}$ and $\mathfrak{g}$ has a simultaneous eigenspace decomposition with respect to the involutions σ and θ given by

$$\mathfrak{g} = (\mathfrak{k} \cap \mathfrak{h}) \oplus (\mathfrak{k} \cap \mathfrak{q}) \oplus (\mathfrak{p} \cap \mathfrak{h}) \oplus (\mathfrak{p} \cap \mathfrak{q}).$$

Here $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the Cartan decomposition of the Lie algebra $\mathfrak{g}$ determined by θ.

Denote by $\mathfrak{a}_{\mathrm{q}}$ a maximal abelian subspace of $\mathfrak{p} \cap \mathfrak{q}$.

Definition 2.3.5. The subgroup H is called *essentially connected* if the condition

$$H = Z_{K \cap H}(\mathfrak{a}_{\mathrm{q}}) H^{\circ}$$

is satisfied, where H° denotes the identity component of H.

Remark 2.3.6. Since all Cartan involutions commuting with σ are conjugate via elements of $\mathrm{Ad}(H)$ and all maximal abelian subspaces of $\mathfrak{p} \cap \mathfrak{q}$ are conjugate via $K \cap H$ ([53, Corollary I.9]), it follows that H being essentially connected is independent of the choice of $\mathfrak{a}_{\mathrm{q}}$.

Example 2.3.7 (The group case)**.** Let G be a real connected semisimple Lie group with finite center, θ be a Cartan involution on G and P a minimal parabolic subgroup of G belonging to $\mathcal{P}(A)$. The Iwasawa decomposition of G with respect to θ and P is given by

$$G = KAN_P.$$

Define

$$G' = G \times G, \quad \theta' = \theta \times \theta, \quad K' = K \times K, \quad A' = A \times A \text{ and } N' = N_P \times N_P.$$

Then G' is a real semisimple connected Lie group with finite center and its fixed point set under the involution

$$\sigma : G' \to G', \quad \sigma(x, y) = (y, x)$$

is given by $H = \mathrm{diag}(G')$. Hence $(G \times G)/\mathrm{diag}(G \times G)$ is a semisimple symmetric space. Moreover, $\mathfrak{a}' = \mathfrak{a} \times \mathfrak{a}$ decomposes as $\mathfrak{a}' = \mathfrak{a}'_{\mathrm{h}} \oplus \mathfrak{a}'_{\mathrm{q}}$ where $\mathfrak{a}'_{\mathrm{h}}$ is given by $\{(X, X) \,|\, X \in \mathfrak{a}\}$ and $\mathfrak{a}'_{\mathrm{q}}$ by $\{(X, -X) \,|\, X \in \mathfrak{a}\}$. According to Proposition 7.33 in [36], $Z_K(\mathfrak{a})$ meets every connected component of G and hence, $Z_{\mathrm{diag}(K \times K)}(\mathfrak{a}'_{\mathrm{q}})$ meets every connected component of $\mathrm{diag}(G \times G)$. It follows that H is essentially connected.

The map from $G'/\mathrm{diag}(G')$ to G, given by

$$(g_1, g_2)H \mapsto g_1 g_2^{-1},$$

is a diffeomorphism. Hence, every semisimple Lie group G can viewed in this way as a semisimple symmetric space. We will refer to this example as *the group case*.

2.4 Groups of the Harish-Chandra class $\mathcal{H}$

In the next chapters, 3 and 4, we will state and prove a convexity theorem for semisimple symmetric spaces. This theorem generalizes the non-linear convexity theorem of Kostant to this type of symmetric spaces. The proof of the theorem will at some point use induction on the real rank of the Lie group G. More specifically, we will look at centralizers $G_X := Z_G(X)$ for certain elements X in $\mathfrak{a}$ and apply the induction hypothesis to these subgroups. For such ideas to work one needs that the subgroup G_X has the same properties as the Lie group G. For real connected semisimple Lie groups with finite center, the main properties of the group are not necessarily inherited by this type of subgroups, as the next example clearly illustrates.

Example 2.4.1. Let G be the Lie group $SL(3,\mathbb{R})$ with Iwasawa decomposition as in Example 2.1.23. Take $X_1 \in \mathfrak{a}$ to be the diagonal matrix $(1,1,-2)$. Then $SL(3,\mathbb{R})_{X_1}$ is isomorphic to the Lie group $GL(2,\mathbb{R})$, which is neither connected nor semisimple.

As a second example take $X_2 \in \mathfrak{a}$ to be the diagonal matrix $(1,-1,0)$. Then $SL(3,\mathbb{R})_{X_2}$ is isomorphic to the abelian group $(\mathbb{R}^*,\cdot) \times (\mathbb{R}^*,\cdot)$ which has four connected components; it as well is neither connected nor semisimple.

Hence, we need to consider a larger class of Lie groups: reductive Lie groups of the Harish-Chandra class. We will denote this class of groups by $\mathcal{H}$.

The theory presented in Sections 2.1, 2.2 and 2.3 above can be extended to the Harish-Chandra class. We will not repeat this theory here, instead we will give the definition of this type of groups and state a few of their many interesting properties.

Definition 2.4.2. A real Lie group G is said to be a *reductive Lie group of the Harish-Chandra class* (we write $G \in \mathcal{H}$) if

i) its Lie algebra $\mathfrak{g}$ is reductive,

ii) the index of the identity component G° of G in G is finite,

iii) $\mathrm{A}d(G)$ is contained in the connected complex adjoint group $\mathrm{Aut}(\mathfrak{g}_\mathbb{C})^\circ$ of $\mathfrak{g}_\mathbb{C}$,

iv) the commutator subgroup of G has finite center; $|Z([G^\circ, G^\circ])| < \infty$.

Example 2.4.3. Let G be a real connected semisimple Lie group with finite center. Then $G \in \mathcal{H}$.

Example 2.4.4. If P is a parabolic subgroup of the Lie group G, then its θ-stable Levi component L_P is a reductive Lie group of the Harish-Chandra class, according to [55, Theorem II.6.3.13].

The next proposition and its proof can be found in [55, Proposition II.1.1].

Proposition 2.4.5.

(a) *Let $G_i \in \mathcal{H}$, $1 \le i \le n$. Then $G_1 \times \ldots \times G_n \in \mathcal{H}$.*

(b) *Let $G \in \mathcal{H}$ and let H be a subgroup of G, s.t. $G^\circ \subset H \subset G$. Then $H \in \mathcal{H}$.*

(c) *Let $G \in \mathcal{H}$. Then $\mathrm{Ad}(G) \in \mathcal{H}$.*

The next proposition contains the main reason why we choose to work with this class of groups. Namely, by passing to centralizers of suitable elements we obtain subgroups, which are in their own right groups of the Harish-Chandra class. This will allow us to use a certain induction on the dimension or the rank of the group.

Proposition 2.4.6 ([36, Proposition 7.25])**.** *Let G be a Lie group belonging to $\mathcal{H}$ and θ a Cartan involution on G. Denote by $K := G^\theta$ the maximal compact subgroup of G and by A the unique θ-stable split component. Let $X \in \mathfrak{a}$, the Lie algebra of A. Then G_X, the centralizer of X in G, is a reductive Lie group of the Harish-Chandra class, with maximal compact subgroup K_X and associated Cartan involution given by the restriction of θ.*

The symmetric space G/H is called a *reductive symmetric space of the Harish-Chandra class* if $G \in \mathcal{H}$ and H is essentially connected.

Example 2.4.7. Let $G' = G \times G$, where G is a reductive group of the Harish-Chandra class. Then the symmetric space $G \times G/\mathrm{diag}(G \times G)$, constructed in Example 2.3.7, is a reductive symmetric space.

Chapter 3

Convexity theorems for semisimple symmetric spaces

In this chapter we generalize the non-linear convexity result of Kostant to the setting of semisimple symmetric spaces (or more generally, reductive symmetric spaces). Let $G = KAN_P$ (where $P \in \mathcal{P}(A)$ is a minimal parabolic subgroup of G) be a reductive Lie group of the Harish-Chandra class and H an essentially connected open subgroup of the fixed point set G^σ of the involution $\sigma : G \to G$ (σ and the Cartan involution θ, determined by K, commute). Fix a in A. It is natural to consider the question whether the image $\mathfrak{H}_P(aH)$ is a convex subset of $\mathfrak{a}$. This question was first answered in [5] for a particular kind of minimal parabolic subgroups of G. The convexity result in [5] represents a special case of our convexity theorem.

In Section 3.1 below, we give a precise formulation of our result. We continue in Section 3.2 with a detailed exposition of the group case. We start this exposition in Subsection 3.2.1 with an example of our convexity result for the case $SL_3(\mathbb{R}) \times SL_3(\mathbb{R})/\mathrm{diag}(SL_3(\mathbb{R}) \times SL_3(\mathbb{R}))$. In Subsection 3.2.2 we present an independent proof for the case of the group. We end the chapter with a consequence of the convexity theorem for the case of the group. Namely, we present an easy proof for a well-known result about the image $\mathfrak{H}_P(N_Q)$, where Q is a minimal parabolic subgroup of G contained in $\mathcal{P}(A)$, see Subsection 3.2.3.

3.1 A precise formulation of the result

In this section we give the precise statement of the convexity result. We start by briefly recalling the notation established above.

G is a reductive Lie group of the Harish-Chandra class, σ and involution on it, and H an open subgroup of the fixed point set G^σ. Let $\theta : G \to G$ be a Cartan involution on G that commutes with σ. As before, K is the maximal compact subgroup of G associated to θ ($K = G^\theta$) and $\mathfrak{g}$ is the Lie algebra of G. With respect to the two corresponding involutions on $\mathfrak{g}$ (denoted σ and θ as well), $\mathfrak{g}$ decomposes into eigenspaces as

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} = \mathfrak{h} \oplus \mathfrak{q}.$$

Note that $\mathfrak{h}$ is the Lie algebra of H and observe that since σ and θ commute, we have that $\mathfrak{p} = \mathfrak{p} \cap \mathfrak{h} \oplus \mathfrak{p} \cap \mathfrak{q}$.

Fix a maximal abelian subspace $\mathfrak{a}_{\mathrm{q}}$ of $\mathfrak{p} \cap \mathfrak{q}$, and $\mathfrak{a}$ a maximal abelian subspace of $\mathfrak{p}$ that contains $\mathfrak{a}_{\mathrm{q}}$. Then $\mathfrak{a}$ is σ-stable and decomposes as

$$\mathfrak{a} = \mathfrak{a}_{\mathrm{h}} \oplus \mathfrak{a}_{\mathrm{q}}, \tag{3.1}$$

where $\mathfrak{a}_{\mathrm{h}}$ denotes the subspace $\mathfrak{a} \cap \mathfrak{h}$. Let $\Sigma(\mathfrak{g}, \mathfrak{a})$ be the set of roots of $\mathfrak{a}$ in $\mathfrak{g}$ and $\Sigma(\mathfrak{g}, \mathfrak{a}_{\mathrm{q}})$ the set of roots of $\mathfrak{a}_{\mathrm{q}}$ in $\mathfrak{g}$. The latter set is a (possibly non-reduced) root system as well, see e.g. [53, Proposition 10]. Its Weyl group is given by

$$W(\mathfrak{a}_{\mathrm{q}}) \simeq N_K(\mathfrak{a}_{\mathrm{q}})/Z_K(\mathfrak{a}_{\mathrm{q}}). \tag{3.2}$$

Let $\Sigma(P)$ be a positive system for $\Sigma(\mathfrak{g}, \mathfrak{a})$ and define

$$\Sigma(P, \sigma\theta) := \{\alpha \in \Sigma(P) : \sigma\theta\alpha \in \Sigma(P)\}$$

and

$$\Sigma(P)_- := \{\alpha \in \Sigma(P, \sigma\theta) \mid \sigma\theta\alpha = \alpha \Rightarrow \sigma\theta|_{\mathfrak{g}_\alpha} \neq \mathrm{id}_{\mathfrak{g}_\alpha}\}. \tag{3.3}$$

We use the notation $\mathrm{pr}_{\mathrm{q}} : \mathfrak{a} \to \mathfrak{a}_{\mathrm{q}}$ for the projection of $\mathfrak{a}$ onto $\mathfrak{a}_{\mathrm{q}}$ along $\mathfrak{a}_{\mathrm{h}}$, see (3.1).

As mentioned before, it is natural to study the more general question of convexity of the set $\mathfrak{H}_P(aH)$, $a \in A$, i.e. if this image is a convex subset of $\mathfrak{a}$. Before we can state the theorem, we need to introduce some more notation and make a few remarks.

Remark 3.1.1. Note that $A = A_{\mathrm{q}} \times A_{\mathrm{h}}$ where $A_{\mathrm{h}} = A \cap H$. Thus, we just need to consider $a \in A_{\mathrm{q}}$.

Remark 3.1.2. Since $\mathfrak{H}_P(aH) = \mathrm{pr}_{\mathrm{q}} \circ \mathfrak{H}_P(aH) + \mathfrak{a}_{\mathrm{h}}$, it suffices to consider the image of aH, $a \in A_{\mathrm{q}}$, under the map

$$\mathfrak{H}_{P,\mathrm{q}} := \mathrm{pr}_{\mathrm{q}} \circ \mathfrak{H}_P : G \to \mathfrak{a}_{\mathrm{q}}.$$

Let B denote an extension of the Killing form on $[\mathfrak{g}, \mathfrak{g}]$ (the semisimple part of the reductive Lie algebra $\mathfrak{g}$) to the entire algebra $\mathfrak{g}$, such that B is an $\mathrm{Ad}(G)$-invariant non-degenerate symmetric bilinear form on $\mathfrak{g}$ which is invariant under both θ and σ, and such that B is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$.

We define a positive definite inner product on $\mathfrak{g}$ by

$$\langle U, V\rangle := -B(U, \theta V) \quad (U, V \in \mathfrak{g}).$$

Note that the root space decomposition and the eigenspace decompositions (with respect to θ and σ) are orthogonal with respect to this inner product. Moreover, the extended Killing form and the inner product coincide if either U or V belongs to $\mathfrak{p}$.

Definition 3.1.3. The Weyl group $W_{K\cap H}$ is defined as

$$W_{K\cap H} := N_{K\cap H}(\mathfrak{a}_{\mathrm{q}})/Z_{K\cap H}(\mathfrak{a}_{\mathrm{q}}).$$

Let α be a root in $\Sigma(\mathfrak{g}, \mathfrak{a})$. We denote by H_α the element in $\mathfrak{a}$ that satisfies the conditions: $\alpha(H_\alpha) = 2$ and $H_\alpha \perp \ker\alpha$ with respect to $\langle\cdot,\cdot\rangle$.

Definition 3.1.4. Let P be a minimal parabolic subgroup of G containing A. Then we define the finitely generated polyhedral cone $\Gamma(P)$ in $\mathfrak{a}_{\mathrm{q}}$ by

$$\Gamma(P) = \sum_{\alpha\in\Sigma(P)_-} \mathbb{R}_{\geq 0}\mathrm{pr}_{\mathrm{q}}(H_\alpha).$$

Main Theorem (Theorem 4.10.1) *Let G be a reductive Lie group of the Harish-Chandra class, σ an involution on G and H an essentially connected open subgroup of G^σ. Let P be any minimal parabolic subgroup containing A and $a \in A_{\mathrm{q}}$. Then*

$$\mathfrak{H}_{P,\mathrm{q}}(aH) = \mathrm{conv}(W_{K\cap H}\cdot \log a) + \Gamma(P),$$

where 'conv' denotes the convex hull.

If the two involutions σ and θ are equal, then $K = H$ and $\Sigma(P, \sigma\theta) = \Sigma(P)$. This implies that $W(\mathfrak{a}) = W_{K\cap H}$ and that $\Sigma(P)_- = \emptyset$. Thus, we obtain that $\Gamma(P) = 0$ and hence, in this case our main theorem coincides with the original non-linear convexity theorem of Kostant [21]. For P satisfying $\Sigma(P, \sigma\theta) = \Sigma(P) \setminus \mathfrak{a}_{\mathrm{h}}^*$ the above result coincides with [5, Theorem 1.1]. This will be explained in detail in Section 4.2.2

3.2 The group case

For this section our main goal is to present an example of the convexity theorem stated above, Theorem 4.10.1, for the case of the group (see Example 2.3.7) and to present an independent proof of the convexity theorem for the case of the group.

The first part of this section should be regarded as a continuation of Example 2.3.7 presented above.

Let G', θ', K', N'_P, P, σ and H be as in Example 2.3.7. For the maximal abelian subspace $\mathfrak{a}'_{\mathrm{q}}$ of $\mathfrak{p}' \cap \mathfrak{q}'$ we may take

$$\mathfrak{a}'_{\mathrm{q}} = \{(X, -X) \mid X \in \mathfrak{a}\},$$

while $\mathfrak{a}'_{\mathrm{h}} = \mathfrak{a}' \cap \mathfrak{h}$ is given by $\mathfrak{a}'_{\mathrm{h}} = \mathrm{diag}(\mathfrak{a} \times \mathfrak{a})$.

The root system of $\mathfrak{a}'$ in $\mathfrak{g}'$ is given by

$$\Sigma(\mathfrak{g}', \mathfrak{a}') = \Sigma(\mathfrak{g}, \mathfrak{a}) \times \{0\} \cup \{0\} \times \Sigma(\mathfrak{g}, \mathfrak{a}).$$

Let Q be a minimal parabolic subgroups of G containing A, i.e. $Q \in \mathcal{P}(A)$. Then $P \times Q$ is a minimal parabolic subgroup of G' containing A'. Moreover any minimal parabolic subgroup of G' containing A' is of this form. The positive system associated to $P \times Q$ is given by

$$\Sigma(P \times Q) = \Sigma(P) \times \{0\} \cup \{0\} \times \Sigma(Q),$$

where $\Sigma(P)$ and $\Sigma(Q)$ are positive systems for $\Sigma(\mathfrak{g}, \mathfrak{a})$ corresponding to the minimal parabolic subgroups P and Q. The corresponding negative systems we denote by $\Sigma(\bar{P})$ and $\Sigma(\bar{Q})$.

For $b = (a, a^{-1})$ an element of A'_{q}, Theorem 4.10.1 tells us that

$$\mathrm{pr}_{\mathrm{q}} \circ \mathfrak{H}_{P\times Q}(bH) = \mathrm{conv}(W_{K'\cap H} \cdot \log b) + \Gamma(P \times Q).$$

In order to understand the cone $\Gamma(P \times Q)$, we have to determine those roots $\gamma \in \Sigma(P \times Q)$ for which $\sigma\theta'\gamma \in \Sigma(P \times Q)$. Let $\gamma = (\alpha, 0)$ be such a root. Then $\alpha \in \Sigma(P)$ and $\sigma\theta'\gamma = (0, -\alpha)$ must be an element of $\{0\} \times \Sigma(Q)$. It follows that

$$\Sigma(P \times Q, \sigma\theta') = (\Sigma(P) \cap \Sigma(\bar{Q})) \times \{0\} \cup \{0\} \times (\Sigma(\bar{P}) \cap \Sigma(Q)).$$

Notice that there are no roots $\gamma \in \Sigma(P \times Q)$ for which $\sigma\theta'\gamma = \gamma$.

Thus,

$$\Gamma(P \times Q) = \sum_{\gamma \in \Sigma(P\times Q, \sigma\theta')} \mathbb{R}_{\geq 0} \mathrm{pr}_{\mathrm{q}} H'_\gamma, \tag{3.4}$$

where $H'_\gamma = (H_\alpha, 0)$, if $\gamma = (\alpha, 0)$, and $H'_\gamma = (0, H_\alpha)$, for $\gamma = (0, \alpha)$. The map $\mathrm{pr}_{\mathrm{q}} : \mathfrak{a}' \to \mathfrak{a}'_{\mathrm{q}}$ is given by

$$\mathrm{pr}_{\mathrm{q}}(U, V) = (\frac{U - V}{2}, \frac{V - U}{2}) \tag{3.5}$$

and we can show that the following Lemma holds.

Lemma 3.2.1. *The following equality holds.*

$$\Gamma(P \times Q) = \Gamma(\{(Y, -Y) \,|\, Y \in \Gamma_{\mathfrak{a}}(\Sigma(P) \cap \Sigma(\bar{Q}))\})$$

Proof. By the definition of the map $\mathrm{pr}_{\mathrm{q}} : \mathfrak{a}' \to \mathfrak{a}'_{\mathrm{q}}$ we have that

$$\begin{aligned}
&\sum_{\gamma\in\Sigma(P\times Q,\sigma\theta')} \mathbb{R}_{\geq 0}\mathrm{pr}_{\mathrm{q}} H'_\gamma = \\
&= \sum_{\alpha\in\Sigma(P)\cap\Sigma(\bar{Q})} \mathbb{R}_{\geq 0}(\frac{H_\alpha}{2}, \frac{-H_\alpha}{2}) + \sum_{\alpha\in\Sigma(\bar{P})\cap\Sigma(Q)} \mathbb{R}_{\geq 0}(\frac{-H_\alpha}{2}, \frac{H_\alpha}{2}) = \\
&= \sum_{\alpha\in\Sigma(P)\cap\Sigma(\bar{Q})} \mathbb{R}_{\geq 0}(\frac{H_\alpha}{2}, \frac{-H_\alpha}{2}) + \sum_{\alpha\in\Sigma(\bar{P})\cap\Sigma(Q)} \mathbb{R}_{\geq 0}(\frac{H_{-\alpha}}{2}, \frac{-H_{-\alpha}}{2}) = \\
&= \sum_{\alpha\in\Sigma(P)\cap\Sigma(\bar{Q})} \mathbb{R}_{\geq 0}(\frac{H_\alpha}{2}, \frac{-H_\alpha}{2}) + \sum_{-\alpha\in\Sigma(P)\cap\Sigma(\bar{Q})} \mathbb{R}_{\geq 0}(\frac{H_{-\alpha}}{2}, \frac{-H_{-\alpha}}{2}) = \\
&= \sum_{\alpha\in\Sigma(P)\cap\Sigma(\bar{Q})} \mathbb{R}_{\geq 0}(H_\alpha, -H_\alpha),
\end{aligned}$$

which is precisely the cone $\Gamma(\{(Y,-Y) \,|\, Y \in \Gamma_{\mathfrak{a}}(\Sigma(P)\cap\Sigma(\bar{Q}))\})$. □

Hence, by identifying $\mathfrak{a}'_{\mathrm{q}} \simeq \mathfrak{a}$ via the map $(X,-X) \mapsto X$, we obtain (for $b = (a, a^{-1}) \in A'_{\mathrm{q}}$)

$$\mathrm{pr}_{\mathrm{q}} \circ \mathfrak{H}_{P\times Q}(bH) = \mathrm{conv}(W_K \cdot \log a) + \sum_{\alpha\in\Sigma(P)\cap\Sigma(\bar{Q})} \mathbb{R}_{\geq 0} H_\alpha. \tag{3.6}$$

3.2.1 A particular case of the group case

We specialize to the case where $G' = SL_3(\mathbb{R}) \times SL_3(\mathbb{R})$ and the infinitesimal involution $\theta : sl_3(\mathbb{R}) \to sl_3(\mathbb{R})$ is given by $\theta(X) = -X^T$. We will consider different minimal parabolic subgroups $P \times Q$ of G' containing A' and investigate what our convexity theorem gives in each case.

We may identify $\mathfrak{a} \simeq \mathfrak{a}^*$ via the inner product $\langle\cdot,\cdot\rangle$. With this identification Equation (3.6) can be rewritten as

$$\mathrm{pr}_{\mathrm{q}} \circ \mathfrak{H}_{P\times Q}(bH) = \mathrm{conv}(W_K \cdot \log a) + \sum_{\alpha\in\Sigma(P)\cap\Sigma(\bar{Q})} \mathbb{R}_{\geq 0}\alpha.$$

Here $\Sigma(P)$ and $\Sigma(Q)$ are positive systems of the root system $\Sigma(sl_3(\mathbb{R}), \mathfrak{a})$, as depicted in Figure 2.1.26. Fix $\Sigma(P) = \{\alpha, \beta, \alpha+\beta\}$.

First we consider the case $\Sigma(Q) = \Sigma(P)$. Then $P = Q$ and $\Sigma(P)\cap\Sigma(\bar{Q}) = \emptyset$. Hence, there is no cone. We obtain that the image $\mathrm{pr}_{\mathrm{q}} \circ \mathfrak{H}_{P\times P}(bH)$ is the convex set $\mathrm{conv}(W_K \cdot \log a)$.

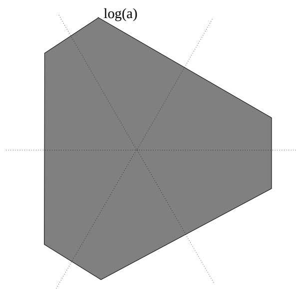


Figure 3.1: The convex hull of $W_K \cdot \log a$

Since every semisimple Lie group G, can be realized as the semisimple symmetric space $G \times G/\mathrm{diag}(G)$, we recover in this fashion the convexity theorem of Kostant.

Let $\Sigma(Q) = \{\alpha + \beta, \beta, -\alpha\}$. Then $\Sigma(P) \cap \Sigma(\bar{Q}) = \{\alpha\}$. Thus, we obtain in this case $\mathrm{conv}(W_K . \log(a)) + \mathbb{R}_{\geq 0}\alpha$, as can be seen in the figure below.

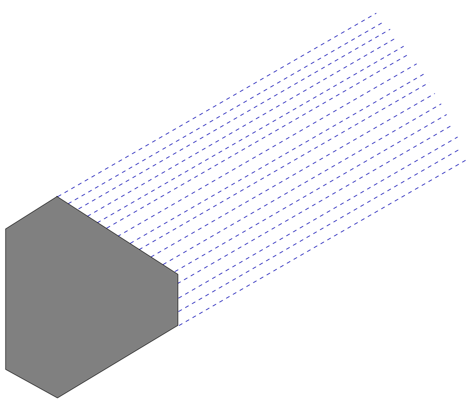


Figure 3.2: The convex set $\mathrm{conv}(W_K \cdot \log a) + \mathbb{R}_{\geq 0}\alpha$

If $\Sigma(Q) = \{\beta, -\alpha, -(\alpha+\beta)\}$, then $\Sigma(P) \cap \Sigma(\bar{Q}) = \{\alpha, \alpha+\beta\}$. Thus, the convex cone will be generated by these roots and we obtain the result $\mathrm{conv}(W_K . \log(a)) + \mathbb{R}_{\geq 0}\alpha + \mathbb{R}_{\geq 0}(\alpha + \beta)$.

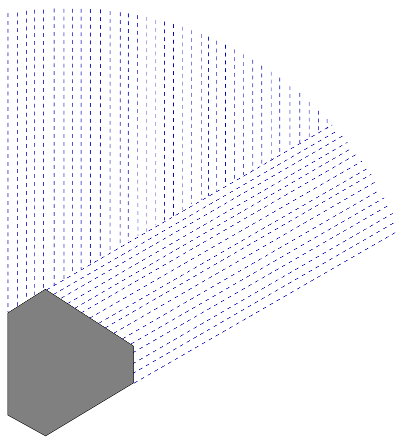


Figure 3.3: The convex set $\mathrm{conv}(W_K \cdot \log a) + \mathbb{R}_{\geq 0}\alpha + \mathbb{R}_{\geq 0}(\alpha + \beta)$

Finally, we analyze the case $\Sigma(Q) = \Sigma(\bar{P})$. This positive system corresponds to precisely the type of minimal parabolic subgroups of G' considered in [5]. Then $\Sigma(\bar{Q}) \cap \Sigma(P) = \{\alpha, \alpha + \beta, \beta\}$, and the convex cone will be spanned by all these roots.

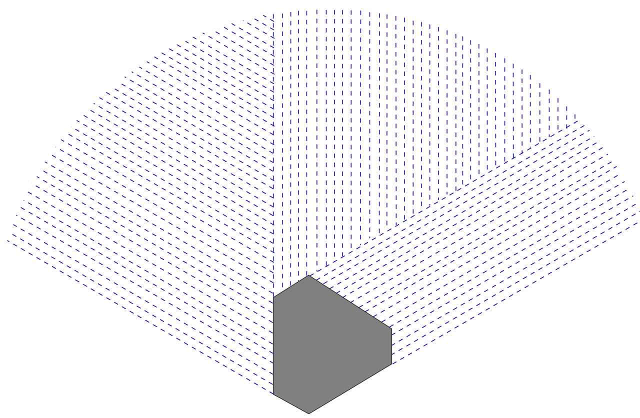


Figure 3.4: The convex set $\mathrm{conv}(W_K \cdot \log a) + \Sigma_{\gamma \in \Sigma(P)} \mathbb{R}_{\geq 0}\gamma$

3.2.2 An independent proof for the case of the group

In this section we present a simpler, computational proof for the case of the group. This proof is inspired by the independent proof of Theorem A.1 in [5].

As in Kostant's non-linear convexity theorem, we will investigate the image under the composition of maps $\mathrm{pr}_{\mathfrak{q}} \mathfrak{H}_{P \times Q}$ of a left translate of H. We first show that it suffices to consider elements of $A'_{\mathfrak{q}}$. Namely, let $(u, v) \in G \times G$. Then according to the decomposition

$$G \times G = (K \times K) \times A'_{\mathfrak{q}} \times \mathrm{diag}(K \times K)$$

(see [17]), we may write $(u,v) = (k_1 a g', k_2 a^{-1} g')$. Then

$$\begin{aligned}
&\mathrm{pr_q}\mathfrak{H}_{P\times Q}((u,v)\mathrm{diag}(G\times G)) = \\
&= \mathrm{pr_q}\mathfrak{H}_{P\times Q}(\{(k_1 a g' g, k_2 a^{-1} g' g) \,|\, g\in G\}) \\
&= \mathrm{pr_q}(\{(\mathfrak{H}_P(k_1 a g''), \mathfrak{H}_Q(k_2 a^{-1} g'')) \,|\, g''\in G\}) \\
&= \mathrm{pr_q}(\{(\mathfrak{H}_P(a g''), \mathfrak{H}_Q(a^{-1} g'')) \,|\, g''\in G\})
\end{aligned} \tag{3.7}$$

which is equal to $\mathrm{pr_q}\mathfrak{H}_{P\times Q}((a,a^{-1})\mathrm{diag}(G\times G))$ and we conclude that indeed it suffices to consider elements of A'_{q}. Observe that

$$\mathrm{pr_q}\mathfrak{H}_{P\times Q}((a,a^{-1})\mathrm{diag}(G\times G)) = \mathrm{pr_q}\mathfrak{H}_{P\times Q}((a^2,e)\mathrm{diag}(G\times G)).$$

Let $a^2\in A$ and write $g = ka'n$ as given by the Iwasawa decomposition $G = KAN_P$. Thus, (3.7) is now equal to

$$\begin{aligned}
&= \mathrm{pr_q}(\{(\mathfrak{H}_P(a^2 k a' n), \mathfrak{H}_Q(k a' n)) \,|\, k\in K, a'\in A, n\in N_P\}) \\
&= \mathrm{pr_q}(\{(\mathfrak{H}_P(a^2 k a'), \mathfrak{H}_Q(a' n)) \,|\, k\in K, a'\in A, n\in N_P\}) \\
&= \mathrm{pr_q}(\{(\mathfrak{H}_P(a^2 k) + \log a', \mathfrak{H}_Q(a' n a'^{-1}) + \log a') \,|\, k\in K, a'\in A, n\in N_P\}).
\end{aligned} \tag{3.8}$$

By the definition of the map $\mathrm{pr_q} : \mathfrak{a}' \to \mathfrak{a}'_{\mathrm{q}}$ given in (3.5) we see that (3.8) equals

$$\begin{aligned}
&= \mathrm{pr_q}(\{(\mathfrak{H}_P(a^2 k), \mathfrak{H}_Q(a' n a'^{-1})) \,|\, k\in K, a'\in A, n\in N_P\}) \\
&= \mathrm{pr_q}(\{(\mathfrak{H}_P(a^2 k), \mathfrak{H}_Q(n')) \,|\, k\in K, n'\in N_P\}) \\
&= \mathrm{pr_q}(\mathfrak{H}_P(a^2 K)\times \mathfrak{H}_Q(N_P)).
\end{aligned}$$

By identifying $\mathfrak{a}'_{\mathrm{q}}$ with $\mathfrak{a}$ via the map $(X,-X)\mapsto X$, we obtain the following theorem.

Theorem 3.2.2.

$$\mathrm{pr_q}\mathfrak{H}_{P\times Q}((a,a^{-1})\mathrm{diag}(G\times G)) = \frac{1}{2}(\mathfrak{H}_P(a^2K) - \mathfrak{H}_Q(N_P)).$$

It is well known, see for instance Lemma 4.4.9, that

$$\mathfrak{H}_Q(N_P) = \Gamma_{\mathfrak{a}}(\Sigma(\bar{P})\cap\Sigma(Q)), \tag{3.9}$$

where $\Gamma_{\mathfrak{a}}(\Sigma(\bar{P})\cap\Sigma(Q))$ denotes the polyhedral cone in $\mathfrak{a}$ spanned by the elements H_α with $\alpha\in\Sigma(\bar{P})\cap\Sigma(Q)$. By Kostant's non-linear convexity theorem and Equation (3.9) we obtain that

$$\begin{aligned}
\mathrm{pr_q}\mathfrak{H}_{P\times Q}((a,a^{-1})\mathrm{diag}(G\times G)) &= \tfrac{1}{2}\mathrm{conv}(W(\mathfrak{a})\cdot\log a^2) - \tfrac{1}{2}\Gamma_{\mathfrak{a}}(\Sigma(\bar{P})\cap\Sigma(Q)) \\
&= \mathrm{conv}(W(\mathfrak{a})\cdot\log a) + \Gamma_{\mathfrak{a}}(\Sigma(P)\cap\Sigma(\bar{Q})).
\end{aligned}$$

Recall that we have identified $\mathfrak{a}'_{\mathrm{q}}$ with $\mathfrak{a}$ via the map $(X, -X) \mapsto X$. Since $-\log a = \log a^{-1}$, we obtain that the image of $(a, a^{-1})H$ under the map $\mathrm{pr}_{\mathrm{q}} \circ \mathfrak{H}_{P\times Q}$ is given by

$$\mathrm{conv}(W_{K'\cap H} \cdot \log(a, a^{-1})) + \Gamma(\{(Y, -Y) \,|\, Y \in \Gamma_{\mathfrak{a}}(\Sigma(P) \cap \Sigma(\bar{Q}))\}),$$

where $\Gamma(\{(Y, -Y) \,|\, Y \in \Gamma_{\mathfrak{a}}(\Sigma(P) \cap \Sigma(\bar{Q}))\})$ is the polyhedral cone in $\mathfrak{a}'_{\mathrm{q}}$ spanned by $\{(Y, -Y) \,|\, Y \in \Gamma_{\mathfrak{a}}(\Sigma(P) \cap \Sigma(\bar{Q}))\}$.

On the other hand, Theorem 4.10.1 tells us that for $(a, a^{-1}) \in A'_{\mathrm{q}}$

$$\mathrm{pr}_{\mathrm{q}} \circ \mathfrak{H}_{P\times Q}((a, a^{-1})H) = \mathrm{conv}(W_{K'\cap H} \cdot \log(a, a^{-1})) + \Gamma(P \times Q). \tag{3.10}$$

The cone $\Gamma(P \times Q)$ is defined in (3.4) as

$$\Gamma(P \times Q) = \sum_{\gamma \in \Sigma(P\times Q, \sigma\theta')} \mathbb{R}_{\geq 0} \mathrm{pr}_{\mathrm{q}} H'_{\gamma},$$

where $H'_{\gamma} = (H_{\alpha}, 0)$, if $\gamma = (\alpha, 0)$, and $H'_{\gamma} = (0, H_{\alpha})$, for $\gamma = (0, \alpha)$.

By Lemma 3.2.1 these two cones coincide.

Remark 3.2.3. Theorem 3.2.2 tells us that our convexity result for the case of the group can be obtained from the original non-linear convexity theorem of Kostant and Equation (3.9).

Conversely, we can obtain both the non-linear convexity theorem of Kostant and Equation (3.9) by assuming that Theorem 4.10.1 holds. Namely, for the minimal parabolic subgroup $P \times P$ ($P \in \mathcal{P}(A)$), Theorem 3.2.2 gives

$$\mathrm{pr}_{\mathrm{q}} \mathfrak{H}_{P\times P}((a, a^{-1})H) = \mathrm{conv}(W(\mathfrak{a}) \cdot \log a),$$

where $(a, a^{-1}) \in A'_{\mathrm{q}}$. This is precisely the non-linear convexity theorem of Kostant.

In the next subsection we will demonstrate a method of obtaining Equation (3.9) from the group case.

3.2.3 A consequence of the group case

Although Lemma 3.2.4 is used in the proof of the convexity theorem, we can recover it by assuming that Theorem 4.10.1 holds. In this subsection we show how this can be done. A different proof of this equality can be found in Lemma 4.4.9 below.

By Theorem 4.10.1 we have that for $(a, a^{-1}) \in A'_{\mathrm{q}}$

$$\mathrm{pr}_{\mathrm{q}} \circ \mathfrak{H}_{P\times Q}((a, a^{-1})\mathrm{diag}(G\times G)) = \mathrm{conv}(W_{K'\cap \mathrm{diag}(G\times G)} \cdot \log(a, a^{-1})) + \Gamma(P\times Q), \tag{3.11}$$

where the cone $\Gamma(P \times Q)$ is defined by (3.4).

We identify $\mathfrak{a}'_{\mathrm{q}}$ with $\mathfrak{a}$ (via the map $(X, -X) \mapsto X$) and thus, Lemma 3.2.1 gives

$$\Gamma(P \times Q) = \Gamma_{\mathfrak{a}}(\Sigma(P) \cap \Sigma(\bar{Q})). \tag{3.12}$$

Hence, for $(a, a^{-1}) = (e, e)$, Equations (3.11) and (3.12) give us

$$\mathrm{pr}_{\mathrm{q}} \circ \mathfrak{H}_{P\times Q}(\mathrm{diag}(G \times G)) = \Gamma_{\mathfrak{a}}(\Sigma(P) \cap \Sigma(\bar{Q})) \tag{3.13}$$

On the other hand, Theorem 3.2.2 with $(a, a^{-1}) = (e, e)$, gives

$$\mathrm{pr}_{\mathrm{q}}\mathfrak{H}_{P\times Q}(\mathrm{diag}(G \times G)) = -\frac{1}{2}\mathfrak{H}_Q(N_P). \tag{3.14}$$

Since $-\frac{1}{2}\mathfrak{H}_Q(N_P) = -\frac{1}{2}\mathfrak{H}_Q((N_P \cap \bar{N}_Q)(N_P \cap N_Q)) = -\frac{1}{2}\mathfrak{H}_Q(N_P \cap \bar{N}_Q)$ (recall that $\bar{N}_Q := \theta(N_Q)$), we conclude that

$$-\frac{1}{2}\mathfrak{H}_Q(N_P \cap \bar{N}_Q) = \Gamma_{\mathfrak{a}}(\Sigma(P) \cap \Sigma(\bar{Q})),$$

which is equivalent to saying that

$$\mathfrak{H}_Q(N_P \cap \bar{N}_Q) = -2\Gamma_{\mathfrak{a}}(\Sigma(P) \cap \Sigma(\bar{Q})) = \Gamma_{\mathfrak{a}}(\Sigma(\bar{P}) \cap \Sigma(Q)).$$

This provides a proof of the following lemma.

Lemma 3.2.4. *Let $S \in \mathcal{P}(A)$. Then the Iwasawa projection $\mathfrak{H}_P : G \to \mathfrak{a}$ restricts to a map $\bar{N}_P \cap N_S \to \mathfrak{a}$, with image equal to the cone*

$$\Gamma_{\mathfrak{a}}(\Sigma(P) \cap \Sigma(\bar{S})).$$

Chapter 4

Proof of the convexity theorem

The present chapter contains the proof of the Main Theorem (Theorem 4.10.1). The proof follows the guideline prescribed in Section 4.1 below and uses ideas in [5]. Each step of the proof is contained in a different section in this chapter. In Section 4.10 all these steps are put together in the final argumentation of the proof.

We conclude this chapter with an appendix, A. In Appendix A we give the proof of Lemma 4.2.10 concerning the decomposition of nilpotent groups in terms of subgroups generated by roots.

4.1 An outline of the proof

The proof of the main theorem follows the line of argument described below, which is an extension of the argumentation of [5], which in turn was inspired by [26].

We first prove the theorem for a regular element $a \in A_{\mathrm{q}}$. Since the map $\mathfrak{H}_P : G \to \mathfrak{a}$ is right $H \cap P$-invariant, see Lemma 4.4.1, the map

$$F_a : H \to \mathfrak{a}_{\mathrm{q}}, \;\; h \mapsto \mathfrak{H}_{P,\mathrm{q}}(ah)$$

factors through a map $\bar{F}_a : H/H \cap P \to \mathfrak{a}_{\mathrm{q}}$ (recall that the map $\mathfrak{H}_{P,\mathrm{q}} : G \to \mathfrak{a}_{\mathrm{q}}$ was defined in Remark 3.1.2). In order for the idea of the proof in [5] to work in the present situation, one needs to establish properness of the map $\bar{F}_a$. This is done in Section 4.4 by reducing the problem to the case of a suitable σ-stable parabolic subgroup R combined with application of results of [5]. The established properness implies that the image $F_a(H)$ is closed in $\mathfrak{a}_{\mathrm{q}}$.

The considerations of Section 4.4 also lead to the constraint on the image $F_a(H)$ that it does not contain any line of $\mathfrak{a}_{\mathrm{q}}$, see Corollary 4.4.15.

In Section 4.5 we introduce the functions $F_{a,X} : H \to \mathbb{R}$, for $X \in \mathfrak{a}_{\mathrm{q}}$, defined by

$$F_{a,X}(h) = \langle X, F_a(h) \rangle = B(X, F_a(h)).$$

Geometrically, these functions test the Iwasawa projection by linear forms on $\mathfrak{a}_{\mathrm{q}}$, and give us constraints on the image of H under F_a. For a more detailed exposition on $F_{a,X}$ we refer the reader to [13]. Our own study of this function follows ideas in [5] and [13].

In Section 4.6 we calculate the critical set $\mathcal{C}_{a,X}$ of the function $F_{a,X}$ explicitly, for $a \in A_{\mathrm{q}}^{\mathrm{reg}}$ and $X \in \mathfrak{a}_{\mathrm{q}}$. In particular, we show that this set is the union of a finite collection $\mathcal{M}_{a,X}$ of injectively immersed connected submanifolds of H. If $\mathcal{C}_{a,X} \subsetneq H$, then all submanifolds in $\mathcal{M}_{a,X}$ are lower dimensional, so that $\mathcal{C}_{a,X}$ is thin in the sense of the Baire theorem, i.e. its closure has empty interior. These considerations allow us to show that in case $\Sigma(\mathfrak{g}, \mathfrak{a}_{\mathrm{q}})$ spans $\mathfrak{a}_{\mathrm{q}}^*$, the set $\mathcal{C}_a$ of points in H where F_a is not submersive, is closed and thin, see Proposition 4.6.7. In particular, we then have that

$$F_a(\mathcal{C}_a) \subsetneq F_a(H). \tag{4.1}$$

In Sections 4.7 and 4.8 we calculate the Hessians of $F_{a,X}$ and their transversal signatures along all manifolds from $\mathcal{M}_{a,X}$. These calculations, which are extensive, in particular allow us to determine all points where the transversal signatures are definite. This in turn gives us all points where $F_{a,X}$ attains local maxima and minima. A main result of Section 4.8 is Lemma 4.8.14 which asserts that for every local minimum m of the function $F_{a,X}$ we have that $\langle X, \cdot \rangle \geq m$ on the set

$$\Omega := \mathrm{conv}(W_{K \cap H} \cdot \log a) + \Gamma(P). \tag{4.2}$$

In Section 4.9 we prepare for the proof of the main theorem by using a limit argument to reduce to the case of a regular element $a \in A_{\mathrm{q}}$.

The proof of the main theorem is finally given in Section 4.10. It proceeds by induction over the rank of the root system $\Sigma(\mathfrak{g}, \mathfrak{a}_{\mathrm{q}})$. More precisely, for $a \in A_{\mathrm{q}}^{\mathrm{reg}}$ the set $\mathcal{C}_{a,X}$ depends on $X \in \mathfrak{a}_{\mathrm{q}}$ through the centralizer $\mathfrak{g}_X$ of X in $\mathfrak{g}$. It is shown that $\mathcal{C}_{a,X} \subsetneq H$ implies that $\mathrm{rk}\,\Sigma(\mathfrak{g}_X, \mathfrak{a}_{\mathrm{q}}) < \mathrm{rk}\,\Sigma(\mathfrak{g}, \mathfrak{a}_{\mathrm{q}})$ so that the induction hypothesis holds for the centralizer G_X of X in G. This allows us to determine the image $F_a(C_{a,X})$ for such X. In particular, this leads to a precise description of the image $F_a(\mathcal{C}_a)$ from which it is seen that the latter image contains the boundary of the set Ω.

In the proof we use this observation, together with the earlier obtained constraint that the image $F_a(H)$ does not contain a line, to conclude that $F_a(H)$ is contained in Ω. In particular, this implies that, for each $X \in \mathfrak{a}_{\mathrm{q}}$, every local minimum of $F_{a,X}$ is global.

For the converse inclusion, we first show that the image of $H \setminus \mathcal{C}_a$ under the map F_a is a union of connected components of $\Omega \setminus F_a(\mathcal{C}_a)$. The established fact that every local minimum of $F_{a,X}$ is global then allows us to show that all connected components appear in the image, thereby completing the proof.

Finally, we wish to mention that many of our calculations have been inspired by [26] and [13].

4.2 Some structure theory for parabolic subgroups

In this section we will construct a (minimal) parabolic subgroup in $\mathcal{P}(A)$, which has a special position relative to the involution σ; it will play an important role in Section 4.4. We will also discuss some structure theory of parabolic subgroups from $\mathcal{P}(A)$ and derive a useful decomposition for their unipotent radicals.

We recall that every parabolic subgroup P from $\mathcal{P}(A)$ has a Langlands decomposition of the form given in Definition 2.2.14. Thus, by the text succeeding Definition 2.2.14, its (θ-stable) Levi component L_P is given by

$$L_P = L = MA$$

and the multiplication map $L \times N_P \to P$ is a diffeomorphism. The opposite parabolic subgroup $\bar{P}$ is defined to be the unique parabolic subgroup from $\mathcal{P}(A)$ with $\Sigma(\bar{P}) = -\Sigma(P)$. It equals $\theta(P)$.

4.2.1 Extremal minimal parabolic subgroups

If τ is any involution of G which leaves A invariant, then its infinitesimal version $\tau : \mathfrak{g} \to \mathfrak{g}$ leaves $\mathfrak{a}$ invariant, and we put

$$\Sigma(P, \tau) := \{\alpha \in \Sigma(P) \ : \ \ \tau\alpha \in \Sigma(P)\}. \tag{4.3}$$

Observe that $\Sigma(P, \tau) = \Sigma(P) \cap \tau\Sigma(P)$.

Definition 4.2.1. A minimal parabolic subgroup $Q \in \mathcal{P}(A)$ is said to be $\mathfrak{h}$-*extreme* if

$$\Sigma(Q, \sigma) = \Sigma(Q) \setminus \mathfrak{a}_{\mathrm{q}}^*. \tag{4.4}$$

Starting with any minimal parabolic subgroup $P \in \mathcal{P}(A)$, we can obtain an $\mathfrak{h}$-extreme minimal parabolic subgroup by changing one simple root at a time. This process is described in Lemma 4.2.6 below.

Lemma 4.2.2. *Let $P \in \mathcal{P}(A)$. Then*

$$\Sigma(P) = \Sigma(P, \sigma) \sqcup \Sigma(P, \sigma\theta) \qquad \text{(disjoint union)}.$$

Proof. Let $\alpha \in \Sigma(P)$. From the fact that $\sigma\theta\alpha = -\sigma\alpha$, the result follows easily. $\square$

Lemma 4.2.3. *Let $P \in \mathcal{P}(A)$ and assume that*

$$\Sigma(P, \sigma) \subsetneq \Sigma(P) \setminus \mathfrak{a}_{\mathrm{q}}^*.$$

Then there exists a P-simple root $\alpha \in \Sigma(P, \sigma\theta)$ with $\alpha \notin \mathfrak{a}_{\mathrm{q}}^*$.

Remark 4.2.4. A root $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a})$ is said to be P-simple if it is simple in the positive system $\Sigma(P)$.

Proof. Assume the contrary. Then for every simple root $\beta \in \Sigma(P)$ we have $\sigma\beta = -\sigma\theta\beta \in \Sigma(P)$ or $\sigma\theta\beta \in \mathfrak{a}_{\mathrm{q}}^*$. In the latter case, $\sigma\theta\beta = \beta$. Thus we see that for any simple root $\beta \in \Sigma(P)$ we have either $\sigma\beta \in \Sigma(P)$ or $\sigma\beta = -\beta$.

The set $\Sigma(P)$ is a positive system for the root system $\Sigma(\mathfrak{g}, \mathfrak{a})$. Hence, there exists an element $X \in \mathfrak{a}$ such that $\alpha(X) > 0$ for all $\alpha \in \Sigma(P)$. Put $X_{\mathrm{h}} := \frac{1}{2}(X + \sigma(X))$. Then for every simple root β in $\Sigma(P)$ we have either $\sigma\beta = -\beta$, in which case $\beta(X_{\mathrm{h}}) = 0$, or $\sigma\beta \in \Sigma(P)$, in which case $\beta(X_{\mathrm{h}}) > 0$. In any case, for each simple $\beta \in \Sigma(P)$, the value $\beta(X_{\mathrm{h}})$ is a nonnegative real number. Moreover, the number is zero if and only if $\sigma\beta = -\beta$. It follows that for all $\alpha \in \Sigma(P)$ we must have $\alpha(X_{\mathrm{h}}) \geq 0$. Moreover, if $\alpha(X_{\mathrm{h}}) = 0$, then we must have $\alpha \in \mathfrak{a}_{\mathrm{q}}^*$. Since $\sigma\theta(X_{\mathrm{h}}) = -X_{\mathrm{h}}$ and $\Sigma(P, \sigma) \subsetneq \Sigma(P) \setminus \mathfrak{a}_{\mathrm{q}}^*$, we easily arrive at a contradiction. $\square$

Corollary 4.2.5. *If P and α are as in Lemma 4.2.3, then $P' := s_\alpha(P)$ has the following properties:*

(a) $\Sigma(P) \cap \mathfrak{a}_{\mathrm{q}}^* = \Sigma(P') \cap \mathfrak{a}_{\mathrm{q}}^*$,

(b) $\Sigma(P, \sigma) \subsetneq \Sigma(P', \sigma)$.

Here, s_α denotes the reflection in α.

In the proof of the above corollary, we will follow the convention established in Remark 2.1.14 to write

$$R_\circ := \{\alpha \in R : \tfrac{1}{2}\,\alpha \notin R\}$$

for the set of indivisible roots of any (possibly non-reduced) root system R. Furthermore, if $S \subseteq R$ is any subset, we will write $S_\circ := S \cap R_\circ$. Finally, we agree to write $\Sigma_\circ(P)$ for $\Sigma(P)_\circ$.

Proof. It suffices to prove (a) and (b) with everywhere Σ replaced by $\Sigma_\circ$. Since $P' := s_\alpha(P)$ with α simple in $\Sigma(P)$, we have

$$\Sigma_\circ(P') = (\Sigma_\circ(P) \setminus \{\alpha\}) \cup \{-\alpha\},$$

which implies (a).

Let $\beta \in \Sigma_\circ(P) \cap \sigma\Sigma_\circ(P)$. Then $\beta \neq \alpha$ and $\sigma\beta \neq \sigma\alpha$. Since β and $\sigma\beta$ both belong to $\Sigma_\circ(P')$, it follows that $\beta \in \Sigma_\circ(P') \cap \sigma\Sigma_\circ(P')$. This proves the inclusion in (b). We still need to show that equality cannot hold. This follows from the fact that $\theta\alpha = -\alpha \in \Sigma(P', \sigma)$. $\square$

Lemma 4.2.6. *Let $P \in \mathcal{P}(A)$. Then there exists a minimal parabolic subgroup $Q_{\mathrm{h}} \in \mathcal{P}(A)$ such that the following conditions hold:*

(a) $\Sigma(Q_{\mathrm{h}}) \cap \mathfrak{a}_{\mathrm{q}}^* = \Sigma(P) \cap \mathfrak{a}_{\mathrm{q}}^*$,

(b) $\Sigma(Q_{\mathrm{h}}) \cap \mathfrak{a}_{\mathrm{h}}^* = \Sigma(P) \cap \mathfrak{a}_{\mathrm{h}}^*$,

(c) $\Sigma(P, \sigma) \subseteq \Sigma(Q_{\mathrm{h}}, \sigma)$,

(d) Q_{h} *is* $\mathfrak{h}$*-extreme, see* (4.4).

Proof. If $\alpha \in \Sigma(P) \cap \mathfrak{a}_{\mathrm{q}}^*$, then $\sigma\alpha = -\alpha \notin \Sigma(P)$. Hence

$$\Sigma(P, \sigma) = \Sigma(P) \cap \sigma\Sigma(P) \subseteq \Sigma(P) \setminus \mathfrak{a}_{\mathrm{q}}^*. \tag{4.5}$$

If the above inclusion is an equality, the result holds with $Q_{\mathrm{h}} := P$. If not, then the inclusion in (4.5) is proper and Lemma 4.2.3 guarantees the existence of a simple root $\alpha \in \Sigma(P) \setminus \mathfrak{a}_{\mathrm{q}}^*$ such that $\sigma\theta\alpha \in \Sigma(P)$. By applying Corollary 4.2.5 we see that the minimal parabolic subgroup $P' := s_\alpha(P)$ satisfies the above conditions (a) and (b), and

$$\Sigma(P, \sigma) \subsetneq \Sigma(P', \sigma). \tag{4.6}$$

Put $P_0 = P$ and $P_1 = P'$. By applying the above process repeatedly, we obtain a sequence of parabolic subgroups $P = P_0, P_1, \ldots, P_k$ satisfying

(a) $\Sigma(P_i) \cap \mathfrak{a}_{\mathrm{q}}^* = \Sigma(P_{i+1}) \cap \mathfrak{a}_{\mathrm{q}}^*$,

(b) $\Sigma(P_i) \cap \mathfrak{a}_{\mathrm{h}}^* = \Sigma(P_{i+1}) \cap \mathfrak{a}_{\mathrm{h}}^*$,

(c) $\Sigma(P_i, \sigma) \subsetneq \Sigma(P_{i+1}, \sigma)$,

for $0 \leq i < k$. The process ends when for some $k > 0$ the condition $\Sigma(P_k) \cap \sigma\Sigma(P_k) = \Sigma(P_k) \setminus \mathfrak{a}_{\mathrm{q}}^*$ is satisfied. The parabolic subgroup $Q_{\mathrm{h}} = P_k$ satisfies all assertions of the lemma. $\square$

Remark 4.2.7. In analogy with Definition 4.2.1, a parabolic subgroup $Q \in \mathcal{P}(A)$ is said to be $\mathfrak{q}$-*extreme* if $\Sigma(Q, \sigma\theta) = \Sigma(Q) \setminus \mathfrak{a}_{\mathrm{h}}^*$. With obvious modifications in the proof, Lemma 4.2.6 is valid with $\mathfrak{q}$-extreme in place of $\mathfrak{h}$-extreme.

4.2.2 The convexity theorem for a q-extreme parabolic subgroup

We shall now explain why the result of [5] is a special case of the Main Theorem. We keep the notation as above and impose that $P \in \mathcal{P}(A)$ is $\mathfrak{q}$-extreme, see Remark 4.2.7. Then $\Sigma(P, \sigma\theta) = \Sigma(P) \setminus \mathfrak{a}_{\mathrm{h}}^*$, so that

$$\Delta^+ := \Sigma(P, \sigma\theta)|_{\mathfrak{a}_{\mathrm{q}}}$$

is a positive system for $\Sigma(\mathfrak{g}, \mathfrak{a}_{\mathrm{q}})$. For $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a}_{\mathrm{q}})$, the root space $\mathfrak{g}_\alpha$ is $\sigma\theta$-invariant; we write $\mathfrak{g}_{\alpha,\pm}$ for the ± 1 eigenspaces of $\sigma\theta|_{\mathfrak{g}_\alpha}$. Put

$$\Delta^+_- = \{\alpha \in \Delta^+ \ : \ \ \mathfrak{g}_{\alpha,-} \neq 0\}.$$

Then [5, Theorem 1.1] asserts that

$$\mathfrak{H}_{P,\mathrm{q}}(aH) = \mathrm{conv}(W_{K\cap H} \cdot \log a) + \Upsilon(P),$$

where $\Upsilon(P)$ is the finitely generated polyhedral cone in $\mathfrak{a}_{\mathrm{q}}$ defined by

$$\Upsilon(P) = \sum_{\alpha \in \Delta^+_-} \mathbb{R}_{\geq 0} H_\alpha;$$

here H_α denotes the element of $\mathfrak{a}_{\mathrm{q}}$ with $H_\alpha \perp \ker \alpha$ and $\alpha(H_\alpha) = 2$.

Thus, our main theorem coincides with [5, Theorem 1.1] if $\Gamma(P) = \Upsilon(P)$. The latter is asserted by the following lemma.

Lemma 4.2.8. *Let $P \in \mathcal{P}(A)$ be $\mathfrak{q}$-extreme. Then $\Upsilon(P) = \Gamma(P)$.*

Proof. Let $\alpha \in \Sigma(P, \sigma\theta)$. Then $\alpha|_{\mathfrak{a}_{\mathrm{q}}}$ is non-zero hence belongs to $\Sigma(\mathfrak{g}, \mathfrak{a}_{\mathrm{q}})$. Moreover,

$$\mathrm{pr}_{\mathrm{q}}(H_\alpha) = H_{\alpha|_{\mathfrak{a}_{\mathrm{q}}}}. \tag{4.7}$$

As $\sigma\theta$ restricts to the identity on $\mathfrak{a}_{\mathrm{q}}$, the $\mathfrak{a}$-roots α and $\sigma\theta\alpha$ have the same restriction to $\mathfrak{a}_{\mathrm{q}}$ giving the root $\alpha|_{\mathfrak{a}_{\mathrm{q}}}$ of Δ^+. If the given $\mathfrak{a}$-roots are different, then the sum $\mathfrak{g}_\alpha + \sigma\theta(\mathfrak{g}_\alpha)$ is direct and contained in $\mathfrak{g}_{\alpha|_{\mathfrak{a}_{\mathrm{q}}}}$ and we see that $\mathfrak{g}_{\alpha|_{\mathfrak{a}_{\mathrm{q}}},-} \neq 0$, so that $\alpha \in \Sigma(P)_-$ and $\alpha|_{\mathfrak{a}_{\mathrm{q}}} \in \Delta^+_-$. On the other hand, if $\alpha = \sigma\theta\alpha$, then $\mathfrak{g}_\alpha = \mathfrak{g}_{\alpha|_{\mathfrak{a}_{\mathrm{q}}}}$ and we see that $\alpha \in \Sigma(P)_-$ if and only if $\alpha|_{\mathfrak{a}_{\mathrm{q}}} \in \Delta^+_-$. It follows from this argument that $\Sigma(P)_-|_{\mathfrak{a}_{\mathrm{q}}} = \Delta^+_-$. Using (4.7) we now see that

$$\Gamma(P) = \sum_{\alpha \in \Sigma(P)_-} \mathbb{R}_{\geq 0} H_{\alpha|_{\mathfrak{a}_{\mathrm{q}}}} = \sum_{\alpha \in \Delta^+_-} \mathbb{R}_{\geq 0} H_\alpha = \Upsilon(P).$$

$\square$

4.2.3 Decompositions of nilpotent Lie groups

In this subsection we give a brief survey of a number of useful results on decompositions of nilpotent Lie groups that will be needed in this chapter.

Lemma 4.2.9 ([29, Lemma IV.6.8])**.** *Let N be a connected, simply connected nilpotent Lie group with Lie algebra $\mathfrak{n}$. Let $(\mathfrak{n}_i)_{0\leq i\leq k}$ be a strictly decreasing sequence of ideals of $\mathfrak{n}$ such that $\mathfrak{n}_0 = \mathfrak{n}$, $\mathfrak{n}_k = 0$ and*

$$[\mathfrak{n}, \mathfrak{n}_i] \subseteq \mathfrak{n}_{i+1} \qquad \text{for all} \quad 0 \leq i < k.$$

Let $\mathfrak{b}_1$ and $\mathfrak{b}_2$ be two mutually complementary subspaces of $\mathfrak{n}$ such that $\mathfrak{n}_i = \mathfrak{b}_1 \cap \mathfrak{n}_i + \mathfrak{b}_2 \cap \mathfrak{n}_i$, for all $0 \leq i \leq k$. Then the mapping

$$\varphi : (X, Y) \to \exp X \exp Y$$

is an analytic diffeomorphism of $\mathfrak{b}_1 \times \mathfrak{b}_2$ onto N.

Lemma 4.2.10. *Let N_P be the nilpotent radical of a minimal parabolic subgroup in $\mathcal{P}(A)$, $\mathfrak{n}_P$ its Lie algebra and $\mathfrak{n}_1, \ldots, \mathfrak{n}_k \subset \mathfrak{n}_P$ linearly independent subalgebras of $\mathfrak{n}_P$ that are direct sums of $\mathfrak{a}$-root spaces. Assume that $\mathfrak{n} = \mathfrak{n}_1 \oplus \ldots \oplus \mathfrak{n}_k$ is a subalgebra of $\mathfrak{n}_P$. Denote by $N := \exp \mathfrak{n}$ and by $N_i := \exp \mathfrak{n}_i$, $i \in \{1, \ldots, k\}$, the corresponding closed subgroups of N_P. Then the multiplication map*

$$\mu : N_1 \times \ldots \times N_k \to N$$

is a diffeomorphism.

This result is stated in [13, Lemma 2.3] for $\mathfrak{n} = \mathfrak{n}_P$, with reference to [41]. We need the present slightly more general version with $\mathfrak{n}$ a subalgebra of $\mathfrak{n}_P$. A proof of this result can be found in Appendix A.

4.2.4 Fixed points for the involution in minimal parabolic subgroups

Let $P \in \mathcal{P}(A)$. The decomposition $P = LN_P$ induces a similar decomposition for the intersection $P \cap H$. In the present subsection we present a proof for this fact, see the lemma below.

Lemma 4.2.11. $P \cap H \simeq (L \cap H) \times (N_P \cap H)$

Proof. Let p be an element in $P \cap H$. According to the decomposition $P = LN_P$, we write $p = ln$. Then, $\sigma(ln) = \sigma(l)\sigma(n) = ln$ and we obtain that $\sigma(n)n^{-1} = \sigma(l)^{-1}l \in L$. Since $\sigma(N_P) = (\sigma(N_P) \cap \bar{N}_P) \times (\sigma(N_P) \cap N_P)$ we conclude that $\sigma(n)n^{-1} \in \bar{N}_P N_P$. Now, by [36, Lemma 7.64] it follows that $\bar{N}_P N_P \cap L = e$ and thus $\sigma(n) = n$ and $\sigma(l) = l$. $\square$

4.2.5 Decomposition of nilpotent radicals induced by the involution

In this subsection, we assume that $P \in \mathcal{P}(A)$. We will show that the unipotent radical N_P decomposes as the product of $N_P \cap H$ and a suitable closed subgroup $N_{P,+}$ of N_P. To describe this subgroup, we need the existence of suitable elements of $\mathfrak{a}_{\mathrm{q}}$. As usual, an element $X \in \mathfrak{a}_{\mathrm{q}}$ is said to be *regular* for the root system $\Sigma(\mathfrak{g}, \mathfrak{a}_{\mathrm{q}})$ if no root of this system vanishes on it. The set of such regular elements is denoted by $\mathfrak{a}_{\mathrm{q}}^{\mathrm{reg}}$. We observe that in terms of the system $\Sigma(\mathfrak{g}, \mathfrak{a})$ this set may be described as

$$\mathfrak{a}_{\mathrm{q}}^{\mathrm{reg}} = \{X \in \mathfrak{a}_{\mathrm{q}} : \ (\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a}) \text{ with } \alpha(X) = 0) \Rightarrow \alpha|_{\mathfrak{a}_{\mathrm{q}}} = 0\}. \qquad (4.8)$$

Lemma 4.2.12.

(a) *There exists an element $Z_{\mathrm{q}} \in \mathfrak{a}_{\mathrm{q}}^{\mathrm{reg}}$ such that $\alpha(Z_{\mathrm{q}}) > 0$ for all $\alpha \in \Sigma(P, \sigma\theta)$.*

(b) *There exists an element $Z_{\mathrm{h}} \in \mathfrak{a}_{\mathrm{h}}$ such that $\alpha(Z_{\mathrm{h}}) > 0$ for all $\alpha \in \Sigma(P, \sigma)$.*

Proof. The set

$$\mathfrak{a}' := \{X \in \mathfrak{a} : \ (\alpha, \beta \in \Sigma(\mathfrak{g}, \mathfrak{a}) \text{ with } \alpha(X) = \beta(X)) \Rightarrow \alpha = \beta\}$$

is the complement of finitely many hyperplanes in $\mathfrak{a}$, hence open and dense. Let $\mathfrak{a}^+(P)$ denote the positive chamber associated with the positive system $\Sigma(P)$ for $\Sigma(\mathfrak{g}, \mathfrak{a})$. Fix $Z_P \in \mathfrak{a}^+(P) \cap \mathfrak{a}'$. Then it is readily verified that $Z_{\mathrm{q}} := Z_P + \sigma\theta(Z_P)$ satisfies the requirements of (a). Likewise, the element $Z_{\mathrm{h}} = Z_P + \sigma(Z_P)$ satisfies the requirements of (b). $\square$

Given $Z_{\mathrm{q}} \in \mathfrak{a}_{\mathrm{q}}^{\mathrm{reg}}$ we put $\Sigma(P, +) := \{\alpha \in \Sigma(P) : \alpha(Z_{\mathrm{q}}) > 0\}$. Then

$$\mathfrak{n}_{P,+} := \bigoplus_{\alpha \in \Sigma(P,+)} \mathfrak{g}_\alpha$$

is a subalgebra of $\mathfrak{n}_P$. Let $N_{P,+} := \exp \mathfrak{n}_+$ be the corresponding closed subgroup of N_P, see Lemma 2.1.24. Define

$$\mathfrak{n}_{P,\sigma} := \sum_{\alpha \in \Sigma(P,\sigma)} \mathfrak{g}_\alpha$$

and $N_{P,\sigma}$ as the corresponding closed subgroup.

Proposition 4.2.13. *Let $Z_{\mathrm{q}} \in \mathfrak{a}_{\mathrm{q}}^{\mathrm{reg}}$ be as in Lemma 4.2.12 (a) and let $N_{P,+}$ be defined as above. Then the multiplication map*

$$N_{P,+} \times (N_P \cap H) \to N_P$$

is a diffeomorphism.

The proof of this result relies on the following lemma.

Lemma 4.2.14. *Let $P \in \mathcal{P}(A)$ and let $Z_{\mathrm{q}} \in \mathfrak{a}_{\mathrm{q}}^{\mathrm{reg}}$ be as in Lemma 4.2.12 (a). Put*

$$\Sigma(P, \sigma, +) := \{\alpha \in \Sigma(P, \sigma) \ : \ \ \alpha(Z_{\mathrm{q}}) > 0\}.$$

Then the following statements hold:

(a) $\mathfrak{n}_{P,\sigma,+} := \sum_{\alpha \in \Sigma(P,\sigma,+)} \mathfrak{g}_\alpha$ *is a subalgebra of* $\mathfrak{n}_{P,\sigma}$*,*

(b) $N_{P,\sigma,+} := \exp \mathfrak{n}_{P,\sigma,+}$ *is a closed subgroup of* $N_{P,\sigma}$*,*

(c) $\mathfrak{n}_{P,\sigma} = \mathfrak{n}_{P,\sigma,+} \oplus (\mathfrak{n}_P \cap \mathfrak{h})$*,*

(d) *the multiplication map*

$$\mu : N_{P,\sigma,+} \times (N_P \cap H) \to N_{P,\sigma}$$

is a diffeomorphism.

Proof. (a): Assume that $\alpha, \beta \in \Sigma(P, \sigma, +)$ and $\alpha + \beta \in \Sigma(\mathfrak{g}, \mathfrak{a})$. Then $\alpha + \beta \in \Sigma(P, \sigma)$ and $(\alpha + \beta)(Z_{\mathrm{q}}) > 0$ so that $\alpha + \beta \in \Sigma(P, \sigma, +)$. This implies (a).

Assertion (b) follows from (a) by application of Lemma 2.1.24.

Next, we prove (c). Let $\alpha \in \Sigma(P, \sigma, +)$. Then $\sigma\alpha(Z_{\mathrm{q}}) < 0$, which implies that $\mathfrak{n}_{P,\sigma,+} \cap \mathfrak{h} = \{0\}$. It follows that

$$\mathfrak{n}_{P,\sigma,+} \cap (\mathfrak{n}_P \cap \mathfrak{h}) = \{0\}.$$

It remains to be shown that any $X \in \mathfrak{n}_{P,\sigma}$ can be written as

$$X = X_+ + X_{\mathrm{h}},$$

with $X_+ \in \mathfrak{n}_{P,\sigma,+}$ and $X_{\mathrm{h}} \in \mathfrak{n}_P \cap \mathfrak{h}$. It suffices to prove this for $X \in \mathfrak{g}_\alpha \subset \mathfrak{n}_{P,\sigma}$. If $\alpha(Z_{\mathrm{q}}) > 0$, then $X \in \mathfrak{n}_{P,\sigma,+}$ by definition. On the other hand if $\alpha(Z_{\mathrm{q}}) = 0$, then by regularity of Z_{q} we have that $\alpha \in \mathfrak{a}_{\mathrm{h}}^*$ and thus $\mathfrak{g}_\alpha \subseteq \mathfrak{h}$, which implies that $X \in \mathfrak{n}_P \cap \mathfrak{h}$. Finally, if $\alpha(Z_{\mathrm{q}}) < 0$, then

$$X = (X + \sigma(X)) - \sigma(X)$$

with $X + \sigma(X) \in \mathfrak{n}_P \cap \mathfrak{h}$ and $-\sigma(X) \in \mathfrak{n}_{P,\sigma,+}$, and we are done.

For (d) fix $Z_{\mathfrak{h}}$ as in Lemma 4.2.12 (b). Then for all $\alpha \in \Sigma(P, \sigma)$ we have that $v_\alpha := \alpha(Z_{\mathfrak{h}}) > 0$. Let the set of positive real numbers thus obtained be ordered by $v_{\alpha_1} < v_{\alpha_2} < \cdots < v_{\alpha_m}$. We define $\mathfrak{n}_0 = \mathfrak{n}_{P,\sigma}$, $\mathfrak{n}_m = 0$, and for $1 \leq i < m$,

$$\mathfrak{n}_i := \sum_{\substack{\alpha \in \Sigma(P,\sigma) \\ \alpha(Z_{\mathrm{h}}) > v_{\alpha_i}}} \mathfrak{g}_\alpha.$$

Then $\mathfrak{n}_1, \ldots, \mathfrak{n}_m$ is a strictly decreasing sequence of ideals in $\mathfrak{n}_{P,\sigma}$ with $[\mathfrak{n}, \mathfrak{n}_i] \subseteq \mathfrak{n}_{i+1}$ for $0 \leq i < m$. We note that each $\mathfrak{n}_i$ is invariant under σ. Hence, by the same argument as in the proof of (c) we obtain that

$$\mathfrak{n}_i = (\mathfrak{n}_i \cap \mathfrak{n}_{P,\sigma,+}) \oplus (\mathfrak{n}_i \cap (\mathfrak{n}_P \cap \mathfrak{h}))$$

for all $0 \leq i \leq m$. Thus, we may apply Lemma 4.2.9 to conclude that

$$N_{P,\sigma} \simeq N_{P,\sigma,+} \times \exp(\mathfrak{n}_P \cap \mathfrak{h}).$$

It remains to show that $\exp(\mathfrak{n}_P \cap \mathfrak{h}) = N_P \cap H$. This follows from the fact that

$$N_P \cap H \subseteq \{n \in N_P \,:\, \sigma(n) = n\} = \{\exp X \,:\, X \in \mathfrak{n}_P \cap \mathfrak{h}\} \subseteq N_P \cap H.$$

This proves assertion (d). $\square$

Proof of Proposition 4.2.13. Let

$$\mathfrak{n}_{P,\sigma\theta} := \sum_{\alpha \in \Sigma(P,\sigma\theta)} \mathfrak{g}_\alpha$$

and let $N_{P,\sigma\theta}$ be the corresponding subgroup of N_P. Then $\mathfrak{n}_P = \mathfrak{n}_{P,\sigma\theta} \oplus \mathfrak{n}_{P,\sigma}$ and by Lemma 4.2.10 we obtain that

$$N_P \simeq N_{P,\sigma\theta} \times N_{P,\sigma}. \tag{4.9}$$

We apply Lemma 4.2.14 to the second component and conclude that

$$N_P \simeq N_{P,\sigma\theta} \times N_{P,\sigma,+} \times (N_P \cap H).$$

On the other hand, $\mathfrak{n}_{P,+} = \mathfrak{n}_{P,\sigma,+} \oplus \mathfrak{n}_{P,\sigma\theta}$. From this we infer by application of Lemma 4.2.10 that

$$N_{P,\sigma\theta} \times N_{P,\sigma,+} \simeq N_{P,+}$$

The result now follows. $\square$

Remark 4.2.15. For the case of an $\mathfrak{h}$-extreme parabolic subgroup, Proposition 4.2.13 is due to [1], where, for this special case, a different proof of the result is given.

4.3 Auxiliary results in convex linear algebra

In this section we present a few results in convex linear algebra which will be used in Section 4.4.

Lemma 4.3.1. *Let V be a finite dimensional real linear space and $B \subseteq V$ a closed subset, star-shaped about the origin. If B is non-compact, then there exists a $v \in V \setminus \{0\}$ such that $\mathbb{R}_{\geq 0} v \subseteq B$.*

Proof. Since B is star-shaped, we have $sB = t(s/t)B \subseteq tB$ for all $0 < s < t$. Fix a positive definite inner product on V and let S be the associated unit sphere centered at the origin. For $s > 0$ we define the compact set $C_s := s^{-1}B \cap S$. Then $s < t \implies C_s \supset C_t$. As B is unbounded and starshaped, each of the sets C_s is non-empty. It follows that the intersection

$$C := \cap_{s>0} C_s$$

is non-empty. Let v be a point in this intersection. Then $v \neq 0$ and for all $s > 0$ we have $sv \in sC_s \subseteq B$. Hence, $\mathbb{R}_{\geq 0} v \subseteq B$. $\square$

Lemma 4.3.2. *Let V and W be two finite dimensional real linear spaces, $p : V \to W$ a linear map and $\Gamma \subseteq V$ a closed convex cone. Then the following assertions are equivalent.*

(a) *$p|_\Gamma$ is a proper map.*

(b) $\ker p \cap \Gamma = \{0\}$.

Proof. First we prove that (a) implies (b). Assume (b) doesn't hold, i.e. there exists $v \in \ker p \cap \Gamma$, $v \neq 0$. Then $\mathbb{R}_{\geq 0} v \subseteq \ker p \cap \Gamma = (p|_\Gamma)^{-1}(0)$ and we obtain that $(p|_\Gamma)^{-1}(0)$ is not compact and hence $p|_\Gamma$ is not a proper map.

For the converse implication, assume that (a) does not hold. Then there exists a compact set $K \subseteq W$, such that the set $p^{-1}(K) \cap \Gamma$ is not compact. As the latter set is closed, it is unbounded in V. Let $\bar{K}$ be the convex hull of $K \cup \{0\}$. Then $\bar{K}$ is compact and $p^{-1}(\bar{K}) \cap \Gamma$ is convex, contains 0 and is unbounded in V, hence not compact. We apply Lemma 4.3.1 and obtain that there exists $v \neq 0$ such that $\forall t \geq 0 : tv \in p^{-1}(\bar{K}) \cap \Gamma$. Hence, $t \cdot p(v) \in \bar{K}$ for every $t \geq 0$. Since $\bar{K}$ is compact, it follows that $p(v) = 0$ and $v \in \ker p \cap \Gamma$, which implies that (b) cannot hold. □

Lemma 4.3.3. *Let V be a finite dimensional real linear space, and Γ a closed convex cone in V such that there exists a linear functional $\xi \in V^*$ with $\xi > 0$ on $\Gamma \setminus \{0\}$. Then the following holds.*

(a) *For every $R > 0$ the set $\{x \in \Gamma : \ \xi(x) \leq R\}$ is compact.*

(b) *The addition map $a : (x, y) \mapsto x + y, \ \Gamma \times \Gamma \to V$, is proper.*

Proof. Let $R > 0$. The set $\Gamma_R := \{x \in \Gamma : \ \xi(x) \leq R\}$ is closed and convex and it contains the origin. If $v \in \Gamma_R \setminus \{0\}$ then the half line $\mathbb{R}_{\geq 0} v$ is not contained in Γ_R. By Lemma 4.3.1 we infer that Γ_R is compact, hence (a).

We turn to (b). Assume $\mathcal{K} \subseteq V$ is compact. Then there exist an $R > 0$ such that $\xi \leq R$ on $\mathcal{K}$. Let $(x, y) \in a^{-1}(\mathcal{K})$. Then it follows that $\xi(x + y) \leq R$, hence $\xi(x) \leq R$ and $\xi(y) \leq R$, so that (x, y) belongs to the compact set $\Gamma_R \times \Gamma_R$. We conclude that $a^{-1}(\mathcal{K})$ is a closed subset of $\Gamma_R \times \Gamma_R$, hence compact. □

If S is a subset of $\Sigma(\mathfrak{g}, \mathfrak{a})$ then the convex cone

$$\Gamma_{\mathfrak{a}}(S) := \sum_{\alpha \in S} \mathbb{R}_{\geq 0} H_\alpha.$$

is finitely generated, hence closed in $\mathfrak{a}$. Likewise,

$$\Gamma_{\mathfrak{a}_{\mathrm{q}}}(S) := \mathrm{pr}_{\mathrm{q}} \Gamma_{\mathfrak{a}}(S) = \sum_{\alpha \in S} \mathbb{R}_{\geq 0} \, \mathrm{pr}_{\mathrm{q}}(H_\alpha)$$

is a closed and convex cone in $\mathfrak{a}_{\mathrm{q}}$.

Corollary 4.3.4. *Let $P \in \mathcal{P}(A)$. Then the following assertions are valid.*

(a) *The map* $\mathrm{pr}_{\mathrm{q}} : \Gamma_{\mathfrak{a}}(\Sigma(P, \sigma\theta)) \to \mathfrak{a}_{\mathrm{q}}$ *is proper.*

(b) *The addition map* $\; a : \Gamma_{\mathfrak{a}_{\mathrm{q}}}(\Sigma(P, \sigma\theta)) \times \Gamma_{\mathfrak{a}_{\mathrm{q}}}(\Sigma(P, \sigma\theta)) \to \mathfrak{a}_{\mathrm{q}}$ *is proper.*

Proof. We start with (a). In view of Lemma 4.3.2 it suffices to establish the claim that $\Gamma_{\mathfrak{a}}(\Sigma(P, \sigma\theta)) \cap \mathfrak{a}_{\mathrm{h}} = 0$. This can be done as follows. There exists a $Y \in \mathfrak{a}$ such that $\alpha(Y) > 0$ for all $\alpha \in \Sigma(P)$. Put $X := Y + \sigma\theta Y = Y - \sigma(Y)$, then $X \in \mathfrak{a}_{\mathrm{q}}$ and $\langle X, H_\alpha \rangle = \alpha(X)/2 = (\alpha + \sigma\theta\alpha)(Y)/2 > 0$ for all $\alpha \in \Sigma(P, \sigma\theta)$. It follows that the linear functional $\xi = \langle X, \cdot \rangle \in \mathfrak{a}^*$ has strictly positive values on $\Gamma_{\mathfrak{a}}(\Sigma(P, \sigma\theta)) \setminus \{0\}$. Now $\xi = 0$ on $\mathfrak{a}_{\mathrm{h}}$ and we see that the claim is valid. Hence, (a).

For (b) we proceed as follows. Let ξ be as above, then $\ker \mathrm{pr}_{\mathrm{q}} \subseteq \ker \xi$ and we see that $\xi > 0$ on $\Gamma_{\mathfrak{a}_{\mathrm{q}}}(\Sigma(P, \sigma\theta)) \setminus \{0\}$. Now use Lemma 4.3.3. $\square$

4.4 Properness of the Iwasawa projection

Let $P \in \mathcal{P}(A)$ and let $\mathfrak{H}_P : G \to \mathfrak{a}$ be the Iwasawa projection defined by (2.8). Let $\mathfrak{H}_{P,\mathrm{q}} : G \to \mathfrak{a}_{\mathrm{q}}$ be defined as in Remark 3.1.2. The purpose of this section is to prove that the restriction of $\mathfrak{H}_{P,\mathrm{q}}$ to H factors through a proper map $H/H \cap P \to \mathfrak{a}_{\mathrm{q}}$.

We start with a simple lemma.

Lemma 4.4.1. *The map* $\mathfrak{H}_{P,\mathrm{q}}|_H : H \to \mathfrak{a}_{\mathrm{q}}$ *is left* $K \cap H$*- and right* $(P \cap H)$*-invariant.*

Proof. Let $h \in H$, $k_H \in K \cap H$ and $p \in P \cap H$. By the Iwasawa decomposition, the element h may be decomposed as $h = kan$, with $k \in K$, $a \in A$ and $n \in N_P$. In view of Lemma 4.2.11 we may decompose $p = mbn'$, with $m \in M \cap H$, $b \in A \cap H$ and $n' \in N_P \cap H$. Since MA normalizes N_P and centralizes A we find

$$k_H hp = k_H kanmbn' = (k_H km)ab((mb)^{-1}n(mb))n' \in KabN_P.$$

From this we deduce that

$$\mathfrak{H}_{P,\mathrm{q}}(k_H hp) = \mathrm{pr}_{\mathrm{q}}\,(\log a + \log b) = \mathrm{pr}_{\mathrm{q}}\, \log a = \mathfrak{H}_{P,\mathrm{q}}(h).$$ $\square$

It follows from the above lemma that the restriction of $\mathfrak{H}_{P,\mathrm{q}}$ to H induces a smooth map

$$\overline{\mathfrak{H}}_{P,\mathrm{q}} : H/H \cap P \to \mathfrak{a}_{\mathrm{q}}. \tag{4.10}$$

The following proposition is the main result of this section.

Proposition 4.4.2. *The induced map (4.10) is proper.*

In order to prove the proposition, we will reduce it to another result, Proposition 4.4.7, establishing some useful lemmas along the way.

We fix Q_{h} in $\mathfrak{h}$-extreme position and related to P as in Lemma 4.2.6. Let $Z_G(\mathfrak{a}_{\mathrm{h}})$ denote the centralizer of $\mathfrak{a}_{\mathrm{h}}$ in G and define the parabolic subgroup

$$R := Z_G(\mathfrak{a}_{\mathrm{h}})N_{Q_{\mathrm{h}}}. \tag{4.11}$$

Let $\mathfrak{n}_R$ be the sum of the root spaces $\mathfrak{g}_\alpha$ for $\alpha \in \Sigma(Q_{\mathrm{h}}, \sigma) = \Sigma(Q_{\mathrm{h}}) \setminus \mathfrak{a}_{\mathrm{q}}^*$ and put $N_R := \exp(\mathfrak{n}_R)$. Then N_R is σ-stable. It is readily seen that R has the Levi decomposition $R = L_R N_R$ where $L_R = Z_G(\mathfrak{a}_{\mathrm{h}})$ is σ-stable. Hence, R is σ-stable. Let $\Sigma(R)$ denote the set of $\mathfrak{a}$-roots that appear in $\mathfrak{n}_R$.

Lemma 4.4.3. $\Sigma(P) \cap \Sigma(\bar{R}) \subseteq \Sigma(P, \sigma\theta)$.

Proof. Let $\alpha \in \Sigma(P) \cap \Sigma(\bar{R})$. Then $\alpha \in \Sigma(\bar{Q}_{\mathrm{h}})$, hence $\alpha \notin \Sigma(P, \sigma)$, see Lemma 4.2.6. This implies that $\alpha \in \Sigma(P, \sigma\theta)$. □

Let $R = M_R A_R N_R$ be the Langlands decomposition of R. Then $L_R = M_R A_R$.

Lemma 4.4.4. *The multiplication map*

$$\mu : (K \cap H) \times (M_R \cap H) \times (N_R \cap H)/(N_R \cap H \cap P) \longrightarrow H/H \cap P,$$

given by $(k, m, [n]) \mapsto km[n]$ *is surjective.*

Proof. The map $K \times (\mathfrak{l}_R \cap \mathfrak{p}) \times N_R \to G$ given by $(k, X, n) \mapsto k \exp X n$ is a diffeomorphism. Since K, $\mathfrak{l}_R \cap \mathfrak{p}$ and N_R are σ-stable, it follows that

$$H = (K \cap H)(L_R \cap H)(N_R \cap H). \tag{4.12}$$

Now $L_R = M_R A_R$ with M_R and A_R both σ-stable. Since $A_R \cap H$ normalizes $N_R \cap H$, we have that

$$\begin{aligned} H &= (K \cap H)(M_R \cap H)(A_R \cap H)(N_R \cap H) \\ &= (K \cap H)(M_R \cap H)(N_R \cap H)(A_R \cap H). \end{aligned}$$

This implies the result. □

We equip $M_R \cap H$ with the natural right-action of the closed subgroup $M_R \cap H \cap P$. The latter group acts on $N_R \cap H$ by conjugation. Moreover, since M_R normalizes N_R and P normalizes N_P, the conjugation action leaves the closed subgroup $N_R \cap H \cap P$ invariant. Accordingly, we have an induced right-action of $M_R \cap H \cap P$ on $(N_R \cap H)/(N_R \cap H \cap P)$ given by

$$[n] \cdot m = [m^{-1}nm], \qquad (m \in M_R \cap H \cap P, n \in N_R \cap H).$$

We equip $(M_R \cap H) \times (N_R \cap H)/(N_R \cap H \cap P)$ with the product action by $M_R \cap H \cap P$. This action is proper and free, so that the associated quotient space $(M_R \cap H) \times_{M_R \cap H \cap P} (N_R \cap H)/(N_R \cap H \cap P)$ is smooth.

Lemma 4.4.5. *The multiplication map of Lemma 4.4.4 induces a surjective smooth map*

$$\bar{\mu} : (K \cap H) \times (M_R \cap H) \times_{M_R \cap H \cap P} (N_R \cap H)/(N_R \cap H \cap P) \to H/H \cap P.$$

Proof. Let $k \in K \cap H$, $m \in M_R \cap H$ and $n \in N_R \cap H$. Then for $p \in M_R \cap H \cap P$ we have

$$\mu(k, (m, [n])\cdot p) = \mu(k, mp, [p^{-1}np]) = kmp(p^{-1}np)[e] = kmn[e] = \mu(k, m, [n]).$$

This implies that μ induces a smooth map $\bar{\mu}$ as described. The surjectivity of $\bar{\mu}$ follows from the surjectivity of μ. $\square$

Proposition 4.4.2 will follow from the result that the composition $\mathfrak{H}_{P,\mathrm{q}} \circ \bar{\mu}$ is proper. The latter map is left-invariant under the left action of $K \cap H$ on the first component. Thus, Proposition 4.4.2 will already follow from the following result.

Lemma 4.4.6. *The map $(m, n) \mapsto \mathfrak{H}_{P,\mathrm{q}}(mn)$ induces a smooth map*

$$\varphi : \ (M_R \cap H) \times_{M_R \cap H \cap P} (N_R \cap H)/(N_R \cap H \cap P) \to \mathfrak{a}_{\mathrm{q}}$$

which is proper.

The inclusion map $N_R \cap H \to N_R$ induces an embedding of $(N_R \cap H)/(N_R \cap H \cap P)$ onto a closed submanifold of $N_R/N_R \cap P$. This embedding is equivariant for the conjugation action of $M_R \cap H \cap P$. Accordingly, we may view

$$(M_R \cap H) \times_{M_R \cap H \cap P} (N_R \cap H)/(N_R \cap H \cap P)$$

as a closed submanifold of

$$(M_R \cap H) \times_{M_R \cap H \cap P} N_R/(N_R \cap P).$$

Thus, for the proof of Lemma 4.4.6 it suffices to establish the following result.

Proposition 4.4.7. *The map $\psi : (m, n) \mapsto \mathfrak{H}_{P,\mathrm{q}}(mn)$ induces a smooth map*

$$\bar{\psi} : (M_R \cap H) \times_{M_R \cap H \cap P} N_R/(N_R \cap P) \to \mathfrak{a}_{\mathrm{q}}. \tag{4.13}$$

This map is proper.

Before we proceed with the proof of Proposition 4.4.7 we will first study the maps $M_R \cap H/M_R \cap H \cap P \to \mathfrak{a}_{\mathrm{q}}$ and $N_R/(N_R \cap P) \to \mathfrak{a}_{\mathrm{q}}$ induced by $\mathfrak{H}_{P,\mathrm{q}}$.

Lemma 4.4.8. *The map $\mathfrak{H}^R_{P,\mathrm{q}} := \mathfrak{H}_{P,\mathrm{q}}|_{M_R \cap H}$ induces a smooth map $\bar{\mathfrak{H}}^R_{P,\mathrm{q}} : (M_R \cap H)/(M_R \cap H \cap P) \to \mathfrak{a}_{\mathrm{q}}$ which is proper and has image equal to the cone $\Gamma_{\mathfrak{a}_{\mathrm{q}}}(\Sigma^R_-)$, where*

$$\Sigma^R_- = \{\alpha \in \Sigma(P) \cap \mathfrak{a}^*_{\mathrm{q}} : \ \mathfrak{g}_\alpha \not\subset \ker(\sigma\theta - I)\}.$$

In particular, the image is contained in the cone $\Gamma_{\mathfrak{a}_{\mathrm{q}}}(\Sigma(P, \sigma\theta))$.

Proof. We start by noting that $(M_R, M_R \cap H)$ is a reductive symmetric pair of the Harish-Chandra class, which is invariant under the Cartan involution θ. Furthermore, ${}^*\mathfrak{a}_R := \mathfrak{m}_R \cap \mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{m}_R \cap \mathfrak{p}$ (contained in $\mathfrak{a}_{\mathrm{q}}$) and $M_R \cap P$ is a minimal parabolic subgroup of M_R containing ${}^*A_R := \exp {}^*\mathfrak{a}_R$. Accordingly, by restriction the Iwasawa projection map $\mathfrak{H}_{P,\mathrm{q}} : H \to \mathfrak{a}_{\mathrm{q}}$ induces the similar projection map $\mathfrak{H}^R_{P,\mathrm{q}} : M_R \cap H \to \mathfrak{a}_{\mathrm{q}}$ which is the analogue of $\mathfrak{H}_{P,\mathrm{q}}$ defined relative to the data $M_R, M_R \cap K, P \cap M_R, H \cap M_R$, in place of G, K, P, H.

The ${}^*\mathfrak{a}_R$-roots in $N_P \cap M_R$ are precisely the restrictions of the roots from $\Sigma(P) \cap \mathfrak{a}^*_{\mathrm{q}}$. From this we see that the minimal parabolic subgroup $P \cap M_R$ of M_R is $\sigma\theta$-stable. Hence, in view of [5, Lemma 3.3], the map $\bar{\mathfrak{H}}^R_{P,\mathrm{q}}$ is proper and has image equal to the cone $\Gamma_{\mathfrak{a}_{\mathrm{q}}}(\Sigma^R_-)$ given above. The final assertion now follows from the observation that $\Sigma(P) \cap \mathfrak{a}^*_{\mathrm{q}} \subseteq \Sigma(P, \sigma\theta)$. $\square$

The following lemma is well known. For completeness of the exposition, we provide the proof.

Lemma 4.4.9. *The Iwasawa map $\mathfrak{H}_P|_{\bar{N}_P} : \bar{N}_P \to \mathfrak{a}$ is proper. If $Q \in \mathcal{P}(A)$, then*

$$\mathfrak{H}_P(N_Q \cap \bar{N}_P) = \Gamma_{\mathfrak{a}}(\Sigma(P) \cap \Sigma(\bar{Q})).$$

Proof. For the first assertion, let $(\bar{n}_j)$ be sequence in $\bar{N}_P$ such that $\mathfrak{H}_P(\bar{n}_j)$ converges. Then $\bar{n}_j = k_j a_j n_j$, with $k_j \in K$, $a_j = \exp \mathfrak{H}_P(\bar{n}_j)$ and $n_j \in N_P$. By passing to a converging subsequence, we may arrange that in addition the sequence (k_j) converges in K. It follows that $n_j^{-1} a_j^{-1} \bar{n}_j = k_j$ converges in G. By [25, Lemma 39], the sequence $(\bar{n}_j)$ converges.

For the second assertion, we may assume $\Sigma(\bar{Q}) \cap \Sigma(P) \neq \emptyset$ and use the idea due to S. Gindikin and F. Karpelevic [22], to decompose $N_Q \cap \bar{N}_P$ by using a P-simple root in $\Sigma(\bar{Q}) \cap \Sigma(P)$. Let α be such a root. Let $\mathfrak{n}_\alpha = \mathfrak{g}_\alpha + \mathfrak{g}_{2\alpha}$ and $N_\alpha = \exp \mathfrak{n}_\alpha$. Put $Q' = s_\alpha Q s_\alpha$. Then, with the notation of Subsection 4.2.1,

$$\Sigma_\circ(\bar{Q}) \cap \Sigma_\circ(P) = \{\alpha\} \sqcup (\Sigma_\circ(\bar{Q}') \cap \Sigma_\circ(P)),$$

so that

$$N_Q \cap \bar{N}_P = \bar{N}_\alpha(N_{Q'} \cap \bar{N}_P) \simeq \bar{N}_\alpha \times (N_{Q'} \cap \bar{N}_P).$$

Let $\bar{n} \in N_Q \cap \bar{N}_P$. Then according to the above decomposition we may write $\bar{n} = \bar{n}_\alpha \bar{n}'$, where $\bar{n}_\alpha \in \bar{N}_\alpha$ and $\bar{n}' \in N_{Q'} \cap N_P$. Let $\mathfrak{g}(\alpha)$ be the semisimple subalgebra generated by $\mathfrak{n}_\alpha$ and $\bar{\mathfrak{n}}_\alpha$ and let $G(\alpha)$ be the corresponding analytic subgroup of G. By the Iwasawa decomposition of $G(\alpha)$ for the minimal parabolic subgroup $P \cap G(\alpha)$ we may write $\bar{n}_\alpha = k_\alpha a_\alpha n_\alpha$ with $k_\alpha \in G(\alpha) \cap K$, $a_\alpha \in \exp(\mathbb{R}H_\alpha)$ and $n_\alpha \in N_\alpha$. From the fact that

$$N_{Q'} \cap \bar{N}_P \simeq N_{Q'}/(N_{Q'} \cap N_P),$$

we see that there exists a diffeomorphism τ_{n_α} of $N_{Q'} \cap \bar{N}_P$ onto itself, such that

$$n_\alpha \bar{n}' \in \tau_{n_\alpha}(\bar{n}') N_P, \quad \text{for all } \bar{n}' \in N_{Q'} \cap \bar{N}_P.$$

This implies that

$$\mathfrak{H}_P(\bar{n}_\alpha n') = \mathfrak{H}_P(a_\alpha \tau_{n_\alpha}(\bar{n}')a_\alpha^{-1}) + \log a_\alpha,$$

and we see that

$$\mathfrak{H}_P(\bar{N}_\alpha(N'_Q \cap \bar{N}_P)) = \mathfrak{H}_P(N'_Q \cap \bar{N}_P) + \mathfrak{H}_P(\bar{N}_\alpha).$$

Now $\mathfrak{H}_P(\bar{N}_\alpha)$ equals the image of $\bar{N}_\alpha$ under the Iwasawa projection $\mathfrak{H}_\alpha$ for the split rank 1 group $G(\alpha)$ and the minimal parabolic subgroup $P \cap G(\alpha)$. By [28, Theorem IX.3.8], which is based on $\mathrm{SU}(2,1)$-reduction, we see that $\mathfrak{H}_\alpha(\bar{N}_\alpha) = \mathbb{R}_{\geq 0} H_\alpha$. It follows that

$$\mathfrak{H}_P(\bar{N}_\alpha(N'_Q \cap \bar{N}_P)) = \mathfrak{H}_P(N'_Q \cap \bar{N}_P) + \mathbb{R}_{\geq 0} H_\alpha.$$

The proof is completed by induction on the number of elements in $\Sigma_\circ(\bar{Q}) \cap \Sigma_\circ(P)$. □

The following lemma is the second ingredient for the proof of Proposition 4.4.7.

Lemma 4.4.10. *The Iwasawa map $\mathfrak{H}_{P,\mathrm{q}}|_{N_R} : N_R \to \mathfrak{a}_\mathrm{q}$ factors through a proper map $N_R/N_R \cap N_P \to \mathfrak{a}_\mathrm{q}$ with image equal to the cone*

$$\Gamma_{\mathfrak{a}_\mathrm{q}}(\Sigma(P) \cap \Sigma(\bar{R})). \tag{4.14}$$

In particular, the image is contained in the cone $\Gamma_{\mathfrak{a}_\mathrm{q}}(\Sigma(P, \sigma\theta))$.

Proof. We denote the induced map by $\mathfrak{H}$. It follows by application of Lemma 4.2.10 that the multiplication map $(N_R \cap \bar{N}_P) \times (N_R \cap N_P) \to N_R$ is a diffeomorphism. Let $\nu : N_R \cap \bar{N}_P \to N_R/N_R \cap N_P$ denote the induced diffeomorphism. Then $\mathfrak{H} \circ \nu$ equals $\mathrm{pr}_\mathrm{q} \circ \mathfrak{H}_{P,R}$, where $\mathfrak{H}_{P,R}$ denotes the restriction of $\mathfrak{H}_P$ to $N_R \cap \bar{N}_P$. This restriction is proper with image $\Gamma_\mathfrak{a}(\Sigma(P) \cap \Sigma(\bar{R}))$, by Lemma 4.4.9 above. In particular, the image is contained in the cone $\Gamma_\mathfrak{a}(\Sigma(P, \sigma\theta))$, by Lemma 4.4.3. In view of Corollary 4.3.4 (a) it now follows that $\mathfrak{H} \circ \nu = \mathrm{pr}_\mathrm{q} \circ \mathfrak{H}_{P,R}$ is proper with image equal to (4.14). This implies the result. □

We proceed with a final lemma needed for the proof of Proposition 4.4.7.

Lemma 4.4.11. *Let $\bar{\psi}$ be as in (4.13) and let*

$$\bar{\mathrm{pr}}_1 : (M_R \cap H) \times_{M_R \cap H \cap P} N_R/(N_R \cap P) \to (M_R \cap H)/(M_R \cap H \cap P)$$

denote the map induced by projection onto the first component.

Let $C \subseteq \mathfrak{a}_\mathrm{q}$ be a compact set. Then the set $\bar{\mathrm{pr}}_1(\bar{\psi}^{-1}(C))$ is relatively compact in $(M_R \cap H)/(M_R \cap H \cap P)$.

Proof. Let $m \mapsto [m]$ denote the canonical projection $M_R \cap H \to (M_R \cap H)/(M_R \cap H \cap P)$. Let (m_j) and (n_j) be sequences in $M \cap H$ and N_R, respectively, such that $\mathfrak{H}_P(m_j n_j) \in C$ for all j. Then it suffices to show that the sequence $([m_j])$ in $(M_R \cap H)/(M_R \cap H \cap P)$ has a converging subsequence.

In accordance with the Iwasawa decomposition $M_R = (K \cap M_R)(A \cap M_R)(N_P \cap M_R)$, we may decompose $m_j = k_j a_j \nu_j$. Since $\mathfrak{a}_{\mathrm{h}} \subseteq \mathfrak{a}_R = \mathrm{center}(\mathfrak{l}_R) \cap \mathfrak{p}$, we have $\mathfrak{m}_R \cap \mathfrak{a} = \mathfrak{a}_R^{\perp} \cap \mathfrak{a} \subseteq \mathfrak{a}_{\mathrm{q}}$, so that $\log a_j = \mathfrak{H}_{P,\mathrm{q}}^R(m_j)$.

The element $t_j = a_j \nu_j$ belongs to M_R, hence $n_j' := t_j n_j t_j^{-1} \in N_R$, for all j. From $m_j n_j = k_j n_j' a_j \nu_j$ it follows that

$$\mathfrak{H}_{P,\mathrm{q}}(m_j n_j) = \mathfrak{H}_{P,\mathrm{q}}(k_j n_j') + \log a_j = \mathfrak{H}_{P,\mathrm{q}}(n_j') + \mathfrak{H}_{P,\mathrm{q}}^R(m_j).$$

We now note that both $\mathfrak{H}_{P,\mathrm{q}}(n_j')$ and $\mathfrak{H}_{P,\mathrm{q}}(m_j)$ belong to $\Gamma_{\mathfrak{a}_{\mathrm{q}}}(P, \sigma\theta)$ by Lemma 4.4.10 and Lemma 4.4.8. By application of Corollary 4.3.4 we infer that the sequence $\mathfrak{H}_{P,\mathrm{q}}(m_j)$ is contained in a relatively compact subset of $\mathfrak{a}_{\mathrm{q}}$. By application of Lemma 4.4.8 it now follows that $([m_j])$ is contained in a relatively compact subset of $(M_R \cap H)/(M_R \cap H \cap P)$, hence contains a convergent subsequence. $\square$

Completion of the proof of Proposition 4.4.7. Let C be a compact subset of $\mathfrak{a}_{\mathrm{q}}$ and let (m_j) be a sequence in $M_R \cap H$ and (n_j) a sequence in N_R such that $\bar{\psi}([(m_j, n_j)]) \in C$ for all j. Then it suffices to show that the sequence of points

$$[(m_j, n_j)] \in (M_R \cap H) \times_{M_R \cap H \cap P} N_R/(N_R \cap N_P)$$

has a converging subsequence.

In view of Lemma 4.4.11 we may pass to a subsequence of indices and assume that the sequence $([m_j])$ in $D := (M_R \cap H)/(M_R \cap H \cap P)$ converges. Since the canonical projection $M_R \cap H \to D$ determines a principal fiber bundle, we may invoke a local trivialization to obtain a converging sequence $({}^{\backprime}m_j)$ in $M_R \cap H$ such that ${}^{\backprime}m_j \in m_j(M_R \cap H \cap P)$ for all j. Let $p_j \in M_R \cap H \cap P$ be such that $m_j = {}^{\backprime}m_j p_j$ for all j. Then

$$[(m_j, n_j)] = [({}^{\backprime}m_j, {}^{\backprime}n_j)],$$

with ${}^{\backprime}n_j = p_j n_j p_j^{-1} \in N_R$.

Replacing the original sequence of points (m_j, n_j) in this fashion if necessary, we may as well assume that the original sequence (m_j) converges in $M_R \cap H$. Let $m \in M_R \cap H$ be the limit of this sequence. As in the proof of Lemma 4.4.11 we may decompose $m_j = k_j a_j \nu_j$ and $m = ka\nu$ in accordance with the Iwasawa decomposition $M_R = (M_R \cap K)(M_R \cap A)(M_R \cap N)$. Then $k_j \to k$, $a_j \to a$ and $\nu_j \to \nu$, for $j \to \infty$. Put $t_j = a_j \nu_j$ and $n_j' = t_j n_j t_j^{-1}$. As in the proof of Lemma 4.4.11 it follows that

$$\bar{\psi}([m_j, n_j]) = \log a_j + \mathfrak{H}_{P,\mathrm{q}}(n_j').$$

Since (a_j) converges, it follows that the sequence $\mathfrak{H}_{P,\mathrm{q}}(n_j')$ is contained in a compact subset $C' \subseteq \mathfrak{a}_{\mathrm{q}}$. By Lemma 4.4.10 it follows that the sequence $([n_j'])$ in $N_R/N_R \cap N_P$

is contained in a compact subset. Passing to a suitable subsequence of indices we may as well assume that the sequence $([n_j'])$ converges to a point $[n]$, for some $n \in N_R$. It follows that

$$[n_j] = [t_j^{-1} n_j' t_j] = t_j^{-1} \cdot [n_j'] \to t^{-1} \cdot [n] = [t^{-1} n t], \quad (j \to \infty),$$

where $t = a\nu$. We conclude that the sequence $[(m_j, n_j)]$ converges with limit equal to $[(m, t^{-1}n)]$. $\square$

We finish this section with a number of results that will be needed in Section 4.10.

Corollary 4.4.12. *Let $\mathcal{A}$ be a compact subset of A_{q}. Then the map*

$$(a, h) \mapsto \mathfrak{H}_{P,\mathrm{q}}(ah)$$

induces a proper map $\mathcal{A} \times H/H \cap P \to \mathfrak{a}_{\mathrm{q}}$.

Proof. Let C be a compact subset of $\mathfrak{a}_{\mathrm{q}}$. Let C' be the compact convex hull of the union of the sets $w(\log \mathcal{A})$, for w in the Weyl group $W(\mathfrak{a}) = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$.

Let $a \in \mathcal{A}$ and $h \in H$ and assume that $\mathfrak{H}_{P,\mathrm{q}}(ah) \in C$. We may decompose $h = kbn$ with $k \in K$, $b \in A$ and $n \in N_P$. By Kostant's convexity theorem, $ak = k'a'n'$, with $k' \in K$, $n' \in N_P$ and $\log a'$ contained in the (compact) convex hull of $W(\mathfrak{a}) \cdot \log a$ hence in C'. Now

$$ah = akbn = k'a'n'bn = k'a'bn''$$

with $n'' = b^{-1}nbn' \in N_P$. It follows that

$$\mathfrak{H}_{P,\mathrm{q}}(ah) = \mathrm{pr}_{\mathrm{q}}(\log b + \log a') = \mathfrak{H}_{P,\mathrm{q}}(h) + \mathrm{pr}_{\mathrm{q}}(\log a'),$$

so that $\mathfrak{H}_{P,\mathrm{q}}(h)$ is contained in the compact set $C'' = C + \mathrm{pr}_{\mathrm{q}}(-C')$. By Proposition 4.4.2, the preimage C_H of C'' in $H/H \cap P$ is compact. It follows from the above that $(a, [h]) \in \mathcal{A} \times C_H$. Hence the preimage of C in $\mathcal{A} \times H/H \cap P$ is compact, and the result follows. $\square$

Corollary 4.4.13. *Let $P \in \mathcal{P}(A)$. Then $\mathfrak{H}_{P,\mathrm{q}}(H) \subseteq \Gamma_{\mathfrak{a}_{\mathrm{q}}}(\Sigma(P, \sigma\theta))$.*

Proof. By (4.12) we have

$$H = (H \cap K)(H \cap N_R)(H \cap L_R) \subseteq KN_R(H \cap L_R).$$

Fix $h \in H$, then we may write $h = kn_R h_L$ with $k \in K$, $n_R \in N_R$ and $h_L \in (H \cap L_R)$. The group $P \cap L_R$ is a minimal parabolic subgroup of L_R, containing A. In accordance with the associated Iwasawa decomposition for L_R, we may write $h_L = k_L a_L n_L$ with $k_L \in K \cap L_R$, $a_L \in A$ and $n_L \in N_P \cap L_R$. Since L_R normalizes N_R, it follows that

$$h = kn_R k_L a_L n_L \in Kn_R' a_L n_L$$

with $n'_R \in N_R$. We now observe that $n'_R \in KbN_P$ with $b = \exp \mathfrak{H}_P(n'_R)$. Thus, $h \in Kba_LN_P$. It follows that

$$\mathfrak{H}_{P,\mathrm{q}}(h) = \mathrm{pr}_\mathrm{q}(\log b + \log a_L) \in \mathfrak{H}_{P,\mathrm{q}}(N_R) + \mathfrak{H}_{P,\mathrm{q}}(H \cap L_R). \tag{4.15}$$

The result now follows by combining the fact that $\mathfrak{H}_{P,\mathrm{q}}(H \cap M_R) = \mathfrak{H}_{P,\mathrm{q}}(H \cap L_R)$ with Lemmas 4.4.8 and 4.4.10. □

Lemma 4.4.14. *Let Γ_1 and Γ_2 be two closed cones inside some vector space V and $B \subset V$ a compact subset. If $\Gamma_1 \subseteq B + \Gamma_2$ then $\Gamma_1 \subseteq \Gamma_2$.*

Proof. Let $\gamma_1 \in \Gamma_1$. Then for all positive (or negative) $n \in \mathbb{R}$ we have that $n\gamma_1 \in \Gamma_1 \subseteq B + \Gamma_2$ (we may assume $n > 0$). Thus,

$$n\gamma_1 = b_n + \gamma_{2n}$$

for $b_n \in B$ and $\gamma_{2n} \in \Gamma_2$. It follows that

$$\gamma_1 = b_n/n + \gamma_{2n}/n$$

and as n goes to infinity b_n/n goes to zero (since B is compact), while γ_{2n}/n converges to some $\gamma_2 \in \Gamma_2$ (Γ_2 being closed). Thus, $\gamma_1 = \gamma_2 \in \Gamma_2$. □

Corollary 4.4.15. *Let $P \in \mathcal{P}(A)$. Then for each $a \in A_\mathrm{q}$, the set $\mathfrak{H}_{P,\mathrm{q}}(aH)$ does not contain any line of $\mathfrak{a}_\mathrm{q}$.*

Proof. Let $h \in H$. We may write $h = kbn$ with $k \in K$, $n \in N_P$ and $b = \exp \mathfrak{H}_P(h)$. Furthermore, $ak = k'a'n'$ in accordance with $G = KAN_P$. Then $a' = \exp \mathfrak{H}_P(ak)$. It follows that

$$ah = k'a'n'bn \in Ka'bN_P,$$

so that

$$\begin{aligned} \mathfrak{H}_{P,\mathrm{q}}(ah) = \mathrm{pr}_\mathrm{q}(\log a' + \log b) &\in \mathfrak{H}_{P,\mathrm{q}}(aK) + \mathfrak{H}_{P,\mathrm{q}}(h) \\ &\subseteq \mathfrak{H}_{P,\mathrm{q}}(aK) + \Gamma_{\mathfrak{a}_\mathrm{q}}(\Sigma(P, \sigma\theta)) \end{aligned} \tag{4.16}$$

by Lemma 4.4.13.

For the completion of the proof we will argue by contradiction. Suppose that $\mathfrak{H}_{P,\mathrm{q}}(ah)$ contains a line of the form $Z + \mathbb{R}Y$, with $Y \in \mathfrak{a}_\mathrm{q} \setminus \{0\}$. There exists an element $X \in \mathfrak{a}$ such that $\alpha(X) > 0$ for all $\alpha \in \Sigma(P)$ and such that $\langle X, Y\rangle \neq 0$. Then it follows that $\alpha(X + \sigma\theta X) > 0$ for all $\alpha \in \Sigma(P, \sigma\theta)$. Hence $\langle X, \mathrm{pr}_\mathrm{q} H_\alpha\rangle > 0$ for all $\alpha \in \Sigma(P, \sigma\theta)$. Since $\mathfrak{H}_{P,\mathrm{q}}(aK)$ is compact, it follows that $\langle X, \cdot\,\rangle$ is bounded from below on the set in the right-hand side of (4.16), hence also on the line $Z + \mathbb{R}Y$. This implies that $\langle X, Y\rangle = 0$, contradiction. □

4.5 Critical points of components of the Iwasawa map

In this section we assume that $P \in \mathcal{P}(A)$ is a fixed minimal parabolic subgroup and that a is a fixed element of A_{q}. We will investigate the critical sets of components of the map $h \mapsto \mathfrak{H}_{P,\mathrm{q}}(ah)$, $H \to \mathfrak{a}_{\mathrm{q}}$. For this, let $X \in \mathfrak{a}_{\mathrm{q}}$, and consider the function $F_{a,X} : H \to \mathbb{R}$ defined by

$$F_{a,X}(h) = \langle X, \mathfrak{H}_P(ah)\rangle = \langle X, \mathfrak{H}_{P,\mathrm{q}}(ah)\rangle = B(X, \mathfrak{H}_{P,\mathrm{q}}(ah)). \tag{4.17}$$

The second equality is valid because $\mathfrak{a}_{\mathrm{h}}$ and $\mathfrak{a}_{\mathrm{q}}$ are perpendicular with respect to the inner product $\langle \cdot\,, \cdot\rangle$, while the third holds because $\mathfrak{H}_{P,\mathrm{q}}(ah) \in \mathfrak{a}_{\mathrm{q}} \subset \mathfrak{p}$. We start with a result on derivatives of the function

$$F_X : G \to \mathbb{R}, \quad g \mapsto \langle X, \mathfrak{H}_P(g)\rangle. \tag{4.18}$$

In order to formulate it, we need a bit of additional notation. If $F \in C^\infty(G)$ and $U \in \mathfrak{g}$, we define:

$$F(g; U) = R_U F(g) := \left.\frac{d}{dt}\right|_{t=0} F(g \exp(tU)).$$

The following result and its proof can be found in [13, Corollary 5.2]. See also [5, Corollary 4.2].

Lemma 4.5.1. *Let $g \in G$ and $U \in \mathfrak{g}$. Then*

$$F_X(g; U) = B(\mathrm{Ad}(\tau(g))U, X) = B(U, \mathrm{Ad}(\nu(g)^{-1})X),$$

where we have used the decompositions $g = k(g)\tau(g)$ and $\tau(g) = a(g)\nu(g)$, according to the Iwasawa decomposition $G = KAN_P$.

We define the set of regular elements in A_{q} by $A_{\mathrm{q}}^{\mathrm{reg}} := \exp(\mathfrak{a}_{\mathrm{q}}^{\mathrm{reg}}))$, see (4.8). If $X \in \mathfrak{a}_{\mathrm{q}}$ we denote by G_X the centralizer of X in G and put

$$N_{P,X} := N_P \cap G_X. \tag{4.19}$$

Lemma 4.5.2. *Let $a \in A_{\mathrm{q}}$ and let $X \in \mathfrak{a}_{\mathrm{q}}$. The point $h \in H$ is a critical point for the function $F_{a,X}$ if and only if $ah = kbn$ for certain $k \in K$, $b \in A$ and $n \in N_{P,X}(N_P \cap H)$.*

Proof. Let $h \in H$. Then h is a critical point for the function $F_{a,X}$ if and only if

$$\forall U \in \mathfrak{h} : \quad 0 = F_{a,X}(h; U) = B(U, \mathrm{Ad}(\nu(ah)^{-1})X). \tag{4.20}$$

Since $\mathfrak{h}$ and $\mathfrak{q}$ are perpendicular with respect to B, see text above Definition 3.1.3, the condition (4.20) is equivalent to the assertion that $\mathrm{Ad}(\nu(ah)^{-1})X \in \mathfrak{q}$. Write $n = \nu(ah)$ and decompose $n = n_+ n_H$ according to the decomposition $N_P = N_{P,+}(N_P \cap H)$ of Proposition 4.2.13. Since $\mathrm{Ad}(n_H)$ normalizes $\mathfrak{q}$, the above condition is equivalent to $\mathrm{Ad}(n_+)X \in \mathfrak{q}$. Now apply the lemma below to see that the latter is equivalent to $n_+ \in N_{P,+} \cap N_{P,X}$. It follows that (4.20) is equivalent to $n \in N_{P,X}(N_P \cap H)$. $\square$

Lemma 4.5.3. *Let $n \in N_{P,+}$ (cf. Proposition 4.2.13) and $X \in \mathfrak{a}_{\mathrm{q}}$. Then*

$$\mathrm{Ad}(n)X \in \mathfrak{q} \iff \mathrm{Ad}(n)X = X.$$

Proof. The implication '$\Leftarrow$' is obvious. Thus, assume that $\mathrm{Ad}(n)X \in \mathfrak{q}$. We may write $n = \exp(U)$, where $U \in \mathfrak{n}_{P,+}$. Then by nilpotence of $\mathfrak{n}_{P,+}$,

$$\mathrm{Ad}(n)X = e^{\mathrm{ad}(U)}X \in X + \mathfrak{n}_{P,+}$$

By assumption, $\mathrm{Ad}(n)X - X \in \mathfrak{q}$. Since obviously $\sigma(\mathfrak{n}_{P,+}) \cap \mathfrak{n}_{P,+} = 0$, it follows that $\mathfrak{n}_{P,+} \cap \mathfrak{q} = 0$ and we infer that $\mathrm{Ad}(n)X = X$. $\square$

Given $X \in \mathfrak{a}_{\mathrm{q}}$ we agree to denote by $\mathcal{C}_{a,X}$ the set of critical points for the function $F_{a,X}$. The remainder of this section will be dedicated to proving the following description of this set in case a is regular. We recall the definitions of the Weyl groups $W(\mathfrak{a}_{\mathrm{q}})$ and $W_{K\cap H}$ from (3.2) and Definition 3.1.3.

Remark 4.5.4. In the following we will use the notation

$$a^w := w^{-1} \cdot a$$

for $a \in A_{\mathrm{q}}$ and $w \in W(\mathfrak{a}_{\mathrm{q}})$. This notation has the advantage that $(a^v)^w = a^{vw}$ and $(a^w)^\beta = a^{w\beta}$, for $v, w \in W(\mathfrak{a}_{\mathrm{q}})$ and $\beta \in \Sigma(\mathfrak{g}, \mathfrak{a}_{\mathrm{q}})$. In particular, $\mathrm{Ad}(a^w) = a^{w\beta}I$ on $\mathfrak{g}_\beta$.

We will use the similar notation for $a \in A$ and $w \in W(\mathfrak{a})$.

Lemma 4.5.5. *Let $a \in A_{\mathrm{q}}^{\mathrm{reg}}$ and $X \in \mathfrak{a}_{\mathrm{q}}$. Then*

$$\mathcal{C}_{a,X} = \bigcup_{w \in W_{K\cap H}} wH_X(N_P \cap H). \tag{4.21}$$

Proof. Let x_w be a representative of w in $N_{K\cap H}(\mathfrak{a}_{\mathrm{q}})$, let $h \in H_X$ and $n_P \in N_P \cap H$. Then

$$\nu(ax_w h n_P) = \nu(x_w^{-1} a x_w h n_P) = \nu(a^w h n_P) = \nu(a^w h) n_P.$$

The element $a^w h$ belongs to G_X, and according to [13, Equation 2.6],

$$G_X \simeq K_X A N_{P,X}.$$

Thus, $\nu(a^w h) \in N_{P,X}$ and it follows that $\nu(ax_w h n_P) \in N_{P,X}(N_P \cap H)$. This proves that the set on the right-hand side of (4.21) is included in the set on the left-hand side. It remains to prove the converse inclusion.

Let $h \in \mathcal{C}_{a,X}$. Then by Lemma 4.5.2 we may write $ah = kbn_X n_H$ with $k \in K$, $b \in A$, $n_X \in N_{P,X}$ and $n_H \in N_P \cap H$. From this we see that $k^{-1}ahn_H^{-1} = bn_X \in G_X$. The element $h' := hn_H^{-1}$, belongs to H. In view of the Cartan decomposition

$H = (K \cap H) \times \exp(\mathfrak{p} \cap \mathfrak{h})$, we may write $h' = h_1 h_2$, where $h_1 \in K \cap H$ and $h_2 \in \exp(\mathfrak{p} \cap \mathfrak{h})$. Then

$$k^{-1} a h_1 h_2 = k^{-1} h_1 (h_1^{-1} a h_1) h_2 \in G_X. \tag{4.22}$$

By [45], the group G decomposes as

$$G \simeq K \times \exp(\mathfrak{p} \cap \mathfrak{q}) \times \exp(\mathfrak{p} \cap \mathfrak{h}).$$

According to [45, Theorem 5], G_X has a similar decomposition

$$G_X \simeq K_X \times \exp(\mathfrak{p} \cap \mathfrak{q}_X) \times \exp(\mathfrak{p} \cap \mathfrak{h}_X).$$

By the uniqueness properties of the latter decomposition it follows from (4.22) that $k^{-1} h_1 \in K_X$, $h_1^{-1} a h_1 \in \exp(\mathfrak{p} \cap \mathfrak{q}_X)$ and $h_2 \in \exp(\mathfrak{p} \cap \mathfrak{h}_X)$.

We note that $\sigma\theta$ fixes X hence leaves the centralizer G_X invariant. The fixed point group $G_{X,+}$ of this involution in G_X admits the Cartan decomposition

$$G_{X,+} \simeq (K \cap H_X) \times \exp(\mathfrak{p} \cap \mathfrak{q}_X).$$

Obviously $\mathfrak{a}_{\mathrm{q}}$ is a maximal abelian subspace of $\mathfrak{p} \cap \mathfrak{q}_X$. Hence, every element of the latter space is conjugate to an element of $\mathfrak{a}_{\mathrm{q}}$ under the group $(K \cap H_X)^\circ$. We infer that there exists an element $l \in (K \cap H_X)^\circ$ such that

$$l^{-1} h_1^{-1} a h_1 l \in A_{\mathrm{q}}. \tag{4.23}$$

Since a was assumed to be regular for $\Sigma(\mathfrak{g}, \mathfrak{a}_{\mathrm{q}})$, it follows that a is regular for $\Sigma(\mathfrak{g}_+, \mathfrak{a}_{\mathrm{q}})$ as well. Hence, (4.23) implies that the element $h_1 l \in K \cap H$ normalizes $\mathfrak{a}_{\mathrm{q}}$. It follows that $h_1 \in N_{K \cap H}(\mathfrak{a}_{\mathrm{q}})(K \cap H_X)$. Then,

$$h' = h_1 h_2 \in N_{K \cap H}(\mathfrak{a}_{\mathrm{q}})(K \cap H_X) \exp(\mathfrak{p} \cap \mathfrak{h}_X) = N_{K \cap H}(\mathfrak{a}_{\mathrm{q}}) H_X$$

and we conclude that $h n_H^{-1} \in N_{K \cap H}(\mathfrak{a}_{\mathrm{q}}) H_X$. This finally implies that

$$h \in N_{K \cap H}(\mathfrak{a}_{\mathrm{q}}) H_X (N_P \cap H),$$

which concludes the proof. $\square$

4.6 Properties of the set of critical points

As in the previous section, we assume that $P \in \mathcal{P}(A)$ and that a is a regular point in A_{q}. In the previous section we defined the function $F_{a,X} : H \to \mathbb{R}$, for $X \in \mathfrak{a}_{\mathrm{q}}$, by (4.17) and we determined its set of critical points $\mathcal{C}_{a,X}$, see (4.21). The purpose of the present section is to study this set in more detail.

We start with the following lemma.

Lemma 4.6.1. *The map $\varphi : H_X \times (N_P \cap H) \to H$ given by $(h, n) \mapsto hn$ induces an injective immersion*

$$\bar{\varphi} : H_X \times_{N_P \cap H_X} (N_P \cap H) \to H$$

with image $H_X(N_P \cap H)$.

Proof. The group $H_X \times (N_P \cap H)$ has a natural left action on H given by the formula: $(h, n) \cdot x = hxn^{-1}$. The set $H_X(N_P \cap H)$ is the orbit for this action through the identity element e of H. Let F be the stabilizer of e for this action. Then it follows that the map $(h, n) \mapsto (h, n) \cdot e = hn^{-1}$ factors through an injective immersion $(H_X \times (N_P \cap H))/F \to H$ with image $H_X(N_P \cap H)$. The stabilizer F consists of the elements (h, h) with $h \in H_X \cap N_P$. To complete the proof of the lemma, we note that the map $(h, n) \mapsto (h, n^{-1})$ induces a diffeomorphism $H_X \times_{N_P \cap H_X} (N_P \cap H) \to (H_X \times (N_P \cap H))/F$. □

Lemma 4.6.2. *Let $X \in \mathfrak{a}_{\mathrm{q}}$. Then the set $\mathcal{C}_{a,X}$ is closed in H. Moreover, the following holds.*

(a) *If $\mathfrak{h}_X + (\mathfrak{n}_P \cap \mathfrak{h}) = \mathfrak{h}$ then $\mathcal{C}_{a,X} = H$.*

(b) *If $\mathfrak{h}_X + (\mathfrak{n}_P \cap \mathfrak{h}) \subsetneq \mathfrak{h}$ then $\mathcal{C}_{a,X}$ is a finite union of lower dimensional injectively immersed submanifolds.*

Proof. Since $\mathcal{C}_{a,X}$ is the set of critical points of the smooth function $F_{a,X}$, it is closed.

From Lemma 4.5.5 combined with Lemma 4.6.1 it follows that $\mathcal{C}_{a,X}$ is a finite union of injectively immersed submanifolds of dimension $d_X := \dim(\mathfrak{h}_X + (\mathfrak{n}_P \cap \mathfrak{h}))$. From this, (b) is immediate.

For (a) we assume the hypothesis to be fulfilled, or equivalently, that $d_X = \dim(H)$. Then $\mathcal{C}_{a,X}$ is open in H. Since this set is also closed in H, and contains $H_X(N_P \cap H)$, it follows that $\mathcal{C}_{a,X} \supset H^\circ$. From Lemma 4.5.5 it follows that $\mathcal{C}_{a,X}$ is left $N_{K \cap H}(\mathfrak{a}_{\mathrm{q}})$-invariant, so that $\mathcal{C}_{a,X} \supset Z_{K \cap H}(\mathfrak{a}_{\mathrm{q}})H^\circ$. Since H is essentially connected, the latter set equals H. □

Lemma 4.6.3. *Let $X \in \mathfrak{a}_{\mathrm{q}}$. Then the following assertions are equivalent:*

(a) $\mathfrak{h} = \mathfrak{h}_X + (\mathfrak{n}_P \cap \mathfrak{h})$;

(b) $\forall \alpha \in \Sigma(\mathfrak{g}, \mathfrak{a}) : \alpha(X) = 0$.

Proof. First assume (b). Then $\mathfrak{g}_X = \mathfrak{g}$ and (a) follows. We will prove the converse implication by contraposition. Thus, assume that (b) does not hold. Then there exists a root $\beta \in \Sigma(\mathfrak{g}, \mathfrak{a})$ such that $\beta(X) \neq 0$. By changing sign if necessary, we may in addition arrange that $\beta \in \Sigma(P)$.

Given a subset $\mathcal{O} \subseteq \Sigma(\mathfrak{g}, \mathfrak{a}) \cup \{0\}$, we agree to write

$$\mathfrak{g}_{\mathcal{O}} = \oplus_{\alpha \in \mathcal{O}}\, \mathfrak{g}_\alpha. \tag{4.24}$$

In particular, we see that $\mathfrak{n}_P = \mathfrak{g}_{\Sigma(P)}$. We also agree to write $\mathcal{O}^\sigma := \mathcal{O} \cap \sigma(\mathcal{O})$. Then using $\sigma(\mathfrak{g}_\alpha) = \mathfrak{g}_{\sigma\alpha}$ we readily see that

$$\mathfrak{g}_{\mathcal{O}} \cap \mathfrak{h} = (\mathfrak{g}_{\mathcal{O}^\sigma})^\sigma = \oplus_{\omega \in \mathcal{O}^\sigma/\{1,\sigma\}} \; (\mathfrak{g}_\omega)^\sigma; \tag{4.25}$$

here $\mathcal{O}^\sigma/\{1,\sigma\}$ denotes the set of orbits for the action on $\mathcal{O}^\sigma$ of the subgroup $\{1,\sigma\}$ of $\mathrm{Aut}(\mathfrak{g})$. If we apply (4.25) to the set $\mathcal{O}_X := \{\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a}) : \;\; \alpha(X) = 0\} \cup \{0\}$, we find

$$\mathfrak{h}_X = \oplus_{\omega \in \mathcal{O}_X/\{1,\sigma\}} \; (\mathfrak{g}_\omega)^\sigma.$$

We note that $\Sigma(P)^\sigma = \Sigma(P, \sigma)$, so that

$$\mathfrak{n}_P \cap \mathfrak{h} = \mathfrak{g}_{\Sigma(P,\sigma)} \cap \mathfrak{h}.$$

We now consider the set $\mathcal{O}_\beta := \{\beta, \sigma\beta, -\beta, -\sigma\beta\}$. Since $\mathcal{O}_X \cap \mathcal{O}_\beta = \emptyset$, it follows from the above that

$$(\mathfrak{h}_X + (\mathfrak{n}_P \cap \mathfrak{h})) \cap \mathfrak{g}_{\mathcal{O}_\beta} = \mathfrak{n}_P \cap \mathfrak{h} \cap \mathfrak{g}_{\mathcal{O}_\beta} = (\mathfrak{g}_{\Sigma(P,\sigma) \cap \mathcal{O}_\beta})^\sigma. \tag{4.26}$$

On the other hand,

$$\mathfrak{h} \cap \mathfrak{g}_{\mathcal{O}_\beta} = (\mathfrak{g}_{\mathcal{O}_\beta})^\sigma.$$

From $\beta(X) \neq 0$ it follows that $\beta \notin \mathfrak{a}_{\mathrm{h}}^*$. If $\beta \in \mathfrak{a}_{\mathrm{q}}^*$ then $\Sigma(P, \sigma) \cap \mathcal{O}_\beta = \emptyset$ and if $\beta \notin \mathfrak{a}_{\mathrm{q}}^*$ then $\Sigma(P, \sigma) \cap \mathcal{O}_\beta \subseteq \{\beta, \sigma\beta\}$. In any case, $\Sigma(P, \sigma) \cap \mathcal{O}_\beta$ is a proper σ-invariant subset of $\mathcal{O}_\beta$. By application of (4.25) it now follows that

$$(\mathfrak{g}_{\Sigma(P,\sigma) \cap \mathcal{O}_\beta})^\sigma \subsetneq (\mathfrak{g}_{\mathcal{O}_\beta})^\sigma.$$

Using (4.26) we infer that (a) is not valid. $\square$

We agree to write

$$S := \mathfrak{a}_{\mathrm{q}} \setminus \cap_{\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a}_{\mathrm{q}})} \ker \alpha. \tag{4.27}$$

Remark 4.6.4. If $\Sigma(\mathfrak{g}, \mathfrak{a}_{\mathrm{q}})$ spans $\mathfrak{a}_{\mathrm{q}}$ then it follows that $S = \mathfrak{a}_{\mathrm{q}} \setminus \{0\}$.

Corollary 4.6.5. $S = \{X \in \mathfrak{a}_{\mathrm{q}} : \;\; \mathcal{C}_{a,X} \subsetneq H\}$.

Proof. Let $X \in \mathfrak{a}_{\mathrm{q}}$. In the situation of Lemma 4.6.2 (b) the set $\mathcal{C}_{a,X}$ is a countable union of lower dimensional submanifolds, hence nowhere dense by the Baire category theorem. Thus, by application of Lemmas 4.6.2 and 4.6.3 it follows that $\mathcal{C}_{a,X} \subsetneq H \iff X \in S$. $\square$

For each $Z \in \mathfrak{a}_{\mathrm{q}}$, let $\Sigma(Z)$ denote the collection of roots in $\Sigma(\mathfrak{g}, \mathfrak{a}_{\mathrm{q}})$ vanishing on Z. We define the equivalence relation $\sim$ on $\mathfrak{a}_{\mathrm{q}}$ by

$$X \sim Y \iff \Sigma(X) = \Sigma(Y).$$

Then clearly, $\sim$ has finitely many equivalence classes in $\mathfrak{a}_{\mathrm{q}}$ and

$$X \sim Y \iff G_X = G_Y.$$

The class of 0 is given by $[0] = \cap_{\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a}_{\mathrm{q}})} \ker \alpha$ and S is the union of the remaining finitely many equivalence classes for $\sim$. Furthermore, the set $\mathcal{C}_{a,X}$ depends on $X \in S$ through the centralizer G_X, hence through the equivalence class $[X]$ for $\sim$. Accordingly, we will also write $\mathcal{C}_{a,[X]}$ for this set.

We define

$$\mathcal{C}_a := \cup_{X \in S} \mathcal{C}_{a,X}. \tag{4.28}$$

Lemma 4.6.6.

(a) *There exists a finite subset $S_0 \subseteq S$ such that (4.28) is valid for the union over S_0 in place of S.*

(b) *The set $\mathcal{C}_a$ is closed and a finite union of lower dimensional injectively immersed submanifolds of H.*

(c) *The set $\mathcal{C}_a$ is nowhere dense in H.*

Proof. By the discussion preceding the lemma, $\mathcal{C}_a$ is the union of the sets $\mathcal{C}_{a,[X]}$, for $[X] \in S/\sim$. Since the latter set is finite, assertion (a) follows with S_0 a complete set of representatives for $S/\sim$. Assertion (b) now follows by application of Corollary 4.6.5 and Lemma 4.6.2. Assertion (c) follows from (b) by application of the Baire category theorem. $\square$

The following result illustrates the importance of the set $\mathcal{C}_a$.

Proposition 4.6.7. *The set $H \setminus \mathcal{C}_a$ is open and dense in H. Assume that $\Sigma(\mathfrak{g}, \mathfrak{a}_{\mathrm{q}})$ spans $\mathfrak{a}_{\mathrm{q}}^*$. Then the map $F_a : h \mapsto \mathfrak{H}_{P,\mathrm{q}}(ah)$, $H \to \mathfrak{a}_{\mathrm{q}}$ is submersive at all points of $H \setminus \mathcal{C}_a$.*

Proof. The first assertion is a consequence of Lemma 4.6.6.

Let $h_0 \in H \setminus \mathcal{C}_a$. Then for every $X \in S$ the point h_0 is not critical for the function $F_{a,X}$. As $S = \mathfrak{a}_{\mathrm{q}} \setminus \{0\}$, see Remark 4.6.4, it follows that $F_a : h \mapsto \mathfrak{H}_{P,\mathrm{q}}(ah)$ is submersive at h_0. $\square$

Lemma 4.6.8. *Let $P \in \mathcal{P}(A)$ and $a \in A_{\mathrm{q}}^{\mathrm{reg}}$. Then the following assertions are valid.*

(a) *The sets $\mathfrak{H}_{P,\mathrm{q}}(aH)$ and $\mathfrak{H}_{P,\mathrm{q}}(a\mathcal{C}_a)$ are closed in $\mathfrak{a}_{\mathrm{q}}$.*

(b) *If $\Sigma(\mathfrak{g}, \mathfrak{a}_{\mathrm{q}})$ spans $\mathfrak{a}_{\mathrm{q}}^*$ then the set $\mathfrak{H}_{P,\mathrm{q}}(aH) \setminus \mathfrak{H}_{P,\mathrm{q}}(a\mathcal{C}_a)$ is open and closed in $\mathfrak{a}_{\mathrm{q}} \setminus \mathfrak{H}_{P,\mathrm{q}}(a\mathcal{C}_a)$.*

Proof. For $\mathcal{A} \subseteq A$ compact, the map $\mathcal{A} \times H/(H \cap P) \to \mathfrak{a}_{\mathrm{q}}$, $(b, [h]) \mapsto \mathfrak{H}_{P,\mathrm{q}}(bh)$ is proper, hence closed; see Corollary 4.4.12. In particular, it follows that $\mathfrak{H}_{P,\mathrm{q}}(aH)$ is closed in $\mathfrak{a}_{\mathrm{q}}$.

It follows from Lemma 4.6.6 that $\mathcal{C}_a$ is closed in H. Moreover, $\mathcal{C}_a$ is a countable union of lower dimensional submanifolds of H. Thus, by the Baire property, $\mathcal{C}_a$ has empty interior in H. In particular, it is a proper subset of H.

Furthermore, the set $\mathcal{C}_a$ is right $H \cap P$-invariant, hence has closed image in $H/H \cap P$. It follows that $\mathfrak{H}_{P,\mathrm{q}}(a\mathcal{C}_a)$ is closed in $\mathfrak{a}_{\mathrm{q}}$. This establishes (a).

By Proposition 4.6.7 the map $F_a : h \mapsto \mathfrak{H}_{P,\mathrm{q}}(ah)$ is submersive at the points of $H \setminus \mathcal{C}_a$. Hence $\mathfrak{H}_{P,\mathrm{q}}(a(H \setminus \mathcal{C}_a))$ is open in $\mathfrak{a}_{\mathrm{q}}$. It follows that

$$\mathfrak{H}_{P,\mathrm{q}}(aH) \setminus \mathfrak{H}_{P,\mathrm{q}}(a\mathcal{C}_a) = \mathfrak{H}_{P,\mathrm{q}}(a(H \setminus \mathcal{C}_a)) \setminus \mathfrak{H}_{P,\mathrm{q}}(a\mathcal{C}_a) \tag{4.29}$$

is open in $\mathfrak{a}_{\mathrm{q}}$ hence in $\mathfrak{a}_{\mathrm{q}} \setminus \mathfrak{H}_{P,\mathrm{q}}(a\mathcal{C}_a)$. Finally, since $\mathfrak{H}_{P,\mathrm{q}}(aH)$ is closed, the first set in (4.29) is closed in $\mathfrak{a}_{\mathrm{q}} \setminus \mathfrak{H}_{P,\mathrm{q}}(a\mathcal{C}_a)$. We conclude that the set (4.29) is both open and closed in $\mathfrak{a}_{\mathrm{q}} \setminus \mathfrak{H}_{P,\mathrm{q}}(a\mathcal{C}_a)$. □

Lemma 4.6.9. *Assume that $\Sigma(\mathfrak{g}, \mathfrak{a}_{\mathrm{q}})$ spans $\mathfrak{a}_{\mathrm{q}}^*$. Then $\mathfrak{H}_{P,\mathrm{q}}(aH) \setminus \mathfrak{H}_{P,\mathrm{q}}(a\mathcal{C}_a) \neq \emptyset$.*

Proof. Under the assumption that $\Sigma(\mathfrak{g}, \mathfrak{a}_{\mathrm{q}})$ spans $\mathfrak{a}_{\mathrm{q}}^*$, the map $\mathfrak{H}_{P,\mathrm{q}} : aH \to \mathfrak{a}_{\mathrm{q}}$ is submersive except at points of $\mathcal{C}_a$. The set $H \setminus \mathcal{C}_a$ is open and non-empty. Thus, $\mathfrak{H}_{P,\mathrm{q}}(a(H \setminus \mathcal{C}_a))$ is open and non-empty. By Sard's Theorem, $\mathfrak{H}_{P,\mathrm{q}}(a\mathcal{C}_a)$ has measure zero. This implies that

$$\mathfrak{H}_{P,\mathrm{q}}(a(H \setminus \mathcal{C}_a)) \setminus \mathfrak{H}_{P,\mathrm{q}}(a\mathcal{C}_a) \neq \emptyset,$$

and hence

$$\mathfrak{H}_{P,\mathrm{q}}(aH) \setminus \mathfrak{H}_{P,\mathrm{q}}(a\mathcal{C}_a) \neq \emptyset.$$

□

Remark 4.6.10. The lemma can readily be extended to the case that $\Sigma(\mathfrak{g}, \mathfrak{a}_{\mathrm{q}})$ does not span $\mathfrak{a}_{\mathrm{q}}^*$, but we will not need this.

4.7 The computation of Hessians

We retain the assumption that $P \in \mathcal{P}(A)$. Furthermore, we assume that $a \in A_{\mathrm{q}}^{\mathrm{reg}}$ and $X \in \mathfrak{a}_{\mathrm{q}}$. In this section we will compute the Hessian of the function $F_{a,X} : H \to \mathbb{R}$, defined in (4.17), at all points of its critical locus $\mathcal{C}_{a,X}$.

Given $U \in \mathfrak{h}$, we denote by R_U the associated left-invariant vector field on H defined by

$$R_U(h) = dl_h(e)U = \frac{\partial}{\partial t}(h \exp tU)|_{t=0}, \qquad (h \in H).$$

The associated derivation on $C^\infty(H)$ is denoted by the same symbol.

If $f : H \to \mathbb{R}$ is a C^2-function with critical point at h, then its Hessian at h is the symmetric bilinear form $H(f)(h) = H(f)_h$ on T_hH given by

$$\mathrm{H}(f)_h(R_U(h), R_V(h)) := R_U R_V f(h) = \partial_s \partial_t f(h \exp sU \exp tV)|_{s=t=0},$$

for $U, V \in \mathfrak{h}$.

Lemma 4.7.1. *Let $a \in A_{\mathrm{q}}$, $X \in \mathfrak{a}_{\mathrm{q}}$ and $h \in H$. Then for all $U, V \in \mathfrak{h}$ we have:*

$$R_U R_V F_{a,X}(h) = B(U, L_{a,X,h}(V)) = -\langle U, \theta L_{a,X,h}(V)\rangle,$$

where $L_{a,X,h} : \mathfrak{h} \to \mathfrak{h}$ is the linear map given by

$$L_{a,X,h}(V) = -\mathrm{Ad}(h^{-1}) \circ \pi_{\mathfrak{h}} \circ \mathrm{Ad}(a^{-1}) \circ \mathrm{Ad}(k_a(h)) \circ \mathrm{ad}(X) \circ E_{\mathfrak{k}} \circ \mathrm{Ad}(\tau(ah))V. \tag{4.30}$$

Here $\pi_{\mathfrak{h}} : \mathfrak{g} \to \mathfrak{h}$ denotes the projection according to the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ and $E_{\mathfrak{k}} : \mathfrak{g} \to \mathfrak{k}$ is the projection associated with the Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}_P$. The notation $k_a(h)$ is used to express the K-part of the element ah with respect to the Iwasawa decomposition $G = KAN_P$. Finally, $\tau(ah)$ denotes the (AN_P)-part of ah with respect to the same Iwasawa decomposition.

Proof. By [5, Lemma 5.1], see also [13], we obtain that for $x \in G$ and $U, V \in \mathfrak{g}$,

$$R_U R_V F_X(x) = B([\mathrm{Ad}(\tau)U,\ E_{\mathfrak{k}} \circ \mathrm{Ad}(\tau)V], X),$$

where F_X is the function defined in (4.18) and where $\tau := \tau(x)$. Therefore,

$$\begin{aligned} R_U R_V F_X(x) &= -B(\mathrm{Ad}(\tau)U,\ \mathrm{ad}X \circ E_{\mathfrak{k}} \circ \mathrm{Ad}(\tau)V) \\ &= \ -B(U,\ \mathrm{Ad}(\tau)^{-1} \circ \mathrm{ad}X \circ E_{\mathfrak{k}} \circ \mathrm{Ad}(\tau)V). \end{aligned}$$

We can restrict now to the case where $x = ah$ and $U, V \in \mathfrak{h}$. Since $F_{a,X}(h) = F_X(ah)$, we obtain

$$R_U R_V F_{a,X}(h) = R_U R_V F_X(ah) = B(U,\ -\pi_{\mathfrak{h}} \circ \mathrm{Ad}(\tau)^{-1} \circ \mathrm{ad}X \circ E_{\mathfrak{k}} \circ \mathrm{Ad}(\tau)V). \tag{4.31}$$

Since $ah = k_a(h)\tau(ah)$, it follows that $\tau^{-1} = \tau(ah)^{-1} = h^{-1}a^{-1}k_a(h)$ and by applying Ad to this equality we obtain

$$\mathrm{Ad}(\tau^{-1}) = \mathrm{Ad}(h^{-1})\mathrm{Ad}(a^{-1})\mathrm{Ad}(k_a(h)).$$

We complete the proof by substituting this equality in (4.31) and observing that $\pi_{\mathfrak{h}}$ commutes with $\mathrm{Ad}(h^{-1})$. $\square$

4.8 The transversal signature of the Hessian

In this section we fix $P \in \mathcal{P}(A)$, $a \in A_{\mathrm{q}}^{\mathrm{reg}}$ and $X \in \mathfrak{a}_{\mathrm{q}}$. We will study the behavior of the Hessian $H(F_{a,X})_h$ of the function $F_{a,X} : H \to \mathbb{R}$ defined in (4.17) at each point h of its critical set $\mathcal{C}_{a,X}$. This Hessian is a symmetric bilinear form on T_hH. Its kernel at h is by definition equal to the following linear subspace of T_hH,

$$\ker(H(F_{a,X})(h)) := \{V \in T_hH \ : \ \ H(F_{a,X})(h)(V, \cdot\,) = 0\}.$$

By symmetry, the Hessian induces a non-degenerate symmetric bilinear form on the quotient space $T_hH/\ker(H(F_{a,X})(h))$, which we will denote by $\bar{H}(F_{a,X})(h)$. For each $w \in W_{K\cap H}$ we select a representative $x_w \in N_{K\cap H}(\mathfrak{a}_{\mathrm{q}})$. The set

$$\mathcal{C}_{a,X,w} := x_w H_X(H \cap N_P)$$

is an injectively immersed submanifold of H, see Lemma 4.6.1. In particular this set has a well-defined tangent space at each of its points. We will show that the Hessian of $F_{a,X}$ is transversally non-degenerate along $\mathcal{C}_{a,X,w}$.

Lemma 4.8.1. *Let $w \in W_{K\cap H}$. Then at each point $\bar{h} \in \mathcal{C}_{a,X,w}$ the kernel of the Hessian $H(F_{a,X})(\bar{h})$ equals the tangent space $T_{\bar{h}}\mathcal{C}_{a,X,w}$.*

The proof of this lemma will make use of Lemma 4.8.2 below. In that lemma, $L_{a,X,h} \in \mathrm{End}(\mathfrak{h})$ is defined as in (4.30). Let $\bar{k}_a := \pi \circ k_a : H \to K/M$, where $k_a : H \to K$ is defined as in Lemma 4.7.1 and where π denotes the canonical projection $K \to K/M$.

Lemma 4.8.2. *Let $h \in H_X^\circ$ and $V \in \mathfrak{h}$. Then the following statements are equivalent.*

(a) $V \in \ker L_{a,X,h}$,

(b) $\mathrm{d}(l_{k_a(h)^{-1}} \circ \bar{k}_a \circ l_h)(e)(V) \in \mathfrak{k}_X/\mathfrak{m}$,

(c) $V \in \mathfrak{h}_X + (\mathfrak{h} \cap \mathfrak{n}_P)$.

Proof. First, we prove that (a) $\Longrightarrow$ (b). Assume (a) holds. In view of (4.30) this is equivalent to

$$\mathrm{Ad}(a^{-1}) \circ \mathrm{Ad}(k_a(h)) \circ \mathrm{ad}(X) \circ E_{\mathfrak{k}} \circ \mathrm{Ad}(\tau(ah))V \in \mathfrak{q}. \tag{4.32}$$

Observe that $\mathrm{Ad}(k_a(h)) \circ \mathrm{ad}(X) \circ E_{\mathfrak{k}} \circ \mathrm{Ad}(\tau(ah))V \in \mathfrak{p}$. In view of [5, Lemma 5.7] we see that (4.32) implies that

$$\mathrm{Ad}(k_a(h)) \circ \mathrm{ad}(X) \circ E_{\mathfrak{k}} \circ \mathrm{Ad}(\tau(ah))V \in \mathfrak{a}_{\mathrm{q}}. \tag{4.33}$$

Since $h \in H_X$ and $G_X = K_X A N_{P,X}$, see (4.19), it follows that $k_a(h)$ centralizes X. Thus, $\mathrm{Ad}(k_a(h))$ and $\mathrm{ad}(X)$ commute. Now $\mathrm{Ad}(k_a(h)) \circ E_{\mathfrak{k}} \circ \mathrm{Ad}(\tau(ah))V$ is an element in $\mathfrak{k}$, which decomposes as

$$\mathfrak{k} = \mathfrak{k}_X + \bigoplus_{\substack{\alpha \in \Sigma(P) \\ \alpha(X) \neq 0}} (I + \theta)\mathfrak{g}_\alpha.$$

Furthermore, by (4.33), we know that $\mathrm{ad}(X)$ maps this element to an element of $\mathfrak{a}_{\mathrm{q}}$. This implies that

$$\mathrm{Ad}(k_a(h)) \circ E_{\mathfrak{k}} \circ \mathrm{Ad}(\tau(ah))V \in \mathfrak{k}_X.$$

Since $k_a(h) \in K_X$, we obtain that

$$E_{\mathfrak{k}} \circ \mathrm{Ad}(\tau(ah))V \in \mathfrak{k}_X. \tag{4.34}$$

By the use of [5, Lemma 5.2], we may rewrite

$$E_{\mathfrak{k}} \circ \mathrm{Ad}(\tau(ah)) = \mathrm{d}l_{k_a(h)}(e)^{-1} \circ \mathrm{d}k_a(h) \circ \mathrm{d}l_h(e) = \mathrm{d}(l_{k_a(h)^{-1}} \circ k_a \circ l_h)(e).$$

Hence, (4.34) implies

$$\mathrm{d}(l_{k_a(h)^{-1}} \circ k_a \circ l_h)(e)(V) \in \mathfrak{k}_X. \tag{4.35}$$

Observe that $\mathrm{d}\pi(e) : \mathfrak{k}_X \to \mathfrak{k}_X/\mathfrak{m}$ is given by the canonical projection and that the maps π and $l_{k_a(h)^{-1}}$ commute. Hence, equation (4.35) is equivalent to

$$\mathrm{d}(l_{k_a(h)^{-1}} \circ \bar{k}_a \circ l_h)(e)(V) \in \mathfrak{k}_X/\mathfrak{m} \tag{4.36}$$

and (b) follows.

Next, we prove that (b) $\Longrightarrow$ (c). Assume (b) and denote by φ the diffeomorphism $\varphi : K/M \to G/P$ arising from the Iwasawa decomposition $G = KAN_P$. The inclusion $H \hookrightarrow G$ induces the map $\psi : H \to G/P$. It is easy to check that the diagram given below commutes.

$$\begin{tikzcd} H \arrow[r, "\psi"] \arrow[d, "\bar{k}_a"'] & G/P \arrow[d, "l_a"] \\ K/M \arrow[r, "\varphi"] & G/P \end{tikzcd} \tag{4.37}$$

The map ψ commutes with the left multiplication by an element $h \in H$, viewed either as the map $l_h : H \to H$ or as the map $l_h : G/P \to G/P$. On the other hand, the diffeomorphism φ introduced above, commutes with the left multiplication $l_k : K/M \to K/M$, where $k \in K$. Hence, the commutative diagram (4.37) gives rise to the following commutative diagram. We use the notation $k := k_a(h)$.

$$\begin{tikzcd} H \arrow[r, "\psi"] \arrow[d, "l_k^{-1} \circ \bar{k}_a \circ l_h"'] & G/P \arrow[d, "l_{k^{-1}ah}"] \\ K/M \arrow[r, "\varphi"] & G/P \end{tikzcd} \tag{4.38}$$

Note that under each of the four maps in diagram (4.38), the origin of the domain is mapped to the origin of the codomain. Taking derivatives at the origins we obtain the commutative diagram given below.

$$\begin{tikzcd} \mathfrak{h} \arrow[r, "\psi_\star"] \arrow[d, "T"'] & \mathfrak{g}/\underline{\mathfrak{p}} \arrow[d, "\mathrm{d}(l_{k^{-1}ah})(eP)"] \\ \mathfrak{k}/\mathfrak{m} \arrow[r, "\varphi_\star"] & \mathfrak{g}/\underline{\mathfrak{p}} \end{tikzcd} \tag{4.39}$$

Here

$$\underline{\mathfrak{p}} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}_P$$

denotes the Lie algebra of P and T denotes the map $\mathrm{d}(l_k^{-1} \circ \bar{k}_a \circ l_h)(e) : \mathfrak{h} \to \mathfrak{k}/\mathfrak{m}$. Furthermore, $\varphi_* = d\varphi(eM)$ and $\psi_* = d\psi(e)$.

Observe that $k^{-1}ah = \tau := \tau(ah)$. Since h belongs to H_X, it follows that τ and τ^{-1} belong to $AN_{P,X} \subseteq P$. This in turn implies that $\mathrm{Ad}(\tau^{-1})$ is a bijection from $\mathfrak{g}_X$ to $\mathfrak{g}_X$ which normalizes $\underline{\mathfrak{p}}$. Let $\overline{\mathrm{Ad}(\tau)} : \mathfrak{g}/\underline{\mathfrak{p}} \to \mathfrak{g}/\underline{\mathfrak{p}}$ be the map induced by $\mathrm{Ad}(\tau) : \mathfrak{g} \to \mathfrak{g}$. Then

$$\mathrm{d}(l_{k^{-1}ah})(eP) = \overline{\mathrm{Ad}(\tau)}.$$

We use the commutativity of diagram (4.39) to compute the pre-image of $\mathfrak{k}_X/\mathfrak{m}$ under the map T :

$$\begin{aligned} T^{-1}(\mathfrak{k}_X/\mathfrak{m}) &= \psi_\star^{-1} \circ \overline{\mathrm{Ad}(\tau^{-1})} \circ \varphi_\star(\mathfrak{k}_X/\mathfrak{m}) \\ &= \psi_\star^{-1}(\overline{\mathrm{Ad}(\tau^{-1})}(\mathfrak{k}_X + \underline{\mathfrak{p}})) \\ &= \psi_\star^{-1}(\overline{\mathrm{Ad}(\tau^{-1})}(\mathfrak{g}_X + \underline{\mathfrak{p}})) \\ &= \psi_\star^{-1}((\mathrm{Ad}(\tau^{-1})\mathfrak{g}_X) + \underline{\mathfrak{p}}) \\ &= \psi_\star^{-1}(\mathfrak{g}_X + \underline{\mathfrak{p}}) \\ &= \{U \in \mathfrak{h} : \ U + \underline{\mathfrak{p}} \in \mathfrak{g}_X + \underline{\mathfrak{p}}\} \\ &= \mathfrak{h}_X + (\mathfrak{h} \cap \underline{\mathfrak{p}}). \end{aligned}$$

Since $\mathfrak{h} \cap \underline{\mathfrak{p}} = (\mathfrak{m} \oplus \mathfrak{a}) \cap \mathfrak{h} \oplus (\mathfrak{n}_P \cap \mathfrak{h})$, see Subsection 4.2.4, and $(\mathfrak{m} \oplus \mathfrak{a}) \cap \mathfrak{h} \subseteq \mathfrak{h}_X$, we obtain that $\mathfrak{h}_X + (\mathfrak{h} \cap \underline{\mathfrak{p}}) = \mathfrak{h}_X + (\mathfrak{h} \cap \mathfrak{n}_P)$. Thus if (b) holds, then $T(V) \in \mathfrak{k}_X/\mathfrak{m}$ and we infer that $V \in \mathfrak{h}_X + (\mathfrak{h} \cap \underline{\mathfrak{p}})$ hence (c).

Finally, the implication (c) $\Longrightarrow$ (a) is easy. $\square$

Proof of Lemma 4.8.1. Recall that H is essentially connected. By [5, Proposition 2.3], the centralizer H_X is essentially connected as well (relative to G_X).

Assume first that $\bar{h} = h \in H_X^\circ$. Then, by Lemma 4.8.2 above, we have that

$$\ker L_{a,X,h} = \mathfrak{h}_X + (\mathfrak{n}_P \cap \mathfrak{h}).$$

Since $dl_h(e)$ is a linear isomorphism $\mathfrak{g} \to T_hG$, mapping $T_e[H_X(N_P \cap H)]$ onto $T_h[H_X(N_P \cap H)]$, we obtain that

$$\ker H(F_{a,X})(h) = dl_h(e)(\ker L_{a,X,h}) = T_h[H_X(N_P \cap H)],$$

which establishes the assertion for $\bar{h} = h \in H_X^\circ$.

Let now $\bar{h} = hn$, with $n \in N_P \cap H$. Then the right-multiplication $r_n : H \to H$ is a diffeomorphism and $F_{a,X} \circ r_n = F_{a,X}$, so that

$$\ker H(F_{a,X})(hn) = dr_n(h)[\ker H(F_{a,X})(h)] = dr_n(h)T_h[H_X(N_P \cap H)].$$

As the latter space equals $T_{hn}[H_X(N_P \cap H)]$ this proves the assertion for $\bar{h} \in H_X^\circ(N_P \cap H)$.

Finally, we discuss the general case $\bar{h} \in wH_X(N_P \cap H)$. Since H, respectively H_X, is essentially connected we may write $\bar{h} = x_w hn$, where $h \in H_X^\circ$, $n \in N_P \cap H$ and x_w is a representative of w in $N_{K\cap H}(\mathfrak{a}_{\mathrm{q}})$ chosen accordingly. Since x_w normalizes A_{q},

$$F_{a,X} \circ l_{x_w} = F_{w^{-1}a,X}.$$

Furthermore, from $a \in A_{\mathrm{q}}^{\mathrm{reg}}$ it follows that $w^{-1}a \in A_{\mathrm{q}}^{\mathrm{reg}}$. Since l_{x_w} is a diffeomorphism from H to itself, it follows that $dl_{x_w}(hn)$ is a linear isomorphism from $T_{hn}H$ onto $T_{\bar{h}}H$ and that

$$\begin{aligned}
\ker H(F_{a,X})(\bar{h}) &= \ker H(F_{a,X})(x_w hn) \\
&= dl_{x_w}(hn)[\ker H(F_{x_w^{-1}a,X})(hn)] \\
&= dl_{x_w}(hn)T_{hn}[H_X(N_P \cap H)] \\
&= T_{\bar{h}}[x_w H_X(N_P \cap H)] \\
&= T_{\bar{h}}\mathcal{C}_{a,X,w}.
\end{aligned}$$

□

We will now determine the set of critical points where the Hessian is transversally positive definite. For the description of our next result we define the following subsets of $\Sigma(P)$. If $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a}) \cap \mathfrak{a}_{\mathrm{q}}^*$, then the associated root space $\mathfrak{g}_\alpha$ is $\sigma\theta$-invariant. Hence, for such a root α,

$$\mathfrak{g}_\alpha = \mathfrak{g}_{\alpha,+} \oplus \mathfrak{g}_{\alpha,-},$$

where

$$\mathfrak{g}_{\alpha,\pm} = \{U \in \mathfrak{g}_\alpha \;:\; \sigma\theta U = \pm U\}.$$

Accordingly, we define

$$\Sigma(\mathfrak{g}, \mathfrak{a}_{\mathrm{q}})_\pm := \{\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a}_{\mathrm{q}}) \;:\; \mathfrak{g}_{\alpha,\pm} \neq 0\}.$$

In order to formulate the first main result of this section, we need to specify particular subsets of $\Sigma(P)$.

Definition 4.8.3.

(a) $\Sigma(P)_+ := \{\alpha \in \Sigma(P) : \ \alpha \in \mathfrak{a}_{\mathrm{q}}^* \implies \mathfrak{g}_{\alpha,+} \neq 0\}$.

(b) $\Sigma(P)_- := \{\alpha \in \Sigma(P, \sigma\theta) : \ \alpha \in \mathfrak{a}_{\mathrm{q}}^* \implies \mathfrak{g}_{\alpha,-} \neq 0\}$.

Note that (b) in this definition is consistent with (3.3).

Proposition 4.8.4. *Let $w \in W_{K\cap H}$. Then the Hessian $H(F_{a,X})(x_w)$ is positive definite transversally to $\mathcal{C}_{a,X,w}$ if and only if the following two conditions are fulfilled*

(a) $\forall \alpha \in \Sigma(P)_+ : \ \ \alpha(X)\alpha(w^{-1}(\log a)) \leq 0$*;*

(b) $\forall \alpha \in \Sigma(P)_- : \ \ \alpha(X) \geq 0$.

Remark 4.8.5. For the geometric meaning of these conditions we refer to Lemma 4.8.14, towards the end of this section.

Proof. We will prove the proposition in a number of steps. As a first step, let $l_w := l_{x_w}$ denote left multiplication by x_w on H. Then the tangent space of $\mathcal{C}_{a,X,w}$ at x_w is the image of $\mathfrak{h}_X + (\mathfrak{h} \cap \mathfrak{n}_P)$ under the tangent map $dl_w(e) : \mathfrak{h} \to T_{x_w}H$. We will denote by H_w the pull-back of the Hessian $H(F_{a,X})(x_w)$ under $dl_w(e)$. Then

$$\ker H_w = \mathfrak{h}_X + (\mathfrak{n}_P \cap \mathfrak{h}) \tag{4.40}$$

and the following conditions are equivalent:

(a) the Hessian $H(F_{a,w})(x_w)$ is positive definite transversally to $\mathcal{C}_{a,X,w}$;

(b) the bilinear form H_w is positive definite transversally to $\mathfrak{h}_X + (\mathfrak{h} \cap \mathfrak{n}_P)$.

Accordingly, we will concentrate on deriving necessary and sufficient conditions for (b) to be valid.

Lemma 4.8.6. *The bilinear form H_w on $\mathfrak{h}$ is given by*

$$\mathrm{H}_w(U, V) = \langle U, L_w V\rangle, \qquad (U, V \in \mathfrak{h}),$$

where $L_w : \mathfrak{h} \to \mathfrak{h}$ is the linear map given by

$$L_w = -\pi_{\mathfrak{h}} \circ \mathrm{ad}(X) \circ \mathrm{Ad}(a^w) \circ E_{\mathfrak{k}} \circ \mathrm{Ad}(a^w).$$

Proof. Let $U, V \in \mathfrak{h}$. Then in view of Lemma 4.7.1 we have

$$\mathrm{H}_w(U, V) = R_U R_V F_{a,X}(x_w) = B(U, L_{a,X,h}V) = -\langle U, \theta L_{a,X,h} V\rangle$$

with $h = x_w$ and $L_{a,X,h}$ defined as in Lemma 4.7.1. Now $ah = ax_w = x_w a^w$ and we see that $\tau = \tau(ah) = a^w$ and $k_a(h) = x_w$. Hence,

$$-\theta \circ L_{a,X,h}(V) = \theta \circ \mathrm{Ad}(x_w^{-1}) \circ \pi_{\mathfrak{h}} \circ \mathrm{Ad}(a^{-1}) \circ \mathrm{Ad}(x_w) \circ \mathrm{ad}(X) \circ E_{\mathfrak{k}} \circ \mathrm{Ad}(a^w)V$$

$$= \theta \circ \pi_{\mathfrak{h}} \circ \mathrm{Ad}(x_w^{-1}) \circ \mathrm{Ad}(a^{-1}) \circ \mathrm{Ad}(x_w) \circ \mathrm{ad}(X) \circ E_{\mathfrak{k}} \circ \mathrm{Ad}(a^w)V$$
$$= \theta \circ \pi_{\mathfrak{h}} \circ \mathrm{Ad}(a^w)^{-1} \circ \mathrm{ad}(X) \circ E_{\mathfrak{k}} \circ \mathrm{Ad}(a^w)V$$
$$= -\pi_{\mathfrak{h}} \circ \mathrm{Ad}(a^w) \circ \mathrm{ad}(X) \circ E_{\mathfrak{k}} \circ \mathrm{Ad}(a^w)V.$$

The result now follows since $\mathrm{Ad}(a^w)$ and $\mathrm{ad}(X)$ commute. □

In the sequel it will be useful to consider the finite subgroup

$$F = \{1, \sigma, \theta, \sigma\theta\} \subseteq \mathrm{Aut}(\mathfrak{g}).$$

The natural left action of F on $\mathfrak{g}$ leaves $\mathfrak{a}$ invariant, and induces natural left actions on $\mathfrak{a}^*$ and on $\Sigma(\mathfrak{g}, \mathfrak{a})$. Accordingly, if $\tau \in F$ and $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a})$, then

$$\tau(\mathfrak{g}_\alpha) = \mathfrak{g}_{\tau\alpha}$$

If $\mathcal{O}$ is an orbit for the F-action on $\Sigma(\mathfrak{g}, \mathfrak{a})$, we write, in accordance with (4.24),

$$\mathfrak{g}_{\mathcal{O}} = \bigoplus_{\alpha \in \mathcal{O}} \mathfrak{g}_\alpha.$$

Then obviously,

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\mathcal{O} \in \Sigma(\mathfrak{g},\mathfrak{a})/F} \mathfrak{g}_{\mathcal{O}}, \tag{4.41}$$

with mutually orthogonal summands. Each of the summands is F-invariant, hence σ-invariant. In particular, if we write $\mathfrak{h}_0 = \mathfrak{h} \cap \mathfrak{g}_0$ and $\mathfrak{h}_{\mathcal{O}} = \mathfrak{h} \cap \mathfrak{g}_{\mathcal{O}}$, then

$$\mathfrak{h} = \mathfrak{h}_0 \oplus \bigoplus_{\mathcal{O} \in \Sigma(\mathfrak{g},\mathfrak{a})/F} \mathfrak{h}_{\mathcal{O}}, \tag{4.42}$$

with F-stable orthogonal summands.

Lemma 4.8.7.

(a) *The decomposition* (4.42) *is orthogonal for* $\langle \cdot\,, \cdot \rangle$.

(b) *The decomposition* (4.42) *is preserved by* L_w.

(c) *The decomposition* (4.42) *is orthogonal for* H_w.

Proof. The validity of (a) follows immediately from the fact that relative to the given inner product, the root spaces are mutually orthogonal, as well as orthogonal to $\mathfrak{g}_0$.

For (b) we note that the decomposition (4.41) is preserved by $\mathrm{Ad}(A)$, $\mathrm{ad}(\mathfrak{a})$, $E_{\mathfrak{k}}$ and $\pi_{\mathfrak{h}}$. Finally, in view of Lemma 4.8.6, the validity of (c) follows from (a) and (b). □

It follows from the above lemma that the kernel of H_w decomposes in accordance with (4.42). Let $\mathfrak{v}_{P,X} := (\ker \mathrm{H}_w)^\perp \cap \mathfrak{h}$. Then in view of (4.40) we have

$$\mathfrak{v}_{P,X} = \mathfrak{h}_X^\perp \cap (\mathfrak{h} \cap \mathfrak{n}_P)^\perp \cap \mathfrak{h} = \sum_{\mathcal{O} \in \Sigma(\mathfrak{g},\mathfrak{a})/F} \mathfrak{v}_{\mathcal{O}}, \tag{4.43}$$

with $\mathfrak{v}_{\mathcal{O}} = \mathfrak{v}_{P,X} \cap \mathfrak{h}_{\mathcal{O}}$. From these definitions it follows that H_w is non-degenerate on each of the spaces $\mathfrak{v}_{\mathcal{O}}$. Moreover, H_w is positive definite if and only if the restriction of H_w to $\mathfrak{v}_{\mathcal{O}}$ is positive definite for every $\mathcal{O} \in \Sigma(\mathfrak{g}, \mathfrak{a})/F$. This in turn is equivalent to the condition that the symmetric map $L_w : \mathfrak{h} \to \mathfrak{h}$ has a positive definite restriction to each of the spaces $\mathfrak{v}_{\mathcal{O}}$ (if $\mathfrak{v}_{\mathcal{O}}$ is zero, we agree that the latter is automatic). We will now systematically discuss the types of orbits $\mathcal{O}$ for which $\mathfrak{v}_{\mathcal{O}}$ is non-trivial.

First of all, we note that $\alpha \in \mathcal{O} \implies -\alpha = \theta\alpha \in \mathcal{O}$. Therefore, we see that $\mathcal{O} \cap \Sigma(P) \neq \emptyset$ for all $\mathcal{O} \in \Sigma(\mathfrak{g}, \mathfrak{a})/F$. Let $\sim$ denote the equivalence relation on $\Sigma(P)$ defined by

$$\alpha \sim \beta \iff F\alpha = F\beta,$$

then the map $\alpha \mapsto F\alpha$ induces a bijection from $\Sigma(P)/\sim$ onto $\Sigma(\mathfrak{g}, \mathfrak{a})/F$. The following lemma summarizes all possibilities for the spaces $\mathfrak{v}_{\mathcal{O}}$, as $\mathcal{O} \in \Sigma(\mathfrak{g}, \mathfrak{a})/F$.

Lemma 4.8.8. *Let $\alpha \in \Sigma(P)$, and put $\mathcal{O} = F\alpha$.*

(a) *If $\alpha(X) = 0$ then $\mathfrak{v}_{\mathcal{O}} = 0$.*

(b) *If $\alpha(X) \neq 0$ then we are in one of the following two cases* (b.1) *and* (b.2).

  (b.1) $\alpha \in \Sigma(P, \sigma)$; *in this case* $\mathfrak{v}_{\mathcal{O}} = \{V + \sigma(V) \ : \ V \in \mathfrak{g}_{-\alpha}\}$.

  (b.2) $\alpha \in \Sigma(P, \sigma\theta)$; *in this case* $\mathfrak{v}_{\mathcal{O}} = \mathfrak{h}_{\mathcal{O}}$.

Proof. (a) If $\alpha(X) = 0$ then $\mathfrak{h}_{\mathcal{O}} \subseteq \mathfrak{g}_X$, so that $\mathfrak{v}_{\mathcal{O}} = \{0\}$.

(b) Assume that $\alpha(X) \neq 0$. Then it follows that $\alpha \notin \mathfrak{a}_{\mathrm{h}}^*$, so that $\alpha \neq \sigma\alpha$. By Lemma 4.2.2 we are in one of the cases (b.1) and (b.2).

We first discuss case (b.1). Then $\sigma\alpha \in \Sigma(P)$ so that $\sigma\alpha \neq -\alpha$ and $\mathcal{O} = F\alpha$ consists of the four distinct elements $\alpha, \theta\alpha = -\alpha, \sigma\alpha$ and $\sigma\theta\alpha = -\sigma\alpha$. We see that $\mathfrak{h}_{\mathcal{O}}$ consists of sums of elements of the form $U + \sigma(U)$ and $V + \sigma(V)$ with $U \in \mathfrak{g}_\alpha$ and $V \in \mathfrak{g}_{-\alpha}$. The elements $U + \sigma(U)$ belong to $\mathfrak{h} \cap \mathfrak{n}_P$, whereas the elements $V + \sigma(V)$ belong to $\mathfrak{h}_X^\perp \cap (\mathfrak{h} \cap \mathfrak{n}_P)^\perp$. In view of (4.43) this implies the assertion of (b.1).

Next, we discuss case (b.2). Then $\mathcal{O} \cap \Sigma(P) = \{\alpha, -\sigma\alpha\}$ so that $\mathfrak{h}_{\mathcal{O}} \perp (\mathfrak{n}_P \cap \mathfrak{h})$. Since obviously $\mathfrak{h}_{\mathcal{O}} \perp \mathfrak{h}_X$, we infer the assertion of (b.2). $\square$

We will now proceed by explicitly calculating the restrictions $L_w|_{\mathfrak{v}_{\mathcal{O}}}$ for all these cases. The following lemma will be instrumental in our calculations.

Lemma 4.8.9. *Let $T_w : \mathfrak{g} \to \mathfrak{g}$ be defined by*

$$T_w = \mathrm{ad}(X) \circ \mathrm{Ad}(a^w) \circ E_{\mathfrak{k}} \circ \mathrm{Ad}(a^w).$$

Let $\beta \in \Sigma(\mathfrak{g}, \mathfrak{a})$ and $U_\beta \in \mathfrak{g}_\beta$.

(a) *If* $\beta \in \Sigma(P)$ *then* $T_w(U_\beta) = 0$.

(b) *If* $-\beta \in \Sigma(P)$ *then* $T_w(U_\beta) = \beta(X)\,(a^{2w\beta}U_\beta - \theta U_\beta)$.

Proof. Assume $\beta \in \Sigma(P)$. Then $\mathfrak{g}_\beta \subseteq \mathfrak{n}_P \subseteq \ker E_{\mathfrak{k}}$. Since $\mathrm{Ad}(a^w)$ preserves $\mathfrak{g}_\beta$, (a) follows.

For (b), assume that $-\beta \in \Sigma(P)$. Then U_β equals $U_\beta + \theta U_\beta$ modulo $\mathfrak{n}_P$, so that $E_{\mathfrak{k}}(U_\beta) = U_\beta + \theta U_\beta$. Hence,

$$\begin{aligned} T_w(U_\beta) &= \mathrm{ad}(X) \circ \mathrm{Ad}(a^w)[a^{w\beta}(U_\beta + \theta U_\beta)] \\ &= \mathrm{ad}(X)(a^{2w\beta}U_\beta + \theta U_\beta) \\ &= \beta(X)(a^{2w\beta}U_\beta - \theta U_\beta). \end{aligned}$$

□

In our calculations of $L_w|_{\mathfrak{v}_{\mathcal{O}}}$, we will distinguish between the cases described in Lemma 4.8.8. Case (a) is trivial.

Lemma 4.8.10 (Case b.1)**.** *Let* $\mathcal{O} = F\alpha$ *with* $\alpha \in \Sigma(P,\sigma)$ *and* $\alpha(X) \neq 0$. *Then*

$$L_w|_{\mathfrak{v}_{\mathcal{O}}} = \frac{\alpha(X)}{2}(a^{-2w\alpha} - a^{2w\alpha})I.$$

In particular, this restriction is positive definite if and only if $\alpha(X)\alpha(w^{-1}\log a) < 0$.

Proof. Let $V \in \mathfrak{g}_{-\alpha}$ and put $Z := V + \sigma(V)$. Since $-\alpha, -\sigma\alpha \in -\Sigma(P)$, it follows from Lemma 4.8.9 that

$$\begin{aligned} T_w(Z) &= -\alpha(X)(a^{-2w\alpha}V - \theta V) - \alpha(\sigma X)(a^{2w\alpha}\sigma V - \theta\sigma V) \\ &= \alpha(X)[-a^{-2w\alpha}V + a^{2w\alpha}\sigma V + \theta V - \theta\sigma V] \end{aligned}$$

so that

$$L_w(Z) = -\pi_{\mathfrak{h}} \circ T_w(Z) = \frac{\alpha(X)}{2}(a^{-2w\alpha} - a^{2w\alpha})Z.$$

It follows that L_w restricts to multiplication by a scalar on $\mathfrak{v}_{\mathcal{O}}$. The sign of this scalar equals the sign of $-\alpha(X)\alpha(w^{-1}\log a)$. The result follows. □

We now turn to the calculation of $L_w|_{\mathfrak{v}_{\mathcal{O}}}$ in case (b.2), where $\mathcal{O} = F\alpha$, with $\alpha \in \Sigma(P,\sigma\theta)$ and $\alpha(X) \neq 0$. There are two possibilities between which we will distinguish:

(b.2.1) $\alpha \in \Sigma(P,\sigma\theta) \setminus \mathfrak{a}_{\mathrm{q}}^*$,

(b.2.2) $\alpha \in \Sigma(P) \cap \mathfrak{a}_{\mathrm{q}}^*$.

In each of these cases, $\mathfrak{v}_{\mathcal{O}} = \mathfrak{h}_{\mathcal{O}}$ by Lemma 4.8.10. We will use the notation

$$\mathfrak{v}(U) = \mathfrak{h}(U) = \mathfrak{h} \cap \operatorname{span}(F \cdot U),$$

for $U \in \mathfrak{g}_\alpha$. In case (b.2.1), the orbit $\mathcal{O} = F\alpha$ consists of the four distinct roots $\alpha, \sigma\alpha, \theta\alpha$ and $\sigma\theta\alpha$, and

$$\mathfrak{v}(U) = \mathbb{R}(U + \sigma(U)) \oplus \mathbb{R}(\sigma\theta(U) + \theta(U)).$$

In case (b.2.2), $\mathcal{O} = F\alpha = \{\alpha, -\alpha\}$, and we see that

$$\mathfrak{v}(U) = \mathbb{R}(U + \sigma(U)).$$

In all of these cases, we see that if $U_1, \dots, U_m$ is an orthonormal basis of $\mathfrak{g}_\alpha$, then

$$\mathfrak{v}_{\mathcal{O}} = \bigoplus_{j=1}^{m} \mathfrak{v}(U_j), \tag{4.44}$$

with mutually orthogonal summands.

Lemma 4.8.11 (Case (b.2.1))**.** *Let $\mathcal{O} = F\alpha$, with $\alpha \in \Sigma(P, \sigma\theta) \setminus \mathfrak{a}_{\mathrm{q}}^*$ and $\alpha(X) \neq 0$. Then $L_w|_{\mathfrak{v}_{\mathcal{O}}}$ is positive definite if and only if $\alpha(X) > 0$ and $\alpha(X)\alpha(w^{-1}\log a) < 0$.*

Proof. Fix an element $U \in \mathfrak{g}_\alpha$ and put $Z_1 = U + \sigma(U)$ and $Z_2 = \theta Z_1 = \theta U + \sigma\theta U$. Then $T_w(U) = 0$ by Lemma 4.8.9, hence

$$\begin{aligned} T_w(Z_1) &= T_w(\sigma U) \\ &= \sigma\alpha(X)(a^{2w\sigma\alpha}\sigma(U) - \theta\sigma(U)) \\ &= \alpha(X)(\theta\sigma(U) - a^{-2w\alpha}\sigma(U)), \end{aligned}$$

from which we see that

$$L_w(Z_1) = -\pi_{\mathfrak{h}} T_w(Z_1) = \frac{\alpha(X)}{2}(a^{-2w\alpha}Z_1 - Z_2).$$

Likewise,

$$L_w(Z_2) = \frac{\alpha(X)}{2}(a^{-2w\alpha}Z_2 - Z_1).$$

It follows that L_w preserves the subspace $\mathfrak{v}(U)$ of $\mathfrak{v}_{\mathcal{O}}$ spanned by the orthogonal vectors Z_1, Z_2 and that the restriction $L_w|_{\mathfrak{v}(U)}$ has the following matrix with respect to this basis:

$$\operatorname{mat}(L_w|_{\mathfrak{v}(U)}) = \frac{\alpha(X)}{2}\begin{pmatrix} a^{-2w\alpha} & -1 \\ -1 & a^{-2w\alpha} \end{pmatrix}$$

This matrix is positive definite if and only if both its trace and determinant are positive. This is equivalent to

$$\alpha(X) > 0 \quad \text{and} \quad \alpha(X)(a^{-4w\alpha} - 1) > 0.$$

It follows that L_w is positive definite on the subspace $\mathfrak{v}(U)$ if and only if $\alpha(X) > 0$ and $\alpha(X)\alpha(w^{-1}\log a) < 0$.

Let $U_1, \ldots, U_m$ be an orthonormal basis for $\mathfrak{g}_\alpha$. Then by (4.44) we see that map L_w is positive definite if and only if all restrictions $L_w|_{\mathfrak{v}(U_j)}$ are positive definite. This is true if and only if $\alpha(X) > 0$ and $\alpha(X)\alpha(w^{-1}\log a) < 0$. $\square$

Lemma 4.8.12 (Case (b.2.2)). *Let $\mathcal{O} = F\alpha$ with $\alpha \in \Sigma(P) \cap \mathfrak{a}_{\mathrm{q}}^*$ and $\alpha(X) \neq 0$. Then $L_w|_{\mathfrak{v}_\mathcal{O}}$ is positive definite if and only if the following two conditions are fulfilled.*

(a) $\alpha \in \Sigma(P)_+ \cap \mathfrak{a}_{\mathrm{q}}^* \implies \alpha(X)\alpha(w^{-1}\log a)) < 0.$

(b) $\alpha \in \Sigma(P)_- \cap \mathfrak{a}_{\mathrm{q}}^* \implies \alpha(X) > 0.$

Proof. We write $\mathfrak{v}_{\mathcal{O},+} = \mathfrak{v}_\mathcal{O} \cap \mathfrak{k}$ and $\mathfrak{v}_{\mathcal{O},-} = \mathfrak{v}_\mathcal{O} \cap \mathfrak{p}$. Then

$$\mathfrak{v}_\mathcal{O} = \mathfrak{v}_{\mathcal{O},+} \oplus \mathfrak{v}_{\mathcal{O},-},$$

with orthogonal summands. We will show that L_w preserves this decomposition, and determine when both restrictions $L_w|_{\mathfrak{v}_{\mathcal{O},\pm}}$ are positive definite.

Let $U_\pm \in \mathfrak{g}_{\alpha,\pm}$ and put $Z_\pm = U_\pm + \sigma(U_\pm)$. Then $Z_\pm \in \mathfrak{g}_{\mathcal{O},\pm}$, and every element of $\mathfrak{v}_{\mathcal{O},\pm}$ can be expressed in this way.

By a straightforward computation, involving Lemma 4.8.9, we find

$$L_w(Z_\pm) = \frac{1}{2}\alpha(X)(a^{-2w\alpha} \mp 1)Z_\pm.$$

This shows that L_w acts by a real scalar $C_\pm$ on $\mathfrak{v}_{\mathcal{O},\pm}$. The restriction of L_w to $\mathfrak{v}_{\mathcal{O}_\pm}$ is positive definite if and only if the restrictions of L_w to both subspaces $\mathfrak{v}_{\mathcal{O},\pm}$ are positive definite. The latter condition is equivalent to

$$\mathfrak{v}_{\mathcal{O},+} \neq 0 \implies C_+ > 0 \quad \text{and} \quad \mathfrak{v}_{\mathcal{O},-} \neq 0 \implies C_- > 0.$$

The space $\mathfrak{v}_{\mathcal{O},\pm}$ is non-trivial if and only if $\mathfrak{g}_{\alpha,\pm} \neq 0$, which in turn is equivalent to $\alpha \in \Sigma(P)_\pm \cap \mathfrak{a}_{\mathrm{q}}^*$. On the other hand, the sign of C_+ equals that of $-\alpha(X)\alpha(w^{-1}\log a)$ whereas the sign of C_- equals that of $\alpha(X)$. From this the desired result follows. $\square$

Completion of the proof of Proposition 4.8.4. First assume that H_w is positive definite. Then L_w restricts to a positive definite symmetric map on each of the spaces $\mathfrak{v}_\mathcal{O}$ for $\mathcal{O} = F\alpha$, $\alpha \in \Sigma(P)$. First assume that $\alpha \in \Sigma(P)_+$. If $\alpha(X) = 0$, then

$$\alpha(X)\alpha(w^{-1}\log a) \leq 0 \tag{4.45}$$

holds. If $\alpha(X) \neq 0$, we are in one of the cases (b.1) or (b.2) of Lemma 4.8.8. In the latter case, we are either in the subcase (b.2.1) or in (b.2.2) with $\alpha \in \Sigma(P)_+ \cap \mathfrak{a}_{\mathrm{q}}^*$. In all of these cases, inequality (4.45) is valid. We conclude that assertion (a) of the proposition is valid.

For the validity of assertion (b), assume that $\alpha \in \Sigma(P)_-$. If $\alpha(X) = 0$, then

$$\alpha(X) \geq 0. \tag{4.46}$$

If $\alpha(X) \neq 0$, then we must be in case (b.2) of Lemma 4.8.8, since $\Sigma(P)_- \cap \Sigma(P,\sigma) = \emptyset$. We are either in subcase (b.2.1) or in subcase (b.2.2) with $\alpha \in \Sigma(P)_+ \cap \mathfrak{a}_{\mathrm{q}}^*$. In both subcases, (4.46) holds. This establishes condition (b) of the proposition, and the implication in one direction.

For the converse implication, assume that conditions (a) and (b) of the proposition hold. Let $\alpha \in \Sigma(P)$ and put $\mathcal{O} = F\alpha$. Then it suffices to show that H_w is positive definite on $\mathfrak{v}_{\mathcal{O}}$.

If $\alpha(X) = 0$, then $\mathfrak{v}_{\mathcal{O}} = 0$ by Lemma 4.8.8 and it follows that H_w is positive definite on $\mathfrak{v}_{\mathcal{O}}$. Thus, assume that $\alpha(X) \neq 0$. Then by regularity of $\log a$, the expression $\alpha(X)\alpha(w^{-1}\log a)$ is different from zero. Hence if any of the inequalities (4.45) or (4.46) holds, it holds as a strict inequality.

In case (b.1), $\alpha \in \Sigma(P,\sigma) \subseteq \Sigma(P)_+$ so that $\mathrm{H}_w|_{\mathfrak{v}_{\mathcal{O}}}$ is positive definite by Lemma 4.8.10. In case (b.2.1), $\alpha \in \Sigma(P,\sigma\theta) \setminus \mathfrak{a}_{\mathrm{q}}^* \subseteq \Sigma(P)_+ \cap \Sigma(P)_-$ so that (4.45) and (4.46) are both valid. Hence, $\mathrm{H}_w|_{\mathfrak{v}_{\mathcal{O}}}$ is positive definite by Lemma 4.8.11.

Finally, assume we are in case (b.2.2). Since $\Sigma(P)_\pm \cap \mathfrak{a}_{\mathrm{q}}^* \subseteq \Sigma(P)_\pm$, it follows from the hypotheses (a) and (b) of the proposition and from Lemma 4.8.12 that $\mathrm{H}_w|_{\mathfrak{v}_{\mathcal{O}}}$ is positive definite. $\square$

Corollary 4.8.13. *Let $w \in W_{K\cap H}$. Then the function $F_{a,X}$ as well as the signature and rank of its Hessian are constant on the immersed submanifold $wH_X(N_P \cap H)$.*

Proof. As the group H is essentially connected, $H_X = Z_{K\cap H}(\mathfrak{a}_{\mathrm{q}})H_X^\circ$. Let x_w be a representative of w in $N_{K\cap H}$. Since $Z_{K\cap H}(\mathfrak{a}_{\mathrm{q}})$ is normal in $N_{K\cap H}(\mathfrak{a}_{\mathrm{q}})$, it follows that

$$wH_X(N_P \cap H) = x_w Z_{K\cap H}(\mathfrak{a}_{\mathrm{q}})H_X^\circ(N_P \cap H) = Z_{K\cap H}(\mathfrak{a}_{\mathrm{q}})x_w H_X^\circ(N_P \cap H).$$

The function $F_{a,X} : H \to \mathbb{R}$ is left $Z_{K\cap H}(\mathfrak{a}_{\mathrm{q}})$- and right $(N_P \cap H)$-invariant. Hence, it suffices to prove the assertions for the set $x_w H_X^\circ$ of critical points. This set is connected, so that $F_{a,X}$ is constant on it. From Lemma 4.8.1 it follows that rank and signature of its Hessian remain constant along this set as well. $\square$

As in (4.2) we define

$$\Omega := \mathrm{conv}(W_{K\cap H} \cdot \log a) + \Gamma(P).$$

Lemma 4.8.14. *Let $a \in A_{\mathrm{q}}^{\mathrm{reg}}$ and $X \in \mathfrak{a}_{\mathrm{q}}$. Assume that the function $F_{a,X}$ has a local minimum at the critical point $h \in \mathcal{C}_{a,X}$. Then for every $U \in \Omega$*

$$\langle X, U\rangle \geq \langle X, \mathfrak{H}_{P,\mathrm{q}}(ah)\rangle.$$

In particular, Ω lies on one side of the hyperplane $\mathfrak{H}_{P,\mathrm{q}}(ah) + X^\perp$.

Proof. The critical point h belongs to a connected immersed submanifold of the form $x_w H_X^\circ (H \cap N_P)$. All points of this submanifold are critical for $F_{a,X}$, so that $F_{a,X}$ is constant along it. We see that

$$F_{a,X}(h) = F_{a,X}(x_w) = \langle X, \mathfrak{H}_{P,\mathrm{q}}(x_w^{-1} a x_w)\rangle = \langle X, w^{-1}\log a\rangle.$$

The Hessian of $F_{a,X}$ at the critical point h must be positive semidefinite. It now follows from Proposition 4.8.4 that

(a) $\forall \alpha \in \Sigma(P)_+ : \quad \alpha(X)\alpha(w^{-1}(\log a)) \leq 0;$

(b) $\forall \alpha \in \Sigma(P)_- : \quad \alpha(X) \geq 0.$

By (a) and Lemma 4.8.17 below (applied to $-X$), it follows that

$$\langle X, U_1\rangle \geq \langle X, w^{-1}\log a\rangle = F_{a,X}(h),$$

for all $U_1 \in \mathrm{conv}(W_{K\cap H} \cdot w^{-1}\log a)$. From (b) it follows that $\langle X, H_\alpha\rangle = \alpha(X) \geq 0$ for all $\alpha \in \Sigma(P)_-$, so that

$$\langle X, U_2\rangle \geq 0 \qquad (\forall\, U_2 \in \Gamma(P)).$$

Since every element $U \in \Omega$ may be decomposed as $U = U_1 + U_2$ with U_1 and U_2 as above, the assertion follows. $\square$

Remark 4.8.15. It can be readily shown that the converse implication also holds, namely if for every $U \in \Omega$

$$\langle X, U\rangle \geq \langle X, w^{-1}(\log a))\rangle,$$

then the two conditions of Proposition 4.8.4 hold.

Lemma 4.8.16. *The set $\Sigma(P)_+$ consists of all roots $\alpha \in \Sigma(P)$ with $\alpha \in \mathfrak{a}_\mathrm{h}^*$ or $\alpha|_{\mathfrak{a}_\mathrm{q}} \in \Sigma(\mathfrak{g}, \mathfrak{a}_\mathrm{q})_+$.*

Proof. In view of Definition 4.8.3 it suffices to show that for $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a}) \setminus (\mathfrak{a}_\mathrm{h}^* \cup \mathfrak{a}_\mathrm{q}^*)$ we have $\alpha|_{\mathfrak{a}_\mathrm{q}} \in \Sigma(\mathfrak{g}, \mathfrak{a}_\mathrm{q})_+$. Assume $\alpha \notin \mathfrak{a}_\mathrm{h}^* \cup \mathfrak{a}_\mathrm{q}^*$. Then α and $\sigma\theta\alpha$ are distinct roots that restrict to the same root $\bar{\alpha}$ of $\Sigma(\mathfrak{g}, \mathfrak{a}_\mathrm{q})$. Thus, the sum $\mathfrak{g}_\alpha + \sigma\theta\mathfrak{g}_\alpha$ is direct and contained in $\mathfrak{g}_{\bar{\alpha}}$ and we see that $\mathfrak{g}_{\bar{\alpha},+} \neq 0$. $\square$

Lemma 4.8.17. *Let $P \in \mathcal{P}(A)$. Let $X, Y \in \mathfrak{a}_\mathrm{q}$ and assume that $\alpha(X)\alpha(Y) \geq 0$ for all $\alpha \in \Sigma(P)_+$. Then*

$$\langle X, U\rangle \leq \langle X, Y\rangle, \qquad \text{for all} \quad U \in \mathrm{conv}(W_{K\cap H} \cdot Y).$$

Proof. In view of Lemma 4.8.16, the hypothesis is equivalent to

$$\alpha(X)\,\alpha(Y) \geq 0$$

for all roots $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a}_\mathrm{q})_+$. We may now fix a Weyl chamber $\mathfrak{a}_\mathrm{q}^+$ for the root system $\Sigma(\mathfrak{g}, \mathfrak{a}_\mathrm{q})_+$ such that X and Y belong to the closure of $\mathfrak{a}_\mathrm{q}^+$. Then it is well known that $\langle X, wY\rangle \leq \langle X, Y\rangle$ for all w in the reflection group $W(\Sigma(\mathfrak{g}, \mathfrak{a}_\mathrm{q})_+)$ generated by $\Sigma(\mathfrak{g}, \mathfrak{a}_\mathrm{q})_+$. Since this reflection group is equal to $W_{K\cap H}$, by Proposition 2.2 in [5], the result follows. $\square$

4.9 Reduction by a limit argument

Before turning to the proof of our main theorem, Theorem 4.10.1, we will first prove a lemma that reduces the validity of the theorem to its validity under the additional assumption that the element a be regular in A_{q}. We assume that $P \in \mathcal{P}(A)$ and recall the definition of the closed convex polyhedral cone $\Gamma(P)$ given in Definition 3.1.4.

Lemma 4.9.1. *Assume that the assertion*

$$\mathrm{pr}_{\mathrm{q}} \circ \mathfrak{H}_P(aH) = \mathrm{conv}(W_{K\cap H} \cdot \log a) + \Gamma(P) \tag{4.47}$$

is valid for all $a \in A_{\mathrm{q}}^{\mathrm{reg}}$. Then assertion (4.47) holds for all $a \in A_{\mathrm{q}}$.

Proof. Assume the assertion is valid for all $a \in A_{\mathrm{q}}^{\mathrm{reg}}$, and let $a \in A_{\mathrm{q}}$ be an arbitrary fixed element. Fix a sequence $(a_j)_{j\geq 1}$ in $A_{\mathrm{q}}^{\mathrm{reg}}$ with limit a. Let $h \in H$. By the validity of (4.47) for a_j in place of a, there exist, for each $j \geq 1$, elements $\lambda_{w,j} \in [0,1]$ with $\sum_{w\in W_{K\cap H}} \lambda_{w,j} = 1$ and elements $\gamma_j \in \Gamma(P)$ such that

$$\mathfrak{H}_{P,\mathrm{q}}(a_j h) = \sum_{w\in W_{K\cap H}} \lambda_{w,i} w(\log a_j) + \gamma_j.$$

By passing to a subsequence of indices we may arrange that the sequence $(\lambda_{w,j})_j$ converges with limit $\lambda_w \in [0,1]$ for each $w \in W_{K\cap H}$. It follows that the sequence (γ_j) must have a limit $\gamma \in \mathfrak{a}_{\mathrm{q}}$ such that

$$\mathfrak{H}_{P,\mathrm{q}}(ah) = \lim_{j\to\infty} \mathfrak{H}_{P,\mathrm{q}}(a_j h) = \sum_{w\in W_{K\cap H}} \lambda_w w(\log a) + \gamma.$$

By taking the limit we see that $\sum_w \lambda_w = 1$ and since $\Gamma(P)$ is closed, $\gamma \in \Gamma(P)$. Hence, $\mathfrak{H}_{P,\mathrm{q}}(ah) \in \mathrm{conv}(W_{K\cap H} \cdot \log a) + \Gamma(P)$, and we find that

$$\mathfrak{H}_{P,\mathrm{q}}(aH) \subseteq \mathrm{conv}(W_{K\cap H} \cdot \log a) + \Gamma(P).$$

For the converse inclusion, assume that $Y \in \mathrm{conv}(W_{K\cap H} \cdot \log a) + \Gamma(P)$. Then there exist $\gamma \in \Gamma(P)$ and $\lambda_w \in [0,1]$ with $\sum_{w\in W_{K\cap H}} \lambda_w = 1$ such that

$$Y = \sum_{w\in W_{K\cap H}} \lambda_w w(\log a) + \gamma.$$

Put

$$Y_j = \sum_{w\in W_{K\cap H}} \lambda_w w(\log a_j) + \gamma.$$

Then there exist $h_j \in H$ such that $\mathfrak{H}_{P,\mathrm{q}}(a_j h_j) = Y_j$ for every j. The sequence (Y_j) is convergent, hence contained in a compact set of $\mathfrak{a}_{\mathrm{q}}$. Likewise, the sequence (a_j) is contained in a compact subset $\mathcal{A} \subseteq A_{\mathrm{q}}$. By Corollary 4.4.12 there exists a compact subset $\mathcal{K}$ of $H/H\cap P$ such that $h_j(H\cap P) \in \mathcal{K}$ for all j. By passing to a subsequence we see that we may arrange that the sequence $h_j(H \cap P)$ converges in $H/H \cap P$. By replacing the h_j with suitable other representatives, we may arrange that the sequence (h_j) converges to an element $h \in H$. It then follows that

$$Y = \lim_{j \to \infty} Y_j = \lim_{j \to \infty} \mathfrak{H}_{P,\mathrm{q}}(a_j h_j) = \mathfrak{H}_{P,\mathrm{q}}(ah) \in \mathfrak{H}_{P,\mathrm{q}}(aH).$$

□

4.10 Proof of the main theorem

In this section we will prove our main result. For $P \in \mathcal{P}(A)$ we recall the definition of the closed convex polyhedral cone $\Gamma(P)$ given in Definition 3.1.4.

Theorem 4.10.1. *Let P be a minimal parabolic subgroup of G containing A and let $a \in A_{\mathrm{q}}$. Then*

$$\mathrm{pr}_{\mathrm{q}} \circ \mathfrak{H}_P(aH) = \mathfrak{H}_{P,\mathrm{q}}(aH) = \mathrm{conv}(W_{K\cap H} \cdot \log a) + \Gamma(P). \tag{4.48}$$

The proof of our main theorem proceeds by induction, for whose induction step the following lemma is a key ingredient.

If $X \in \mathfrak{a}_{\mathrm{q}}$, we denote by G_X the centralizer of X in G. This group belongs to the Harish-Chandra class and is σ-stable. Moreover, by [5, Proposition 2.3], the centralizer $H_X := H \cap G_X$ is an essentially connected open subgroup of $(G_X)^\sigma$. From

$$P \cap G_X = (Z_K(\mathfrak{a}) A N_P) \cap (K_X A N_{P,X}) = Z_K(\mathfrak{a}) A N_{P,X},$$

see (4.19) for notation, we see that $P_X := P \cap G_X$ is a minimal parabolic subgroup of G_X.

We agree to write $\Gamma(P_X)$ for the cone in $\mathfrak{a}_{\mathrm{q}}$ spanned by $\mathrm{pr}_{\mathrm{q}} H_\alpha$, for $\alpha \in \Sigma(P)_-$ with $\alpha(X) = 0$. Furthermore, for a given $a \in A_{\mathrm{q}}$, we define $\Omega_{a,X} = \Omega_X$ by

$$\Omega_X := \bigcup_{w \in W_{K\cap H}} \Omega_{X,w}, \qquad \text{where} \tag{4.49}$$

$$\Omega_{X,w} := \left(\mathrm{conv}(W_{K\cap H_X} \cdot w^{-1} \log a) + \Gamma(P_X)\right). \tag{4.50}$$

Remark 4.10.2. It is clear from the definition that the set $\Omega_{X,w}$, for $w \in W_{K\cap H}$, is a closed convex polyhedral set, contained in the affine subset $w^{-1} \log a + \mathrm{span}\,\{H_\alpha : \alpha \in \Sigma(\mathfrak{g}_X, \mathfrak{a}_{\mathrm{q}})\}$ of $\mathfrak{a}_{\mathrm{q}}$. In particular,

$$\Omega_{X,w} \subseteq w^{-1} \log a + X^\perp.$$

Lemma 4.10.3. *Let $X \in S$, $a \in A_{\mathrm{q}}^{\mathrm{reg}}$ and let $\mathcal{C}_{a,X} \subseteq H$ be the set of critical points of the function $F_{a,X} : H \to \mathbb{R}$; cf. Lemma 4.5.5 and* (4.27). *If the analogue of the assertion of Theorem 4.10.1 holds for the data G_X, H_X, K_X and P_X in place of G, H, K and P then*

$$\mathfrak{H}_{P,\mathrm{q}}(a\mathcal{C}_{a,X}) = \Omega_X. \tag{4.51}$$

Proof. Using the characterization of $\mathcal{C}_{a,X}$ given in Lemma 4.5.5, we obtain

$$\begin{aligned} \mathfrak{H}_{P,\mathrm{q}}(a\mathcal{C}_{a,X}) &= \bigcup_{w\in W_{K\cap H}} \mathfrak{H}_{P,\mathrm{q}}(awH_X(N_P\cap H)) \\ &= \bigcup_{w\in W_{K\cap H}} \mathfrak{H}_{P,\mathrm{q}}(a^w H_X), \end{aligned} \tag{4.52}$$

where $a^w = w^{-1}aw$ is regular in A_q, for each $w \in W_{K\cap H}$.

By the compatibility of the Iwasawa decompositions for the two groups G and G_X we see that the restriction of $\mathfrak{H}_{P,\mathrm{q}} : G \to \mathfrak{a}_\mathrm{q}$ to G_X equals the similar projection $G_X \to \mathfrak{a}_\mathrm{q}$ associated with P_X; we denote the latter by $\mathfrak{H}_{P_X,\mathrm{q}}$. Hence,

$$\mathfrak{H}_{P,\mathrm{q}}(a^w H_X) = \mathfrak{H}_{P_X,\mathrm{q}}(a^w H_X).$$

In view of the hypothesis that the convexity theorem holds for the data G_X, H_X, P_X, we infer that

$$\mathfrak{H}_{P,\mathrm{q}}(a^w H_X) = \mathrm{conv}(W_{K\cap H_X}\cdot \log a^w) + \Gamma(P_X) = \Omega_{X,w}.$$

In view of (4.52) and (4.49) we now obtain (4.51). □

Proof of Theorem 4.10.1. The proof relies on an inductive procedure, with induction over the rank of the root system $\Sigma(\mathfrak{g},\mathfrak{a}_\mathrm{q})$. The legitimacy of this procedure has been discussed at length in [5, Sect. 2].

We start the induction with $\mathrm{rk}\,\Sigma(\mathfrak{g},\mathfrak{a}_\mathrm{q}) = 0$. Then, for every root $\alpha \in \Sigma(\mathfrak{g},\mathfrak{a})$ we have that $\alpha|_{\mathfrak{a}_\mathrm{q}} = 0$. Thus, $\alpha \in \Sigma(\mathfrak{g},\mathfrak{a}) \cap \mathfrak{a}_\mathrm{h}^*$, which implies that $\mathfrak{g}_\alpha$ is stable under σ, so that

$$\mathfrak{g}_\alpha = (\mathfrak{g}_\alpha \cap \mathfrak{h}) \oplus (\mathfrak{g}_\alpha \cap \mathfrak{q}).$$

An easy computation now shows that $\mathfrak{g}_\alpha \cap \mathfrak{q} = \{0\}$ which implies that $\mathfrak{g}_\alpha \subseteq \mathfrak{h}$. As this holds for any $\mathfrak{a}$-root α, it follows that $\mathfrak{n}_P \subseteq \mathfrak{h}$, hence

$$\mathfrak{h} = \bar{\mathfrak{n}}_P \oplus (\mathfrak{m}\cap\mathfrak{h}) \oplus \mathfrak{a}_\mathrm{h} \oplus \mathfrak{n}_P$$

and we see that $\mathfrak{h}$ centralizes $\mathfrak{a}_\mathrm{q}$. It follows that H° centralizes $\mathfrak{a}_\mathrm{q}$. Since H is essentially connected, this implies that H centralizes $\mathfrak{a}_\mathrm{q}$. In particular, $W_{K\cap H} = \{1\}$. It also follows that $\Sigma(P)_- = \emptyset$, so that $\Gamma(P) = \{0\}$. Hence, in this case, the set on the right-hand side of (4.48) equals $\{\log a\}$.

On the other hand, the Iwasawa decomposition $G = KAN_P$ is σ-stable so that

$$H = (K\cap H)(A\cap H)N_P.$$

We thus see that

$$\mathfrak{H}_{P,\mathrm{q}}(aH) = \mathfrak{H}_{P,\mathrm{q}}(Ha) = \mathfrak{H}_{P,\mathrm{q}}(H\cap A) + \log a = \log a.$$

Hence, the equality (4.48) holds in case $\mathrm{rk}\,\Sigma(\mathfrak{g},\mathfrak{a}_\mathrm{q}) = 0$.

Now assume that m is a positive integer, that $\mathrm{rk}\,\Sigma(\mathfrak{g}, \mathfrak{a}_{\mathrm{q}}) = m$ and that the assertion of the theorem has already been established for the case that $\mathrm{rk}\,\Sigma(\mathfrak{g}, \mathfrak{a}_{\mathrm{q}}) < m$.

By Lemma 4.9.1 it suffices to prove the validity of (4.48) under the assumption that $a \in A_{\mathrm{q}}^{\mathrm{reg}}$. We will first do so under the additional assumption that $\Sigma(\mathfrak{g}, \mathfrak{a}_{\mathrm{q}})$ spans $\mathfrak{a}_{\mathrm{q}}^*$. In the end, the general case will be reduced to this.

Our assumption that $\Sigma(\mathfrak{g}, \mathfrak{a}_{\mathrm{q}})$ is spanning guarantees that for each non-zero $X \in \mathfrak{a}_{\mathrm{q}}$ not all roots of $\Sigma(\mathfrak{g}, \mathfrak{a}_{\mathrm{q}})$ vanish on X. Therefore, the rank of $\Sigma(\mathfrak{g}_X, \mathfrak{a}_{\mathrm{q}})$ is strictly smaller than $m = \mathrm{rk}\Sigma(\mathfrak{g}, \mathfrak{a}_{\mathrm{q}})$. By the induction hypothesis, the convexity theorem holds for (G_X, H_X, K_X, P_X). Hence, by Lemma 4.10.3, the set $\mathfrak{H}_{P,\mathrm{q}}(a\mathcal{C}_{a,X})$ equals Ω_X. By Remark 4.10.2 the complement $\mathfrak{a}_{\mathrm{q}} \setminus \Omega_X$ is open and dense in $\mathfrak{a}_{\mathrm{q}}$.

Let $S_0 \subseteq S$ be a finite subset as in Lemma 4.6.6. Then it follows by application of Lemma 4.10.3 that

$$\mathfrak{H}_{P,\mathrm{q}}(a\mathcal{C}_a) = \cup_{X \in S_0} \Omega_X. \tag{4.53}$$

In particular, the complement of this set in $\mathfrak{a}_{\mathrm{q}}$ is dense. Moreover, it follows from (4.53) that

$$\mathfrak{H}_{P,\mathrm{q}}(a\mathcal{C}_a) \subseteq \mathrm{conv}(W_{K\cap H} \cdot \log a) + \Gamma(P) = \Omega. \tag{4.54}$$

From Lemma 4.6.8 we see that $\mathfrak{H}_{P,\mathrm{q}}(aH)$ and $\mathfrak{H}_{P,\mathrm{q}}(a\mathcal{C}_a)$ are closed subsets of $\mathfrak{a}_{\mathrm{q}}$ and that $\mathfrak{H}_{P,\mathrm{q}}(aH) \setminus \mathfrak{H}_{P,\mathrm{q}}(a\mathcal{C}_a)$ is an open and closed subset of the (open and dense) subset $\mathfrak{a}_{\mathrm{q}} \setminus \mathfrak{H}_{P,\mathrm{q}}(a\mathcal{C}_a)$, hence a union of connected components of the latter set. Lemma 4.6.9 ensures that at least one connected component of $\mathfrak{a}_{\mathrm{q}} \setminus \mathfrak{H}_{P,\mathrm{q}}(a\mathcal{C}_a)$ must belong to $\mathfrak{H}_{P,\mathrm{q}}(aH) \setminus \mathfrak{H}_{P,\mathrm{q}}(a\mathcal{C}_a)$.

From (4.54) it follows that

$$\mathfrak{a}_{\mathrm{q}} \setminus \Omega \subseteq \mathfrak{a}_{\mathrm{q}} \setminus \mathfrak{H}_{P,\mathrm{q}}(a\mathcal{C}_a).$$

Now $\mathfrak{a}_{\mathrm{q}} \setminus \Omega$ is connected hence must be contained in a connected component Λ of $\mathfrak{a}_{\mathrm{q}} \setminus \mathfrak{H}_{P,\mathrm{q}}(a\mathcal{C}_a)$.

There are two possibilities:

(a) $\Lambda \subseteq \mathfrak{H}_{P,\mathrm{q}}(aH) \setminus \mathfrak{H}_{P,\mathrm{q}}(a\mathcal{C}_a)$;

(b) $\Lambda \cap (\mathfrak{H}_{P,\mathrm{q}}(aH) \setminus \mathfrak{H}_{P,\mathrm{q}}(a\mathcal{C}_a)) = \emptyset$.

From its definition, one sees that Ω is strictly contained in half-space, which implies that $\mathfrak{a}_{\mathrm{q}} \setminus \Omega$, and therefore Λ, must contain a line of $\mathfrak{a}_{\mathrm{q}}$. From Corollary 4.4.15 we know that $\mathfrak{H}_{P,\mathrm{q}}(aH)$ does not contain such a line, so that we may exclude case (a) above. From (b) it follows that

$$(\mathfrak{a}_{\mathrm{q}} \setminus \Omega) \cap \mathfrak{H}_{P,\mathrm{q}}(aH) \setminus \mathfrak{H}_{P,\mathrm{q}}(a\mathcal{C}_a) = \emptyset,$$

which implies that $\mathfrak{H}_{P,\mathrm{q}}(aH) \setminus \mathfrak{H}_{P,\mathrm{q}}(a\mathcal{C}_a) \subseteq \Omega$. Combining this with (4.54) we conclude that

$$\mathfrak{H}_{P,\mathrm{q}}(aH) \subseteq \Omega. \tag{4.55}$$

We now turn to the proof of the converse inclusion.

In the above we concluded that the set $\mathfrak{H}_{P,\mathrm{q}}(aH) \setminus \mathfrak{H}_{P,\mathrm{q}}(a\mathcal{C}_a)$ is open and closed as a subset of $\mathfrak{a}_\mathrm{q} \setminus \mathfrak{H}_{P,\mathrm{q}}(a\mathcal{C}_a)$. In view of (4.55) the set is also open and closed as a subset of $\Omega \setminus \mathfrak{H}_{P,\mathrm{q}}(a\mathcal{C}_a)$. Thus, $\mathfrak{H}_{P,\mathrm{q}}(aH) \setminus \mathfrak{H}_{P,\mathrm{q}}(a\mathcal{C}_a)$ is a union of connected components of $\Omega \setminus \mathfrak{H}_{P,\mathrm{q}}(a\mathcal{C}_a)$. We will establish the converse of (4.55) by showing that all connected components of $\Omega \setminus \mathfrak{H}_{P,\mathrm{q}}(a\mathcal{C}_a)$ are contained in $\mathfrak{H}_{P,\mathrm{q}}(aH)$.

Again by the use of Lemma 4.6.9 we infer that at least one connected component Λ_1 of $\Omega \setminus \mathfrak{H}_{P,\mathrm{q}}(a\mathcal{C}_a)$ is contained in $\mathfrak{H}_{P,\mathrm{q}}(a\mathcal{C}_a)$. Arguing by contradiction, assume this were not the case for all components. Then there exists a second connected component Λ_2 of $\Omega \setminus \mathfrak{H}_{P,\mathrm{q}}(a\mathcal{C}_a) = \Omega \setminus \cup_{X \in S_0} \Omega_X$ such that $\Lambda_2 \cap \mathfrak{H}_{P,\mathrm{q}}(aH) = \emptyset$. In view of Remark 4.10.2, we may apply Lemma 4.10.4 below to the set Ω and the finite collection of subsets $\Omega_{X,w}$, where $X \in S_0$ and $w \in W_{K\cap H}$, and obtain a line segment with the properties of Lemma 4.10.4, connecting Λ_1 and Λ_2. By following intersections along this line segment, we see that we may assume that the connected components Λ_1 and Λ_2 exist with the additional property that they are adjacent, i.e., there exists a codimension 1 subset $\Omega_{X,w} \subseteq \Omega$ together with a point $Y \in \Omega_{X,w}$ and a positive number $\epsilon > 0$ such that $B(Y;\epsilon) \setminus \Omega_{X,w}$ consists of two connected components Λ_1' and Λ_2' such that $\Lambda_j' \subseteq \Lambda_j$ for $j = 1, 2$. In particular, this implies that Λ_1' and Λ_2' are on different sides of the hyperplane $\mathrm{aff}(\Omega_{X,w}) = Y + X^\perp$. We may replace X by $-X$ if necessary, to arrange that $Y + tX \in \Lambda_1$ for $t \downarrow 0$.

By (4.53) there exists a point $h \in \mathcal{C}_{a,X}$ such that $\mathfrak{H}_{P,\mathrm{q}}(ah) = Y$. For a sufficiently small neighborhood U of h in H we have $\mathfrak{H}_{P,\mathrm{q}}(aU) \subseteq B(Y;\epsilon)$, hence $\mathfrak{H}_{P,\mathrm{q}}(aU) \subseteq \Lambda_1 \cap B(Y;\epsilon)$. It follows that $F_{a,X} \geq \langle X, Y\rangle = F_{a,X}(h)$ on U. Hence, $F_{a,X}$ has a local minimum at h. By what we established in Lemma 4.8.14 this implies that Ω should be on one side of the hyperplane $Y + X^\perp$, contradicting the observation that Λ_1' and Λ_2' are non-empty open subsets on different sides of this hyperplane, but both contained in Ω.

In view of this contradiction we conclude that all components of $\Omega \setminus \mathfrak{H}_{P,\mathrm{q}}(a\mathcal{C}_a)$ are contained in $\mathfrak{H}_{P,\mathrm{q}}(aH)$.

This finishes the proof in case $\Sigma(\mathfrak{g}, \mathfrak{a}_\mathrm{q})$ has rank m and spans $\mathfrak{a}_\mathrm{q}^*$. We finally consider the case with $\mathrm{rk}\,\Sigma(\mathfrak{g}, \mathfrak{a}_\mathrm{q}) = m$ in general.

Let $\mathfrak{c}$ be the intersection of the root hyperplanes $\ker\alpha \subseteq \mathfrak{a}_\mathrm{q}$ for $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a}_\mathrm{q})$. Then $\mathfrak{c}$ is contained in $\mathfrak{a}_\mathrm{q}$ and central in $\mathfrak{g}$. Let ${}^\backprime\mathfrak{p}$ be the orthocomplement of $\mathfrak{c}$ in $\mathfrak{p}$. Then ${}^\backprime\mathfrak{g} := \mathfrak{k} \oplus {}^\backprime\mathfrak{p}$ is an ideal of $\mathfrak{g}$ which is complementary to $\mathfrak{c}$.

By the Cartan decomposition and the fact that $\mathfrak{c}$ is central, it follows that the map $K \times {}^\backprime\mathfrak{p} \times \mathfrak{c} \to G$, $(k, X, Z) \mapsto k \exp X \exp Z$ is a diffeomorphism onto. It readily follows that ${}^\backprime G = K \exp {}^\backprime\mathfrak{p}$ is a group of the Harish-Chandra class, with the indicated Cartan decomposition for the Cartan involution ${}^\backprime\theta = \theta|_{{}^\backprime G}$. The restricted map ${}^\backprime\sigma := \sigma|_{{}^\backprime G}$ is an involution of ${}^\backprime G$ which commutes with ${}^\backprime\theta$. The group ${}^\backprime H := H$ is an open subgroup of $({}^\backprime G)^{{}^\backprime\sigma}$, which is essentially connected. Furthermore, ${}^\backprime\mathfrak{a}_\mathrm{q} := {}^\backprime\mathfrak{p} \cap \mathfrak{a}_\mathrm{q}$ is maximal abelian in ${}^\backprime\mathfrak{p} \cap \mathfrak{q}$ and ${}^\backprime\mathfrak{a} = {}^\backprime\mathfrak{p} \cap \mathfrak{a}$ is maximal abelian in ${}^\backprime\mathfrak{p}$. The root system $\Sigma({}^\backprime\mathfrak{g}, {}^\backprime\mathfrak{a}_\mathrm{q})$ consists of the restrictions of the roots from $\Sigma(\mathfrak{g}, \mathfrak{a}_\mathrm{q})$, hence spans the dual of ${}^\backprime\mathfrak{a}_\mathrm{q}$.

The group ${}^\backprime P = {}^\backprime G \cap P$ is a minimal parabolic subgroup of ${}^\backprime G$ containing ${}^\backprime A$. We note that ${}^\backprime P = M\,{}^\backprime A N_P$.

We note that $A_{\mathrm{q}}^{\mathrm{reg}} \simeq {}^\backprime A_{\mathrm{q}}^{\mathrm{reg}} \times C$. Let $a \in A_{\mathrm{q}}^{\mathrm{reg}}$. Then we may write $a = {}^\backprime a \cdot c$, with ${}^\backprime a \in {}^\backprime A_{\mathrm{q}}^{\mathrm{reg}}$ and $c \in C$. By the convexity theorem for ${}^\backprime G$ and since c is central in G, it now follows that

$$\begin{aligned}
\mathfrak{H}_{P,\mathrm{q}}(aH) &= \mathfrak{H}_{P,\mathrm{q}}({}^\backprime aHc) \\
&= \mathfrak{H}_{{}^\backprime P,\mathrm{q}}({}^\backprime aH) + \log c \\
&= \mathrm{conv}(W_{K\cap H} \cdot \log {}^\backprime a) + \Gamma({}^\backprime P) + \log c \\
&= \mathrm{conv}(W_{K\cap H} \cdot \log a) + \Gamma(P).
\end{aligned}$$

□

We recall that the relative interior of a convex subset S of a finite dimensional real linear space is defined to be the interior of S in its affine span $\mathrm{aff}(S)$.

Lemma 4.10.4. *Let V be a finite dimensional real linear space and $C \subseteq V$ a closed convex polyhedral subset with non-empty interior. Let C_i ($i \in \{1, \ldots, n\}$) be closed convex polyhedral subsets of C, of positive codimension. Then the following statements are true.*

(a) *The complement $C' := C \setminus \cup_{i=1}^n C_i$ is dense in C.*

(b) *Let A and B be open subsets of V contained in C'. Then for each $a \in A$ there exists $b \in B$ such that for each i with $C_i \cap [a,b] \neq \emptyset$ the following assertions are valid,*

  (1) $\mathrm{codim}(C_i) = 1$;

  (2) *$[a,b] \cap C_i$ consists of a single point p which belongs to the relative interior of C_i. Furthermore, if $p \in C_j$ for some $1 \leq j \leq n$, then $\mathrm{aff}(C_j) = \mathrm{aff}(C_i)$.*

Proof. (a) Clearly, for every $1 \leq i \leq n$, the set $C \setminus C_i$ is open and dense in C. The same holds for their intersection, which equals $C \setminus \cup_{1\leq i\leq n} C_i$.

(b) We begin by enlarging the set $\{C_i \,|\, 1 \leq i \leq n\}$ of closed convex polyhedral subsets. We add to this set intersections of elements, $C_i \cap C_j$, and the boundaries of elements, ∂C_i. These sets are closed convex polyhedral sets as well. We repeat this step as many times as necessary, until nothing more gets added to the set. We denote the new set similarly: $\{C_i \,|\, 1 \leq i \leq m\}$, $m \geq n$.

Observe that $\mathrm{codim}(\partial C_i) > \mathrm{codim}(C_i)$ (since C_i has non-empty interior) and that the same holds for $C_i \cap C_j$, namely $\mathrm{codim}(C_i \cap C_j) > \mathrm{codim}(C_i)$ and $\mathrm{codim}(C_i \cap C_j) > \mathrm{codim}(C_j)$, unless $\mathrm{aff}(C_i) = \mathrm{aff}(C_j)$ or one set is contained in the other, $C_j \subset C_i$. These observations insure that the second assertion of (b) will follow automatically from the first.

By (a) we may choose $a \in A \setminus \cup_{i=1}^m C_i$. Denote by

$$\Omega_i := \begin{cases} C_i & , \quad \text{if } codim(C_i) = 1 \\ \{a + r(c-a) \,:\, c \in C_i, r \geq 0\} & , \quad \text{otherwise.} \end{cases}$$

Then the sets Ω_i ($1 \leq i \leq m$) satisfy the conditions of the lemma and thus, by (a) we conclude that the complement $C \setminus \cup_{i=1}^m \Omega_i$ is open and dense in C. Hence, there exists $b \in B \setminus \cup_{i=1}^m \Omega_i$ which satisfies $b \notin C_i$ for all $1 \leq i \leq m$. It remains to show that the segment $[a, b]$ does not intersect any of the convex polyhedral cones C_j of codimension greater than 1.

Assume $[a, b] \cap C_j \neq \emptyset$ for some C_j of codimension greater than 1. Then, there exists $t \in [0, 1]$ such that $c := (1-t)a + tb \in C_j$. Since $a \notin C_j$ it follows that $t \neq 0$ and thus

$$b = a + \frac{1}{t}(c - a) \in \Omega_j.$$

We arrived at a contradiction. $\square$

Appendix A

Proof of Lemma 4.2.10

Finally, we prove Lemma 4.2.10.

We begin by showing that the result holds for the case that G is a complex semi-simple Lie group, connected with trivial center. That proof will be based on the following general lemma, inspired by [41, Proposition 1].

Let $\mathfrak{h}$ be a Cartan subalgebra of the Lie algebra $\mathfrak{g}$ and let $\mathcal{N}$ be the class of complex finite dimensional nilpotent Lie algebras $\mathfrak{n}$, equipped with a representation of $\mathfrak{h}$ by derivations, such that the following conditions are fulfilled

(a) the representation of $\mathfrak{h}$ in $\mathfrak{n}$ is semi-simple;

(b) all weight spaces of $\mathfrak{h}$ in $\mathfrak{n}$ have complex dimension one.

If $\mathfrak{n}$ belongs to the class $\mathcal{N}$, we write $\Lambda(\mathfrak{n})$ for the set of $\mathfrak{h}$-weights in $\mathfrak{n}$. If $\lambda \in \Lambda(\mathfrak{n})$, then the associated weight space is denoted by $\mathfrak{n}_\lambda$.

Lemma A.0.5. *Let $\mathfrak{n} \in \mathcal{N}$ and let N be the connected, simply-connected Lie group with Lie algebra $\mathfrak{n}$. Let $\lambda_1, \ldots, \lambda_m$ be the distinct weights of $\mathfrak{h}$ in $\mathfrak{n}$. Then the map*

$$\psi : (X_1, \ldots, X_m) \mapsto \exp X_1 \cdot \cdots \cdot \exp X_m$$

defines a diffeomorphism

$$\mathfrak{n}_{\lambda_1} \times \ldots \times \mathfrak{n}_{\lambda_m} \xrightarrow{\simeq} N.$$

Proof. We will use induction on $\dim_{\mathbb{C}}(\mathfrak{n})$. If $\dim_{\mathbb{C}} \mathfrak{n} = 1$ then $\mathfrak{n}$ is abelian and the result holds trivially.

Next, assume that $m > 1$ and assume that the result has been established for $\mathfrak{n}$ with $\dim_{\mathbb{C}} \mathfrak{n} < m$. Assume that $\mathfrak{n} \in \mathcal{N}$ has dimension m.

Denote by $\mathfrak{n}_1$ the center of $\mathfrak{n}$, which is non-trivial. If $\mathfrak{n}_1 = \mathfrak{n}$ then $\mathfrak{n}$ is abelian and the result is trivially true. Thus, we may as well assume that $0 \subsetneq \mathfrak{n}_1 \subsetneq \mathfrak{n}$. In particular, this implies that both $\mathfrak{n}_1$ and $\mathfrak{n}/\mathfrak{n}_1$ have dimensions at most $m - 1$. Put $l := \dim \mathfrak{n}_1$.

The ideal $\mathfrak{n}_1$ is stable under the action of $\mathfrak{h}$ and it is readily verified that $\mathfrak{n}_1$ and $\mathfrak{n}/\mathfrak{n}_1$ with the natural $\mathfrak{h}$-representations belong to $\mathcal{N}$. Furthermore, since all weight spaces are 1-dimensional, we see that

$$\Lambda(\mathfrak{n}) = \Lambda(\mathfrak{n}_1) \sqcup \Lambda(\mathfrak{n}/\mathfrak{n}_1).$$

We will first prove that ψ is a diffeomorphism under the assumption that the $\mathfrak{h}$-weights in $\mathfrak{n}$ are numbered in such a way that

$$\Lambda(\mathfrak{n}_1) = \{\lambda_1, \dots, \lambda_l\} \quad \text{and} \quad \Lambda(\mathfrak{n}/\mathfrak{n}_1) = \{\lambda_{l+1}, \dots, \lambda_m\}.$$

Since N is simply-connected, the map $\exp : \mathfrak{n} \to N$ is a diffeomorphism so that $N_1 := \exp(\mathfrak{n}_1)$ is the connected subgroup of N with Lie algebra $\mathfrak{n}_1$. In particular, N_1 is simply connected as well. Since $\mathfrak{n}_1$ is an ideal, N/N_1 has a unique structure of Lie group for which the natural map $N \to N/N_1$ is a Lie group homomorphism. We now observe that $N \to N/N_1$ is a principal fiber bundle with fiber N_1. By standard homotopy theory we have a natural exact sequence

$$\pi_1(N) \to \pi_1(N/N_1) \to \pi_0(N_1).$$

Since N is simply-connected, and N_1 connected, we conclude that N/N_1 is the simply connected group with Lie algebra $\mathfrak{n}/\mathfrak{n}_1$.

By the induction hypothesis, the maps

$$\begin{aligned} \psi_{\mathfrak{n}_1} &: \mathfrak{n}_{\lambda_1} \times \dots \times \mathfrak{n}_{\lambda_l} \to N_1 \\ \psi_{\mathfrak{n}/\mathfrak{n}_1} &: (\mathfrak{n}/\mathfrak{n}_1)_{\lambda_{l+1}} \times \dots \times (\mathfrak{n}/\mathfrak{n}_1)_{\lambda_m} \to N/N_1 \end{aligned}$$

are diffeomorphisms. For every $j \in \{l+1, \dots, m\}$ the canonical projection $\mathfrak{n} \to \mathfrak{n}/\mathfrak{n}_1$ induces the isomorphisms of weight spaces $\mathfrak{n}_{\lambda_j} \to (\mathfrak{n}/\mathfrak{n}_1)_{\lambda_j}$, for $j > l$. Let $\bar{\psi} : \mathfrak{n}_{\lambda_{l+1}} \times \dots \times \mathfrak{n}_{\lambda_m} \to N/N_1$ be defined by $\bar{\psi}(X_{l+1}, \dots, X_m) = \exp X_{l+1} \cdot \dots \cdot \exp X_m N_1$. Then the following diagram commutes:

$$\begin{tikzcd} \mathfrak{n}_{\lambda_{l+1}} \times \dots \times \mathfrak{n}_{\lambda_m} \arrow[r, "\bar{\psi}"] \arrow[d, "\simeq"'] & N/N_1 \arrow[d, equal] \\ (\mathfrak{n}/\mathfrak{n}_1)_{\lambda_{l+1}} \times \dots \times (\mathfrak{n}/\mathfrak{n}_1)_{\lambda_m} \arrow[r, "\psi_{\mathfrak{n}/\mathfrak{n}_1}"] & N/N_1 \end{tikzcd}$$

From this we infer that $\bar{\psi}$ is a diffeomorphism. We now obtain that the map $\tilde{\psi} : \mathfrak{n}_{\lambda_{l+1}} \times \dots \times \mathfrak{n}_{\lambda_m} \times N_1 \to N$,

$$(X_{l+1}, \dots, X_m, n_1) \mapsto (\exp X_{l+1} \cdot \dots \cdot \exp X_m) n_1,$$

is a diffeomorphism onto N. Since

$$\psi(X_1, \dots, X_l, X_{l+1}, \dots, X_m) = \tilde{\psi}(X_{l+1}, \dots, X_m, \psi_{\mathfrak{n}_1}(X_1, \dots, X_l))$$

it follows that ψ is a diffeomorphism as well. Clearly, the above proof works for every enumeration of the weights in $\Lambda(\mathfrak{n}/\mathfrak{n}_1)$. Since the weight spaces $(\mathfrak{n}_1)_\lambda$ for $\lambda \in \Lambda(\mathfrak{n}_1)$ are all central in $\mathfrak{n}$, we conclude that the result holds for any enumeration of the weights in $\Lambda(\mathfrak{n})$. $\square$

Corollary A.0.6. *Let G be a connected complex semi-simple Lie group and $\mathfrak{n}_B$ the nilpotent radical of a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$. Let $\mathfrak{h}$ be a Cartan subalgebra contained in $\mathfrak{b}$. Let $\mathfrak{n}_1, \ldots, \mathfrak{n}_k$ be linearly independent subalgebras of $\mathfrak{n}_B$, each of which is a direct sum of $\mathfrak{h}$-root spaces, and assume that their direct sum $\mathfrak{n} := \mathfrak{n}_1 \oplus \ldots \oplus \mathfrak{n}_k$ is again a subalgebra. Put $N := \exp \mathfrak{n}$, $N_1 := \exp \mathfrak{n}_1, \ldots, N_k := \exp \mathfrak{n}_k$.*

Then the multiplication map

$$\mu : N_1 \times \ldots \times N_k \to N$$

is a diffeomorphism.

Proof. This is an immediate consequence of Lemma A.0.5. $\square$

Proof of Lemma 4.2.10. We assume that G is a real reductive Lie group of the Harish-Chandra class. Define

$$\mathfrak{g}_1 := [\mathfrak{g}, \mathfrak{g}],$$

the semi-simple part of the Lie algebra of G. Let G_1 be the analytic subgroup of G with Lie algebra $\mathfrak{g}_1$. Since the nilpotent radical N_P of P is completely contained in G_1, we may assume from the start that $G = G_1$, i.e. G is connected semi-simple with finite center.

Since Ad is a Lie group diffeomorphism from G onto $\mathrm{Aut}(\mathfrak{g})^\circ$, a real form of $\mathrm{Aut}(\mathfrak{g}_\mathbb{C})^\circ$, and $\mathrm{Ad}(N_P)$ is diffeomorphic to N_P, we may assume that G is a real form of a complex semi-simple Lie group $G_\mathbb{C}$, which is connected with trivial center. Let τ be the conjugation on $G_\mathbb{C}$, such that

$$G = (G_\mathbb{C}^\tau)^\circ.$$

Let $\mathfrak{g}_\mathbb{C}$ denote the Lie algebra of $G_\mathbb{C}$, then $\mathfrak{g}_\mathbb{C} = \mathfrak{g} \oplus i\mathfrak{g}$. Note that the complexification $\mathfrak{n}_{P\mathbb{C}}$ of $\mathfrak{n}_P$ equals $\mathfrak{n}_P \oplus i\mathfrak{n}_P$ and that

$$N_P = (N_{P\mathbb{C}})^\tau.$$

Take a Cartan subalgebra of $\mathfrak{g}_\mathbb{C}$, containing $\mathfrak{a}_\mathbb{C} = \mathfrak{a} \oplus i\mathfrak{a}$. It is of the form

$$\mathfrak{h}_\mathbb{C} = \mathfrak{t}_\mathbb{C} \oplus \mathfrak{a}_\mathbb{C},$$

where $\mathfrak{t}$ is a maximal abelian subspace of $\mathfrak{m} := Z_\mathfrak{k}(\mathfrak{a})$. Since $\mathfrak{t}$ centralizes $\mathfrak{a}$, all $\mathfrak{a}$-root spaces are invariant under $\mathrm{ad}(\mathfrak{t})$. This implies that the subalgebras $\mathfrak{n}_{j\mathbb{C}} := \mathfrak{n}_j \oplus i\mathfrak{n}_j$ ($j \in \{1, \ldots, k\}$) of $\mathfrak{n}_{P\mathbb{C}}$ are direct sums of $\mathfrak{h}_\mathbb{C}$-root spaces. Furthermore, their direct sum equals $\mathfrak{n}_{\mathbb{C}} = \mathfrak{n} \oplus i\mathfrak{n}$, hence is a subalgebra. Finally, there exists a Borel subalgebra containing $\mathfrak{h}_{\mathbb{C}} + \mathfrak{n}_{\mathbb{C}}$. By Corollary A.0.6, the multiplication map

$$\mu_{\mathbb{C}} : N_{1\mathbb{C}} \times \ldots \times N_{k\mathbb{C}} \to N_{\mathbb{C}}$$

is a diffeomorphism. It readily follows that $\mu_{\mathbb{C}}$ restricts to a bijection from $(N_{1\mathbb{C}})^{\tau} \times \cdots \times (N_{k\mathbb{C}})^{\tau}$ onto $(N_{\mathbb{C}})^{\tau}$. Since

$$(N_{\mathbb{C}})^{\tau} = N \quad \text{and} \quad (N_{j\mathbb{C}})^{\tau} = N_j \quad \text{for all } 1 \leq j \leq k,$$

it follows that μ is a bijective embedding from $N_1 \times \cdots \times N_k$ onto N, hence a diffeomorphism.

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Summary

This thesis presents two new results in Lie theory, and accordingly is composed of two parts. The first part deals with n-Lie algebras, and is more algebraic in nature, whereas the second part generalizes a well known result of Kostant to the setting of symmetric spaces, and is of a geometric nature. These results represent the outcome of two independent research projects.

Lie theory is the branch of mathematics concerned with symmetries and transformations of a large variety of mathematical objects, and is named after the Norwegian mathematician Sophus Lie (1842-1899). Shapes, bodies, mechanical systems, and even equations can have 'few' or 'many' symmetries. An equilateral triangle, for instance, has only six symmetries (three rotations and three reflections) whereas a circle has infinitely many: a reflection with respect to any diameter of the circle gives a symmetry and any rotation around the center of the circle gives a symmetry as well. The mathematical notion that encodes the symmetries of an object is that of a group, and if the object has a continuous family of symmetries (like the circle), then the group of symmetries is called a Lie group. Lie algebras have been introduced as the infinitesimal counterparts of Lie groups; they consist of infinitesimally small symmetries of an object, and form the tangent directions in the Lie group at the identity (the symmetry which fixes every point of the object). Lie algebras and Lie groups have been studied extensively and are central to contemporary mathematics. One of the greatest achievements, which came as early as the end of the 19th century, is the classification of the simple Lie algebras. Later, the development of representation theory of Lie groups and Lie algebras had a strong impact on quantum mechanics.

A generalization of the notion of a Lie algebra is that of a n-Lie algebra (here $n = 2, 3, 4, \ldots$ is a natural number). The first part of this thesis (Chapter 1) studies the representation theory of simple n-Lie algebras.

The theory of n-Lie algebras was first introduced in 1985 by Filippov. Algebraically, a Lie algebra is a binary operation (called the Lie bracket) on a vector space which satisfies certain axioms: bilinearity, skew-symmetry and the famous Jacobi identity. Generalizing this notion, an n-Lie algebra is an operation which is not binary, but n-ary, and satisfies similar conditions as a Lie bracket. Many of the classical concepts in the theory of Lie algebras have very natural counter-parts in the theory of n-Lie algebras; for instance Ling classified in his PhD thesis the simple n-Lie algebras and showed that for $n \geq 3$ there is up to isomorphism a unique n-Lie

algebra, which has dimension $n + 1$, and is the higher dimensional vector product.

In the first part of this thesis we classify the irreducible highest weight representations of the simple n-Lie algebra. A partial answer to this problem was given by Dzhumadil'daev, who classified such representations which are finite dimensional. The theorem in this thesis treats both finite and infinite dimensional irreducible highest weight representations.

In the second part of this thesis (Chapters 2,3 and 4) we focus on Lie groups. In particular, we present here a generalization of the famous non-linear convexity theorem of Kostant.

As an example, Kostant's theorem can be used to prove that the diagonal part of all Hermitian matrixes with fixed eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$ is the convex polytope in $\mathbb{R}^n$ whose vertices are the permutations of the eigenvalues. This theorem was proven first by Schur and Horn, but Kostant's theorem adds a geometric perspective to the problem, and can be applied to more general situations. Taking the diagonal part of a matrix is a special case of the infinitesimal Iwasawa projection, which can be defined for all semisimple Lie algebras, and can be lifted on the Lie group to the so-called Iwasawa projection. Kostant's (non-linear) convexity theorem says that the image under the Iwasawa projection of a left-translate of the maximal compact subgroup is a convex polytope.

The second part of the thesis presents an extension of Kostant's convexity theorem to semisimple symmetric spaces, generalizing also a result of van den Ban. As the name suggests, a symmetric space is a space which has 'many' symmetries, in particular, there are central reflections in each of its points. Such spaces come with an Iwasawa projection, as they can be described also via Lie groups. The theorem in the thesis shows that, also in this setting, the Iwasawa projection of certain translates of groups are convex sets.

Samenvatting

In dit proefschrift presenteren we twee resultaten in de Lie-theorie. Het is bijgevolg opgebouwd uit twee delen. In het eerste deel behandelen we n-Lie-algebra's, een onderwerp dat een algebraïsch karakter heeft. In het tweede deel veralgemeniseren we een beroemd resultaat van Kostant naar symmetrische ruimtes. Dit deel heeft een meetkundig karakter. De resultaten zijn voortgekomen uit twee onafhankelijke onderzoeksprojecten.

Lie-theorie is een tak van de wiskunde die de symmetriën en verschuivingen van veel meetkundige objecten beschrijft. Het is vernoemd naar de Noorse wiskundige Sophus Lie (1842 – 1899). Vormen, hemellichamen, mechanische systemen en zelfs vergelijkingen hebben symmetriën. Een gelijkzijdige driehoek heeft bijvoorbeeld zes symmetriën (drie rotaties en drie spiegelingen), terwijl een cirkel er oneindig veel heeft: Elke rotatie rond het middelpunt geeft namelijk een symmetrie.

Het 'groep' is het wiskunige begrip dat de symmetriën van een object beschrijft. Als het object een cotinuë familie van symmetriën heeft, dan geeft dit een Lie-groep. Lie-algebra's zijn de 'infinitesimale' tegenhangers van Lie-groepen. Zij bestaan uit infinitesimaal kleine symmetriën van een object en geven alle richtingen aan waarin het object symmetriën heeft.

Lie-groepen en Lie-algebra's zijn door de jaren heen uitvoerig bestudeerd en spelen een centrale rol in de moderne wiskunde. Eén van de belangrijkste resultaten, welke terug gaat naar het einde van de negentiende eeuw, is de classificatie van enkelvoudige Lie-algebra's. Later heeft de theorie van voorstellingen van Lie-groepen en Lie-algebra's een grote invloed gehad op de ontwikkeling van de kwantummechanica.

n-Lie-algebra's, waar n een positief geheel getal is ($n \geq 3$), veralgemeniseren Lie-algebra's. In het eerste deel van dit proefshrift worden de voorstellingen van n-Lie-algebra's bestudeerd. Het concept van een n-Lie-algebra werd in 1985 door Filippov geïntroduceerd. Algebraïsch gezien bestaat een Lie-algebra uit een binaire operatie, een operatie waar twee waardes ingevuld kunnen worden, genaamd het Lie-haakje dat aan bepaalde axioma's voldoet. Een n-Lie-algebra heeft een operatie waar n waardes ingevuld kunnen worden en die aan vergelijkbare axioma's voldoet. Maar het is meer dan een analogie; Veel klassieke concepten laten zich naar n-Lie-algebra's vertalen. In het bijzonder geeft Ling in zijn proefschrift een classificatie van alle enkelvoudige n-Lie-algebra's. Voor $n \geq 3$ toont hij aan dat er slechts één enkelvoudige n-Lie-algebra bestaat, dat deze dimensie $n + 1$ heeft en dat de operatie gegeven wordt door een hoger-dimensionaal vectorproduct.

In het eerste deel van het proefschrift (hoofdstuk 1) classificeren we de irreducibele voorstellingen van het hoogste gewicht voor de enkelvoudige n-Lie-algebra. Een gedeeltelijk antwoord op dit vraagstuk werd reeds gegeven door Dzhumadil'daev. Hij classificeerde namelijk alle zulke voorstellingen die eindigdimensionaal zijn. De stelling in dit proefschrift behandelt eindig- en oneindigdimensionale voorstellingen op gelijke voet.

In het tweede deel van dit proefschrift (hoofdstukken 2, 3 en 4) concentreren we ons op Lie-groepen. In het bijzonder presenteren we een veralgemenisering van de beroemde niet-lineaire convexiteitsstelling van Kostant. Kostant's stelling kan bijvoorbeeld gebruikt worden om te bewijzen dat de diagonalen van alle Hermietse matrices met gegeven eigenwaarden $\{\lambda_1, \dots, \lambda_n\}$ een convexe veelhoek vormen wiens hoekpunten gegeven worden door alle mogelijke permutaties van de eigenwaarden. Dit was al reeds bewezen door Schur en Horn, maar Kostant's stelling voegt er een meetkundige interpretatie aan toe en is toepasbaar in veel algemenere situaties. Het uitlezen van de diagonaal van een matrix is een speciaal geval van de infinitesimale Iwasawa-projectie, die gedefinieerd is voor alle halfenkelvoudige Lie-algebra's. De Iwasawa-projectie kent ook een definitie voor de bijbehorende Lie-groepen. Kostant's niet-lineaire convexiteitsstelling vertelt ons dat het beeld onder de Iwasawa-projectie van het linksverschovene van de maximale compacte deelgroep een convexe veelhoek is.

We hebben Kostant's convexiteitsstelling naar halfenkelvoudige symmetrische ruimtes veralgemeniseerd. Dit is tevens een veralgemenisering van een stelling van Van den Ban. Zoals de naam mogelijk suggereert is een symmetrische ruimte een ruimte met veel symmetriën. In het bijzonder zijn er zogenaamde centrale spiegelingen in elk punt. De Iwasawa-projectie kent óók een definitie voor deze ruimtes en de stelling in dit proefschrift laat zien dat ook hier bepaalde linksverschoven groepen een convexe verzameling als beeld hebben.

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Curriculum Vitae

Dana Bălibanu was born in Sibiu, Romania on the 31st of March 1984. During 1999 and 2003 she studied at the German-language high school Samuel von Brukenthal Gymnasium in Sibiu.

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