Partial differential equations from integrable vertex models

W. Galleas

Institute for Theoretical Physics and Spinoza Institute,
Utrecht University, Leuvenlaan 4, 3584 CE Utrecht,
The Netherlands

`w.galleas@uu.nl`

Abstract

In this work we propose a mechanism for converting the spectral problem for vertex models transfer matrices into the solution of certain linear partial differential equations. This mechanism is illustrated for the $U_q[\widehat{\mathfrak{sl}}(2)]$ invariant six-vertex model and the resulting partial differential equation is studied for particular values of the lattice length.

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1 Introduction

Vertex models of Statistical Mechanics are prominent examples where the computation of the model partition function can be described as an eigenvalue problem. This possibility dates back to Kramers and Wannier transfer matrix technique [1,2] originally devised for the Ising model. Within that approach the partition function of the model is expressed in terms of the eigenvalues of a matrix usually refereed to as *transfer matrix*. This technique has been successfully applied to a large variety of models although one has no guarantee a priori that the diagonalization of the transfer matrix can be achieved.

An important class of non-trivial models whose transfer matrix has been exactly diagonalized is formed by those possessing a parameter λ such that its transfer matrix $T(\lambda)$ satisfies the commutativity condition $[T(\lambda_1), T(\lambda_2)] = 0$ for general values of λ_1 and λ_2 [3]. This latter property paves the way for a variety of non-perturbative methods such as the coordinate Bethe ansatz [4], the algebraic Bethe ansatz [5], T - Q relations [6], analytical Bethe ansatz [7] and separation of variables [8,9] among others.

Within the approach of the coordinate Bethe ansatz for instance, the transfer matrices diagonalization process involves the explicit computation of the action of the transfer matrix on a finite dimensional vector in terms of its components. However, suppose one would like to study the spectral problem for an operator constituted by generators of the $\mathfrak{sl}(2)$ algebra. In that case we could also consider a differential representation of the $\mathfrak{sl}(2)$ algebra [10] and study the same eigenvalue problem through the corresponding differential equation. It is worth remarking that we use this same methodology in Quantum

Mechanics when we convert the spectral problem for a Hamiltonian in the Heisenberg formulation into the solution of a stationary Schrödinger equation.

In this work we devise an analogous approach for the transfer matrix of the $U_q[\widehat{\mathfrak{sl}}(2)]$ six-vertex model. More precisely, we obtain a partial differential equation describing the spectral problem of the aforementioned operator. The main ingredient of our derivation is the Yang-Baxter algebra employed within the lines of the algebraic-functional approach introduced in [11] and subsequently refined in the series of papers [12–15]. It is also worth remarking that a connection between the spectral problem of transfer matrices and differential equations had appeared previously in the literature under the name ODE/IM correspondence. See for instance the review [16] and references therein. However, it is not clear if there is any relation between the ODE/IM correspondence and the approach considered here. The main reason for that lies in the fact that the ODE/IM correspondence describes a relation between ordinary differential equations and integrable models while here we shall obtain partial differential equations. Moreover, our results are valid for finite lattices while the ODE/IM correspondence emerges in the continuum limit.

This paper is organized as follows. In Section 2 we describe the $U_q[\widehat{\mathfrak{sl}}(2)]$ transfer matrix and its spectral problem. Section 3 is devoted to the analysis of the aforementioned spectral problem in terms of a functional equation derived as a direct consequence of the Yang-Baxter algebra. This functional equation is converted into a partial differential equation in Section 4 and its analysis is performed for particular values of the lattice length. Concluding remarks are presented in Section 5 and extra results for the case of domain wall boundaries are given in Appendix A.

2 The transfer matrix spectral problem

In this section we shall briefly recall some standard definitions and introduce a convenient notation to describe the eigenvalue problem for the transfer matrix associated with the $U_q[\widehat{\mathfrak{sl}}(2)]$ solution of the Yang-Baxter equation. Although this case corresponds to the well known trigonometric six-vertex model, here we shall consider it from the perspective described in [15].

Monodromy and transfer matrices. Let $\mathcal{T} \in \mathrm{End}(\mathbb{V}_\mathcal{A} \otimes \mathbb{V}_\mathcal{Q})$ be an operator which we shall refer to as monodromy matrix. Here we shall consider the $U_q[\widehat{\mathfrak{sl}}(2)]$ vertex model and in that case we have $\mathbb{V}_\mathcal{A} \cong \mathbb{C}^2$ and $\mathbb{V}_\mathcal{Q} \cong (\mathbb{C}^2)^{\otimes L}$ for $L \in \mathbb{N}$. Hence the monodromy matrix $\mathcal{T}$ can be recast as

$$\mathcal{T} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{2.1}$$

with entries $A, B, C, D \in \mathbb{V}_\mathcal{Q}$. We then define the transfer matrix T as the following operator,

$$T = A + D \,. \tag{2.2}$$

Yang-Baxter algebra. Integrable vertex models in the sense of Baxter are characterized by a monodromy matrix $\mathcal{T}$ satisfying the following algebraic relation,

$$\mathcal{R}(x-y)\left[\mathcal{T}(x)\otimes\mathcal{T}(y)\right]=\left[\mathcal{T}(y)\otimes\mathcal{T}(x)\right]\mathcal{R}(x-y)\ . \tag{2.3}$$

In (2.3) we have spectral parameters $x,y\in\mathbb{C}$ and $\mathcal{R}\in\mathrm{End}(\mathbb{C}^2\otimes\mathbb{C}^2)$ in the case of the $U_q[\widehat{\mathfrak{sl}}(2)]$ vertex model. The algebraic relation (2.3) is associative for $\mathcal{R}$-matrices satisfying the Yang-Baxter equation, namely

$$\left[\mathcal{R}(x)\otimes\mathbb{1}\right]\left[\mathbb{1}\otimes\mathcal{R}(x+y)\right]\left[\mathcal{R}(y)\otimes\mathbb{1}\right]=\left[\mathbb{1}\otimes\mathcal{R}(y)\right]\left[\mathcal{R}(x+y)\otimes\mathbb{1}\right]\left[\mathbb{1}\otimes\mathcal{R}(x)\right]\ , \tag{2.4}$$

with symbol $\mathbb{1}$ denoting the 2×2 identity matrix. The matrix $\mathcal{R}$ plays the role of structure constant for the Yang-Baxter algebra (2.3) and the solution of (2.4) invariant under the $U_q[\widehat{\mathfrak{sl}}(2)]$ algebra explicitly reads

$$\mathcal{R}(x)=\begin{pmatrix} a(x) & 0 & 0 & 0 \\ 0 & c(x) & b(x) & 0 \\ 0 & b(x) & c(x) & 0 \\ 0 & 0 & 0 & a(x) \end{pmatrix}\ . \tag{2.5}$$

The non-null entries of (2.5) are given by functions $a(x)=\sinh\left(x+\gamma\right)$, $b(x)=\sinh\left(x\right)$ and $c(x)=\sinh\left(\gamma\right)$.

Remark 1. *Since the $\mathcal{R}$-matrix (2.5) is invertible, the relation (2.3) implies that the transfer matrix (2.2) forms a commutative family, i.e. $[T(x),T(y)]=0$.*

Now let $\mathcal{M}$ be the set $\mathcal{M}(x)=\{A,B,C,D\}(x)$ parameterized by a continuous complex variable x. The elements of $\mathcal{M}$ obey the Yang-Baxter algebra and among the commutation rules encoded in (2.3) we shall make use of the following ones:

$$\begin{aligned} A(x_1)B(x_2) &= \frac{a(x_2-x_1)}{b(x_2-x_1)}B(x_2)A(x_1)-\frac{c(x_2-x_1)}{b(x_2-x_1)}B(x_1)A(x_2) \\ D(x_1)B(x_2) &= \frac{a(x_1-x_2)}{b(x_1-x_2)}B(x_2)D(x_1)-\frac{c(x_1-x_2)}{b(x_1-x_2)}B(x_1)D(x_2) \\ B(x_1)B(x_2) &= B(x_2)B(x_1)\ . \end{aligned} \tag{2.6}$$

Remark 2. *The elements of $\mathcal{M}$ are subjected to the Yang-Baxter algebra (2.3) which is in general non-abelian. In this way the 2-tuple $(\xi_1,\xi_2):\ \xi_i\in\mathcal{M}(\lambda_i)$ originated from the Cartesian product $\mathcal{M}(\lambda_1)\times\mathcal{M}(\lambda_2)$ will be simply understood as the non-commutative product $\xi_1\xi_2$. This convention is naturally extended for the n-tuples $(\xi_1,\dots,\xi_n)$ generated by the products $\mathcal{M}(\lambda_1)\times\cdots\times\mathcal{M}(\lambda_n)$.*

Monodromy matrix representation. The ordered product

$$\mathcal{T}(\lambda)=\overrightarrow{\prod_{1\leq j\leq L}}\mathrm{P}_{\mathcal{A}j}\mathcal{R}_{\mathcal{A}j}(\lambda-\mu_j) \tag{2.7}$$

with $\mathcal{R}$-matrix given by (2.5) is a representation of (2.3) due to the Yang-Baxter relation (2.4). In (2.7) P denotes the standard permutation matrix $\mathrm{P}_{lm}:\mathbb{V}_l\otimes\mathbb{V}_m\mapsto\mathbb{V}_m\otimes\mathbb{V}_l$ for $\mathbb{V}_{l,m}\cong\mathbb{C}^2$, while $\lambda,\mu_j\in\mathbb{C}$ are respectively the spectral and inhomogeneity parameters. In its turn the subscripts in $\mathcal{R}_{\mathcal{A}j}$ indicate that we have a $\mathcal{R}$-matrix acting on the tensor product space $\mathbb{V}_{\mathcal{A}}\otimes\mathbb{V}_j$. More precisely, we have $\mathcal{R}_{\mathcal{A}j}\in\mathrm{End}(\mathbb{V}_{\mathcal{A}}\otimes\mathbb{V}_j)$.

Highest weight vectors. The vector $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}^{\otimes L}$ is a $\mathfrak{sl}(2)$ highest weight vector and the action of $\mathcal{M}$ built from the representation (2.7) can be straightforwardly computed due to the structure of (2.5). They are given by the following expressions:

$$A(\lambda)|0\rangle = \prod_{j=1}^{L} a(\lambda - \mu_j)|0\rangle \qquad\qquad B(\lambda)|0\rangle \neq 0$$

$$C(\lambda)|0\rangle = 0 \qquad\qquad D(\lambda)|0\rangle = \prod_{j=1}^{L} b(\lambda - \mu_j)|0\rangle \quad . \qquad (2.8)$$

Yang-Baxter relations of higher degrees. The spectrum of the transfer matrix T can be encoded in a set of functional relations along the lines described in [11, 15, 17]. Those functional relations are derived as a direct consequence of the Yang-Baxter algebra. In order to simplify our notation we introduce the symbol $[\lambda_1, \ldots, \lambda_n]$ defined as the following product of operators,

$$[\lambda_1, \ldots, \lambda_n] = \overrightarrow{\prod_{1 \leq j \leq n}} B(\lambda_j) . \qquad (2.9)$$

Remark 3. *Due to the last relation of (2.6) we have that $[\lambda_1, \ldots, \lambda_n]$ is symmetric under the permutation of variables, i.e. $[\ldots, \lambda_i, \ldots, \lambda_j, \ldots] = [\ldots, \lambda_j, \ldots, \lambda_i, \ldots]$. This property motivates the set theoretic notation $X^{a,b} = \{\lambda_j : a \leq j \leq b\}$ and we can write $[\lambda_1, \ldots, \lambda_n] = [X^{1,n}]$. We shall also employ the notation $X_\lambda^{a,b} = X^{a,b} \backslash \{\lambda\}$.*

Now the products $A(\lambda_0)[X^{1,n}]$ and $D(\lambda_0)[X^{1,n}]$ can be investigated under the light of (2.6) taking into account the previous definitions. By doing so we are left with the following Yang-Baxter relations of degree $n+1$,

$$A(\lambda_0)[X^{1,n}] = \prod_{\lambda \in X^{1,n}} \frac{a(\lambda - \lambda_0)}{b(\lambda - \lambda_0)}[X^{1,n}]A(\lambda_0)$$

$$- \sum_{\lambda \in X^{1,n}} \frac{c(\lambda - \lambda_0)}{b(\lambda - \lambda_0)} \prod_{\tilde{\lambda} \in X_\lambda^{1,n}} \frac{a(\tilde{\lambda} - \lambda)}{b(\tilde{\lambda} - \lambda)}[X_\lambda^{0,n}]A(\lambda)$$

$$D(\lambda_0)[X^{1,n}] = \prod_{\lambda \in X^{1,n}} \frac{a(\lambda_0 - \lambda)}{b(\lambda_0 - \lambda)}[X^{1,n}]D(\lambda_0)$$

$$- \sum_{\lambda \in X^{1,n}} \frac{c(\lambda_0 - \lambda)}{b(\lambda_0 - \lambda)} \prod_{\tilde{\lambda} \in X_\lambda^{1,n}} \frac{a(\lambda - \tilde{\lambda})}{b(\lambda - \tilde{\lambda})}[X_\lambda^{0,n}]D(\lambda) . \qquad (2.10)$$

Here we intend to explore the relations (2.10) in order to describe the spectrum of the transfer matrix (2.2). With that goal in mind we add up both expressions in (2.10) to obtain the identity

$$T(\lambda_0)[X^{1,n}] = [X^{1,n}](M_0^A A(\lambda_0) + M_0^D D(\lambda_0))$$

$$- \sum_{\lambda \in X^{1,n}} [X_\lambda^{0,n}](M_\lambda^A A(\lambda) + M_\lambda^D D(\lambda)) . \qquad (2.11)$$

The coefficients in (2.11) explicitly read

$$M_0^A = \prod_{\lambda \in X^{1,n}} \frac{a(\lambda - \lambda_0)}{b(\lambda - \lambda_0)} \qquad\qquad M_0^D = \prod_{\lambda \in X^{1,n}} \frac{a(\lambda_0 - \lambda)}{b(\lambda_0 - \lambda)}$$

$$M_\lambda^A = \frac{c(\lambda - \lambda_0)}{b(\lambda - \lambda_0)} \prod_{\tilde\lambda \in X_\lambda^{1,n}} \frac{a(\tilde\lambda - \lambda)}{b(\tilde\lambda - \lambda)} \qquad M_\lambda^D = \frac{c(\lambda_0 - \lambda)}{b(\lambda_0 - \lambda)} \prod_{\tilde\lambda \in X_\lambda^{1,n}} \frac{a(\lambda - \tilde\lambda)}{b(\lambda - \tilde\lambda)} \ . \qquad (2.12)$$

Within the framework of the algebraic Bethe ansatz [5] one would then consider the action of (2.11) on the highest weight vector $|0\rangle$ and try to fix the set of parameters $X^{1,n}$ in such a way that the transfer matrix eigenvalues can be directly read from the resulting expression. Here we follow a different strategy and we shall demonstrate how the Yang-Baxter relation (2.11) can be converted into a functional equation.

3 Algebraic-functional approach

In this section we aim to show that the eigenvalue problem for the transfer matrix (2.2) can be described in terms of certain functional equations originated from the Yang-Baxter algebra. This statement is made precise in Theorem 1 and its proof will require the following definitions.

Definition 1 (Algebras into functions)**.** *Considering the mechanism described in [15] we define the following continuous and additive map π,*

$$\pi_{n+1} : \quad \mathcal{M}(\lambda_0) \times \mathcal{M}(\lambda_1) \times \cdots \times \mathcal{M}(\lambda_n) \mapsto \mathbb{C}[\lambda_0^{\pm 1}, \lambda_1^{\pm 1}, \ldots, \lambda_n^{\pm 1}] \ . \qquad (3.1)$$

The map π_{n+1} essentially associates a complex function to the elements of $\mathcal{M}(\lambda_0) \times \mathcal{M}(\lambda_1) \times \cdots \times \mathcal{M}(\lambda_n)$.

The proof of the announced Theorem 1 will require the application of the map π_{n+1} over the higher order Yang-Baxter relation (2.11). We shall also need to build a suitable realization of (3.1) and here we consider the recipe given in [17].

Lemma 1. *Let $|\Lambda\rangle \in \mathrm{span}(\mathbb{V}_\mathcal{Q})$ be an eigenvector of the transfer matrix (2.2) and let $\langle\Lambda|$ denote its dual. Also consider the $\mathfrak{sl}(2)$ highest weight vector $|0\rangle$ as previously defined and $\mathcal{W}_{n+1} = \mathcal{M}(\lambda_0) \times \mathcal{M}(\lambda_1) \times \cdots \times \mathcal{M}(\lambda_n)$. Hence we have that*

$$\pi_{n+1}(\mathcal{A}) = \langle\Lambda| \, \mathcal{A} \, |0\rangle \qquad\qquad \forall \, \mathcal{A} \in \mathcal{W}_{n+1} \qquad (3.2)$$

is a realization of (3.1).

Proof. The proof is straightforward and follows from the fact that both $|0\rangle$ and $|\Lambda\rangle$ do not depend on the variables λ_j. The independence of $|0\rangle$ with λ_j is clear from its definition while this same property for $|\Lambda\rangle$ is due to the fact that the transfer matrix (2.2) forms a commutative family. $\qquad\square$

Theorem 1 (Functional equation). *Let Λ be an eigenvalue of the transfer matrix T associated with the eigenvector $|\Lambda\rangle \in \text{span}(\mathbb{V}_{\mathcal{Q}})$, i.e. $T(\lambda)|\Lambda\rangle = \Lambda(\lambda)|\Lambda\rangle$. Then $\exists\, \mathcal{F}_n :$ $\mathbb{C}^n \mapsto \mathbb{C}$ characterizing the eigenvalue Λ through the equation*

$$J_0 \mathcal{F}_n(X^{1,n}) - \sum_{\lambda \in X^{1,n}} K_\lambda \mathcal{F}_n(X^{0,n}_\lambda) = \Lambda(\lambda_0)\mathcal{F}_n(X^{1,n}) \tag{3.3}$$

with coefficients

$$J_0 = \prod_{j=1}^{L} a(\lambda_0 - \mu_j)\, M_0^A + \prod_{j=1}^{L} b(\lambda_0 - \mu_j)\, M_0^D$$

$$K_\lambda = \prod_{j=1}^{L} a(\lambda - \mu_j)\, M_\lambda^A + \prod_{j=1}^{L} b(\lambda - \mu_j)\, M_\lambda^D\ . \tag{3.4}$$

Proof. The realization (3.2) exhibits useful properties which will aid us in extracting information about the transfer matrix spectrum from the higher degree relation (2.11). For instance, the application of π_{n+1} to the LHS of (2.11) will produce the term $\pi_{n+1}(T(\lambda_0)[X^{1,n}])$ which simplifies to

$$\pi_{n+1}(T(\lambda_0)[X^{1,n}]) = \Lambda(\lambda_0)\pi_n([X^{1,n}])\ . \tag{3.5}$$

Similarly, we find that the application of the map π_{n+1} over the RHS of (2.11) only yields terms of the form $\pi_{n+1}([Z^{1,n}]A(x))$ and $\pi_{n+1}([Z^{1,n}]D(x))$ for generic variables x and $Z^{1,n} = \{z_j \in \mathbb{C}\ :\ 1 \leq j \leq n\}$. Due to (2.8) and (3.2) those terms exhibit the following reduction properties,

$$\pi_{n+1}([Z^{1,n}]A(x)) = \prod_{j=1}^{L} a(x - \mu_j)\pi_n([Z^{1,n}])$$

$$\pi_{n+1}([Z^{1,n}]D(x)) = \prod_{j=1}^{L} b(x - \mu_j)\pi_n([Z^{1,n}])\ . \tag{3.6}$$

In their turn the relations (3.5) and (3.6) tell us that the map π given by (3.2) obeys recurrence relations of type $\pi_{n+1} \mapsto \pi_n$ over the elements of (2.11). Next we introduce the notation $\mathcal{F}_n(X^{1,n}) = \pi_n([X^{1,n}])$ in such a way that the application of (3.2) on (2.11), taking into account the properties (3.5) and (3.6), yields the functional relation

$$J_0 \mathcal{F}_n(X^{1,n}) - \sum_{\lambda \in X^{1,n}} K_\lambda \mathcal{F}_n(X^{0,n}_\lambda) = \Lambda(\lambda_0)\mathcal{F}_n(X^{1,n})\ , \tag{3.7}$$

with coefficients

$$J_0 = \prod_{j=1}^{L} a(\lambda_0 - \mu_j)\, M_0^A + \prod_{j=1}^{L} b(\lambda_0 - \mu_j)\, M_0^D$$

$$K_\lambda = \prod_{j=1}^{L} a(\lambda - \mu_j)\, M_\lambda^A + \prod_{j=1}^{L} b(\lambda - \mu_j)\, M_\lambda^D\ . \tag{3.8}$$

This completes the proof of Theorem 1. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3.1 Operatorial description

The transfer matrix eigenvalue problem has been described in Theorem 1 as the solution of a functional equation. In this section we intend to show that the obtained functional equation, namely (3.3), can be recasted in an operatorial form which allows us to identify the action of the transfer matrix (2.2) on a particular function space. For that we introduce the operator $D_{z_i}^{z_\alpha}$ whose properties are described as follows.

Definition 2. *Let $n \in \mathbb{N}$ and $\alpha \notin \{k \in \mathbb{N} : 1 \leq k \leq n\}$. Also, let f be a complex function $f(z) \in \mathbb{C}[z]$ where $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$. Then we define the action of the operator $D_{z_i}^{z_\alpha}$ on $\mathbb{C}[z]$ as,*

$$D_{z_i}^{z_\alpha} : \qquad f(z_1, \ldots, z_i, \ldots, z_n) \ \mapsto \ f(z_1, \ldots, z_\alpha, \ldots, z_n) . \qquad (3.9)$$

The operator $D_{z_i}^{z_\alpha}$ has been previously introduced in [18] and it basically replaces a given variable z_i by a variable z_α.

In terms of the operator $D_{z_i}^{z_\alpha}$ we can rewrite Eq. (3.3) as

$$\mathfrak{L}(\lambda_0)\mathcal{F}_n(X^{1,n}) = \Lambda(\lambda_0)\mathcal{F}_n(X^{1,n}) \qquad (3.10)$$

with operator $\mathfrak{L}$ reading

$$\mathfrak{L}(\lambda_0) = J_0 - \sum_{\lambda \in X^{1,n}} K_\lambda D_\lambda^{\lambda_0} . \qquad (3.11)$$

Now we can immediately recognize Eq. (3.10) as an eigenvalue equation and some comments are in order at this stage. For instance, the introduction of the operator $D_{z_i}^{z_\alpha}$ is able to localize the whole dependence of the LHS of (3.3) with the spectral parameter λ_0 in the operator $\mathfrak{L}$. In fact, we can identify $\mathfrak{L}$ with the transfer matrix (2.2) acting on a particular function space spanned by functions $\mathcal{F}_n$. This function space will be described in the next section.

3.2 The function space $\Xi(\mathbb{C}^n)$

The functions $\mathcal{F}_n$ solving Eq. (3.3) consist of the projection of the dual transfer matrix eigenvector onto a particular set of vectors usually refereed to as Bethe vectors. Although we are mainly interested in the eigenvalues Λ we still need to restrict our solutions $\mathcal{F}_n$ to a class of functions preserving certain representation theoretic properties exhibited by the elements involved in the derivation of (3.3). This fact motivates the definition of the function space $\Xi(\mathbb{C}^n)$ whose properties will be discussed in what follows.

Definition 3. *Let the functions $\mathcal{F}_n : \ \mathbb{C}^n \mapsto \mathbb{C}$ be of the form*

$$\mathcal{F}_n(X^{1,n}) = \langle \Lambda | [\lambda_1, \ldots, \lambda_n] |0\rangle \ , \qquad (3.12)$$

where each operator B in the product $[\lambda_1, \ldots, \lambda_n]$ is built according to (2.1), (2.5) and (2.7). In its turn $\langle \Lambda |$ is a dual eigenvector of the transfer matrix (2.2) whilst $|0\rangle$ is the $\mathfrak{sl}(2)$ highest weight vector as previously defined.

We can see from Lemma 1 that the whole dependence of $\mathcal{F}_n$ with a given variable λ_j comes from the operator $B(\lambda_j)$. Thus the characterization of $\Xi(\mathbb{C}^n)$ can be performed with the help of the following Proposition.

Proposition 1 (Polynomial structure). *The operator $B(\lambda_i)$ is of the form*

$$B(\lambda_i) = x_i^{\frac{1-L}{2}} P_B(x_i) \qquad (3.13)$$

where $x_i = e^{2\lambda_i}$ and P_B is a polynomial of degree $L - 1$.

Proof. The proof follows from induction and it can be found with details in [11]. $\square$

Definition 4. *Let $\mathbb{K}[x_1, x_2, \ldots, x_n]$ be the polynomial ring in n variables $x_1, \ldots, x_n$ which we shall simply denote as $\mathbb{K}[x]$. Then we define $\mathbb{K}^m[x] \subset \mathbb{K}[x]$ as the subset of $\mathbb{K}[x]$ formed by polynomials of degree m in each variable x_i.*

Due to the Proposition 1 we can conclude that $\mathcal{F}_n$ are of the form

$$\mathcal{F}_n(X^{1,n}) = \prod_{i=1}^{n} x_i^{\frac{1-L}{2}} \, \bar{\mathcal{F}}_n(x_1, x_2, \ldots, x_n) \,, \qquad (3.14)$$

where $\bar{\mathcal{F}}_n$ is a multivariate polynomial of degree $L - 1$ in each one of its variables. Taking into account the Definition 4 we can write $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$ and conclude that $\bar{\mathcal{F}}_n = \bar{\mathcal{F}}_n(x) \in \mathbb{K}^{L-1}[x]$. The function space Ξ is then defined as follows.

Definition 5 (Space Ξ). *The function space $\Xi(\mathbb{C}^n)$ consists of the following set of functions,*

$$\Xi(\mathbb{C}^n) = \left\{ \mathcal{F}_n \ : \ \mathfrak{L}(\lambda)\mathcal{F}_n = \Lambda(\lambda)\mathcal{F}_n, \ \mathcal{F}_n = \mathbf{x}^{\frac{1-L}{2}} \, \bar{\mathcal{F}}_n(x), \ \bar{\mathcal{F}}_n \in \mathbb{K}^{L-1}[x] \right\} \,, \qquad (3.15)$$

where $\mathbf{x}^{\frac{1-L}{2}} = \prod_{i=1}^{n} x_i^{\frac{1-L}{2}}$.

Remark 4. *We can readily see from Definition 4 that $\Xi(\mathbb{C}^n) \subset \mathbf{x}^{\frac{1-L}{2}} \mathbb{K}^{L-1}[x]$.*

4 Partial differential equations

The operator $\mathfrak{L}$ defined in (3.11) corresponds to the transfer matrix in the function space $\Xi(\mathbb{C}^n)$. In its turn $\mathfrak{L}$ is given in terms of operators $D_{z_i}^{z_\alpha}$ and here we intend to demonstrate that those operators admit a differential realization when their action is restricted to the set $\mathbb{K}^m[x]$. This realization could not be immediately employed for (3.11) as we are interested in solutions $\mathcal{F}_n \in \Xi(\mathbb{C}^n)$. Nevertheless, in what follows we shall see how this differential structure can still be incorporated into Eq. (3.10). This approach has been previously employed in [15] where we have derived a set of partial differential equations satisfied by the partition function of the six-vertex model with domain wall boundaries.

Lemma 2 (Differential realization)**.** *Let $\mathbb{K}^m[z] \subset \mathbb{K}[z]$ with $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ be a subset of the polynomial ring according to the Definition 4. The operator $D_{z_i}^{z_\alpha}$ in $\mathbb{K}^m[z]$ is then given by*

$$D_{z_i}^{z_\alpha} = \sum_{k=0}^{m} \frac{(z_\alpha - z_i)^k}{k!} \frac{\partial^k}{\partial z_i^k} \ . \tag{4.1}$$

Proof. A detailed proof is given in [15]. $\qquad\qquad\square$

As we have previously remarked we can not immediately substitute the realization (4.1) into (3.11) as the functions $\mathcal{F}_n$ belong to the function space $\Xi(\mathbb{C}^n)$. However, as the non-polynomial part of $\Xi(\mathbb{C}^n)$ consists of an overall multiplicative factor, we can still rewrite (3.10) in terms of functions $\bar{\mathcal{F}}_n \in \mathbb{K}^{L-1}[x]$ defined through (3.14). For that we introduce the variable $z = e^{2\lambda}$ and define the functions

$$\bar{J}_0 = J_0 x_0^{\frac{L}{2}} \ , \qquad \bar{K}_z = K_\lambda x_0^{\frac{1}{2}} z^{\frac{L-1}{2}} \quad \text{and} \quad \bar{\Lambda}(x_0) = \Lambda(\lambda_0) x_0^{\frac{L}{2}} \ . \tag{4.2}$$

By doing so we are left with the equation

$$\bar{\mathfrak{L}}(x_0) \bar{\mathcal{F}}_n(\bar{X}^{1,n}) = \bar{\Lambda}(x_0) \bar{\mathcal{F}}_n(\bar{X}^{1,n}) \tag{4.3}$$

where $\bar{X}^{a,b} = \{x_k \ : \ a \le k \le b\}$. In its turn the operator $\bar{\mathfrak{L}}$ reads

$$\bar{\mathfrak{L}}(x_0) = \bar{J}_0 - \sum_{x \in \bar{X}^{1,n}} \bar{K}_x D_x^{x_0} \tag{4.4}$$

and now it acts on functions $\bar{\mathcal{F}}_n \in \mathbb{K}^{L-1}[x]$. Hence we can employ the realization (4.1) and this procedure reveals that $\bar{\mathfrak{L}}$ is of the form

$$\bar{\mathfrak{L}}(x_0) = \sum_{k=0}^{L} x_0^k \, \Omega_k \ . \tag{4.5}$$

Here $\{\Omega_k\}$ is a set of differential operators and the expression (4.5) implies that the LHS of (4.3) is a polynomial of degree L in the variable x_0.

Remark 5. *The operator $\bar{\mathfrak{L}}$ corresponds to the transfer matrix T in the function space $\mathbb{K}^{L-1}[x]$. Thus, since $[T(\lambda_1), T(\lambda_2)] = 0$ as matricial operators, we can conclude that $[\bar{\mathfrak{L}}(x_1), \bar{\mathfrak{L}}(x_2)] = 0$ which implies the condition $[\Omega_i, \Omega_j] = 0$.*

The RHS of (4.3) is also a polynomial of degree L in the variable x_0 and this feature prevents that any operator Ω_k vanishes identically by construction. This property can be demonstrated with the help of Proposition 2.

Proposition 2. *The operators $A(\lambda)$ and $D(\lambda)$ are of the form*

$$A(\lambda) = z^{-\frac{L}{2}} P_A(z) \qquad and \qquad D(\lambda) = z^{-\frac{L}{2}} P_D(z) \ , \tag{4.6}$$

where P_A and P_D are polynomials of degree L.

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Proof. The proof follows from induction and the details can be found in [11]. $\qquad\square$

Now from equations (2.2) and (4.6) we can conclude that $T(\lambda) = z^{-\frac{L}{2}} P_T(z)$ where P_T is a polynomial of degree L. Moreover, since our transfer matrix forms a commutative family we can conclude that its eigenvalues will be of the form $\Lambda(\lambda) = z^{-\frac{L}{2}} P_\Lambda(z)$ where P_Λ is also a polynomial of degree L. Hence, the RHS of (4.3) consists of a polynomial of degree L in the variable x_0 and we can write

$$\bar{\Lambda}(x_0) = \sum_{k=0}^{L} x_0^k \, \Delta_k \, . \tag{4.7}$$

In this way Eq. (4.3) must be satisfied independently by each power in x_0 and we are left with the following system of differential equations,

$$\Omega_k \bar{\mathcal{F}}_n(x_1, \ldots, x_n) = \Delta_k \bar{\mathcal{F}}_n(x_1, \ldots, x_n) \qquad 0 \le k \le L \, . \tag{4.8}$$

The system of Eqs. (4.8) comprises a total of $L+1$ eigenvalue problems, i.e. an eigenvalue equation for each operator Ω_k, being solved by the same eigenfunction $\bar{\mathcal{F}}_n$. Moreover, the direct inspection of (4.8) for small values of n and L shows that each equation is solely able to determine the eigenfunctions $\bar{\mathcal{F}}_n$ in addition to its eigenvalue Δ_k. Thus the system of differential equations (4.8) can be simultaneously integrated.

The explicit form of the operators Ω_k can be straightforwardly obtained from (4.1) and (4.4). Although their form for general values of n and L can be rather cumbersome we still find compact expressions for some of them. For instance, the operator Ω_L is trivial in consonance with the fact that the leading term coefficient of the transfer matrix corresponds to the Cartan element of the $U_q[\widehat{\mathfrak{sl}}(2)]$ algebra [7]. Fortunately, the situation is more interesting for the operator Ω_{L-1} and we find a compact structure containing only derivatives $\frac{\partial^{L-1}}{\partial x_i^{L-1}}$. In what follows we present the explicit differential equation associated with the operator Ω_{L-1} spectral problem.

The operator Ω_{L-1}. Equation (4.8) for $k = L - 1$ and arbitrary values of n and L corresponds to the following partial differential equation,

$$\left[\mathcal{V}^{(n)} + \sum_{i=1}^{n} \mathcal{Q}_i^{(n)} \frac{\partial^{L-1}}{\partial x_i^{L-1}} \right] \bar{\mathcal{F}}_n(\bar{X}^{1,n}) = \Delta_{L-1} \bar{\mathcal{F}}_n(\bar{X}^{1,n}) \, , \tag{4.9}$$

with functions $\mathcal{V}^{(n)} = \mathcal{V}^{(n)}(\bar{X}^{1,n})$ and $\mathcal{Q}_i^{(n)} = \mathcal{Q}^{(n)}(x_i; \bar{X}_i^{1,n})$ defined over the sets $\bar{X}^{a,b}$ and $\bar{X}_i^{a,b} = \bar{X}^{a,b} \backslash \{x_i\}$. By writing $\mathcal{V}^{(n)} = -2^{-L} \prod_{k=1}^{L} y_k^{-\frac{1}{2}} [\mathcal{V}_1^{(n)} + \mathcal{V}_2^{(n)}]$ we then have

$$\mathcal{V}_1^{(n)} = (q^n + q^{L-n-2}) \sum_{k=1}^{L} y_k \, , \tag{4.10}$$

and

$$\mathcal{V}_2^{(n)} = \begin{cases} q^{n-2}(q-1)^2(q+1) \sum_{k=0}^{L+1-2n} q^k \sum_{i=1}^{n} x_i & L \ge 2(n-1) \\ -q^{L-n}(q-1)^2(q+1) \sum_{k=0}^{2n-3-L} q^k \sum_{i=1}^{n} x_i & L < 2(n-1) \end{cases} \, . \tag{4.11}$$

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In their turn the functions $\mathcal{Q}_i^{(n)}$ are given by

$$\mathcal{Q}_i^{(n)} = \frac{(q-1)^2(q+1)}{2^L q^{L+n}(L-1)!} \frac{\prod_{k=1}^L y_k^{-\frac{1}{2}}}{\prod_{\substack{j=1 \\ j \neq i}}^n (x_j - x_i)} \left[\sum_{m=0}^L \mathcal{G}_m^{(n)}(x_i; \bar{X}_i^{1,n}) \sum_{1 \leq j_1 < \cdots < j_m \leq L} \prod_{\alpha=1}^m y_{j_\alpha} \right],$$

where

$$\mathcal{G}_{L-d}^{(n)}(x_i; \bar{X}_i^{1,n}) = x_i^d \sum_{l=0}^{n-1} x_i^l \, \psi_{l,d} \sum_{\substack{1 \leq j_1 < \cdots < j_{n-1-l} \leq n \\ j_\alpha \neq i}} \prod_{\alpha=1}^{n-1-l} x_{j_\alpha} \qquad (4.12)$$

and

$$\psi_{l,d} =$$
$$\begin{cases} (-1)^{L+d+l} q^{L+2l} \sum_{k=0}^{2d+2n-3-L-4l} q^k & d > L-(n+1)+2l \\ (-1)^{3l-n-1} q^{L+2l} \sum_{k=0}^{L-5} q^k & d = L-(n+1)+2l, \ L \geq 5 \\ (-1)^{3l-n} q^{2L+2l-4} \sum_{k=0}^{3-L} q^k & d = L-(n+1)+2l, \ L < 5 \\ (-1)^{L+d+l+1} q^{2d+2n-2-2l} \sum_{k=0}^{L-2d-2n+1+4l} q^k & d < L-(n+1)+2l \end{cases}.$$
$$(4.13)$$

Clearly the terms $\sum_{k=0}^l q^k$ in (4.11) and (4.14) can be simplified with the help of the geometric sum formula $\sum_{k=0}^l q^k = \frac{1-q^{l+1}}{1-q}$. However, we prefer to keep the summation symbol in order to make more explicit that the summation vanishes for $l < 0$. Eq. (4.9) is a partial differential equation of order $L-1$ and in what follows we shall demonstrate how it can be translated into a system of first order equations. We shall also discuss its solutions and properties for particular values of n and L.

4.1   Reduction of order

Linear differential equations of higher order can be conveniently written as a system of first order equations. In our case we have (4.9), which is a linear partial differential equation of order $L-1$, and here we intend to embed that equation into a system of first order equations. The resulting system of partial differential equations is explicitly given in Lemma 3.

Lemma 3. *Let* $\partial_i \equiv \frac{\partial}{\partial x_i}$ *and let* $\vec{\psi}$ *be a* $(L-2)n+1$ *dimensional vector denoted as*

$$\vec{\psi} = \begin{pmatrix} \psi^{(0)} \\ \psi^{(1)} \\ \vdots \\ \psi^{(L-2)} \end{pmatrix}. \qquad (4.14)$$

Also define $\psi^{(0)} = \psi_0 = \bar{\mathcal{F}}_n(\bar{X}^{1,n})$ *while the remaining entries* $\psi^{(k)}$ *are* n-*dimensional column vectors with components* $\psi_i^{(1)} = \partial_i \psi_0$ *and* $\psi_i^{(k)} = \partial_i \psi_i^{(k-1)}$ *for* $k > 1$. *The equation* *(4.9) then reads*

$$(\mathcal{Q}^{(n)} - \Delta_{L-1})\psi_0 + \sum_{i=1}^n \mathcal{V}_i^{(n)} \partial_i \psi_i^{(L-2)} = 0. \qquad (4.15)$$

Proof. Straightforward substitution of (4.14) into (4.9). □

Matricial form. The system of equations described in Lemma 3 can be conveniently written as a matrix equation. More precisely, the aforementioned system consists of the following equations

$$
\begin{aligned}
\partial_i \psi_0 - \psi_i^{(1)} &= 0 \\
\partial_i \psi_i^{(k-1)} - \psi_i^{(k)} &= 0 \qquad 1 < k \leq L-2 \, ,
\end{aligned}
\tag{4.16}
$$

in addition to (4.15). Here we intend to rewrite (4.15) and (4.16) as $\Upsilon \vec{\psi} = 0$ for a given matrix Υ. For that we introduce the n-dimensional vectors

$$
\vec{\omega}_0 = \left(\mathcal{Q}_1^{(n)} \partial_1, \ldots, \mathcal{Q}_n^{(n)} \partial_n \right) \quad \text{and} \quad \vec{\nabla}_0 = \begin{pmatrix} \partial_1 \\ \vdots \\ \partial_n \end{pmatrix} .
\tag{4.17}
$$

Also let $\mathbf{0}_{r \times s}$ denote the null matrix with dimensions $r \times s$ and define vectors

$$
\vec{\omega} = \left(\mathbf{0}_{1 \times n(L-3)}, \vec{\omega}_0 \right) \quad \text{and} \quad \vec{\nabla} = \begin{pmatrix} \vec{\nabla}_0 \\ \mathbf{0}_{n(L-3) \times 1} \end{pmatrix} .
\tag{4.18}
$$

Next we define $\hat{\mathfrak{D}}$ as a matrix of dimensions $(L-2) \times (L-2)$ with entries

$$
\hat{\mathfrak{D}}_{ij} = \begin{cases} -\mathbb{1}_{n \times n} & i = j \quad ; \ 1 \leq j \leq L-2 \\ \mathcal{D} & i = j+1 \, ; \ 1 \leq j < L-2 \\ \mathbf{0}_{n \times n} & \text{otherwise} \end{cases} ,
\tag{4.19}
$$

where $\mathbb{1}_{n \times n}$ denotes the $n \times n$ identify matrix and $\mathcal{D}$ is also a $n \times n$ diagonal matrix given by $\mathcal{D} = \text{diag}(\partial_i)$. In this way the system of Eqs. formed by (4.15) and (4.16) can be written as $\Upsilon \vec{\psi} = 0$ with matrix Υ given by

$$
\Upsilon = \begin{pmatrix} \mathcal{Q}^{(n)} - \Delta_{L-1} & \vec{\omega} \\ \vec{\nabla} & \hat{\mathfrak{D}} \end{pmatrix} .
\tag{4.20}
$$

4.2 Some particular solutions

In this section we study the solutions of Eq. (4.9) for the cases $n = 0, 1, 2$ and particular values of the lattice length L. Interestingly, for the case $L = 2$ and $n = 2$ some geometric features of our equation appear through the method of characteristics [19].

4.2.1 Case $n = 0$

Although the case $n = 0$ is trivial, we present it here for completeness reasons. By definition we have that $\bar{\mathcal{F}}_0$ is a constant and we can conclude that $\Delta_{L-1} = \mathcal{V}^{(0)}$. Thus from (4.10) and (4.11) we find

$$
\Delta_{L-1} = -\frac{(1 + q^{L-2}) \sum_{k=1}^{L} y_k}{2^L \prod_{k=1}^{L} y_k^{\frac{1}{2}}} .
\tag{4.21}
$$

4.2.2 Case $n = 1$

Equation (4.9) for $n = 1$ is actually an ordinary differential equation reading

$$\frac{d^{L-1}\bar{\mathcal{F}}_1}{dx_1^{L-1}} = \left(\frac{\Delta_{L-1} - \mathcal{V}^{(1)}}{\mathcal{Q}_1^{(1)}}\right)\bar{\mathcal{F}}_1 \ . \tag{4.22}$$

Moreover, when $L = 2$ we can see that (4.22) is a first order equation and the general solution can be obtained by direct integration. In that case we obtain

$$\bar{\mathcal{F}}_1(x_1) = \mathcal{C}_1(q^2x_1^2 - y_1y_2)^{\frac{1}{2}}\exp\left\{-\frac{q}{(q^2-1)^2}\left[(1+q^2)\frac{(y_1+y_2)}{(y_1y_2)^{\frac{1}{2}}} + 4q\Delta_{L-1}\right]\xi(x_1)\right\} , \tag{4.23}$$

where $\mathcal{C}_1$ is an integration constant and $\xi(x_1) = \operatorname{arctanh}(qx_1(y_1y_2)^{-\frac{1}{2}})$. We are interested in solutions $\bar{\mathcal{F}}_1 \in \mathbb{K}^1[x_1]$ while (4.23) consists of a square root multiplied by an exponential. At first glance this structure does not resemble the desired class of solutions but we then notice that

$$\exp[\xi(x_1)] = -\mathrm{i}\frac{[qx_1 + (y_1y_2)^{\frac{1}{2}}]}{(q^2x_1^2 - y_1y_2)^{\frac{1}{2}}} \ . \tag{4.24}$$

In this way we find that the condition

$$-\frac{q}{(q^2-1)^2}\left[(1+q^2)\frac{(y_1+y_2)}{(y_1y_2)^{\frac{1}{2}}} + 4q\Delta_{L-1}\right] = \pm 1 \tag{4.25}$$

leave us with the desired type of solution. Hence, the requirement $\bar{\mathcal{F}}_1 \in \mathbb{K}^1[x_1]$ yields a constraint for the eigenvalues Δ_{L-1} which is solved by

$$\Delta_{L-1} = -\frac{(1+q^2)}{4q}\frac{(y_1+y_2)}{(y_1y_2)^{\frac{1}{2}}} \mp \frac{(q^2-1)^2}{4q^2} \ . \tag{4.26}$$

4.2.3 Case $n = 2$

Here we shall address the case $n = 2$ and $L = 2$ where (4.9) reads

$$\mathcal{Q}_1^{(2)}\frac{\partial\bar{\mathcal{F}}_2}{\partial x_1} + \mathcal{Q}_2^{(2)}\frac{\partial\bar{\mathcal{F}}_2}{\partial x_2} = (\Delta_{L-1} - \mathcal{V}^{(2)})\bar{\mathcal{F}}_2 \ . \tag{4.27}$$

For this particular case it is worth remarking that $\mathcal{V}^{(2)}$ does not depend on the variables x_1 and x_2. Now let $\mathcal{S} = \{(x_1, x_2, \bar{\mathcal{F}}_2)\}$ be the surface generated by the solution of (4.27) and let $\mathcal{C}$ be a curve lying on $\mathcal{S}$. Also, let s be a variable parameterizing the curve $\mathcal{C}$ such that the vector

$$\left(\mathcal{Q}_1^{(2)}(x_1(s), x_2(s)), \mathcal{Q}_2^{(2)}(x_1(s), x_2(s)), (\Delta_{L-1} - \mathcal{V}^{(2)})\bar{\mathcal{F}}_2(x_1(s), x_2(s))\right) \tag{4.28}$$

is tangent to $\mathcal{C}$ at each point of the curve. Then the curve

$$\mathcal{C} = \{(x_1(s), x_2(s), \bar{\mathcal{F}}_2(x_1(s), x_2(s)))\} \tag{4.29}$$

satisfy the following system of ordinary differential equations,

$$
\begin{aligned}
\frac{dx_1}{ds} &= \mathcal{Q}_1^{(2)}(x_1(s), x_2(s)) \\
\frac{dx_2}{ds} &= \mathcal{Q}_2^{(2)}(x_1(s), x_2(s)) \\
\frac{d\bar{\mathcal{F}}_2}{ds} &= (\Delta_{L-1} - \mathcal{V}^{(2)})\bar{\mathcal{F}}_2(x_1(s), x_2(s)) \,.
\end{aligned}
\tag{4.30}
$$

The curve $\mathcal{C}$ is called characteristic curve for the vector field (4.28) and it is determined by the solution of the system (4.30). The characteristic equations (4.30) can also be written without fixing a particular parameterization variable as

$$\frac{dx_1}{\mathcal{Q}_1^{(2)}} = \frac{dx_2}{\mathcal{Q}_2^{(2)}} = \frac{d\bar{\mathcal{F}}_2}{(\Delta_{L-1} - \mathcal{V}^{(2)})\bar{\mathcal{F}}_2} \,. \tag{4.31}$$

Now we can form any two equations by combining the terms of (4.31) and for convenience we choose

$$
\begin{aligned}
\frac{dx_1}{dx_2} &= \frac{\mathcal{Q}_1^{(2)}}{\mathcal{Q}_2^{(2)}} = -\frac{x_1}{x_2} \\
\frac{d\bar{\mathcal{F}}_2}{dx_2} &= (\Delta_{L-1} - \mathcal{V}^{(2)})\frac{\bar{\mathcal{F}}_2}{\mathcal{V}_2^{(2)}} \,.
\end{aligned}
\tag{4.32}
$$

The integration of (4.32) yields the solution

$$\bar{\mathcal{F}}_2(x_1, x_2) = \kappa(x_1 x_2) \exp\left\{ \frac{[(1 + q^4)(y_1 + y_2) + 4q^2\sqrt{y_1 y_2}\Delta_{L-1}]}{(q^2 - 1)^2(y_1 + y_2)} \log \zeta(x_1, x_2) \right\}, \tag{4.33}$$

where $\kappa(x_1 x_2)$ is an arbitrary function of the product $x_1 x_2$ and

$$\zeta(x_1, x_2) = q^2(y_1 + y_2)(x_1 + x_2) - (1 + q^2)(y_1 y_2 + q^2 x_1 x_2) \,. \tag{4.34}$$

Hence, in order to having $\bar{\mathcal{F}}_2 \in \mathbb{K}^1[x_1, x_2]$ we choose κ as a constant function and impose the condition

$$\frac{[(1 + q^4)(y_1 + y_2) + 4q^2\sqrt{y_1 y_2}\Delta_{L-1}]}{(q^2 - 1)^2(y_1 + y_2)} = 1 \,. \tag{4.35}$$

The resolution of (4.35) for Δ_{L-1} yields the eigenvalue

$$\Delta_{L-1} = -\frac{(y_1 + y_2)}{2\sqrt{y_1 y_2}} \,. \tag{4.36}$$

5   Concluding remarks

In this work we have presented a mechanism allowing to associate a linear partial differential equation with the eigenvalue problem of the six-vertex model transfer matrix. This mechanism has its roots in the algebraic-functional approach described in details in [15], and a crucial step in this program is the identification of the function space $\Xi(\mathbb{C}^n)$ defined in Section 3.2.

In Section 3.1 we identify the action of the transfer matrix on the function space $\Xi(\mathbb{C}^n)$ and we find that it is given in terms of certain operators $D_{z_i}^{z_\alpha}$ exhibiting a simple action even on larger spaces such as $\mathbb{C}[z]$ with $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$. Also, the operators $D_{z_i}^{z_\alpha}$ play a fundamental role in establishing the aforementioned partial differential equations as they possess a differential realization in the desired function space.

The six-vertex model transfer matrix admits the series expansion $T(\lambda) = T(0)(\mathbb{1} + \mathcal{H}\lambda + \ldots)$ where $T(0)$ is the discrete translation operator, i.e. momentum operator exponentiated, and $\mathcal{H}$ is the hamiltonian of the XXZ spin chain [20]. Although $\mathcal{H}$ contains only next-neighbors interactions, the higher order terms contain highly non-local terms. On the other hand, our transfer matrix in the appropriate variable consists essentially of a polynomial whose degree scales linearly with the lattice length. Thus the number of independent commuting quantities having $T(\lambda)$ as its former are finite and, in particular, we find the operator Ω_{L-1} which exhibits a local structure in terms of differentials.

The partial differential equation corresponding to the operator Ω_{L-1} spectral problem is explicitly given in Section 4 and it consists of a linear equation of order $L-1$. Although we have an equation whose order scales with the lattice length L, in Section 4.1 we also show how this equation can be translated into a system of first order partial differential equations by standard methods. In Section 4.2 we analyze the solutions of our equation for particular values of the lattice length. Interestingly, we find that the spectrum of eigenvalues is fixed by the condition that the eigenfunctions belong to $\Xi(\mathbb{C}^n)$.

6   Acknowledgements

A   Domain wall boundaries

The possibility of extracting partial differential equations from the algebraic-functional method described in Section 3 has been first put forward in [18] and subsequently refined in [15]. Those works have considered the six-vertex model with domain wall boundary

conditions and a compact partial differential equation describing the model partition function has been derived in [15]. This equation reads

$$\left[\sum_{i=1}^{L} \bar{a}(x_i, y_i) - \frac{1}{(L-1)!} \sum_{i=1}^{L} \prod_{j=1}^{L} \bar{a}(x_i, y_j) \prod_{\substack{j=1 \\ j \neq i}}^{L} \frac{\bar{a}(x_j, x_i)}{\bar{b}(x_j, x_i)} \frac{\partial^{L-1}}{\partial x_i^{L-1}}\right] \bar{Z}(\bar{X}^{1,L}) = 0 \,, \quad \text{(A.1)}$$

where $\bar{Z}$ is essentially the partition function of the model. In (A.1) we are considering the conventions $\bar{a}(x, y) = xq - yq^{-1}$ and $\bar{b}(x, y) = x - y$ which is slightly different from the ones used in [15]. Also, the structure of (A.1) is closely related to the structure of (4.9) and to make this feature more apparent we rewrite (A.1) as

$$\left[\mathcal{V}^{DW} + \sum_{i=1}^{L} \mathcal{Q}_i^{DW} \frac{\partial^{L-1}}{\partial x_i^{L-1}}\right] \bar{Z}(\bar{X}^{1,L}) = 0 \,, \quad \text{(A.2)}$$

where

$$\mathcal{V}^{DW} = \sum_{i=1}^{L} \bar{a}(x_i, y_i)$$

$$\mathcal{Q}_i^{DW} = -\frac{1}{(L-1)!} \prod_{j=1}^{L} \bar{a}(x_i, y_j) \prod_{\substack{j=1 \\ j \neq i}}^{L} \frac{\bar{a}(x_j, x_i)}{\bar{b}(x_j, x_i)} \,. \quad \text{(A.3)}$$

Here we intend to translate (A.2) into a system of first order partial differential equations in the same lines of Section 4.1. Since there are no significant modifications compared to the reduction of order used for Eq. (4.9), we restrict ourselves to presenting only the final result.

Lemma 4. *Let $\vec{\phi}$ be the following $L(L-2)+1$ dimensional vector*

$$\vec{\phi} = \begin{pmatrix} \phi^{(0)} \\ \phi^{(1)} \\ \vdots \\ \phi^{(L-2)} \end{pmatrix} \,, \quad \text{(A.4)}$$

where $\phi^{(0)} = \phi_0 = \bar{Z}(\bar{X}^{1,L})$ and the remaining entries are L-dimensional column vectors with components $\phi_i^{(1)} = \partial_i \phi_0$ and $\phi_i^{(k)} = \partial_i \phi_i^{(k-1)}$ for $k > 1$. Then Eq. (A.2) is equivalent to $\Upsilon_{DW} \vec{\phi} = 0$ with Υ_{DW} being obtained from Υ given in (4.20) under the mappings $\Delta_{L-1} \mapsto 0$, $\mathcal{V}^{(n)} \mapsto \mathcal{V}^{DW}$, $\mathcal{Q}_i^{(n)} \mapsto \mathcal{Q}_i^{DW}$ and $n \mapsto L$.

Proof. Direct comparison of (4.9), (A.2) and (4.20). □

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