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Non-renormalization theorems and N=2 supersymmetric backgrounds

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Abstract

The conditions for fully supersymmetric backgrounds of general $N = 2$ locally supersymmetric theories are derived based on the off-shell superconformal multiplet calculus. This enables the derivation of a non-renormalization theorem for a large class of supersymmetric invariants with higher-derivative couplings. The theorem implies that the invariant and its first order variation must vanish in a fully supersymmetric background. The conjectured relation of one particular higher-derivative invariant with a specific five-dimensional invariant containing the mixed gauge-gravitational Chern-Simons term is confirmed.

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1 Introduction

There is increasing interest in locally supersymmetric actions with higher-derivative couplings, whose rigorous study is possible in the context of a consistent off-shell formulation. Such formulations are available when the number of supersymmetries is less than or equal to eight. An off-shell analysis of partially or fully supersymmetric backgrounds is then feasible and the results thereof are relevant for various applications. A first step towards this was made some time ago in [1] in the context of evaluating the corrections to BPS black hole entropy from a specific higher-derivative coupling. More recent results concern the discovery of so-called non-renormalization theorems according to which certain classes of actions as well as their first derivatives with respect to fields or coupling constants must vanish in a fully supersymmetric background [2, 3]. This implies that those actions will not contribute to BPS black hole entropy and neither do they contribute to the field equations when studying supersymmetric field configurations.

In flat space-time the analysis of fully supersymmetric backgrounds is rather straightforward. In that case the supersymmetry algebra generically implies that all component fields are space-time independent, so that all derivative terms in the supersymmetry transformations can be ignored. It then follows that all fields that are in the image of the supercharges must vanish. Therefore only the lowest-dimensional field, which cannot be generated by applying a supersymmetry transformation on yet another field, can take a finite, but constant value. In terms of superfields, this means that full supersymmetry requires any superfield to be constant, i.e. independent of both the bosonic and the fermionic coordinates. In the context of non-trivial space-times, similar results can be derived as long as one is dealing with rigid supersymmetry.

The first part of this paper deals with a systematic analysis of the supersymmetric values that certain supermultiplets can take, but now in the context of local supersymmetry which is somewhat more subtle. When considering a large variety of supersymmetric invariants, we prefer to make use of the (off-shell) superconformal multiplet calculus, where one encounters an extended set of local gauge invariances associated with the superconformal algebra. Proper attention should be paid to all these invariances. This last aspect does not form an impediment for analyzing supersymmetric backgrounds and in fact the presence of the extra conformal (super)symmetries greatly improves the systematics of the analysis. But it is important to appreciate that we are now dealing with *local* gauge invariances which imply a reduction of the physical degrees of freedom. Therefore it does not make sense to just impose gauge invariance on a field configuration and it is natural that a gauge invariant orbit of solutions will remain at the end. In principle this implies that a fully supersymmetric background is only determined up to (small) gauge transformations. In practice this means that we will obtain (conformally) covariant conditions on the field configuration.

This is perhaps the point to briefly introduce the various gauge invariances belonging to the superconformal group. There are two types of supersymmetries, called Q- and S-supersymmetry. Furthermore there are space-time diffeomorphisms, local Lorentz transformations (M), dilatations (D), special conformal boosts (K), and finally the local R-symmetry transformations that constitute the group SU(2) $\times$ U(1). In the superconformal setting a (conformal primary) superfield is characterized by its behaviour under dilatations and the local R-symmetry. The behaviour under dilatations and U(1) transformations is generally characterized by the so-called Weyl and chiral weights, w and c, respectively.

To explain the strategy that we will follow in this paper for establishing supersymmetric backgrounds and to further elucidate some of the conceptual issues, we start in section 2 by discussing a single $N = 2$ vector supermultiplet coupled to a conformal supergravity background (whose covariant quantities comprise the so-called Weyl multiplet). When deriving the consequences of supersymmetry for the resulting field configuration we naturally discover that the conformal supergravity background itself is also subject to constraints. These constraints are identical to the ones that apply to the Weyl multiplet without the presence of the vector multiplet.

In section 3, we briefly present three other short supermultiplets coupled to a conformal supergravity background, namely the tensor multiplet, the non-linear multiplet, and the hypermultiplet. These three multiplets are all characterized by the fact that their lowest-weight scalars transform under the SU(2) R-symmetry group. Requiring supersymmetry in the presence of any of these multiplets turns out to impose a stronger restriction on the Weyl multiplet than when only vector multiplets are present. With this additional restriction the allowed field configurations are equivalent to the ones derived in [1].

Having determined the conditions imposed by supersymmetry we turn to a large class of supersymmetric actions with higher-derivative couplings. We first concentrate on the kinetic multiplet of the logarithm of a conformal primary anti-chiral superfield of Weyl weight w, $\mathbb{T}(\ln\bar\Phi_w)$. This multiplet has been extensively discussed in [3]. The superfield $\bar\Phi_w$ is usually not an elementary multiplet but a composite one, and the kinetic multiplet plays a role in constructing a class of higher-derivative supersymmetric actions that extend the class studied in [2] which corresponds to the case of $w = 0$. One such action seems to emerge upon dimensional reduction from the higher-derivative coupling constructed in five dimensions in [4]. This was first noted in [5] but at that time only the $w = 0$ version of $\mathbb{T}(\ln\bar\Phi_w)$ was known. In [3] the construction of $\mathbb{T}(\ln\bar\Phi_w)$ was presented for arbitrary values of w, and it was concluded that the actual invariant arising from dimensional reduction corresponds to the case with $w = 1$. To exhibit some characteristic features of these couplings one may consider the purely bosonic case, where the relevant expression that appears in the action equals

$$\begin{aligned}\Box_{\rm c}\Box_{\rm c}\ln\phi = {}&\big(\mathcal{D}^2\big)^2\ln\phi - 2\,\mathcal{D}^\mu\big[\big(2\,f_{(\mu}{}^a e_{\nu)a} - f\,g_{\mu\nu}\big)\mathcal{D}^\nu\ln\phi\big]\\ &+ w\big[\mathcal{D}^2 f + 2\,f^2 - 2\,(f_\mu{}^a)^2\big]\,. \end{aligned}\qquad(1.1)$$

The scalar field ϕ can be either an elementary or a composite field, and it scales under local dilatations according to $\phi \to \exp[w\,\Lambda_{\rm D}]\,\phi$, where w denotes the (arbitrary) scaling weight of the field. The derivatives are standard gravitational derivatives and $f_\mu{}^a$ is a composite gauge field associated with special conformal boosts, which, in the simple theory introduced above with a gravitational background, can be expressed in terms of the Riemann tensor. In that case one has the identity

$$\mathcal{D}^2 f + 2\, f^2 - 2\,(f_\mu{}^a)^2 = \tfrac{1}{6}\mathcal{D}^2\mathcal{R} - \tfrac{1}{2}\mathcal{R}^{ab}\,\mathcal{R}_{ab} + \tfrac{1}{6}\mathcal{R}^2 \ , \tag{1.2}$$

where $\mathcal{R}_{ab}$ and $\mathcal{R}$ denote the Ricci tensor and scalar. The crucial property of this expression is that it is conformally invariant irrespective of the value of the Weyl weight and furthermore that it can be easily extended to $N = 2$ supergravity on the basis of chiral supermultiplets. Hence this expression defines a class of actions upon multiplying with any (composite or elementary) scalar of weight $w = 0$.

In section 4 we summarize the salient features of the chiral multiplet $\mathbb{T}(\ln \bar\Phi_w)$ and derive the conditions imposed by full supersymmetry. This then facilitates our task, undertaken in section 5, to establish the existence of the non-renormalization theorem of the type discussed before for this class of couplings. This result thus establishes an extension of the non-renormalization theorem that was initially proven for the more restricted class of higher-derivative couplings with $w = 0$ [2]. Some early indications of this extended non-renormalization theorem were already noted in [3], where some applications were also pointed out.

In section 6, we return to the issue of the dimensional reduction of the supersymmetric $5D$ mixed gauge-gravitational Chern-Simons invariant given in [4]. The resulting $4D$ action has two contributions: one is a holomorphic term involving the square of the Weyl multiplet, and the other involves the new higher-derivative coupling discussed above. Its existence confirmed the observation made in a study of $5D$ BPS black holes and black rings in the context of a Lagrangian with the same $5D$ higher-derivative couplings, that the $5D$ equations of motion do not reduce to the expected $4D$ equations, thus indicating the presence of new $4D$ higher-derivative couplings [6]. In [5] these new $4D$ couplings were identified with those constructed in [2], which involve the $w = 0$ version of $\mathbb{T}(\ln \bar\Phi_w)$. The more general class based on $w \neq 0$ was considered later in [3], and at that point it was noted that actually the new higher-derivative coupling should correspond to the case $w = 1$. However, a comprehensive proof of this correspondence was missing until now, and this is the reason why this topic is addressed in this last section.

For further definitions and notational details, we refer the reader to the literature, and in particular to [2, 3].

2 Vector supermultiplets in a superconformal background

In this section we derive the conditions that follow from imposing full supersymmetry on a field configuration consisting of a single vector supermultiplet in a conformal supergravity background. We first focus on the conditions imposed by supersymmetry on the vector multiplet. This eventually leads to conditions on the Weyl multiplet, the supermultiplet that characterizes the conformal supergravity background. The same analysis for the Weyl supermultiplet without any vector multiplet present turns out to lead to identical conditions. This situation will change in the case that other supermultiplets than the vector one are present, as will be shown in section 3. There we will deal with the remaining short supermultiplets, namely the tensor multiplet, the so-called non-linear multiplet and the hypermultiplet. As it turns out, in the presence of either one of these multiplets, the Weyl multiplet is subject to additional restrictions.

The vector multiplet consists of a complex scalar X, transforming with weights $w = 1$ and $c = -1$ under local dilatations and chiral U(1) transformations, a Majorana spinor doublet decomposed into chiral and anti-chiral components, Ω_i and Ω^i, which are each other's conjugates, an abelian gauge field W_μ and a triplet of auxiliary fields Y^{ij}. The indices $i, j, \ldots = 1, 2$ refer to the components of the doublet representation of the R-symmetry group SU(2). For further definitions we refer the reader to, for instance, [2, 3], where explicit definitions and further details are given in the same notation as employed in this paper. Under Q- and S-supersymmetry the transformation rules of the vector multiplet take the following form:

$$\begin{aligned}
\delta X &= \bar{\epsilon}^i \Omega_i \,, \\
\delta \Omega_i &= 2 \not{D} X \epsilon_i + \tfrac{1}{2} \varepsilon_{ij} \hat{F}^-_{bc} \gamma^{bc} \epsilon^j + Y_{ij} \epsilon^j + 2 X \eta_i \,, \\
\delta W_\mu &= \varepsilon^{ij} \bar{\epsilon}_i (\gamma_\mu \Omega_j + 2\, \psi_{\mu j} X) + \varepsilon_{ij} \bar{\epsilon}^i (\gamma_\mu \Omega^j + 2\, \psi_\mu{}^j \bar{X}) \,, \\
\delta Y_{ij} &= 2\, \bar{\epsilon}_{(i} \not{D} \Omega_{j)} + 2\, \varepsilon_{ik} \varepsilon_{jl}\, \bar{\epsilon}^{(k} \not{D} \Omega^{l)} \,.
\end{aligned} \tag{2.1}$$

The derivatives D_μ are fully covariant with respect to superconformal transformations and thus contain the various connection fields associated with the superconformal gauge symmetries. The parameters of Q- and S-supersymmetry are the chiral spinors ϵ^i and η_i, respectively, and their conjugate (anti-chiral) spinors, ϵ_i and η^i. We should point out that $\hat{F}^\pm_{\mu\nu}$ are the (anti-)selfdual components of the modified field strength tensor associated with the gauge field W_μ,

$$\hat{F}_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu - \tfrac{1}{4} \big[X\, T_{\mu\nu\, ij}\, \varepsilon^{ij} + \bar{X}\, T_{\mu\nu}{}^{ij}\, \varepsilon_{ij} \big] \,, \tag{2.2}$$

up to additional contributions quadratic in fermion fields. The fields $T_{ab\, ij}$ and $T_{ab}{}^{ij}$ are the self-dual and anti-selfdual covariant tensor fields that belong to the Weyl multiplet. Note that we will generally suppress terms that are of higher order in the fermions, because eventually the supersymmetric field configurations will be presented with all fermion fields set to zero.

Before beginning the actual analysis of supersymmetric field configurations, let us recall that the superconformal symmetries are realized as *local* gauge invariances, which makes the analysis conceptually rather different as compared to the rigid case. For instance, imposing rigid supersymmetry requires the scalar field X to be constant. In the present context such a result is not meaningful, because X is subject to local scale and phase transformations, so that any two non-zero values of the field X will be gauge equivalent. A similar comment applies also to the fermions, where one might expect that the fields Ω_i will be required to vanish. But here again one realizes that two different values of Ω_i can be gauge equivalent by S-supersymmetry. Obviously a gauge invariant orbit of solutions must remain, but it is often convenient to choose a particular representative of the gauge orbit, which is equivalent to adopting a gauge condition. However, we prefer to restrict this option to the fermionic symmetries and leave the bosonic superconformal gauge invariances unaffected to keep the structure of our results as transparent as possible.

Let us now point out that in certain cases the analysis of supersymmetric configurations can be more direct, which is an important result that will be relevant throughout this paper. Rather than considering a single vector multiplet, let us briefly consider two such multiplets with fields (X^1, X^2), $(\Omega_i{}^1, \Omega_i{}^2)$, etcetera. Then we may consider a (conformal primary) chiral multiplet with the components

$$\frac{X^1}{X^2}\,, \qquad \frac{X^2\,\Omega_i{}^1 - X^1\,\Omega_i{}^2}{(X^2)^2}\,, \qquad \text{etcetera}\,. \tag{2.3}$$

Now the analysis of full supersymmetry becomes straightforward, because the first (scalar) component is invariant under dilatations and U(1) transformations (it has weights $w = c = 0$), whereas the second fermionic component is invariant under S-supersymmetry. Therefore it is now straightforward to conclude that the scalar must be a constant, while the fermionic component must vanish. Continuing this analysis will show that this multiplet is restricted to a constant, or, equivalently, that in the supersymmetric limit the two multiplets must be proportional to one another. This is an example of a more generic result: if the lowest-weight (scalar) component of a multiplet does not transform under dilatations and U(1) transformations, then the supersymmetry algebra implies that the lowest-weight fermion into which it transforms must be invariant under S-supersymmetry. In the supersymmetric limit, this multiplet is then restricted to a constant. For a general chiral multiplet this result was proven in [2].

From the above result it is therefore clear that nothing will be learned by considering several vector multiplets at once, so we return to the original problem using to a single vector multiplet. Given the fact that the local superconformal gauge invariances will naturally lead to a certain degeneracy, we will define a specific approach based on two guiding principles. First of all, we insist that the bosonic superconformal invariances are preserved so that the final result can be expressed in terms of equations that are manifestly covariant with respect to all these gauge invariances. Secondly we assume that all (supercovariant) fermionic quantities will vanish in the bosonic background. This leaves the bosonic invariance intact. The only equations that are relevant thus follow from the requirement that the supersymmetry variations of the (supercovariant) fermionic quantities should vanish under a particular set of supersymmetry transformations parametrized by eight independent spinorial parameters ϵ^i and ϵ_i. The resulting bosonic covariant equations then characterize all the supersymmetric configurations. As we shall see, this strategy amounts to choosing a certain representative of the fermionic gauge orbit. In principle one can still apply the fermionic gauge transformations, but this will then lead to a different representative for which the fermion fields do not vanish.

Hence, in order that X is invariant under full supersymmetry one naturally assumes that $\Omega_i = 0$. To ensure that the transformation of the fermions will vanish as well, one requires that a linear combination of Q- and S-supersymmetry will vanish on the spinor fields Ω_i, which can be found by expressing the parameter η_i of the S-supersymmetry transformation in terms of the parameters of the Q-supersymmetry transformations, i.e.,

$$\hat\eta_i = -X^{-1}\left[\not{\mathcal{D}}X\epsilon_i + \tfrac{1}{4}\varepsilon_{ij}\hat F^-_{bc}\gamma^{bc}\epsilon^j + \tfrac{1}{2}Y_{ij}\epsilon^j\right]. \tag{2.4}$$

Here we have replaced the supercovariant derivative D_a by a derivative $\mathcal{D}_a$, which is covariant with respect to only the linearly realized bosonic symmetries. We should stress here that special conformal boosts are not realized linearly. Usually this does not lead to additional terms when considering derivatives on quantities that themselves are invariant under these boosts. To avoid confusion we will usually write the conformal gauge connection $f_\mu{}^a$ explicitly in the purely bosonic expressions and not keep it implicit as we do when dealing with supercovariant derivatives.

In this strategy the initial vector multiplet plays a key role, but in due course we will demonstrate that the results will be independent of the choice of the particular supermultiplet from where one starts this procedure. We should also mention that all the constraints can alternatively be obtained by exploiting the observation given below (2.3). Namely, one can start from bosonic expressions constructed from various supermultiplet components that are invariant under dilatations and chiral transformations, and explore the fact that they must vanish under repeated supersymmetry transformations. We shall comment on this aspect when considering the specific results of our calculations.

As explained earlier we subsequently require that all supercovariant fermionic quantities vanish under supersymmetry and so must their supersymmetry variations. Hence the superconformal derivative $D_a\Omega_i$ is assumed to vanish identically. What remains is to ensure that also its variation will vanish under the particular combination of Q- and S-supersymmetry defined by (2.4). To investigate the invariance of $D_a\Omega_i$, let us first define the superconformal derivative,

$$D_a\Omega_i = \mathcal{D}_a\Omega_i - \not{D}X\psi_{ai} - \tfrac{1}{4}\varepsilon_{ij}\hat{F}^{-}_{bc}\,\gamma^{bc}\psi_a{}^j - \tfrac{1}{2}Y_{ij}\psi_a{}^j - X\,\phi_{ai}\,, \tag{2.5}$$

where $\psi_\mu{}^i$ and $\psi_{\mu i}$ denote the chiral and anti-chiral components of the gravitino field that is the gauge field associated with Q-supersymmetry. The gauge fields of S-supersymmetry are not elementary but composite fields denoted by $\phi_{\mu i}$ and $\phi_\mu{}^i$. Its explicit definition can be found in e.g. [2, 3]. The derivative $\mathcal{D}_\mu$ is covariant under all the linearly acting bosonic transformations, namely dilatations, local Lorentz transformations and local R-symmetry transformations. Since we assumed that the fermionic gauge field must also vanish in the supersymmetric limit we indeed have $D_a\Omega_i = 0$.

Now consider the supersymmetry variation of $D_a\Omega_i$, restricting ourselves to the purely bosonic terms, using that the generic supersymmetry variations of the Q- and S-supersymmetry gauge fields are given (up to terms proportional to fermionic bilinears) by

$$\begin{aligned}
\delta\psi_\mu{}^i &= 2\,\mathcal{D}_\mu\epsilon^i - \tfrac{1}{8}T_{ab}{}^{ij}\gamma^{ab}\gamma_\mu\epsilon_j - \gamma_\mu\eta^i\,,\\
\delta\phi_\mu{}^i &= -\,2\,f_\mu{}^a\gamma_a\epsilon^i + \tfrac{1}{4}R(\mathcal{V})_{ab}{}^i{}_j\gamma^{ab}\gamma_\mu\epsilon^j + \tfrac{1}{2}\mathrm{i}R(A)_{ab}\gamma^{ab}\gamma_\mu\epsilon^i - \tfrac{1}{8}\not{D}T^{ab\,ij}\gamma_{ab}\gamma_\mu\epsilon_j + 2\,\mathcal{D}_\mu\eta^i\,,
\end{aligned} \tag{2.6}$$

where $f_\mu{}^a$ is the gauge field of special conformal boosts, which is a composite field whose bosonic terms take the form

$$f_\mu{}^a = \tfrac{1}{2}R(\omega,e)_\mu{}^a - \tfrac{1}{4}\big(D + \tfrac{1}{3}R(\omega,e)\big)e_\mu{}^a - \tfrac{1}{2}\mathrm{i}\tilde{R}(A)_\mu{}^a + \tfrac{1}{16}T_{\mu b}{}^{ij}T^{ab}{}_{ij}\,. \tag{2.7}$$

Here $R(\omega,e)_\mu{}^a$ and $R(\omega,e)$ are the contractions of the curvature tensor associated with the spin connection field $\omega_\mu{}^{ab}$, defined by $R(\omega)_{\mu\nu}{}^{ab} = 2\,\partial_{[\mu}\omega_{\nu]}{}^{ab} - 2\,\omega_{[\mu}{}^{ac}\omega_{\nu]c}{}^b$. Furthermore χ^i and D are a spinor doublet and a real scalar field belonging to the Weyl multiplet, while $R(A)_{\mu\nu}$ and $R(\mathcal{V})_{\mu\nu}{}^i{}_j$

denote the curvature tensors associated with the connections of the U(1) and SU(2) R-symmetry, respectively.

Of course, for consistency one must also determine the constraints from full supersymmetry on the conformal supergravity background. As a first step in that direction we will therefore also include the consequences of the supersymmetry invariance of the spinor χ^i, which belongs to the Weyl multiplet. An independent analysis of the supersymmetry conditions based only on the Weyl multiplet fields will be discussed at the end of this section. Under supersymmetry χ^i transforms as follows,

$$\delta\chi^i = -\tfrac{1}{12}\gamma^{ab}\,\not{D}T_{ab}{}^{ij}\,\epsilon_j + \tfrac{1}{6}R(\mathcal{V})_{\mu\nu}{}^i{}_j\gamma^{\mu\nu}\epsilon^j - \tfrac{1}{3}\mathrm{i}R(A)_{\mu\nu}\gamma^{\mu\nu}\epsilon^i + D\epsilon^i + \tfrac{1}{12}\gamma_{ab}T^{abij}\eta_j\,. \tag{2.8}$$

In evaluating the consequences of the above results one may assume that both X and $T_{ab}{}^{ij}$ are non-vanishing. The reason is that they are the lowest-weight fields of the two multiplets, so that their vanishing would imply that the corresponding multiplets will vanish.

Upon substituting (2.4) it turns out that $\delta(D_a\Omega_i)=0$ and $\delta\chi^i=0$ give rise to the following conditions,

$$\begin{aligned}
R(\mathcal{V})_{\mu\nu}{}^i{}_j &= R(A)_{\mu\nu} = R(D)_{\mu\nu} = Y_{ij} = 0\,,\\
D &= \tfrac{1}{48}\big[X^{-1}\,\varepsilon_{ij}T_{ab}{}^{ij}\,\hat F^{-ab} + \bar X^{-1}\,\varepsilon^{ij}T_{abij}\,\hat F^{+ab}\big]\,,\\
\hat F^{-}{}_a{}^c\,T_{cb}{}^{ij} &= T_{ac}{}^{ij}\,\hat F^{-c}{}_b\,,\\
\bar X\,\varepsilon_{ij}\,T_{ab}{}^{ij}\,\hat F^{-ab} &= X\,\varepsilon^{ij}\,T_{abij}\,\hat F^{+ab}\,.
\end{aligned} \tag{2.9}$$

The third equation implies that $\hat F^{-ab}$ is proportional to $\bar X\,\varepsilon_{ij}\,T_{ab}{}^{ij}$, with a proportionality factor that is invariant under local dilatations and U(1) R-symmetry transformations. Using also the the second and fourth equation in (2.9), one can determine this factor and obtain the relation

$$\hat F^-_{ab} = \frac{24\,D\,X\,T_{ab}{}^{ij}\,\varepsilon_{ij}}{(T^{cdkl}\,\varepsilon_{kl})^2}\,. \tag{2.10}$$

Here we have assumed that $T_{ab}{}^{ij}$ is not null, that is, $(T_{ab}{}^{ij}\varepsilon_{ij})^2\neq 0$. We will continue making this assumption from now on.[1]

Furthermore we also derive the following conditions involving derivatives,

$$\begin{aligned}
\mathcal{D}_a\big(X\,T^{abij}\big) &= 0\,,\\
\mathcal{D}_a\big(X\,T^{ab}{}_{ij}\big) &= 2\,\varepsilon_{ij}\,\mathcal{D}_a\hat F^{-ab}\,,\\
\mathcal{D}_a\hat F^{-ab} &= -\,\mathcal{D}_a\ln(X/\bar X)\,\hat F^{-ab}\,,\\
\mathcal{D}_a\hat F^{-bc} - \mathcal{D}_a\ln X\,\hat F^{-bc} &= -\,2\,\big[\mathcal{D}^{[b}\ln(X\bar X)\,\hat F^{-c]}{}_a - \mathcal{D}_d\ln(X/\bar X)\,\hat F^{-d[b}\,\delta^{c]}{}_a\big]^{[bc]-}\,,\\
X\,D_{(a}D_{b)}X - 2\,\mathcal{D}_aX\,\mathcal{D}_bX &= \frac{X}{2\,\bar X}\,\hat F^{-c}_{a}\hat F^{+}_{cb} - \frac{1}{2}\eta_{ab}\Big[(\mathcal{D}_cX)^2 + \frac{1}{16}X\,\hat F^{-cd}\,T_{cd}{}^{ij}\varepsilon_{ij}\Big]\,,
\end{aligned} \tag{2.11}$$

[1] The case where $(T_{ab}{}^{ij}\varepsilon_{ij})^2$ vanishes (in spite of the fact that $T_{ab}{}^{ij}\neq 0$) is rather special but can still be dealt with by using the same method. Since the results are not substantially different, we ignore this case here.

where, in the last equation, $D_{(a}D_{b)}X \equiv \big(\mathcal{D}_{(a}\,\mathcal{D}_{b)} + \,f_{\mu(a}\,e_{b)}{}^{\mu}\,\big)X$. This equation thus leads to a condition on the field $f_\mu{}^a$ and therefore on $R(\omega,e)_\mu{}^a$. The imaginary part of the second equation is consistent with the Bianchi identity on the field strength associated with the vector gauge field W_μ. The last term in the fourth equation (2.9) involves an anti-selfdual projection on the indices $[bc]$. When this is taken into account, the result takes the form

$$\mathcal{D}_a\hat F^{-bc} - \mathcal{D}_a\ln(X\bar X)\,\hat F^{-bc} + 2\,\mathcal{D}^{[b}\ln X\,\hat F^{-c]}{}_a - 2\,\mathcal{D}_d\ln X\,\hat F^{-d[b}\,\delta_a{}^{c]} = 0\,, \tag{2.12}$$

which is conformally invariant in agreement with our original assumption.

We note one more equation that follows from the first three equations of (2.11), namely

$$\big(\hat F^{-ab} + \tfrac14 X\,T^{ab}{}_{ij}\varepsilon^{ij}\big)\mathcal{A}_b = 0\,, \tag{2.13}$$

where

$$\mathcal{A}_\mu \equiv -\tfrac12\mathrm{i}\mathcal{D}_\mu\ln[X/\bar X] = A_\mu - \tfrac12\mathrm{i}\partial_\mu\ln[X/\bar X] \tag{2.14}$$

Obviously $\mathcal{A}_\mu$ is invariant under chiral U(1) and dilatations. Because $R(A)_{\mu\nu}=0$ it follows that $\partial_{[\mu}\mathcal{A}_{\nu]} = 0$. Substituting (2.10) into (2.13), one derives, after multiplication with the selfdual tensor T_{abij} and making use of the standard identities for products of (anti-)selfdual tensors,

$$\big[\varepsilon^{ij}T_{ab\,ij}\,T^{ackl}\varepsilon_{kl} + 24\,D\,\delta_b{}^c\big]\,\mathcal{A}_c = 0\,, \tag{2.15}$$

The first term in this equation contains the product of a selfdual and an anti-selfdual tensor which is symmetric and traceless, and whose square must be proportional to the identity matrix. In this way one can obtain the following equation,

$$\left(\frac{D^2}{\big|(T^{abij}\varepsilon_{ij})^2\big|^2} - \frac{1}{(96)^2}\right)\mathcal{A}_\mu = 0\,. \tag{2.16}$$

At this point we have not yet evaluated all the constraints of full supersymmetry on the Weyl multiplet. Besides the spinor field χ^i that we have already considered, there exists a supercovariant tensor-spinor, $R(Q)_{ab}{}^i$, which is the superconformal field strength of the gravitini fields. It emerges as the supersymmetry variation of the tensor field T^{abij}, so that it must vanish. Under Q- and S-supersymmetry $R(Q)_{ab}{}^i$ transforms as

$$\delta R(Q)_{ab}{}^i = -\tfrac12\not{D}T_{ab}{}^{ij}\epsilon_j + R(\mathcal{V})^{-}_{ab}{}^i{}_j\,\epsilon^j - \tfrac12\mathcal{R}(M)_{ab}{}^{cd}\,\gamma_{cd}\epsilon^i + \tfrac18 T_{cd}{}^{ij}\gamma^{cd}\gamma_{ab}\,\eta_j\,, \tag{2.17}$$

where $\mathcal{R}(M)_{ab}{}^{cd}$ is a modification of the curvature associated with the spin connection field $\omega_\mu{}^{ab}$.

Requiring $\delta R(Q)_{ab}{}^i = 0$, and using again (2.4), leads to two more equations,

$$\begin{aligned}\mathcal{D}_aT^{bcij} - \mathcal{D}_a\ln X\,T^{bcij} + 2\,\mathcal{D}^{[b}\ln X\,T^{c]}{}_a{}^{ij} - 2\,\mathcal{D}_d\ln X\,T^{d[bij}\,\delta^{c]}{}_a = 0\,,\\ \mathcal{R}(M)^{-}_{ab\,cd} - \frac{1}{2\,|X|^2}\,(\varepsilon_{ij}\bar X\,T_{a[c}{}^{ij})\,\hat F^{-}{}_{d]b}\big|^{[ab]-} = 0\,.\end{aligned} \tag{2.18}$$

From the first equation we derive

$$\varepsilon_{kl} T_{ab}{}^{kl}\, \mathcal{D}_c T^{cbij} \varepsilon_{ij} = -\tfrac{1}{8} \mathcal{D}_a (T^{bckl} \varepsilon_{kl})^2 \,, \tag{2.19}$$

by making use of the identities that hold for contractions of (anti-)selfdual tensors. Furthermore one derives, upon combining (2.10), (2.12) and the first equation of (2.18), that certain ratios of fields must be constant,

$$\frac{X^2}{(T^{abij}\varepsilon_{ij})^2} = \text{constant}\,, \qquad \frac{D}{\big|(T^{abij}\varepsilon_{ij})^2\big|} = \text{constant}\,. \tag{2.20}$$

These expressions can be regarded as the lowest-weight components of a chiral or real supermultiplet, respectively, with $w = c = 0$. According to the theorem discussed earlier in this section, such multiplets must indeed be equal to a constant in the supersymmetric limit. This observation enables an alternative derivation of the same results that we are deriving in this section.

The second equation (2.18) involves an anti-selfdual projection over the the index pair $[ab]$ (because of the symmetry of this term, it is also anti-selfdual in $[cd]$), while $\mathcal{R}(M)^-_{ab\,cd}$ is anti-selfdual in both index pairs $[ab]$ and $[cd]$. Using (2.10) the equation then takes the form

$$\mathcal{R}(M)^-_{ab\,cd} - \frac{12\, D}{(T^{abij}\varepsilon_{ij})^2}\, P^-_{ab,cd} = 0\,, \tag{2.21}$$

where[2]

$$P^-_{ab,cd} \equiv T_{a[c}\, T_{d]b}\Big|^{[ab]-} = \tfrac{1}{8}\big(\delta_{a[c}\, \delta_{d]b} - \tfrac{1}{2}\varepsilon_{abcd}\big)(T^{efij}\varepsilon_{ij})^2 - \tfrac{1}{2}\varepsilon_{ij}\, T_{cd}{}^{ij}\, T_{ab}{}^{kl}\, \varepsilon_{kl}\,. \tag{2.22}$$

By now we have obtained a number of conditions that do not explicitly involve the vector multiplet fields. A relevant question is therefore whether the Weyl multiplet alone (i.e. without being coupled to a vector multiplet) requires the same conditions when imposing supersymmetry. Therefore we repeat the same procedure but now without coupling to a vector multiplet. Hence we start with the supersymmetry variation of the field χ^i shown in (2.8), and choose $\hat\eta_i$ such that its supersymmetry variation vanishes.

At this point the reader may wonder whether a different choice for $\hat\eta_i$ would not affect the results of the previous analysis, so that they would become incompatible with the new ones that we are about to derive. This is actually not the case, as one can simply see by considering the supersymmetry variation of the S-supersymmetric linear combination, $T^{abij}\gamma_{ab}\Omega_j - 24\, X\, \chi^i$, whose vanishing under Q-supersymmetry is obviously independent of whether $\hat\eta_i$ is chosen such that $\delta\Omega_i$ or $\delta\chi^i$ will vanish. To base the analysis on S-supersymmetric combinations of spinors was precisely the approach followed in [1]. Hence it follows that the choice of $\hat\eta_i$ is irrelevant, and it is again obvious that the fermionic gauge orbit associated with S-supersymmetry is not affected, as was emphasized earlier. Our approach of adopting a specific $\hat\eta_i$ associated with a specific supermultiplet is thus a matter of convenience when considering separate configurations of supermultiplets.

[2]Note that we are using Pauli-Källén conventions so that the Levi-Civita symbol is effectively pseudo-real.

Using the expression for $\hat{\eta}_i$ that is found by solving $\delta\chi^i = 0$ directly, one can evaluate the variations of $D_a\chi^i$ and $R(Q)^i_{ab}$, requiring them to vanish also. This calculation is completely similar to the approach followed before. A careful evaluation then shows that all the constraints of the Weyl multiplet imposed by requiring supersymmetry coincide fully with the constraints that we have evaluated before, starting from the vector multiplet (possibly exploiting the first equation of (2.20)).

Let us now return the last equation of (2.11), which involves terms quadratic in derivatives and yields an expression for the composite connection $f_\mu{}^a$ associated with the conformal boosts,

$$\begin{aligned} f_a{}^b = &- \mathcal{D}_a\mathcal{D}^b \ln X + \mathcal{D}_a \ln X\, \mathcal{D}^b \ln X - \tfrac{1}{2}\delta_a{}^b \left(\mathcal{D}_c \ln X\right)^2 - \tfrac{3}{4}\delta_a{}^b D \\ &- \frac{288\, D^2\, \varepsilon_{ij} T_{ac}{}^{ij}\, T^{bc}{}_{kl}\varepsilon^{kl}}{\left|(T^{demn}\varepsilon_{mn})^2\right|^2} \end{aligned} \tag{2.23}$$

Whereas the left-hand side is manifestly real, the right-hand side is not. To analyze this we note that $\mathcal{D}_\mu X = \mathcal{D}_\mu |X| + \mathrm{i}\mathcal{A}_\mu$, where $\mathcal{A}_\mu$ has been defined in (2.14). The reality of (2.23) then implies

$$\mathcal{D}_a\mathcal{A}_b - 2\,\mathcal{A}_{(a}\,\mathcal{D}_{b)} \ln|X| - \eta_{ab}\,\mathcal{A}_c\,\mathcal{D}^c \ln|X| = 0\,, \tag{2.24}$$

where we note that (2.16) implies that $\mathcal{A}_\mu = 0$ for $|D| \neq \frac{1}{96}\,|(T^{abij}\varepsilon_{ij})^2|$. Hence we obtain the following form for the real part of (2.23)

$$\begin{aligned} f_a{}^b = &- \mathcal{D}_a\mathcal{D}^b \ln|X| + \mathcal{D}_a \ln|X|\, \mathcal{D}^b \ln|X| - \mathcal{A}_a\, \mathcal{A}^b \\ &- \tfrac{1}{2}\delta_a{}^b \left[\left(\mathcal{D}_c \ln|X|\right)^2 - \mathcal{A}^c\mathcal{A}_c + \tfrac{3}{2}D\right] - \frac{288\, D^2\, \varepsilon_{ij} T_{ac}{}^{ij}\, T^{bc}{}_{kl}\varepsilon^{kl}}{\left|(T^{demn}\varepsilon_{mn})^2\right|^2} \end{aligned} \tag{2.25}$$

This completes the derivation of a consistent set of covariant equations that characterize the fully supersymmetric configurations consisting of a vector and the Weyl supermultiplet. What remains is to present the results for the components of the Riemann tensor. Up to this point we have fully preserved the covariance with respect to the bosonic symmetries of the superconformal group, so that the spin-connection field $\omega_\mu{}^{ab}$ depends both on the vierbein $e_\mu{}^a$ and on the dilatational gauge field b_μ. Hence the associated curvature $R(\omega)_{\mu\nu}{}^{ab}$ is only identical to the Riemann tensor when b_μ vanishes. For a conformally invariant action b_μ will be absent, while otherwise one still has the option to impose $b_\mu = 0$ as a gauge condition. Comparing (2.25) to (2.7), one derives the following expression for the Ricci tensor and scalar,

$$\begin{aligned} \mathcal{R}(\omega, e)_{ab} = &- 2\,\mathcal{D}_a\mathcal{D}_b \ln|X| + 2\,\mathcal{D}_a \ln|X|\,\mathcal{D}_b \ln|X| - 2\,\mathcal{A}_a\,\mathcal{A}_b \\ &- \eta_{ab}\Big[\mathcal{D}^c\mathcal{D}_c \ln|X| + 2\left(\mathcal{D}_c \ln|X|\right)^2 + 2\,\mathcal{A}^c\mathcal{A}_c + 3\,D\Big] \\ &- \left[\frac{1}{16} + \frac{576\, D^2}{\left|(T^{demn}\varepsilon_{mn})^2\right|^2}\right] \varepsilon_{ij} T_{ac}{}^{ij}\, T_b{}^{ckl}\varepsilon_{kl}\,. \\ \mathcal{R}(\omega, e) = &- 6\,\mathcal{D}^a\mathcal{D}_a \ln|X| - 6\,\mathcal{D}^a \ln|X|\,\mathcal{D}_a \ln|X| + 6\,\mathcal{A}^2 - 12\,D\,. \end{aligned} \tag{2.26}$$

Note that the Ricci tensor is in general not symmetric in the presence of the field b_μ. Finally we note that

$$\mathcal{R}(M)_{ab}{}^{cd} = \mathcal{C}(e,\omega)_{ab}{}^{cd} + D\,\delta_{ab}{}^{cd} + \cdots\,, \tag{2.27}$$

where the suppressed terms are proportional to $R(A)_{\mu\nu}$ and to fermion bilinears, which all vanish in the supersymmetric background. Making use of (2.21) one then derives the expression for the Weyl tensor,

$$\mathcal{C}(e,\omega)_{ab}{}^{cd} = D\left[2\,\delta_{ab}{}^{cd} - \frac{6\,\varepsilon_{ij}T^{ij}_{ab}\,T^{cdkl}\varepsilon_{kl}}{(\varepsilon_{mn}T^{demn})^2} - \frac{6\,\varepsilon^{ij}T_{abij}\,T^{cd}{}_{kl}\varepsilon^{kl}}{(\varepsilon^{mn}T^{de}{}_{mn})^2}\right]\,. \tag{2.28}$$

3 Three other short multiplets

In this section, we consider the remaining $N=2$ short multiplets commonly encountered. They are the tensor multiplet, the non-linear multiplet, and the (on-shell) hypermultiplet. Their distinctive feature is that their lowest-weight components are scalar fields transforming under the SU(2) R-symmetry. For the tensor multiplet these fields are the pseudo-real SU(2) vector L_{ij}, for the non-linear multiplet it is given by a space-time dependent SU(2) element $\Phi^i{}_\alpha$, and for the hypermultiplet they are represented by certain sections $A(\phi)_i{}^\alpha$ of a hyperkähler cone.[3] These quantities will be introduced shortly. We assume that their SU(2) invariant norms are non-vanishing. For the non-linear multiplet, the norm equals $\det[\Phi^i{}_\alpha]=1$; for the tensor and the hypermultiplet, these norms are the length L of the vector L_{ij} and the so-called hyperkähler potential $\chi(\phi)$, respectively, which both have $w=2$. Their precise definitions will be given shortly.

Requiring that the scalars are invariant under supersymmetry leads to the condition that the fermion fields must vanish. We discover that the presence of SU(2) indices on the lowest-dimension scalars generically leads to stronger conditions on the Weyl multiplet than the ones found for the vector multiplet in the previous section. Since all the underlying principles of the analysis have already been exhibited in the previous section, we keep the presentation rather concise. Obviously the conditions on the Weyl multiplet alone may be assumed. In particular, taking $R(\mathcal{V})_{\mu\nu}{}^i{}_j = R(A)_{\mu\nu} = R(D)_{\mu\nu} = 0$ from the start will simplify the analysis. An important condition, which will play a key role in many of the formulae, is

$$\mathcal{D}_a \ln\left|(T_{bc}{}^{ij}\varepsilon_{ij})^2\right| = \mathcal{D}_a \ln(X\bar{X}) = \begin{cases}\mathcal{D}_a \ln L\ , & \text{tensor multiplet}\\ -V_a\ , & \text{non-linear multiplet}\\ \mathcal{D}_a \ln\chi\ , & \text{hypermultiplet}\end{cases} \tag{3.1}$$

where V_a is a vector component of the non-linear multiplet, and L and χ are the two composite real $w=2$ scalar fields introduced above. These conditions are consistent with the (now familiar) observation that any $w=c=0$ scalar field must be constant, and so $|(T_{ab}{}^{ij}\varepsilon_{ij})^2|$ must be proportional to $X\bar{X}$, L and χ for a vector multiplet, tensor multiplet and hypermultiplet, respectively. Note that the vector V_a is not invariant under special conformal boosts.

In contrast with the previous section, we will find that for the three multiplets discussed here, the $w=2$ scalar field D of the Weyl multiplet will be required to vanish. This turns out to have

[3] The indices α for the non-linear multiplet and the hypermultiplet sections are unrelated. For example, the former take the values $\alpha=1,2$ while the latter take the values $\alpha=1,\cdots,r$.

major consequences for both the Weyl multiplet and for any vector multiplet. Invoking (2.10) and (2.21), one derives the following constraints on the Weyl multiplet and any vector multiplet:

$$D=0 \quad\Longrightarrow\quad \mathcal{R}(M)_{ab\,cd}=0\,,\qquad \hat{F}_{ab}=0\,. \tag{3.2}$$

The second equation implies that the Weyl tensor must vanish as a result of (2.28). The third equation of (3.2) leads to a constraint on the vector multiplet field strength,

$$F_{\mu\nu}\equiv 2\,\partial_{[\mu}W_{\nu]}=\tfrac{1}{4}\big[X\,T_{\mu\nu\,ij}\,\varepsilon^{ij}+\bar{X}\,T_{\mu\nu}{}^{ij}\,\varepsilon_{ij}\big]\,. \tag{3.3}$$

Another consequence of $D=0$ is given by (2.16), which implies that

$$\mathcal{A}_\mu=-\tfrac{1}{2}\mathrm{i}\mathcal{D}_\mu\ln(X/\bar{X})=-\tfrac{1}{4}\mathrm{i}\mathcal{D}_\mu\ln\big[(T_{bc}{}^{ij}\varepsilon_{ij})^2/(T^{de}{}_{kl}\varepsilon^{kl})^2\big]=0\ . \tag{3.4}$$

This determines the U(1) gauge connection in terms of the phase of $T_{ab}{}^{ij}$ (or X). The final two conditions we will encounter are the analogues of (2.18) and (2.25), found by making the replacement (3.1) with the additional constraints (3.2) and (3.4).

3.1 The tensor multiplet

The tensor multiplet consists of a pseudo-real SU(2) triplet of scalar fields L_{ij}, which has Weyl weight $w=2$ and satisfies the pseudo-reality constraint $(L^{ij})^*=\varepsilon_{ik}\varepsilon_{jl}L^{kl}$, a doublet of spinors φ^i, a two-form gauge field $E_{\mu\nu}$, and a complex scalar G. Their Q- and S-supersymmetry transformations are

$$\begin{aligned}
\delta L_{ij}=&\,2\,\bar{\epsilon}_{(i}\varphi_{j)}+2\,\varepsilon_{ik}\varepsilon_{jl}\,\bar{\epsilon}^{(k}\varphi^{l)}\ ,\\
\delta\varphi^i=&\,\not{D}L^{ij}\,\epsilon_j+\varepsilon^{ij}\,\hat{\not{E}}^I\,\epsilon_j-G\,\epsilon^i+2L^{ij}\,\eta_j\ ,\\
\delta G=&\,-2\,\bar{\epsilon}_i\not{D}\,\varphi^i\,-\bar{\epsilon}_i(6\,L^{ij}\,\chi_j+\tfrac{1}{4}\,\gamma^{ab}T_{abjk}\,\varphi^l\,\varepsilon^{ij}\varepsilon^{kl})+2\,\bar{\eta}_i\varphi^i\ ,\\
\delta E_{\mu\nu}=&\,\mathrm{i}\bar{\epsilon}^i\gamma_{\mu\nu}\varphi^j\,\varepsilon_{ij}-\mathrm{i}\bar{\epsilon}_i\gamma_{\mu\nu}\varphi_j\,\varepsilon^{ij}\,+2\mathrm{i}\,L_{ij}\,\varepsilon^{jk}\,\bar{\epsilon}^i\gamma_{[\mu}\psi_{\nu]k}-2\mathrm{i}\,L^{ij}\,\varepsilon_{jk}\,\bar{\epsilon}_i\gamma_{[\mu}\psi_{\nu]}{}^k\,,
\end{aligned} \tag{3.5}$$

where D_a are the superconformally covariant derivatives, and $\hat{E}^a$ equals the dual of a supercovariant three-form field strength,

$$\hat{E}^\mu=\tfrac{1}{2}\mathrm{i}\,e^{-1}\,\varepsilon^{\mu\nu\rho\sigma}\Big[\partial_\nu E_{\rho\sigma}-\tfrac{1}{2}\mathrm{i}\bar{\psi}^i_\nu\gamma_{\rho\sigma}\varphi^j\varepsilon_{ij}+\tfrac{1}{2}\mathrm{i}\bar{\psi}_{\nu i}\gamma_{\rho\sigma}\varphi_j\varepsilon^{ij}-\mathrm{i}\,L_{ij}\varepsilon^{jk}\bar{\psi}_\nu{}^i\gamma_\rho\psi_{\sigma k}\Big]\,. \tag{3.6}$$

A supersymmetric field configuration for this multiplet can be found by following the same steps as for the vector multiplet. We note the convenient identity, $L^{ij}L_{jk}=\delta^i{}_k\,L^2$, where the modulus L of the SU(2) triplet is given by $L^2=\tfrac{1}{2}L^{ij}L_{ij}$. We will assume that L is non-vanishing and impose $\delta\varphi^i=0$ by choosing

$$\hat{\eta}_i=-\tfrac{1}{2}L_{ij}\,L^{-2}\big[\not{D}L^{jk}\,\epsilon_k+\varepsilon^{jk}\hat{\not{E}}\,\epsilon_k-G\,\epsilon^j\big]\ , \tag{3.7}$$

where all terms containing fermionic bilinears can be dropped. Next, we impose the conditions $\delta(D_a\varphi^i)=0$ and $\delta\chi^i=\delta R(Q)_{ab}{}^i=0$ and analyze their consequences. Although the latter two conditions have already been investigated separately, it turns out that when combining these with the condition $\delta(D_a\varphi^i)=0$, while using the expression (3.7), one more readily obtains the results (3.2), strongly restricting the Weyl multiplet. Assuming as before that $T_{ab}{}^{ij}$ does not vanish leads to the conditions

$$G=\hat{E}_a=0\,,\qquad L_{ik}\overset{\leftrightarrow}{\mathcal{D}}_a L^{kj}=0\,, \tag{3.8}$$

which force the two-form $E_{\mu\nu}$ to be pure gauge and restrict $\mathcal{D}_a L_{ij}=L_{ij}\,\mathcal{D}_a\ln L$, or

$$\mathcal{D}_a(L_{ij}\,L^{-1})=0\,. \tag{3.9}$$

We find that the derivative of $T_{ab}{}^{ij}$ is given by (2.18) with the replacement $\mathcal{D}_a\ln X\to\frac{1}{2}\mathcal{D}_a\ln L$, implying both (3.4) and (3.1). Similarly, the analogue of (2.25) is reproduced.

3.2 The non-linear multiplet

Next we consider the case of the 'non-linear multiplet' in a conformal supergravity background [7, 8]. This multiplet consists of a scalar SU(2) matrix $\Phi^i{}_\alpha$ with $\alpha=1,2$, a fermion doublet with negative (positive) chirality components λ^i (λ_i), a complex anti-symmetric tensor M^{ij} and a real vector field V^a. Because $\Phi^i{}_\alpha$ is an element of SU(2), it must have vanishing Weyl weight and its inverse matrix is given by its hermitian conjugate denoted by $\Phi^\alpha{}_i$. Under Q- and S-supersymmetry, the fields transform as

$$\begin{aligned}
\delta\Phi^i{}_\alpha &= (2\,\bar\epsilon^i\lambda_j-\delta^i{}_j\,\bar\epsilon^k\lambda_k-\text{h.c.})\,\Phi^j{}_\alpha\,,\\
\delta\lambda^i &= -\tfrac{1}{2}\not{V}\epsilon^i-\tfrac{1}{2}M^{ij}\epsilon_j+\Phi^i{}_\alpha\not{D}\Phi^\alpha{}_j\epsilon^j+\eta^i\,,\\
\delta M^{ij} &= 12\,\bar\epsilon^{[i}\chi^{j]}+\tfrac{1}{2}\bar\epsilon^k\gamma^{ab}\lambda_k\,T_{ab}{}^{ij}-4\bar\epsilon^{[i}\not{V}\lambda^{j]}-2\,\bar\epsilon^k\lambda_k M^{ij}+8\,\bar\epsilon^{[i}\big(\not{D}\lambda^{j]}+\Phi^{j]}{}_\alpha\not{D}\Phi^\alpha{}_k\lambda^k\big)\,,\\
\delta V^a &= \tfrac{3}{2}\bar\epsilon^i\gamma^a\chi_i-\tfrac{1}{8}\bar\epsilon^i\gamma^a\gamma^{bc}\lambda^j\,T_{bc\,ij}-\bar\epsilon^i\gamma^a\not{V}\lambda_i+\bar\epsilon^i\gamma^a\lambda^j M_{ij}+2\,\bar\epsilon^i\gamma^{ab}\mathcal{D}_b\lambda_i\\
&\quad+2\bar\epsilon_i\gamma^a\Phi^i{}_\alpha\not{D}\Phi^\alpha{}_j\lambda^j-\bar\lambda_i\gamma^a\eta^i+\text{h.c.}\,,
\end{aligned} \tag{3.10}$$

where we have suppressed terms explicitly quadratic in the fermion fields. In order for the supersymmetry algebra to close, the vector V^a must obey the non-linear constraint (up to terms quadratic in the fermion fields)

$$D_aV^a-\tfrac{1}{2}V^2-3D-\tfrac{1}{4}M^{ij}M_{ij}+\mathcal{D}_a\Phi^i{}_\alpha\mathcal{D}^a\Phi^\alpha{}_i=0\,, \tag{3.11}$$

which can be interpreted as a condition on the field D of the Weyl multiplet. An unusual feature is that V^a transforms under conformal boosts, $\delta_{\rm K}V^a=2\,\Lambda_{\rm K}{}^a$. Therefore the bosonic terms in the covariant derivative of $D_\mu V^a$ take the form

$$D_\mu V^a=(\partial_\mu-b_\mu)V^a-\omega_\mu{}^{ab}\,V_b-2\,f_\mu{}^a\,. \tag{3.12}$$

Since V^a has Weyl weight $w=1$, it follows that $\delta_{\rm K}(D_aV^a)=2\,\Lambda_{\rm K}{}^a\,V_a$, so that the combination $D_aV^a-\frac{1}{2}V^2$ is conformally invariant.

As before, the condition $\delta\lambda^i = 0$ can be implemented by making a special choice for the S-supersymmetry parameter,

$$\hat{\eta}^i = \tfrac{1}{2}\not{V}\epsilon^i + \tfrac{1}{2}M^{ij}\epsilon_j - \Phi^i{}_\alpha \not{\mathcal{D}}\Phi^\alpha{}_j\,\epsilon^j \ . \tag{3.13}$$

Requiring $\delta(D_a\lambda^i) = 0$ and $\delta\chi^i = \delta R(Q)_{ab}{}^i = 0$ leads to a number of conditions. The Weyl multiplet constraints are obviously implied, and one again finds that (3.2) should hold, along with

$$M^{ij} = 0 \ , \qquad \Phi^i{}_\alpha\,\mathcal{D}_a\Phi^\alpha{}_j = 0 \ . \tag{3.14}$$

The latter equation determines the SU(2) connection in terms of $\Phi^i{}_\alpha\partial_\mu\Phi^\alpha{}_j$. In addition, one finds

$$V_a = -\mathcal{D}_a \ln(T^{bcij}\varepsilon_{ij})^2 = -\mathcal{D}_a \ln(T^{bc}{}_{kl}\varepsilon^{kl})^2 \,, \tag{3.15}$$

implying (3.4) and (3.1). The equations (2.21) and (2.25), upon replacing $\mathcal{D}_a \ln X \to -\tfrac{1}{2}V_a$, are also found.

3.3 The hypermultiplet sector

Unlike the previous supermultiplets, hypermultiplets are realized as an on-shell supermultiplet. Since the multiplet consists only of scalar fields and fermions, without any gauge fields, there does not exist a preferred basis for the fields, which are subject to non-linear redefinitions that take the form of target-space diffeomorphisms and frame transformations of the fermions. For this reason, the hypermultiplets tend to mix under supersymmetry and so it is necessary to consider the entire hypermultiplet sector at once.

For a system of r hypermultiplets, one is dealing with a $4r$-dimensional hyperkähler target space with local coordinates ϕ^A and a target-space metric g_{AB}, $2r$ positive-chirality spinors $\zeta^{\bar\alpha}$ and $2r$ negative-chirality spinors ζ^α. The chiral and anti-chiral spinors are related by complex conjugation as they are Majorana spinors. They are subject to field-dependent reparametrizations of the form $\zeta^\alpha \to S^\alpha{}_\beta(\phi)\,\zeta^\beta$; the fields $\zeta^{\bar\alpha}$ are then redefined with the complex conjugate of $S^\alpha{}_\beta$. The target space is subject to arbitrary diffeomorphisms and has the standard Christoffel connection $\Gamma_{AB}{}^C$. Likewise there exist connections $\Gamma_A{}^\alpha{}_\beta$ and $\Gamma_A{}^{\bar\alpha}{}_{\bar\beta}$ associated with the field-dependent redefinitions noted above. Furthermore supersymmetry implies the existence of an hermitian and a skew-symmetric covariantly constant tensor, $G^{\alpha\bar\beta}$ and $\Omega^{\alpha\beta}$, respectively. The hermitian one appears in the kinetic term for the fermions, and the skew-symmetric one is related to the canonical invariant antisymmetric tensor of Sp(r).

In order to couple the r hypermultiplets to conformal supergravity, their target-space geometry must be a $4r$-dimensional hyperkähler cone [9].[4] The hypermultiplet scalars transform under

[4] Upon fixing the dilatational and SU(2) gauges, conformal supergravity is converted to Poincaré supergravity, and correspondingly the hyperkähler cone is converted into a quaternion-Kähler target space [9, 10], in accordance with [11].

dilatations associated with a homothetic Killing vector, and under the SU(2) R-symmetry, associated with the SU(2) Killing vectors of the hyperkähler cone. The fermions transform under dilatations and the U(1) factor of the R-symmetry by scale transformations and chiral rotations, respectively.

A systematic treatment of hypermultiplets makes use of local sections $A_i{}^\alpha(\phi)$ of an $\mathrm{Sp}(r) \times \mathrm{Sp}(1)$ bundle, where $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$ refers to the corresponding R-symmetry group. These sections transform covariantly under R-symmetry and scale under dilatations with $w = 1$. We refer to [9] for further details. The Q- and S-supersymmetry transformations on the sections and the fermions take the following form,

$$\begin{aligned} \delta A_i{}^\alpha &= 2\bar\epsilon_i \zeta^\alpha + 2\varepsilon_{ij}\, G^{\alpha\bar\beta}\Omega_{\bar\beta\bar\gamma}\bar\epsilon^j\zeta^{\bar\gamma} - \delta_{\mathrm{Q}}\phi^B\, \Gamma_B{}^\alpha{}_\beta A_i{}^\beta \;, \\ \delta\zeta^\alpha &= \not{D} A_i{}^\alpha \epsilon^i + A_i{}^\alpha \eta^i - \delta_{\mathrm{Q}}\phi^B\, \Gamma_B{}^\alpha{}_\beta \zeta^\beta \;, \end{aligned} \tag{3.16}$$

where $\delta_{\mathrm{Q}}\phi^A$ denotes the transformation rule for the target-space scalars whose form is not relevant for what follows. The covariant tensors $G_{\bar\alpha\beta}$ and $\Omega_{\bar\alpha\bar\beta}$ can be expressed as bilinears in the covariant derivatives of the sections,

$$g^{AB}\, D_A A_i{}^\alpha\, D_B A^{j\bar\beta} = \delta_i{}^j\, G^{\alpha\bar\beta} \,, \qquad g^{AB}\, D_A A_i{}^\alpha\, D_B A_j{}^\beta = \varepsilon_{ij}\, \Omega^{\alpha\beta} \,. \tag{3.17}$$

A supersymmetric configuration requires that both the fermions and their supersymmetry variations vanish. For $r > 1$, one cannot find a choice for $\hat\eta^i$ which immediately solves $\delta\zeta^\alpha = 0$ for all α, so it will help to first single out one specific fermion to solve for $\hat\eta^i$. We will follow a similar procedure as in [1] and first single out the $w = 2$ hyperkähler potential χ, defined by

$$\chi = \tfrac{1}{2}\varepsilon^{ij}\, \bar\Omega_{\alpha\beta}\, A_i{}^\alpha\, A_j{}^\beta \,, \tag{3.18}$$

and focus on the composite fermion ζ_i into which it varies,

$$\delta\chi = 2\varepsilon^{ij}\bar\epsilon_j\zeta_i + \text{h.c.} \;, \qquad \zeta_i = \bar\Omega_{\alpha\beta}\, A_i{}^\alpha\, \zeta^\beta \;. \tag{3.19}$$

Solving $\delta\zeta_i = 0$ leads to

$$\hat\eta^i = \varepsilon^{ij}\, \chi^{-1} A_j{}^\beta\, \bar\Omega_{\beta\alpha} \not{D} A_k{}^\alpha\, \epsilon^k \,. \tag{3.20}$$

Subsequently one imposes the conditions $\delta\chi^i = \delta R(Q)_{ab}{}^i = 0$ and $\delta(D_a\zeta_i) = 0$. One confirms again the standard conditions on the Weyl multiplet, including the additional conditions (3.2) and (3.4). The first equation of (2.21) and (2.25) follow with $\mathcal{D}_a \ln X \to \frac{1}{2}\mathcal{D}_a \ln\chi$. In addition to these constraints, one finds

$$A_{(i}{}^\alpha \bar\Omega_{\alpha\beta} \mathcal{D}_a A_{j)}{}^\beta = 0 \;. \tag{3.21}$$

For $r > 1$, one must still satisfy $\delta\zeta^\alpha = 0$. Using (3.21), one finds the additional condition (trivially satisfied for $r = 1$)

$$\mathcal{D}_a A_i{}^\alpha - \tfrac{1}{2}\mathcal{D}_a \ln\chi\, A_i{}^\alpha = \chi^{1/2}\mathcal{D}_a(\chi^{-1/2} A_i{}^\alpha) = 0 \;. \tag{3.22}$$

This implies that the $w=0$ section $\chi^{-1/2}A_i{}^\alpha$ is covariantly constant.

We should draw attention to the fact that the hypermultiplet sector is on-shell and so is associated with a specific Lagrangian. The hyperkähler potential, for instance, captures all the details of a locally supersymmetric two-derivative Lagrangian of hypermultiplets. In closing this section we should also mention that many of the equations obtained here can also be found in [1] where the results were derived in a slightly different context. In the next section we will be discussing a supermultiplet that has never been subjected to this analysis.

4 The chiral $\mathbb{T}(\ln\bar\Phi_w)$ multiplet

In a previous paper [3] a new class of higher-derivative invariants was constructed from the so-called kinetic multiplet. This multiplet, denoted by $\mathbb{T}(\ln\bar\Phi_w)$, is a composite chiral multiplet of weight $w=2$ constructed from the highest component of the logarithm of an anti-chiral multiplet $\bar\Phi_w$ of arbitrary weight w. In this section, we will briefly review that construction and then analyze the conditions for a supersymmetric configuration.

Let us start by recalling that the components of a general (conformal primary) chiral multiplet Φ_w consist of a complex scalar A, a chiral fermion Ψ_i, a complex symmetric SU(2) tensor B_{ij}, an anti-selfdual tensor F_{ab}^-, a second chiral fermion Λ_i, and a complex scalar C, whose Weyl weights range from w to $w+2$.[5] Their supersymmetry transformation rules are [8, 2]

$$\begin{aligned}
\delta A =&\,\bar\epsilon^i\Psi_i\,,\\
\delta\Psi_i =&\,2\,\not{D}A\epsilon_i + B_{ij}\,\epsilon^j + \tfrac12\gamma^{ab}F_{ab}^-\,\varepsilon_{ij}\epsilon^j + 2\,wA\,\eta_i\,,\\
\delta B_{ij} =&\,2\,\bar\epsilon_{(i}\not{D}\Psi_{j)} - 2\,\bar\epsilon^k\Lambda_{(i}\,\varepsilon_{j)k} + 2(1-w)\,\bar\eta_{(i}\Psi_{j)}\,,\\
\delta F_{ab}^- =&\,\tfrac12\varepsilon^{ij}\,\bar\epsilon_i\not{D}\gamma_{ab}\Psi_j + \tfrac12\bar\epsilon^i\gamma_{ab}\Lambda_i - \tfrac12(1+w)\,\varepsilon^{ij}\bar\eta_i\gamma_{ab}\Psi_j\,,\\
\delta\Lambda_i =&\,-\tfrac12\gamma^{ab}\not{D}F_{ab}^-\epsilon_i - \not{D}B_{ij}\varepsilon^{jk}\epsilon_k + C\varepsilon_{ij}\,\epsilon^j + \tfrac14\big(\not{D}A\,\gamma^{ab}T_{abij} + w\,A\,\not{D}\gamma^{ab}T_{abij}\big)\varepsilon^{jk}\epsilon_k\\
&\,-3\,\gamma_a\varepsilon^{jk}\epsilon_k\,\bar\chi_{[i}\gamma^a\Psi_{j]} - (1+w)\,B_{ij}\varepsilon^{jk}\,\eta_k + \tfrac12(1-w)\,\gamma^{ab}\,F_{ab}^-\eta_i\,,\\
\delta C =&\,-2\,\varepsilon^{ij}\bar\epsilon_i\not{D}\Lambda_j - 6\,\bar\epsilon_i\chi_j\;\varepsilon^{ik}\varepsilon^{jl}B_{kl}\\
&\,-\tfrac14\varepsilon^{ij}\varepsilon^{kl}\big((w-1)\,\bar\epsilon_i\gamma^{ab}\not{D}T_{abjk}\Psi_l + \bar\epsilon_i\gamma^{ab}T_{abjk}\not{D}\Psi_l\big) + 2\,w\varepsilon^{ij}\bar\eta_i\Lambda_j\,.
\end{aligned}\qquad(4.1)$$

From these formulae, it is easy to see that if a chiral multiplet has weight $w=0$, then requiring $\delta\Psi_i=0$ amounts to choosing A to be constant and $B_{ij}=F_{ab}^-=\Lambda_i=C=0$, as was argued in [2]. For chiral multiplets of non-zero weight, the situation is more subtle, as we will soon see.

To construct $\mathbb{T}(\ln\bar\Phi_w)$, it is more convenient to deal with the components of $\hat\Phi\equiv\ln\Phi_w$ rather than with Φ_w itself. These are related in a non-linear way: $\hat A=\ln A$, $\hat\Psi_i=A^{-1}\Psi_i$, etc. Because $\hat A$ does not transform homogeneously under local dilatations and U(1) transformations, the superconformal transformations of the higher components will be slightly modified. The Q-

[5] The tensor F_{ab}^-, and likewise $\hat F_{ab}^-$, used in this section should not be confused with the (modified) field strength (2.2) of the vector multiplet. The latter multiplet is related to a *reduced* chiral field, which implies that it is subject to a Bianchi identity.

and S-supersymmetry transformations of the components $\hat{A}$, $\hat{\Psi}_i, \cdots$ are

$$
\begin{aligned}
\delta \hat{A} &= \bar{\epsilon}^i \hat{\Psi}_i \, , \\
\delta \hat{\Psi}_i &= 2\, \not{D} \hat{A} \epsilon_i + \hat{B}_{ij}\, \epsilon^j + \tfrac{1}{2} \gamma^{ab} \hat{F}^-_{ab}\, \varepsilon_{ij} \epsilon^j + 2\, w\, \eta_i \, , \\
\delta \hat{B}_{ij} &= 2\, \bar{\epsilon}_{(i} \not{D} \hat{\Psi}_{j)} - 2\, \bar{\epsilon}^k \hat{\Lambda}_{(i}\, \varepsilon_{j)k} + 2\, \bar{\eta}_{(i} \hat{\Psi}_{j)} \, , \\
\delta \hat{F}^-_{ab} &= \tfrac{1}{2} \varepsilon^{ij}\, \bar{\epsilon}_i \not{D} \gamma_{ab} \hat{\Psi}_j + \tfrac{1}{2} \bar{\epsilon}^i \gamma_{ab} \hat{\Lambda}_i - \tfrac{1}{2}\, \varepsilon^{ij} \bar{\eta}_i \gamma_{ab} \hat{\Psi}_j \, , \\
\delta \hat{\Lambda}_i &= -\tfrac{1}{2} \gamma^{ab} \not{D} \hat{F}^-_{ab} \epsilon_i - \not{D} \hat{B}_{ij} \varepsilon^{jk} \epsilon_k + \hat{C} \varepsilon_{ij}\, \epsilon^j + \tfrac{1}{4} \big(\not{D} \hat{A}\, \gamma^{ab} T_{abij} + w\, \not{D} \gamma^{ab} T_{abij} \big) \varepsilon^{jk} \epsilon_k \\
&\quad - 3\, \gamma_a \varepsilon^{jk} \epsilon_k\, \bar{\chi}_{[i} \gamma^a \hat{\Psi}_{j]} - \hat{B}_{ij} \varepsilon^{jk}\, \eta_k + \tfrac{1}{2}\, \gamma^{ab}\, \hat{F}^-_{ab} \eta_i \, , \\
\delta \hat{C} &= -2\, \varepsilon^{ij} \bar{\epsilon}_i \not{D} \hat{\Lambda}_j - 6\, \bar{\epsilon}_i \chi_j\, \varepsilon^{ik} \varepsilon^{jl} \hat{B}_{kl} + \tfrac{1}{4} \varepsilon^{ij} \varepsilon^{kl} \big(\bar{\epsilon}_i \gamma^{ab} \not{D} T_{abjk} \hat{\Psi}_l - \bar{\epsilon}_i \gamma^{ab} T_{abjk} \not{D} \hat{\Psi}_l \big) \, .
\end{aligned} \tag{4.2}
$$

Note in particular the transformation rule of $\hat{\Psi}_i$, which transforms inhomogeneously under S-supersymmetry into a w-dependent constant. For the special case of $w = 0$, these components transform in the same way as those in (4.1).

Taking the complex conjugate gives the components and transformation rules of the anti-chiral multiplet $\ln \bar{\Phi}_w$. To construct the multiplet $\mathbb{T}(\ln \bar{\Phi}_w)$, one begins by identifying its lowest component with the highest component of $\ln \bar{\Phi}_w$. Subsequent components are defined using supersymmetry. Here we concern ourselves only with the bosonic components and their bosonic constituents. These are given by

$$
\begin{aligned}
A|_{\mathbb{T}(\ln \bar{\Phi})} &= \hat{\bar{C}} \, , \\
B_{ij}|_{\mathbb{T}(\ln \bar{\Phi})} &= -2\, \varepsilon_{ik} \varepsilon_{jl} \big(\Box_{\rm c} + 3\, D \big) \hat{B}^{kl} - 2\, \hat{F}^+_{ab}\, R(\mathcal{V})^{ab\,k}{}_i\, \varepsilon_{jk} \, , \\
F^-_{ab}|_{\mathbb{T}(\ln \bar{\Phi})} &= -\big(\delta_a{}^{[c} \delta_b{}^{d]} - \tfrac{1}{2} \varepsilon_{ab}{}^{cd} \big) \\
&\qquad \times \big[4\, D_c D^e \hat{F}^+_{ed} + (D^e \hat{\bar{A}}\, D_c T_{de}{}^{ij} + D_c \hat{\bar{A}}\, D^e T_{ed}{}^{ij}) \varepsilon_{ij} - w D_c D^e T_{ed}{}^{ij} \varepsilon_{ij} \big] \\
&\quad + \Box_{\rm c} \hat{\bar{A}}\, T_{ab}{}^{ij} \varepsilon_{ij} - R(\mathcal{V})^-{}_{ab}{}^i{}_k\, \hat{B}^{jk}\, \varepsilon_{ij} + \tfrac{1}{8} T_{ab}{}^{ij}\, T_{cdij} \hat{F}^{+cd} \, , \\
C|_{\mathbb{T}(\ln \bar{\Phi})} &= 4(\Box_{\rm c} + 3D) \Box_{\rm c} \hat{\bar{A}} + 6 (D_a D)\, D^a \hat{\bar{A}} - 16\, D^a \big(R(D)^+_{ab} D^b \hat{\bar{A}} \big) \\
&\quad - D^a (T_{abij} T^{cbij} D_c \hat{\bar{A}}) - \tfrac{1}{2} D^a (T_{abij} T^{cbij}) D_c \hat{\bar{A}} + \tfrac{1}{16} (T_{abij} \varepsilon^{ij})^2 \hat{\bar{C}} \\
&\quad + \tfrac{1}{2} D_a D^a (T_{bcij} \hat{F}^{bc+}) \varepsilon^{ij} + 4\, D_a \big(D^b T_{bcij} \hat{F}^{ac+} + D^b \hat{F}^+_{bc} T^{ac}{}_{ij} \big) \varepsilon^{ij} \\
&\quad - w \big[R(\mathcal{V})^+_{ab}{}^i{}_j R(\mathcal{V})^{ab+j}{}_i + 8 R(D)^+_{ab} R(D)^{ab+} \big] \\
&\quad - w \big[D^a T_{abij} D_c T^{cbij} + D^a (T_{abij} D_c T^{cbij}) \big] \, .
\end{aligned} \tag{4.3}
$$

Following the same strategy as before, let us analyze the conditions for a supersymmetric configuration. Requiring $\delta \hat{\Psi}_i = 0$ leads to

$$
\hat{\eta}_i = -\frac{1}{w} \Big[\not{D} \hat{A} \epsilon_i + \tfrac{1}{2} \hat{B}_{ij} \epsilon^j + \tfrac{1}{4} \gamma^{ab} \hat{F}^-_{ab} \varepsilon_{ij} \epsilon^j \Big] \, . \tag{4.4}
$$

Next we sequentially impose $\delta \hat{\Lambda}_i = 0$, $\delta \chi^i = \delta R(Q)_{ab}{}^i = 0$ and finally $\delta (D_a \hat{\Psi}_i) = 0$ using this choice for $\hat{\eta}_i$. We find several algebraic conditions,

$$
\hat{B}_{ij} \hat{F}^-_{ab} = \hat{B}_{ij} T_{ab}{}^{kl} = 0 \, , \qquad \hat{C} = -\tfrac{1}{2w} \hat{F}^-_{ab}\, \hat{F}^{ab-} - \tfrac{1}{4w} \hat{B}_{kl} \hat{B}_{mn} \varepsilon^{kn} \varepsilon^{lm} \, ,
$$

$$\hat{F}^-_{a[b} T_{c]}{}^{a\,ij} = 0 \ , \qquad\qquad D = \tfrac{1}{24w} \hat{F}^{ab-} T_{ab}{}^{ij} \varepsilon_{ij} \ , \tag{4.5}$$

in addition to the first-order differential equations

$$\begin{aligned} \mathcal{D}_\mu \hat{B}_{ij} - \tfrac{1}{w} \mathcal{D}_\mu \hat{A}\, \hat{B}_{ij} &= 0 \ , \\ \mathcal{D}_a T^{bcij} - \tfrac{1}{w} \mathcal{D}_a \hat{A}\, T^{bcij} + \tfrac{2}{w} \mathcal{D}^{[b} \hat{A}\, T^{c]}{}_a{}^{ij} - \tfrac{2}{w} \mathcal{D}_d \hat{A}\, T^{d[bij} \delta^{c]}{}_a &= 0 \ , \\ \mathcal{D}_a \hat{F}^{bc-} - \tfrac{1}{w} \mathcal{D}_a \hat{\bar{A}}\, \hat{F}^{bc-} + \tfrac{2}{w} \mathcal{D}^{[b} \hat{A}\, \hat{F}^{c]}{}_a{}^- - \tfrac{2}{w} \mathcal{D}_d \hat{A}\, \hat{F}^{-d[b} \delta^{c]}{}_a &= 0 \ , \end{aligned} \tag{4.6}$$

and the second-order differential equation

$$\mathcal{D}_a \mathcal{D}_b \hat{A} + w\, e_a{}^\mu f_{\mu b} - \tfrac{1}{w} \mathcal{D}_a \hat{A} \mathcal{D}_b \hat{A} + \tfrac{1}{2w} \mathcal{D}_c \hat{A} \mathcal{D}^c \hat{A}\, \eta_{ab} + \tfrac{3}{4}\, w\, D\, \eta_{ab} - \tfrac{1}{2w} \hat{F}^-_{ac}\, \hat{F}^{+\,c}{}_b = 0 \ . \tag{4.7}$$

One additional condition is also found:

$$\mathcal{D}^c (\hat{A} - \hat{\bar{A}})\, \hat{F}^-_{cb} = -\tfrac{1}{4}\, w\, \mathcal{D}^c (\hat{A} - \hat{\bar{A}})\, T_{cb\,ij}\, \varepsilon^{ij} \ . \tag{4.8}$$

From (4.5), we deduce that

$$\hat{B}_{ij} = 0 \ , \qquad \hat{F}^-_{ab} = \frac{24\, w\, D\, T_{ab}{}^{ij} \varepsilon_{ij}}{(T_{cd}{}^{kl} \varepsilon_{kl})^2} \ , \qquad \hat{C} = -\frac{288\, w\, D^2}{(T_{ab}{}^{ij} \varepsilon_{ij})^2} \ . \tag{4.9}$$

Multiplying the second equation of (4.6) by $T_{bc}{}^{kl}$ leads to $\mathcal{D}_a \big[\hat{A} - \tfrac{1}{2} w \ln(T^{bcij} \varepsilon_{ij})^2\big] = 0$. Because $\hat{A} - \tfrac{1}{2} w \ln(T^{bcij} \varepsilon_{ij})^2$ is inert under dilatations and U(1) rotations, one recovers

$$\mathcal{D}_a \big[\hat{A} - \tfrac{1}{2} w \ln(T^{bcij} \varepsilon_{ij})^2\big] = 0 \qquad \Longrightarrow \qquad \hat{A} = \tfrac{1}{2} w \ln(T_{ab}{}^{ij} \varepsilon_{ij})^2 + \text{const} \ . \tag{4.10}$$

With these choices, the equations (4.5)–(4.8) are identically satisfied, once we use the conditions established for the Weyl multiplet in section 2. At this point we should remark that we could have immediately derived these results by noting that

$$\hat{A} - \tfrac{1}{2} w \ln(T_{ab}{}^{ij} \varepsilon_{ij})^2 = \ln \left(\frac{A}{((T_{ab}{}^{ij} \varepsilon_{ij})^2)^{w/2}} \right) \tag{4.11}$$

is the lowest component of a $w = 0$ chiral multiplet and therefore must be a constant. The higher components of this new $w = 0$ multiplet must vanish, which leads after some algebra to the relations (4.9).

Now we are in a position to evaluate the supersymmetric configuration of $\mathbb{T}(\ln \bar{\Phi}_w)$. From (4.9) one finds that the lowest component of the kinetic multiplet is completely determined to be

$$A|_{\mathbb{T}(\ln \bar{\Phi}_w)} = -\frac{288\, w\, D^2}{(T_{abij} \varepsilon^{ij})^2} \ . \tag{4.12}$$

The remainder of the components of $\mathbb{T}(\ln \bar{\Phi}_w)$ can be found by explicit use of the formulae (4.3), but it is much simpler to note that since $\mathbb{T}(\ln \bar{\Phi}_w)$ is a $w = 2$ chiral multiplet, it must be proportional to the square of the Weyl multiplet, schematically denoted W^2, whose lowest component is $(T_{ab}{}^{ij}\varepsilon_{ij})^2$. For example, we can relate the component B_{ij} of $\mathbb{T}(\ln\bar{\Phi}_w)$ to the same component of W^2,

$$B_{ij}|_{\mathbb{T}(\ln\bar{\Phi}_w)} = B_{ij}|_{W^2} \times \frac{A|_{\mathbb{T}(\ln\bar{\Phi}_w)}}{(T_{cd}{}^{kl}\varepsilon_{kl})^2} = 0 \ . \tag{4.13}$$

In the last equality we have used the fact that in the supersymmetric configuration $B_{ij}|_{W^2}$ is proportional to $\varepsilon_{ik}R(\mathcal{V})_{ab}{}^k{}_j$, which vanishes. In a similar way, one finds

$$F^-_{ab}|_{\mathbb{T}(\ln\bar{\Phi}_w)} = 48D\,T_{ab}{}^{ij}\varepsilon_{ij}\,\frac{A|_{\mathbb{T}(\ln\bar{\Phi}_w)}}{(T_{cd}{}^{kl}\varepsilon_{kl})^2}\ , \quad C|_{\mathbb{T}(\ln\bar{\Phi}_w)} = 576D^2\,\frac{A|_{\mathbb{T}(\ln\bar{\Phi}_w)}}{(T_{cd}{}^{kl}\varepsilon_{kl})^2}\ . \tag{4.14}$$

Note that these higher components are completely determined by the lowest component $A|_{\mathbb{T}(\ln\bar{\Phi}_w)}$, given in (4.12). Two special cases are worthy of note. If Φ_w is actually a weight $w=0$ multiplet, then $\mathbb{T}(\ln\bar{\Phi}_w)$ vanishes completely, as was noted in [2]. Similarly, if we apply the conditions of section 3 (equivalently, the conditions of [1]), then $D=0$ causes the entire kinetic multiplet to vanish for any value of the Weyl weight. This will be a crucial point for the non-renormalization theorem presented in the next section.

5 A new non-renormalization theorem

The preceding sections have mainly been concerned with deriving the conditions of off-shell $N=2$ supersymmetry for various multiplets independently of any action. We devoted particular attention to the chiral multiplet $\mathbb{T}(\ln\bar{\Phi}_w)$, which has been constructed only recently. This multiplet leads to a new class of $4D$ higher-derivative invariants. Our goal in this section is to establish a non-renormalization theorem: in a fully supersymmetric configuration, these higher-derivative invariants always vanish, as do their first derivative with respect to any field or coupling constant. To accomplish this, we will make one assumption. In addition to the apparent field content – a non-vanishing chiral multiplet Φ_w coupled to conformal supergravity – we require at least one multiplet of the set discussed in section 3. The motivation for this last requirement is physical. A Poincaré supergravity action requires both a vector multiplet and at least one other short multiplet. So even if such a multiplet is not present in the specific higher-derivative terms under discussion, it must be present in the sector of the action responsible for generating Poincaré supergravity. This means that it too must take its supersymmetric value. Making this assumption means that the restrictive conditions discussed in section 3 apply. In particular, we will require that $D=0$.

It will be convenient to exploit superfield and superspace terminology as discussed in [3]. Superspace actions generically fall into two classes: they can be integrals over chiral superspace or integrals over the full superspace. Schematically, we can write a chiral superspace action up to a normalization factor as

$$\int \mathrm{d}^4x\,\mathrm{d}^4\theta\,\mathcal{E}\,F \tag{5.1}$$

where F is some quantity built out of chiral multiplets (fundamental or composite) and $\mathcal{E}$ is the chiral superspace measure. The other option is a full superspace integral

$$\int \mathrm{d}^4x\, \mathrm{d}^4\theta\, \mathrm{d}^4\bar{\theta}\, E\, \mathcal{H} \ , \tag{5.2}$$

where $\mathcal{H}$ is real and E is the full superspace measure. In order to satisfy the requirements of superconformal invariance, F must have Weyl weight $w = 2$ and $\mathcal{H}$ must have Weyl weight $w = 0$. In addition, both F and $\mathcal{H}$ must be annihilated by S-supersymmetry.

The distinction between these two types of invariants is not a sharp one. Any full superspace integral can be recast as a chiral one by making use of the so-called $N = 2$ kinetic operator $\mathbb{T}$, normalized here so that[6]

$$\int \mathrm{d}^4x\, \mathrm{d}^4\theta\, \mathrm{d}^4\bar{\theta}\, E\, \mathcal{H} = -\frac{1}{2} \int \mathrm{d}^4x\, \mathrm{d}^4\theta\, \mathcal{E}\, \mathbb{T}(\mathcal{H}) \ . \tag{5.3}$$

Therefore, when we discuss chiral superspace invariants, we usually mean ones which *cannot* be converted back into full superspace invariants by removing a kinetic operator. It will be convenient to call such chiral multiplets *intrinsically chiral*.

A common example of intrinsically chiral integrands are of the form $F(X, A|_{W^2})$ where X^I are vector multiplets and $A|_{W^2} = (T_{ab}{}^{ij}\varepsilon_{ij})^2$ is the lowest component of the square of the Weyl multiplet. This class $F(X, A|_{W^2})$ is actually quite important: it was shown in [14, 15] to accurately describe the subleading corrections to the Wald entropy in the limit of large charges required for matching the degeneracy of the microscopic string and brane states. This precise matching was in retrospect quite surprising since there are in principle a number of higher-derivative actions that do not fall into this class. In fact, this was the motivation in [2] where a non-renormalization theorem established that a large class of full superspace integrals (5.2) do not contribute to the Wald entropy.

It is now important to address what other intrinsically chiral invariants might exist and whether they might possess non-renormalization theorems as well. As discussed in [3], the kinetic multiplet $\mathbb{T}(\ln \bar{\Phi}_w)$ is actually a new contribution to intrinsically chiral functions F. To see why, we note that the naive equality

$$-\frac{1}{2} \int \mathrm{d}^4x\, \mathrm{d}^4\theta\, \mathcal{E}\, \Phi'\, \mathbb{T}(\ln \bar{\Phi}_w) \stackrel{?}{=} \int \mathrm{d}^4x\, \mathrm{d}^4\theta\, \mathrm{d}^4\bar{\theta}\, E\, \Phi'\, \ln \bar{\Phi}_w \tag{5.4}$$

(where Φ' is some $w = 0$ chiral multiplet) does not hold since the integrand on the right-hand side is not actually weight zero due to the inhomogeneous dilatation transformation of $\ln \bar{\Phi}_w$. This means that the left-hand side is actually an intrinsically chiral quantity.

It would seem that this observation might open the door for many new intrinsically chiral contributions, but it turns out this is not the case. The reason is that any two such multiplets

[6] The kinetic operator defined in [2] acts on an anti-chiral multiplet of weight $w = 0$. It can be extended to act on any conformal primary (chiral or not) with $w = -c$ to yield a new chiral multiplet of weight $w + 2$. This is equivalent to the chiral projection operator defined in superspace [12, 13].

are actually related to each other by the kinetic operator of a weight-zero multiplet. Taking Φ'_w and Φ_w to be chiral multiplets of the same nonzero weight (for simplicity), the difference

$$\mathbb{T}(\ln \bar{\Phi}'_w) - \mathbb{T}(\ln \bar{\Phi}_w) = \mathbb{T}(\ln(\bar{\Phi}'_w/\bar{\Phi}_w)) \tag{5.5}$$

is actually the kinetic multiplet of a weight-zero multiplet. This permits, for example, manipulations like

$$\int \mathrm{d}^4x\,\mathrm{d}^4\theta\,\mathcal{E}\;\Phi'\;\mathbb{T}(\ln \bar{\Phi}'_w) = \int \mathrm{d}^4x\,\mathrm{d}^4\theta\,\mathcal{E}\;\Phi'\;\mathbb{T}(\ln \bar{\Phi}_w) - 2\int \mathrm{d}^4x\,\mathrm{d}^4\theta\,\mathrm{d}^4\bar{\theta}\,E\;\Phi'\;\ln(\bar{\Phi}'_w/\bar{\Phi}_w)\;, \tag{5.6}$$

where Φ' is a $w = 0$ chiral multiplet. This allows any operators $\mathbb{T}(\ln \bar{\Phi}'_w)$ to be traded for one universal choice $\mathbb{T}(\ln \bar{\Phi}_w)$ and the rest lifted to full superspace integrals, where the non-renormalization theorem of [2] applies.

We will now establish a new non-renormalization theorem: the contribution of $\mathbb{T}(\ln \bar{\Phi}_w)$ to any chiral integral (5.1) always vanishes as does the first derivative with respect to any field or coupling constant. Using the condition $D = 0$ found in section 3, we find that *the entire kinetic multiplet* $\mathbb{T}(\ln \bar{\Phi}_w)$ *vanishes in a supersymmetric vacuum.* In other words, in a supersymmetric vacuum, we can replace

$$F(\Phi, \mathbb{T}(\ln \bar{\Phi}_w)) \quad\longrightarrow\quad F(\Phi, 0) \tag{5.7}$$

in any chiral superspace integral (5.1). We still must be careful to analyze what happens under *variations* of the fields in a supersymmetric configuration. For simplicity, we consider first the case

$$-2\int \mathrm{d}^4x\,\mathrm{d}^4\theta\,\mathcal{E}\;\Phi'\;\mathbb{T}(\ln \bar{\Phi}_w) \tag{5.8}$$

with a weight-zero chiral multiplet Φ' whose component action was constructed in [3]. (An overall factor of -2 is necessary to match the component action normalization of [3].) In principle, there are three ways in which this quantity could be varied: we may vary either of the two multiplets Φ' and $\bar{\Phi}_w$ explicit in the expression, or we may vary the supergravity fields which are implicit. Variations of Φ' clearly give zero since $\mathbb{T}(\ln \bar{\Phi}_w)$ vanishes in the supersymmetric background. Variations of $\bar{\Phi}_w$ within the kinetic multiplet also give zero. This can be seen by parametrizing the variation as $\delta\bar{\Phi}_w = \bar{\Phi}_w\bar{\Lambda}$ where $\bar{\Lambda}$ is a $w = 0$ anti-chiral multiplet. This leads to $\mathbb{T}(\delta \ln \bar{\Phi}_w) = \mathbb{T}(\bar{\Lambda})$ and so we can write

$$\delta_{\Phi_w}\int \mathrm{d}^4x\,\mathrm{d}^4\theta\,\mathcal{E}\;\Phi'\;\mathbb{T}(\ln \bar{\Phi}_w) = \int \mathrm{d}^4x\,\mathrm{d}^4\theta\,\mathcal{E}\;\Phi'\;\mathbb{T}(\bar{\Lambda}) = \int \mathrm{d}^4x\,\mathrm{d}^4\bar{\theta}\,\bar{\mathcal{E}}\;\bar{\mathbb{T}}(\Phi')\;\bar{\Lambda} \tag{5.9}$$

where we "integrate by parts" the kinetic operator as in [2]. Since Φ' has zero Weyl weight, its supersymmetric value is a constant and so $\bar{\mathbb{T}}(\Phi') = 0$. The last possibility is to vary the components of the Weyl multiplet itself, with Φ' fixed at its supersymmetric value. Taking the result for the component action of (5.8) given in [3] and imposing the supersymmetry conditions on the components of Φ', one finds

$$e^{-1}\mathcal{L} = w\,A'\Big(\tfrac{2}{3}\mathcal{R}^2 - 2\,\mathcal{R}^{ba}\mathcal{R}_{ab} - 6\,D^2 + 2\,R(A)^{ab}R(A)_{ab} - R(\mathcal{V})^{+ab i}{}_j\,R(\mathcal{V})^{+}_{ab}{}^{j}{}_i$$

$$+\tfrac{1}{128}T^{abij}T_{ab}{}^{kl}T^{cd}{}_{ij}T_{cdkl}+T^{acij}\mathcal{D}_a\mathcal{D}^bT_{bcij}-T^{acij}f_a{}^bT_{bcij}\Big)\ , \qquad (5.10)$$

where A' must be a constant. Note already that the terms D^2, $(R(A)_{ab})^2$ and $(R(\mathcal{V})^{+}_{ab}{}^i{}_j)^2$ are quadratic in quantities which vanish in the supersymmetric background, and so any variation of these quantities must vanish. It turns out that the same holds for the remaining terms. The Lagrangian (5.10) can be written as

$$e^{-1}\mathcal{L}=w\,A'\Big(2(Z_{ab}\eta^{ab})^2-2Z^{ba}Z_{ab}-\tfrac{1}{2}Z^1_aZ^{2a}-6\,D^2 \\ +2\,R(A)^{ab}R(A)_{ab}-R(\mathcal{V})^{+abi}{}_j\,R(\mathcal{V})^{+}_{ab}{}^j{}_i+\mathcal{D}^a\mathcal{O}_a\Big) \qquad (5.11)$$

where the three complex quantities

$$Z_{ab}=\mathcal{R}_{ab}-\tfrac{1}{6}\eta_{ab}\mathcal{R}+\tfrac{1}{8}T_{ac\,ij}T_b{}^{cij}+2w^{-1}\mathcal{D}_a\mathcal{D}_b\hat{\bar{A}}-2w^{-2}\mathcal{D}_a\hat{\bar{A}}\mathcal{D}_b\hat{\bar{A}}+w^{-2}\eta_{ab}(\mathcal{D}_c\hat{\bar{A}})^2\ ,$$
$$Z^1_a=\mathcal{D}^bT_{ba\,ij}\,\varepsilon^{ij}+w^{-1}\mathcal{D}^b\hat{\bar{A}}\,T_{ba\,ij}\,\varepsilon^{ij}\ ,$$
$$Z^2_a=\mathcal{D}^bT_{ba}{}^{ij}\,\varepsilon_{ij}+w^{-1}\mathcal{D}^b\hat{\bar{A}}\,T_{ba}{}^{ij}\,\varepsilon_{ij}\ , \qquad (5.12)$$

vanish in a supersymmetric configuration, using the supersymmetry conditions (4.5) – (4.8), along with the additional condition $D=0$ (which implies $\mathcal{D}_a\hat{A}=\mathcal{D}_a\hat{\bar{A}}$). The last term of (5.11), which involves $\mathcal{D}_a\mathcal{O}^a$ for

$$\mathcal{O}_a=T_{ac}{}^{ij}\mathcal{D}_bT^{bc}{}_{ij}+w^{-1}T_{ac\,ij}T^{bc\,ij}\,\mathcal{D}_b\hat{\bar{A}}-4w^{-1}\mathcal{R}\,\mathcal{D}_a\hat{\bar{A}}+8w^{-1}\mathcal{R}_{ba}\mathcal{D}^b\hat{\bar{A}} \\ -8w^{-2}\mathcal{D}_a\hat{\bar{A}}\mathcal{D}^2\hat{\bar{A}}+8w^{-2}\mathcal{D}^b\hat{\bar{A}}\,\mathcal{D}_b\mathcal{D}_a\hat{\bar{A}}-8w^{-3}\mathcal{D}_a\hat{\bar{A}}\,(\mathcal{D}_c\hat{\bar{A}})^2\ , \qquad (5.13)$$

gives a total derivative because A' is constant. The remaining pieces are each quadratic in terms that vanish in the supersymmetric vacuum, so their variation with respect to any of the supergravity fields must vanish.

We have now established a non-renormalization theorem for the expression (5.8). This is straightforwardly extended to the more general class of functions

$$\int \mathrm{d}^4x\,\mathrm{d}^4\theta\,\mathcal{E}\,F(\Phi^I,\mathbb{T}(\ln\bar{\Phi}_w))\ . \qquad (5.14)$$

Here the superfields Φ^I are a set of chiral superfields which may possess any weight. For instance, they may consist of vector multiplets X^I and the chiral supergravity invariant $W^{\alpha\beta}W_{\alpha\beta}$. We have already observed that in a supersymmetric vacuum $\mathbb{T}(\ln\bar{\Phi}_w)$ vanishes. In this context, the functions F should be analytic at $\mathbb{T}(\ln\bar{\Phi}_w)=0$. Therefore, we may construct a series expansion, a characteristic term of which would be

$$\int \mathrm{d}^4x\,\mathrm{d}^4\theta\,\mathcal{E}\ \Phi_{2-2n}\ \big[\mathbb{T}(\ln\bar{\Phi}_w)\big]^n\ . \qquad (5.15)$$

But any such term can always be written as (5.8) for the choice $\Phi'\propto\Phi_{2-2n}\big[\mathbb{T}(\ln\bar{\Phi}_w)\big]^{n-1}$. Since our treatment of (5.8) holds for arbitrary Φ', the non-renormalization theorem applies to this term and therefore to the broad class (5.14).

6 Dimensional reduction of the $5D$ mixed gauge-gravitational CS invariant

The kinetic multiplet $\mathbb{T}(\ln \bar{\Phi}_w)$ discussed in the preceding sections plays a natural role in extending the known classes of chiral superspace higher-derivative invariants. As alluded to in the introduction and discussed briefly in [3], evidence for the existence of a new class of higher-derivative invariants was actually seen in [5] where the dimensional reduction of the supersymmetric version of the $5D$ Chern-Simons action $\mathrm{Tr}(W \wedge R \wedge R)$ was considered. The authors of [5] identified three distinct types of terms in the dimensional reduction: one corresponded to a usual chiral superspace integral of a holomorphic prepotential $F(X, A|_{W^2})$, another was identified as a full superspace integral $\mathcal{H}(X, \bar{X})$, and a third remained a mystery. As discussed in [3], this identification was actually incorrect: the second and third invariants described in [5] are actually part of a single irreducible chiral invariant constructed from a kinetic multiplet $\mathbb{T}(\ln \bar{\Phi}_w)$. Our goal in this section is to back up this claim by keeping a much wider range of terms in the dimensional reduction and checking against the proposed $4D$ action.

The supersymmetric version of the $5D$ Chern-Simons action $\mathrm{Tr}(W \wedge R \wedge R)$, constructed originally in [4], is given in the conventions of [16] by

$$\begin{aligned}
E^{-1}\,\mathcal{L}_{\rm vww} = {}& \tfrac{1}{4} c_I Y_{ij}{}^I\, T^{AB} R_{ABk}{}^j(V)\,\varepsilon^{ki} \\
&+ c_I\sigma^I\Big[\tfrac{1}{64} R_{AB}{}^{CD}(M)\, R_{CD}{}^{AB}(M) + \tfrac{1}{96} R_{ABj}{}^i(V)\, R^{AB}{}_i{}^j(V)\Big] \\
&- \tfrac{1}{128}\mathrm{i}\, E^{-1}\,\varepsilon^{MNPQR}\, c_I W_M{}^I\, \big[R_{NP}{}^{AB}(M)\, R_{QRAB}(M) + \tfrac{1}{3} R_{NPj}{}^i(V)\, R_{QRi}{}^j(V)\big]\Big] \\
&+ \tfrac{3}{16} c_I \big(10\,\sigma^I\, T_{AB} - F_{AB}{}^I\big)\, R(M)_{CD}{}^{AB}\, T^{CD} \\
&+ c_I\sigma^I\Big[3\, T^{AB}\mathcal{D}^C\mathcal{D}_A T_{BC} - \tfrac{3}{2}\big(\mathcal{D}_A T_{BC}\big)^2 + \tfrac{3}{2}\mathcal{D}_C T_{AB}\,\mathcal{D}^A T^{CB}\Big] \\
&+ c_I\sigma^I\Big[\tfrac{8}{3} D^2 + 8\, T^2\, D - \tfrac{33}{8}(T^2)^2 + \tfrac{81}{2}(T^{AC} T_{BC})^2 + \mathcal{R}_{AB}(T^{AC} T^B{}_C - \tfrac{1}{2}\eta^{AB} T^2)\Big] \\
&+ \tfrac{3}{4}\mathrm{i}\,\varepsilon^{ABCDE}\Big[c_I F_{AB}{}^I\big(T_{CF}\mathcal{D}^F T_{DE} + \tfrac{3}{2} T_{CF}\mathcal{D}_D T_E{}^F\big) - 3\, c_I\sigma^I T_{AB} T_{CD}\,\mathcal{D}^F T_{FE}\Big] \\
&- c_I F_{AB}{}^I\Big[T^{AB}\, D + \tfrac{3}{8} T^{AB}\, T^2 - \tfrac{9}{2}\, T^{AC} T_{CD} T^{DB}\Big]\ ,
\end{aligned} \tag{6.1}$$

with $E = \det(E_M{}^A)$, the determinant of the $5D$ vielbein. The fields σ^I, $W_M{}^I$, and $Y_{ij}{}^I$ are the bosonic components of a $5D$ vector multiplet, with field strength $F_{MN}{}^I = 2\partial_{[M} W_{N]}{}^I$. The index I enumerates a number of such multiplets. The fields T_{AB} and D are the covariant bosonic fields of the $5D$ Weyl multiplet. The $5D$ Lorentz and SU(2) curvature tensors are given respectively by $R(M)_{MN}{}^{AB}$ and $R(V)_{MNi}{}^j$.

We will show that the full $4D$ invariant that matches the reduction of (6.1) is given by

$$S_{\rm vww} = \frac{\mathrm{i}}{64}\int \mathrm{d}^4x\, \mathrm{d}^4\theta\, \mathcal{E}\ c_I\, \frac{X^I}{X^0}\Big(W^{\alpha\beta} W_{\alpha\beta} - \tfrac{1}{3}\mathbb{T}(\ln \bar{X}^0)\Big) + \text{h.c.} \tag{6.2}$$

This corresponds to a chiral superspace action where the holomorphic function F is, in the usual normalization convention, given by

$$F = -\frac{1}{64}\,\frac{c_I X^I}{X^0}\Big(\tfrac{1}{32}(T_{ab}{}^{ij}\varepsilon_{ij})^2 - \tfrac{1}{3} A|_{\mathbb{T}(\ln \bar{X}^0)}\Big)\ . \tag{6.3}$$

This expression involves three types of fields: the "matter" vector multiplets X^I, the Kaluza-Klein vector multiplet X^0, and the $4D$ Weyl multiplet superfield $W_{\alpha\beta}$ whose lowest component is $T_{ab}{}^{ij}\varepsilon_{ij}$. The expression within parentheses in (6.2) is composed of two chiral invariants. The first involves the square of the Weyl multiplet, and the second involves the kinetic multiplet $\mathbb{T}(\ln \bar{X}^0)$.

Before proceeding to details of the actual computation, some elucidating comments are necessary about how to organize the Lagrangian. While (6.1) is fairly complicated, we draw attention to one important feature: every term is linear in a component of the $5D$ vector multiplet. Upon dimensional reduction we must retain this feature, so the $4D$ Lagrangian should take the form

$$e^{-1}\mathcal{L}|_{\rm 4D} = -\tfrac{1}{2}c_I Y^{ij\,I}\,L_{ij} - \tfrac{1}{2}{\rm i}\,c_I F_{\mu\nu}{}^I\,\tilde{E}^{\mu\nu} + c_I X^I\,G + c_I \bar{X}^I\,\bar{G} \tag{6.4}$$

for some composite functions L_{ij}, $\tilde{E}_{\mu\nu} \equiv \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}E^{\rho\sigma}$, G and $\bar{G}$. It is natural to write the coefficient of $F_{\mu\nu}{}^I$ as the dual of a two-form $E_{\mu\nu}$ since the Bianchi identity on $F_{\mu\nu}{}^I$ implies that $E_{\mu\nu}$ can be defined only up to a gauge transformation, $E_{\mu\nu} \to E_{\mu\nu} + 2\partial_{[\mu}\Lambda_{\nu]}$.

We have chosen the normalizations of the composite functions in (6.4) in a very particular way. Supersymmetry dictates that the functions L_{ij}, $E_{\mu\nu}$, G, and $\bar{G}$, must correspond to the bosonic components of a (composite) tensor multiplet. This has some deep implications when one compares two expressions of the form (6.4), such as those we plan to derive from (6.1) and (6.2). In particular, to show full equivalence between them, we must only prove that the two expressions for L_{ij} are the same: as these are the lowest components of some (composite) tensor multiplet, the equality of the remaining pieces follows by supersymmetry.

Unfortunately, we cannot fully exploit this observation. A strict proof along these lines requires that the fermionic bilinears of L_{ij} be compared as well, and in the calculation of the Lagrangian (6.1) these would need to be restored. We will instead demonstrate a proof of equivalence between all bosonic terms of L_{ij}, as well as some characteristic bosonic terms of $E_{\mu\nu}$ and G. This establishes beyond any doubt the equivalence between (6.2) and the reduction of (6.1).

We begin by reviewing some key results of the off-shell dimensional reduction formulated in [5]. In order to avoid confusion between $4D$ and $5D$ fields, we henceforth will place a diacritic on all $5D$ quantities (e.g. $E_M{}^A \to \breve{E}_M{}^A$). All bosonic components of the $5D$ Weyl multiplet, ($\breve{E}_M{}^A$, $\breve{b}_M$, $\breve{V}_{Mi}{}^j$, $\breve{T}_{AB}$, and $\breve{D}$), must reduce to expressions involving the $4D$ Weyl multiplet and a Kaluza-Klein vector multiplet X^0. Below we provide a dictionary relating the $5D$ and $4D$ components. To avoid potential confusion the index 5 will refer *only* to the fifth component of the tangent space index A and *never* to the fifth coordinate.

The fundamental bosonic fields of the Weyl multiplet are given by

$$\breve{E}_M{}^A = \begin{pmatrix} e_\mu{}^a & \frac{1}{2}W_\mu{}^0\,|X^0|^{-1} \\ 0 & \frac{1}{2}\,|X^0|^{-1} \end{pmatrix} \quad , \quad \breve{b}_M = \begin{pmatrix} b_m \\ 0 \end{pmatrix} \quad ,$$

$$\breve{V}_{ai}{}^j = \mathcal{V}_a{}^j{}_i \ , \quad \breve{V}_{5i}{}^j = -\frac{1}{2}\varepsilon_{ik}Y^{kj\,0}|X^0|^{-1} \ ,$$

$$\breve{T}_{ab} = -\tfrac{1}{24}{\rm i}\left(\varepsilon_{ij}T_{ab}{}^{ij}\bar{X}^0 - F_{ab}^{-0}\right)|X^0|^{-1} + {\rm h.c.}\ , \qquad \breve{T}_{a5} = \tfrac{1}{12}{\rm i}\,\mathcal{D}_a \ln(X^0/\bar{X}^0)\ ,$$

$$\breve{D} = \tfrac{1}{4}D - \tfrac{1}{16}|X^0|^{-1}(\mathcal{D}^a\mathcal{D}_a + \tfrac{1}{6}\mathcal{R})|X^0| - \tfrac{3}{512}|X^0|^{-2}F_{ab}{}^0F^{ab\,0} \\ + \tfrac{1}{64}|X^0|^{-2}Y^{ij\,0}Y_{ij}{}^0 - \tfrac{3}{8}\breve{T}^{ab}\breve{T}_{ab} - \tfrac{3}{4}\breve{T}^{a5}\breve{T}_{a5} \ . \tag{6.5}$$

Some derived quantities are also useful. The $5D$ spin connection and Riemann tensor can be found in [5], while the $5D$ SU(2) curvature tensor is given by

$$\breve{R}(V)_{ab\,i}{}^j = R(\mathcal{V})_{ab}{}^j{}_i - \tfrac{1}{4}\varepsilon_{ik}Y^{kj\,0}\,F_{ab}{}^0\,|X^0|^{-2} \ , \\ \breve{R}(V)_{a5\,i}{}^j = -\tfrac{1}{2}\varepsilon_{ik}|X^0|\,\mathcal{D}_a\Big(Y^{kj\,0}/|X^0|^2\Big) \ . \tag{6.6}$$

The decomposition of the $5D$ vector multiplet is given by

$$\breve{\sigma}^I = -\mathrm{i}\,|X^0|\,\Big(\frac{X^I}{X^0} - \frac{\bar{X}^I}{\bar{X}^0}\Big) \ , \qquad \breve{Y}^{ij\,I} = -\tfrac{1}{2}Y^{ij\,I} + \tfrac{1}{4}\Big(\frac{X^I}{X^0} + \frac{\bar{X}^I}{\bar{X}^0}\Big)Y^{ij\,0} \ , \\ \breve{W}_a{}^I = W_a{}^I \ , \qquad \breve{W}_5{}^I = -|X^0|\Big(\frac{X^I}{X^0} + \frac{\bar{X}^I}{\bar{X}^0}\Big) \ , \\ \breve{F}_{ab}{}^I = F_{ab}{}^I - \tfrac{1}{2}F_{ab}{}^0\,\Big(\frac{X^I}{X^0} + \frac{\bar{X}^I}{\bar{X}^0}\Big) \ , \qquad \breve{F}_{a5}{}^I = -|X^0|\,\mathcal{D}_a\Big(\frac{X^I}{X^0} + \frac{\bar{X}^I}{\bar{X}^0}\Big) \ . \tag{6.7}$$

It is important to note that all of these equations are invariant under the $4D$ U(1) R-symmetry group. This is because there is no U(1) factor in the $5D$ R-symmetry group; it emerges from the dimensional reduction.

Let us now analyze the first term L_{ij} of the $4D$ Lagrangian (6.4). This arises only from the first term in (6.1), which decomposes as

$$64\,L_{ij} = -\tfrac{1}{3}\varepsilon_{ik}R(\mathcal{V})^{ab\,k}{}_j\,\Big(\mathrm{i}\bar{X}^0T_{ab}{}^{mn}\varepsilon_{mn} - \mathrm{i}F_{ab}^{-0} + \text{h.c.}\Big)\,|X^0|^{-2} \\ + \tfrac{1}{12}Y_{ij}{}^0\,\Big(\mathrm{i}\,\bar{X}^0T^{ab\,kl}\varepsilon_{kl}F_{ab}^{-0} - \mathrm{i}\,(F_{ab}^{-0})^2 + \text{h.c.}\Big)\,|X^0|^{-4} \\ - \tfrac{2}{3}\mathrm{i}\,\mathcal{D}^a\ln(X^0/\bar{X}^0)\,\mathcal{D}_a(Y_{ij}{}^0/|X^0|^2) \ . \tag{6.8}$$

This expression includes all the bosonic contributions to L_{ij}. Now let us calculate the same contribution from the $4D$ superspace action (6.2). It helps to rewrite the action as

$$\frac{\mathrm{i}}{64}\int \mathrm{d}^4x\,\mathrm{d}^4\theta\,\mathcal{E}\,\frac{c_IX^I}{X^0}\Phi \ , \qquad \Phi = W^{\alpha\beta}W_{\alpha\beta} - \tfrac{1}{3}\mathbb{T}(\ln\bar{X}^0) \tag{6.9}$$

and express the component action in terms of the components of Φ. For example, the contribution to L_{ij} is given by

$$64\,L_{ij} = \frac{\mathrm{i}}{2}\frac{Y_{ij}{}^0}{(X^0)^2}\,A|_\Phi - \frac{\mathrm{i}}{2}\frac{1}{X^0}B_{ij}|_\Phi + \text{h.c.} \tag{6.10}$$

The components of Φ can then be calculated as

$$A|_\Phi = \tfrac{1}{32}(T_{ab}{}^{ij}\varepsilon_{ij})^2 - \tfrac{1}{3}A|_{\mathbb{T}(\ln\bar{X}^0)} \\ = \tfrac{1}{96}(T_{ab}{}^{ij}\varepsilon_{ij})^2 + (\bar{X}^0)^{-1}\Big(\tfrac{2}{3}\Box_{\mathrm{c}}X^0 + \tfrac{1}{12}T^{ab\,ij}\varepsilon_{ij}F_{ab}^{-0}\Big) \\ + (\bar{X}^0)^{-2}\Big(\tfrac{1}{6}(F_{ab}^{+0} - \tfrac{1}{4}X^0T_{ab\,ij}\varepsilon^{ij})^2 - \tfrac{1}{12}(Y_{ij}{}^0)^2\Big) \ ,$$

$$B_{ij}|_\Phi = \varepsilon_{ik} R(\mathcal{V})_{ab}{}^k{}_j \Big\{ \tfrac{1}{2} T^{ab\,kl}\varepsilon_{kl} + \tfrac{2}{3}(F^{+0}_{ab} - \tfrac{1}{4} X^0 T_{ab\,kl}\varepsilon^{kl})\,(\bar X^0)^{-1}\Big\}$$
$$+ \tfrac{2}{3}(\Box_{\rm c} + 3D)\bigg(\frac{Y_{ij}{}^0}{\bar X^0}\bigg) \,. \qquad (6.11)$$

A straightforward calculation leads to L_{ij} as in (6.8). As already mentioned, this nearly guarantees equivalence of the final expressions, but we will check some additional terms to marshal further evidence.

Let us now analyze the second term $E_{\mu\nu}$ of the $4D$ Lagrangian (6.4). We will check only a subset of contributions. One obvious source is terms involving $\breve F_{AB}{}^I$ whose decomposition in $4D$ tangent space indices yields $F_{ab}{}^I$. These give contributions to the $4D$ Lagrangian of the form

$$-\tfrac{1}{2}\, c_I\, F_{ab}{}^I \Big[\tfrac{3}{16} \breve R(M)_{CD}{}^{ab}\, \breve T^{CD} + \breve T^{ab}\Big(\breve D + \tfrac{3}{8}(\breve T_{CD})^2\Big) - \tfrac{9}{2}\breve T^{aC}\breve T_{CD}\breve T^{Db}\Big] |X^0|^{-1}$$
$$+ \tfrac{3}{8}\,\mathrm{i}\,\varepsilon^{abCDE} c_I F_{ab}{}^I \Big(\breve T_{DF}\breve{\mathcal{D}}^F\breve T_{DE} + \tfrac{3}{2}\breve T_{CF}\breve{\mathcal{D}}_D\breve T_E{}^F\Big)\,|X^0|^{-1} \,. \qquad (6.12)$$

We will discuss how to simplify this expression shortly. The other contributions come from the Chern-Simons term, which gives

$$-\tfrac{1}{64}\,\mathrm{i}\,\varepsilon^{abcd}\, c_I\, W_a{}^I \Big(\breve R(M)_{bc}{}^{EF}\,\breve R(M)_{d5\,EF} + \frac{1}{3}\breve R(\mathcal{V})_{bci}{}^j\,\breve R(\mathcal{V})_{d5j}{}^i\Big)\,|X^0|^{-1} \,. \qquad (6.13)$$

This can be rearranged to

$$-\tfrac{1}{64}\,\mathrm{i}\,\varepsilon^{abcd} c_I F_{ab}{}^I \Big(\tfrac{1}{8} R_{cd}{}^{ef} F_{ef}{}^0 |X^0|^2 + \tfrac{1}{128}(F_{ef}{}^0)^2 F_{cd}{}^0 + \tfrac{1}{64}F^{ef\,0}F_{ce}{}^0 F_{df}{}^0\Big)|X^0|^{-4}$$
$$+ \tfrac{1}{192}\,\mathrm{i}\,\varepsilon^{abcd} c_I F_{ab}{}^I \Big(\tfrac{1}{4}\,\varepsilon^{jk}\, R(\mathcal{V})_{cd}{}^i{}_k\, Y_{ij}{}^0\,|X^0|^2 + \tfrac{1}{32} F_{cd}{}^0 (Y_{ij}{}^0)^2\Big)\,|X^0|^{-4} \qquad (6.14)$$

up to terms involving derivatives of $|X^0|$, which from now on we will neglect to keep our expressions simpler. It will be useful to neglect other terms in (6.12). For example, expressions involving $\breve T_{a5}$ appear in nearly every term, often in multiple ways (e.g. from the $5D$ spin connection), so it will be convenient to set $\breve T_{a5}$ to zero, which amounts to discarding $\mathcal{D}_a \ln(X^0/\bar X^0)$. We will also ignore all terms involving $F_{ab}{}^0$ that also contain a factor of $T_{cd}{}^{ij}$, T_{cdij} or another $F_{cd}{}^0$. These conditions together allow us to focus on only the first line of (6.12). Proceeding, we find that the first line reduces to

$$-\tfrac{1}{2} c_I\, F_{ab}{}^I \Big[\tfrac{3}{16}\breve R(M)_{cd}{}^{ab}\,\breve T^{cd} + \breve T^{ab}\Big(\breve D + \tfrac{3}{8}(\breve T_{cd})^2\Big) - \tfrac{9}{2}\breve T^{ac}\breve T_{cd}\breve T^{db}\Big]\,|X^0|^{-1} \,. \qquad (6.15)$$

Now we combine this with (6.14) and find the coefficient of $c_I F^{ab\,I}$ to be

$$-64\,\mathrm{i}\,\tilde E_{ab} \sim \tfrac{1}{2}\mathrm{i}\,\mathcal{C}_{abcd}\, T^{cd\,ij}\varepsilon_{ij}\,(X^0)^{-1} + \tfrac{1}{3}\mathrm{i}\,\varepsilon_{ik}\, R(\mathcal{V})^{-}_{ab}{}^k{}_j\, Y^{ij\,0}\,|X^0|^{-2}$$
$$+ \tfrac{4}{3}\mathrm{i}\,(\mathcal{R}_a{}^c - \tfrac{1}{4}\delta_a{}^c\mathcal{R})\,F^{+0}_{cb}\,|X^0|^{-2} + \tfrac{1}{9}\mathrm{i}\,\mathcal{R}(F^{-0}_{ab} + \tfrac{1}{2}\bar X^0 T_{ab}{}^{ij}\varepsilon_{ij})\,|X^0|^{-2}$$
$$- \tfrac{2}{3}\mathrm{i}\,D\,(F^{-0}_{ab} - \bar X^0 T_{ab}{}^{ij}\varepsilon_{ij})\,|X^0|^{-2} - \tfrac{1}{12}\mathrm{i}\,(Y_{ij}{}^0)^2\,\big(F^{-0}_{ab} - \tfrac{1}{2}\bar X^0 T_{ab}{}^{ij}\varepsilon_{ij}\big)\,|X^0|^{-4}$$
$$- \tfrac{1}{192}\mathrm{i}\,T_{ab}{}^{ij}\varepsilon_{ij}\,(T_{cd}{}^{kl}\varepsilon_{kl})^2\,\bar X^0 (X^0)^{-2} - \tfrac{1}{64}\mathrm{i}\,T_{ab}{}^{ij}\varepsilon_{ij}\,(T_{cd\,kl}\varepsilon^{kl})^2\,(\bar X^0)^{-1} + \text{h.c.} \qquad (6.16)$$

up to the terms we neglected. Keep in mind that $\tilde{E}_{ab}$ is imaginary so the above expression is actually real. To extract the corresponding terms from the $4D$ Lagrangian (6.2), we return to (6.9), where

$$-64\,\mathrm{i}\,\tilde{E}_{ab} = -\frac{\mathrm{i}}{X^0} F^-_{ab}|_\Phi + \frac{1}{(X^0)^2}\Big(\mathrm{i}F^{-\,0}_{ab} - \tfrac14 \mathrm{i}\,\bar{X}^0 T_{ab}{}^{ij}\varepsilon_{ij} + \tfrac14 \mathrm{i}\,X^0 T_{ab\,ij}\varepsilon^{ij}\Big)\,A|_\Phi + \text{h.c.} \tag{6.17}$$

The result for $A|_\Phi$ was given in (6.11). The expression for $F^-_{ab}|_\Phi$ is

$$\begin{aligned}F^-_{ab}|_\Phi = & -\tfrac12 \mathcal{R}(M)^{cd}{}_{ab}\,T_{cd}{}^{ij}\varepsilon_{ij} - \tfrac13 \varepsilon_{ij} T_{ab}{}^{ij}\,\Box_{\mathrm{c}} \ln \bar{X}^0 + \tfrac13 R(\mathcal{V})^{-\,i}_{ab}{}_{k} Y^{jk\,0}\varepsilon_{ij}\,(\bar{X}^0)^{-1} \\ & - \tfrac{1}{24} T_{ab}{}^{ij} T_{cd\,ij}(F^{cd+0} - \tfrac14 X^0 T^{cd}{}_{kl}\varepsilon^{kl})(\bar{X}^0)^{-1} \\ & + \tfrac13(\delta_a{}^{[c}\delta_b{}^{d]} - \tfrac12 \varepsilon_{ab}{}^{cd})\Big[4D_c D^e\Big(\frac{F^{+0}_{ed} - \tfrac14 X^0 T_{ab\,ij}\varepsilon^{ij}}{\bar{X}^0}\Big) - D_c D^e T_{ed}{}^{ij}\varepsilon_{ij} \\ & \qquad + D^e \ln \bar{X}^0\, D_c T_{de}{}^{ij}\varepsilon_{ij} + D_c \ln \bar{X}^0 D^e T_{ed}{}^{ij}\varepsilon_{ij}\Big]\end{aligned} \tag{6.18}$$

A careful calculation, keeping only the terms discussed, reproduces (6.16).

Let us now analyze the last term G of the $4D$ Lagrangian (6.4). Because of the complexity of the full expression, we will only look at a small number of characteristic terms. We begin with all terms involving the $4D$ SU(2) curvature tensor, which arise only from the second and third lines of (6.1). These are

$$\begin{aligned}128\,X^0\,G \sim & -\tfrac13 \mathrm{i}\,R(\mathcal{V})^{+\,i}_{ab}{}_{j} R(\mathcal{V})^{ab+j}{}_{i} - \mathrm{i}R(\mathcal{V})^{-\,i}_{ab}{}_{j} R(\mathcal{V})^{ab-j}{}_{i} \\ & + \tfrac18 R(\mathcal{V})_{ab}{}^{j}{}_{k}\varepsilon^{ki}\,Y_{ij}{}^0\Big(\tfrac43 \mathrm{i}\,\bar{X}^0\,T^{ab\,mn}\varepsilon_{mn} + \tfrac83 \mathrm{i}F^{ab-0} + \text{h.c.}\Big)|X^0|^{-2}\end{aligned} \tag{6.19}$$

Next, we collect all terms involving the $4D$ auxiliary field D that do not involve derivatives of X^0 or $\bar{X}^0$. These arise only from $5D$ terms involving $\breve{D}$ and are given by

$$\begin{aligned}128\,X^0\,G \sim & -\tfrac{32}{3}\mathrm{i}\,D^2 + \mathrm{i}\,D\Big[\tfrac16 \frac{\bar{X}^0}{X^0}(T_{ab}{}^{ij}\varepsilon_{ij})^2 + \tfrac16\frac{X^0}{\bar{X}^0}(T_{ab\,ij}\varepsilon^{ij})^2 - \tfrac23 F^{-0}_{ab} T^{abij}\varepsilon_{ij}(X^0)^{-1} \\ & + (F^{-0}_{ab})^2|X^0|^{-2} + \tfrac13(F^{+0}_{ab})^2|X^0|^{-2} + \tfrac89\mathcal{R} - \tfrac43(Y_{ij}{}^0)^2|X^0|^{-2}\Big]\,.\end{aligned} \tag{6.20}$$

Finally, we include all expressions quadratic in the $4D$ Riemann tensor as well as the terms $(Y_{ij}{}^0)^4$ and $\mathcal{R}(Y_{ij}{}^0)^2$. These are easily deduced from the $5D$ Lagrangian because they arise only from the second and third lines as well as the term involving $\breve{D}^2$. The result is

$$128\,X^0\,G \sim -2\mathrm{i}\,\mathcal{C}^{-\,cd}_{ab}\mathcal{C}^{-\,ab}_{cd} - \tfrac23\mathrm{i}\,(\mathcal{R}_{ab})^2 + \tfrac{4}{27}\mathrm{i}\,\mathcal{R}^2 - \tfrac{1}{24}\mathrm{i}(Y_{ij}{}^0)^4|X^0|^{-4} + \tfrac{1}{18}\mathrm{i}\,\mathcal{R}\,(Y_{ij}{}^0)^2|X^0|^{-2}\,. \tag{6.21}$$

These three sets of terms, (6.19)–(6.21), constitute a useful characteristic set. They can be found within the $4D$ Lagrangian (6.9), for which G is given by

$$\begin{aligned}128\,G = & -\frac{\mathrm{i}}{X^0}\,C|_\Phi - \frac{\mathrm{i}}{2(X^0)^2} Y^{ij\,0}\,B_{ij}|_\Phi - \frac{\mathrm{i}}{4\bar{X}^0} T^{ab}{}_{ij}\varepsilon^{ij}\,F^+_{ab}|_\Phi + \frac{\mathrm{i}}{(X^0)^2}\big(F^{ab-0} - \tfrac14 \bar{X}^0 T^{ab\,ij}\varepsilon_{ij}\big)\,F^-_{ab}|_\Phi \\ & - \frac{\mathrm{i}}{(X^0)^2}\Big[2\Box_{\mathrm{c}}\bar{X}^0 + \tfrac14(F^{+\,0}_{ab} - \tfrac14\,X^0 T_{ab\,ij}\varepsilon^{ij})T^{ab}{}_{kl}\varepsilon^{kl} - \frac{1}{2\,X^0} Y_{ij}{}^0\,Y^{ij\,0}\end{aligned}$$

$$+\frac{1}{X^0}(F_{ab}^{-\,0} - \tfrac14 \bar X^0\, T_{ab}{}^{ij}\varepsilon_{ij})^2\Big]\, A|_\Phi$$
$$-\,2\mathrm{i}\,\Box_{\rm c}\Big(\frac{\bar A|_\Phi}{\bar X^0}\Big) + \frac{\mathrm{i}}{4(\bar X^0)^2}\, T^{ab}{}_{ij}\varepsilon^{ij}(F_{ab}^{+0} - \tfrac14 X^0\, T_{ab\,kl}\varepsilon^{kl})\,\bar A|_\Phi\ . \tag{6.22}$$

The expressions for all of the bosonic components of Φ have been given except for $C|_\Phi$. It is rather lengthy, so we refer to [3] where it was evaluated in detail.

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