

New higher-derivative invariants in $N = 2$ supergravity and the Gauss-Bonnet term

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ABSTRACT: A new class of $N = 2$ locally supersymmetric higher-derivative invariants is constructed based on logarithms of conformal primary chiral superfields. They characteristically involve a coupling to $\mathcal{R}_{\mu\nu}{}^2 - \frac{1}{3}\mathcal{R}^2$, which equals the non-conformal part of the Gauss-Bonnet term. Upon combining one such invariant with the known supersymmetric version of the square of the Weyl tensor, one obtains the supersymmetric extension of the Gauss-Bonnet term. The construction is carried out in the context of both conformal superspace and the superconformal multiplet calculus. The new class of supersymmetric invariants resolves two open questions. The first concerns the proper identification of the 4D supersymmetric invariants that arise from dimensional reduction of the 5D mixed gauge-gravitational Chern-Simons term. The second is why the pure Gauss-Bonnet term without supersymmetric completion has reproduced the correct result in calculations of the BPS black hole entropy in certain models.

KEYWORDS: Extended Supersymmetry, Supergravity Models, Black Holes in String Theory, Superspaces

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1 Introduction

More detailed knowledge of supersymmetric higher-derivative terms is becoming increasingly relevant. Although a substantial body of research in supersymmetric field theories, supergravity and string theory is based on supersymmetric invariants that are at most quadratic in space-time derivatives, there are many questions that require knowledge of supersymmetric invariants beyond the two-derivative level. Originally the central question concerned the issue of possible supersymmetric counterterms in the hope of establishing the ultraviolet finiteness of certain supersymmetric gauge and supergravity theories. Hence candidate counterterms were studied whenever possible, motivated by the assumption that supersymmetry must be the crucial element responsible for the finiteness. However, there are also instances where one is actually interested in finite effects corresponding to higher-derivative invariants, such as encountered when determining subleading corrections to black hole entropy.

This paper is directed to an extension of certain classes of higher-derivative invariants in $N = 2$ supergravity. From the technical point of view, such a study is facilitated by the fact that there exist formulations of $N = 2$ supergravity where supersymmetry is realized off-shell, i.e. without involving the equations of motion associated with specific Lagrangians. In that case there exist well-established methods such as superspace and component calculus that enable a systematic study. There exists a healthy variety of approaches: in this paper we will make use of conformal superspace [1] which is closely related to the superconformal multiplet calculus [2, 3] that is carried out in component form.¹ We will be using these methods in parallel. For higher-extended supersymmetry the application of methods such as these becomes problematic for the simple reason that off-shellness is not realized, up to a few notable exceptions such as the Weyl multiplet in $N = 4$ supergravity.

Some higher-derivative invariants in $N = 2$ supersymmetry and supergravity have been known for some time, such as those involving functions of the field strengths for supersymmetric gauge theories [7–11], the chiral invariant containing the square of the Weyl tensor (possibly coupled to matter chiral multiplets) [12] and invariants for tensor multiplets [13]. A full superspace integral has also been used to generate an \mathcal{R}^4 term in the context of “minimal” Poincaré supergravity [14]. More recently, a large class of higher-derivative supersymmetric invariants was constructed using the superconformal multiplet calculus, corresponding to integrals over the full $N = 2$ superspace [15].² This action involved arbitrary chiral multiplets, which could play the role of composite fields consisting of homogeneous functions of vector multiplets. This entire class had the remarkable property that the corresponding invariants and their first derivatives (with respect to the fields or to coupling constants) vanish in a fully supersymmetric background. This result ensures that these invariants do not contribute to either the entropy or the electric charge of BPS black holes. Actions of this class have also been used recently to study supergravity counterterms and the relation between off-shell and on-shell results [16]. Furthermore, in [17], higher-derivative actions were constructed in projective superspace by allowing vector multiplets and/or tensor multiplets to be contained in similar homogeneous functions of other multiplets. Because the invariants derived in [13, 15, 17] can involve several independent homogeneous functions at the same time, they cannot be classified concisely, although this forms no obstacle when considering applications.

Nevertheless, these broad classes do not exhaust the possibilities for higher-derivative invariants. A previously unknown $4D$ higher-derivative term was identified recently in [18] when applying off-shell dimensional reduction to the $5D$ mixed gauge-gravitational Chern-Simons term [19]. It turned out to involve a Ricci-squared term $\mathcal{R}^{ab}\mathcal{R}_{ab}$ multiplied by the ratio of vector multiplets. This curvature combination does not appear in the previous known invariants and is suggestive of the Gauss-Bonnet term, whose $N = 2$ extension has, remarkably, never been constructed before.

¹Other off-shell methods include the $N = 2$ harmonic [4] and projective [5, 6] superspace approaches, which make it possible to realize the most general off-shell supergravity-matter couplings.

²The action considered in [14] can be interpreted within the conformal framework of [15] as the full superspace integral of $\mathcal{H} = (T_{ab\ ij})^2 (T^{cd\ kl})^2 / (X_0 \bar{X}_0)^2$ where X_0 is a compensating vector multiplet, in the presence of an additional non-linear multiplet.

A related issue, also involving the Gauss-Bonnet term, arose several years ago in a different context: the calculation of black hole entropy from higher-derivative couplings in an effective supergravity action. It was observed in a certain model [20] that one could calculate the entropy of a BPS black hole by considering the effective action involving the product of a dilaton field with the Gauss-Bonnet term *without* supersymmetrization. This result agreed with the original calculation based on the square of the Weyl tensor, which depended critically on its full supersymmetrization [21, 22], but it remained unclear why the non-supersymmetric approach of [20] would yield the same answer and whether the outcome was indicative of some deeper result.

Both of these issues would be resolved by a full knowledge of the $N = 2$ Gauss-Bonnet invariant and the broader class of higher-derivative supersymmetric invariants to which it belongs. The goal of this paper is to present this class and to discuss whether it shares the same properties with the previously explored classes of invariants.

Let us first briefly recall some features of the Gauss-Bonnet invariant as well as other invariants quadratic in the Riemann tensor. In this introductory text we restrict ourselves to bosonic fields; the supersymmetric extension will be discussed in the subsequent sections. In four space-time dimensions there are two terms quadratic in the Riemann tensor whose space-time integral defines topological invariants: these are the Pontryagin density,

$$\mathcal{L}_P = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} \mathcal{R}_{\mu\nu}{}^{\lambda\tau} \mathcal{R}_{\rho\sigma\lambda\tau}, \tag{1.1}$$

and the Euler density,

$$e^{-1} \mathcal{L}_\chi = \frac{1}{4} \varepsilon^{\mu\nu\rho\sigma} \mathcal{R}_{\mu\nu}{}^{\lambda\tau} \mathcal{R}_{\rho\sigma}{}^{\delta\epsilon} \varepsilon_{\lambda\tau\delta\epsilon} = \mathcal{R}^{\mu\nu\rho\sigma} \mathcal{R}_{\mu\nu\rho\sigma} - 4 \mathcal{R}^{\mu\nu} \mathcal{R}_{\mu\nu} + \mathcal{R}^2. \tag{1.2}$$

The integral of the Euler density is the Gauss-Bonnet invariant. Their difference can be made more apparent by trading the Riemann tensor for the Weyl tensor, $C_{\mu\nu}{}^{\rho\sigma} = \mathcal{R}_{\mu\nu}{}^{\rho\sigma} - 2 \delta_{[\mu}^{[\rho} \mathcal{R}_{\nu]}^{\sigma]} + \frac{1}{3} \delta_{\mu}^{[\rho} \delta_{\nu]}^{\sigma]} \mathcal{R}$,

$$\mathcal{L}_P = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} C_{\mu\nu}{}^{\lambda\tau} C_{\rho\sigma\lambda\tau}, \quad e^{-1} \mathcal{L}_\chi = C^{\mu\nu\rho\sigma} C_{\mu\nu\rho\sigma} - 2 \mathcal{R}^{\mu\nu} \mathcal{R}_{\mu\nu} + \frac{2}{3} \mathcal{R}^2. \tag{1.3}$$

From the perspective of supersymmetry (1.3) is not a good basis for discussing supersymmetric extensions. Rather, it turns out that the following combinations are more natural,

$$e^{-1} \mathcal{L}_W^\pm = \frac{1}{2} C_{\mu\nu}{}^{ab} C^{\mu\nu cd} \left[\eta_{ac} \eta_{bd} \pm \frac{1}{2} \varepsilon_{abcd} \right] = C_{\mu\nu}{}^{ab\pm} C^{\mu\nu\pm}_{ab},$$

$$e^{-1} \mathcal{L}_{NL} = - \mathcal{R}^{\mu\nu} \mathcal{R}_{\mu\nu} + \frac{1}{3} \mathcal{R}^2. \tag{1.4}$$

The first expression is the square of the anti-selfdual (selfdual) Weyl tensor, which belongs to a chiral (anti-chiral) multiplet, and whose superextension has been known for a long time [12]. The supersymmetric extension of the second term will be one of the results of this paper.

In the expressions (1.4) we made use of tangent-space indices, a, b, \dots , because the metric formulation is not suitable for supersymmetric theories, which necessarily contain

fermions and therefore require vierbein fields e_μ^a . In this paper, we will employ a superconformal description in which the tangent space will be subject to Lorentz transformations (M), dilatations (D), and conformal boosts (K). This implies that we will be dealing with three tangent space connections, namely the spin connection ω_μ^{ab} , the dilatation connection b_μ , and the connection associated with conformal boosts f_μ^a . The connections ω_μ^{ab} and f_μ^a will turn out to be composite, as we will explain momentarily. As our goal will be to construct the supersymmetric extension of the second invariant in (1.4), we must first discuss how this invariant can arise in the framework of conformal gravity.

Under dilations and conformal boosts, the vierbein fields and the various connections transform as follows,

$$\begin{aligned} \delta e_\mu^a &= -\Lambda_D e_\mu^a, & \delta \omega_\mu^{ab} &= 2 \Lambda_K^{[a} e_\mu^{b]}, \\ \delta b_\mu &= \partial_\mu \Lambda_D + \Lambda_K^a e_{\mu a}, & \delta f_\mu^a &= \mathcal{D}_\mu \Lambda_K^a + \Lambda_D f_\mu^a, \end{aligned} \quad (1.5)$$

where we use Lorentz and dilatationally covariant derivatives \mathcal{D}_μ , such as in

$$\mathcal{D}_\mu \Lambda_K^a = (\partial_\mu - b_\mu) \Lambda_K^a - \omega_\mu^{ab} \Lambda_{Kb}. \quad (1.6)$$

The corresponding curvatures take the following form,

$$\begin{aligned} R(P)_{\mu\nu}^a &= 2 \mathcal{D}_{[\mu} e_{\nu]}^a, \\ R(M)_{\mu\nu}^{ab} &= 2 \partial_{[\mu} \omega_{\nu]}^{ab} - 2 \omega_{[\mu}^{ac} \omega_{\nu]c}^b - 4 f_{[\mu}^{[a} e_{\nu]}^{b]}, \\ R(D)_{\mu\nu} &= 2 \partial_{[\mu} b_{\nu]} - 2 f_{[\mu}^a e_{\nu]a}, \\ R(K)_{\mu\nu}^a &= 2 \mathcal{D}_{[\mu} f_{\nu]}^a. \end{aligned} \quad (1.7)$$

In terms of these curvatures one imposes the following conventional constraints,

$$R(P)_{\mu\nu}^a = 0, \quad R(M)_{\mu\nu}^{ab} e_b^\nu = 0. \quad (1.8)$$

Because the constraints (1.8) are invariant under Lorentz transformations, dilatations and conformal boosts, the transformation rules (1.5) remain unaffected. For the supersymmetric extension this will no longer be the case and additional terms will emerge. The Bianchi identities together with the constraints (1.8) imply the following relations,³

$$R(D)_{\mu\nu} = 0, \quad R(M)_{[\mu\nu}{}^{ab} e_{\rho]b} = 0, \quad R(K)_{\mu\nu}^a = D_b R(M)_{\mu\nu}{}^{ba}. \quad (1.9)$$

The constraints (1.8) express the spin connection field ω_μ^{ab} and the K-connection field f_μ^a in terms of e_μ^a and b_μ . The resulting expression for f_μ^a reads as follows,

$$f_\mu^a = \frac{1}{2} \mathcal{R}(e, b)_\mu^a - \frac{1}{12} e_\mu^a \mathcal{R}(e, b), \quad f_\mu^\mu \equiv f = \frac{1}{6} \mathcal{R}(e, b), \quad (1.10)$$

³Here and below we take \mathcal{D}_μ and D_μ to contain also the affine connection $\Gamma_{\mu\nu}^\rho = e_a^\rho \mathcal{D}_\mu e_\nu^a$ when acting on quantities with world indices. D_μ is the conformally covariant derivative and contains the connection f_μ^a in addition to the spin and dilatation connections. In later sections, we will use the same symbol for the supercovariant derivative.

where $\mathcal{R}(e, b)_{\mu\nu}{}^{ab}$ denotes the curvature associated with the spin connection,

$$\mathcal{R}(e, b)_{\mu\nu}{}^{ab} = 2 \partial_{[\mu} \omega(e, b)_{\nu]}{}^{ab} - 2 \omega(e, b)_{[\mu}{}^{ac} \omega(e, b)_{\nu]c}{}^b. \quad (1.11)$$

Note that it is possible to impose the gauge $b_\mu = 0$, so that only the vierbein remains as an independent field. The spin connection is then the standard torsion-free connection, the curvature $\mathcal{R}(e, b)_{\mu\nu}{}^{ab}$ corresponds to the standard Riemann tensor with a symmetric Ricci tensor, while $f_\mu{}^a e_{\nu a}$ is symmetric. However, it is advantageous to not impose such a gauge at this stage. The curvature (1.11) satisfies the Bianchi identity $\mathcal{D}_{[\mu} \mathcal{R}(e, b)_{\nu\rho]}{}^{ab} = 0$. From this identity it follows that $\mathcal{D}_a [2 \mathcal{R}(e, b)_{\mu}{}^a - e_\mu{}^a \mathcal{R}(e, b)] = 0$, where the Ricci tensor is not symmetric. This equation is equivalent to

$$\mathcal{D}_a [f_\mu{}^a - f e_\mu{}^a] = 0. \quad (1.12)$$

To exhibit some salient features of the above formalism and to give an early demonstration of the strategy we intend to follow in this paper, let us consider a scalar field ϕ transforming under dilatations as

$$\delta_D \phi = w \Lambda_D \phi, \quad (1.13)$$

where the constant w is known as the Weyl weight. We stress that ϕ does not have to be an elementary field; it could also be a composite field, as long as it transforms in the prescribed way under dilatations. It is now straightforward (but more and more tedious) to determine explicit expressions for multiple conformally covariant derivatives of ϕ and their transformation behaviour under K-transformations (cf. appendix B of [15]),

$$\begin{aligned} D_\mu \phi &= \mathcal{D}_\mu \phi = \partial_\mu \phi - w b_\mu \phi, \\ D_\mu D_a \phi &= \mathcal{D}_\mu D_a \phi + w f_{\mu a} \phi, \\ D_\mu \square_c \phi &= \mathcal{D}_\mu \square_c \phi + 2(w-1) f_\mu{}^a D_a \phi, \\ \square_c \square_c \phi &= \mathcal{D}_a D^a \square_c \phi + (w+2) f \square_c \phi + 2(w-1) f_{\mu a} D^\mu D^a \phi, \end{aligned} \quad (1.14)$$

whose variations under K-transformations read,

$$\begin{aligned} \delta_K D_a \phi &= -w \Lambda_{K a} \phi, \\ \delta_K D_\mu D_a \phi &= -(w+1) [\Lambda_{K \mu} D_a + \Lambda_{K a} D_\mu] \phi + e_{\mu a} \Lambda_K{}^b D_b \phi, \\ \delta_K \square_c \phi &= -2(w-1) \Lambda_K{}^a D_a \phi, \\ \delta_K D_\mu \square_c \phi &= -(w+2) \Lambda_{K \mu} \square_c \phi - 2(w-1) \Lambda_K{}^a D_\mu D_a \phi, \\ \delta_K \square_c \square_c \phi &= -2(w-1) \Lambda_K{}^a \square_c D_a \phi - 2(w+1) \Lambda_K{}^a D_a \square_c \phi. \end{aligned} \quad (1.15)$$

It turns out that, for specific Weyl weights, $\square_c \phi$ and $\square_c \square_c \phi$ are K-invariant,

$$\begin{aligned} \delta_K \square_c \phi &= 0, & (\text{for } w = 1), \\ \delta_K \square_c \square_c \phi &= 2 \Lambda_K{}^a (\square_c D_a - D_a \square_c) \phi = 0, & (\text{for } w = 0), \end{aligned} \quad (1.16)$$

where, to prove the last part of the second equation, we rewrote $\square_c D_a \phi - D_a \square_c \phi = D^b [D_b, D_a] \phi + [D_b, D_a] D^b \phi$ and made use of the Ricci identity and the curvature constraints. From (1.16) one derives two conformally invariant Lagrangians by multiplying with a similar scalar field ϕ' of the same Weyl weight as ϕ ,

$$\begin{aligned} e^{-1} \mathcal{L} \propto \phi' \square_c \phi &= -\mathcal{D}^\mu \phi' \mathcal{D}_\mu \phi + f \phi' \phi, & (\text{for } w = 1) \\ e^{-1} \mathcal{L} \propto \phi' \square_c \square_c \phi &= \mathcal{D}^2 \phi' \mathcal{D}^2 \phi + 2 \mathcal{D}^\mu \phi' [2 f_{(\mu}{}^a e_{\nu)a} - f g_{\mu\nu}] \mathcal{D}^\nu \phi, & (\text{for } w = 0) \end{aligned} \quad (1.17)$$

up to total derivatives. Note that we have made use here of (1.12). Both the above expressions are symmetric in ϕ and ϕ' .

Let us comment on the two Lagrangians (1.17). In both Lagrangians the dependence on b_μ will cancel as a result of the invariance under conformal boosts. In the first Lagrangian one may then adjust the product $\phi' \phi$ to a constant by means of a local dilatation. In that case the second term of the Lagrangian is just proportional to the Ricci scalar, so that one obtains the Einstein-Hilbert term. The kinetic term for the scalars depends on the choice made for ϕ' and ϕ . For instance, when the two fields are the same, then ϕ equals a constant; when they are not the same (elementary or composite) fields, the kinetic term can be exclusively written in terms of ϕ and will be proportional to $\phi^{-2} (\partial_\mu \phi)^2$. In that case the first Lagrangian describes an elementary or a composite scalar field coupled to Einstein gravity.

The situation regarding the second Lagrangian is fundamentally different, because one cannot adjust the scalar fields to any particular value by local dilatations in view of the vanishing Weyl weight. The scalar fields may be equal to constants (in which case the Lagrangian vanishes) or to homogeneous functions of other fields such that the combined Weyl weight remains zero, without affecting the invariance under local dilatations. We should also mention that the operator $\square_c \square_c$ appearing in this Lagrangian, when acting on a scalar field with $w = 0$, is the same operator Δ_0 given in [23, 24] and has an interesting history in its own right.⁴ It is, of course, possible to construct invariants which also involve the Weyl tensor. For instance, any scalar field of zero Weyl weight times the square of the Weyl tensor will define a conformally invariant Lagrangian. But how to include invariants such as the four-dimensional Gauss-Bonnet term is less obvious. As it turns out, the crucial assumption made in the examples above is that the scalar fields transform linearly under dilatations. To demonstrate how the situation changes when this is not the case, let us repeat the previous construction for $\ln \phi$, which transforms inhomogeneously under dilatations, $\delta_D \ln \phi = w \Lambda_D$. In the same way as above, we derive the following definitions,

$$\begin{aligned} D_\mu \ln \phi &= \mathcal{D}_\mu \ln \phi = \partial_\mu \ln \phi - w b_\mu, \\ D_\mu D_a \ln \phi &= \mathcal{D}_\mu D_a \ln \phi + w f_{\mu a}, \\ D_\mu \square_c \ln \phi &= \mathcal{D}_\mu \square_c \ln \phi - 2 f_\mu{}^a D_a \ln \phi, \\ \square_c \square_c \ln \phi &= \mathcal{D}_a D^a \square_c \ln \phi + 2 f \square_c \ln \phi - 2 f_{\mu a} D^\mu D^a \ln \phi. \end{aligned} \quad (1.18)$$

⁴This operator was discovered by Fradkin and Tseytlin in 1981 [23, 24] and re-discovered by Paneitz in 1983 [25]. In the mathematics literature, it is known as the Paneitz operator. The same operator along with the second Lagrangian in (1.17) was used by Riegert [26] for the purpose of integrating the conformal anomaly. There is a unique generalization to higher dimensions, see e.g. [27] and references therein.

The equations above show an interesting systematics, namely that, after applying a certain number of covariant derivatives on $\ln \phi$, these expressions take the same form as in (1.15) with $w = 0$. However, it is important to realize that the details implicit in the multiple covariant derivatives will still depend on the characteristic features associated with the logarithm. The same observation can be made for the K-transformations of multiple derivatives which also transform as if one were dealing with a $w = 0$ scalar field,

$$\begin{aligned}
\delta_K D_a \ln \phi &= -w \Lambda_{Ka}, & \delta_D D_a \ln \phi &= \Lambda_D D_a \ln \phi, \\
\delta_K D_\mu D_a \ln \phi &= -[\Lambda_{K\mu} D_a + \Lambda_{Ka} D_\mu] \ln \phi + e_{\mu a} \Lambda_K^b D_b \ln \phi, \\
\delta_K \square_c \ln \phi &= 2 \Lambda_K^a D_a \ln \phi, \\
\delta_K D_\mu \square_c \ln \phi &= -2 \Lambda_{K\mu} \square_c \ln \phi + 2 \Lambda_K^a D_\mu D_a \ln \phi, \\
\delta_K \square_c \square_c \ln \phi &= 2 \Lambda_K^a \square_c D_a \ln \phi - 2 \Lambda_K^a D_a \square_c \ln \phi = 0.
\end{aligned} \tag{1.19}$$

In four space-time dimensions the only conformally invariant Lagrangian based on the above expression must be equal to $\square_c \square_c \ln \phi$, possibly multiplied with a scalar field of zero Weyl weight. This constitutes the non-linear version of the second Lagrangian in (1.17), namely $\sqrt{g} \phi' \square_c \square_c \ln \phi$, where ϕ has a non-vanishing, but arbitrary Weyl weight w and ϕ' has zero Weyl weight. Taking the explicit form of $\square_c \square_c \ln \phi$ this Lagrangian is given by

$$\begin{aligned}
\sqrt{g} \phi' \square_c \square_c \ln \phi &= \sqrt{g} \phi' \left\{ (\mathcal{D}^2)^2 \ln \phi - 2 \mathcal{D}^\mu [(2 f_{(\mu}{}^a e_{\nu)a} - f g_{\mu\nu}) \mathcal{D}^\nu \ln \phi] \right. \\
&\quad \left. + w [\mathcal{D}^2 f + 2 f^2 - 2 (f_\mu{}^a)^2] \right\}.
\end{aligned} \tag{1.20}$$

There are two features to note about this Lagrangian. The first is that its dependence on $\ln \phi$ is isolated in the first line on the right-hand side, which is a total derivative when ϕ' is constant. In other words, the action is independent of the choice of $\ln \phi$ when ϕ' is constant. The second feature is that the Lagrangian is K-invariant, so all the b_μ terms must drop out. Equivalently, one can adopt a K-gauge where $b_\mu = 0$. Using (1.10), one finds

$$\mathcal{D}^2 f + 2 f^2 - 2 (f_\mu{}^a)^2 = \frac{1}{6} \mathcal{D}^2 \mathcal{R} - \frac{1}{2} \mathcal{R}^{ab} \mathcal{R}_{ab} + \frac{1}{6} \mathcal{R}^2, \tag{1.21}$$

which is proportional to \mathcal{L}_{NL} (cf. 1.4) up to a total covariant derivative. When combined with the square of the Weyl tensor with an appropriate relative normalization one obtains the Gauss-Bonnet invariant up to a total covariant derivative

$$\begin{aligned}
e^{-1} \mathcal{L}_\chi &= C^{abcd} C_{abcd} + 4 w^{-1} \square_c \square_c \ln \phi \\
&= C^{abcd} C_{abcd} - 2 \mathcal{R}^{ab} \mathcal{R}_{ab} + \frac{2}{3} \mathcal{R}^2 + \frac{2}{3} \mathcal{D}^2 \mathcal{R} \\
&\quad + 4 w^{-1} \left\{ (\mathcal{D}^2)^2 \ln \phi + \mathcal{D}^a \left(\frac{2}{3} \mathcal{R} D_a \ln \phi - 2 \mathcal{R}_{ab} D^b \ln \phi \right) \right\},
\end{aligned} \tag{1.22}$$

where we have taken the gauge $b_\mu = 0$ in the second equality. Discarding the (explicit) total derivatives, this result reduces to the Euler density. Alternatively the dilatation gauge $\phi = 1$ reduces it to

$$e^{-1} \mathcal{L}_\chi = C^{abcd} C_{abcd} - 2 \mathcal{R}^{ab} \mathcal{R}_{ab} + \frac{2}{3} \mathcal{R}^2 + \frac{2}{3} \mathcal{D}^2 \mathcal{R}. \tag{1.23}$$

This differs from the usual Euler density (1.2) by an explicit total derivative. Obviously additional invariants are obtained by multiplying this result with a $w = 0$ independent (composite or elementary) scalar field ϕ' .

The above relatively simple bosonic Lagrangians indicate how higher-derivative couplings will be characterized in this paper. As we shall argue in the next section, all these Lagrangians have an $N = 2$ supersymmetric counterpart based on *chiral superfields*. These include the well-known Lagrangians quadratic in derivatives, the class of higher-derivative Lagrangians discussed in [15], and a new class of Lagrangians based on $\sqrt{g} \phi' \square_c \square_c \ln \phi$, where ϕ' and ϕ are the lowest components of chiral multiplets with $w' = 0$ and $w \neq 0$. This last class must contain the $N = 2$ supersymmetric higher-derivative invariant that was found upon reducing the $5D$ higher-derivative invariant coupling to four dimensions [18]. The main purpose of this paper is to study this new class of invariants.

This paper is organized as follows. In section 2 we explain how to extend the present results to $N = 2$ supersymmetry by assigning the various fields to chiral multiplets. This discussion will be at the level of flat superspace. We introduce the so-called kinetic multiplet, which supersymmetrizes $\square_c \square_c \phi$, and its non-linear version, corresponding to $\square_c \square_c \ln \phi$. In the subsequent section 3 we extend these results to curved superspace. Then, in section 4, we exhibit the component structure of the kinetic multiplet, both in the linear and in the non-linear case. Explicit results are given for a new class of higher-derivative supersymmetric invariants based on the supersymmetrization of $\square_c \square_c \ln \phi$. The result here is the direct extension of the result presented in [15] and it can be used for similar purposes. One application that is typical for this class concerns the supersymmetric Gauss-Bonnet term. Therefore section 5 deals with a number of characteristic features of this term. Conclusions and implications of our results are discussed in section 6. A number of appendices has been included with additional material.

2 The extension to chiral superfields in flat $N=2$ superspace

In the introduction we noted the existence of four different types of conformally invariant Lagrangians and we pointed out that those can rather easily be embedded into $N = 2$ supersymmetric invariants on the basis of chiral superfields. Just as conformal transformations are an invariance in flat space-time, defined by a constant vierbein and vanishing connections $\omega_\mu^{ab}, b_\mu, f_\mu^a$, superconformal transformations leave a flat *superspace* invariant. Furthermore, almost every statement we will make about flat superspace can transparently be lifted to curved superspace although the required calculations are considerably more involved. Therefore we will first discuss flat superspace in this section. Since chiral multiplets are intrinsically complex, the superfields and corresponding invariants involving them are complex as well. We subsequently describe the systematics of these superfields, discuss the notion of an $N = 2$ superconformal *kinetic multiplet*, and present the four types of invariants. In the next section 3 we will extend this analysis to curved superspace.

Superfields can be defined as functions of the flat superspace coordinates $z^A = (x^a, \theta^{\alpha i}, \bar{\theta}_{\dot{\alpha} i})$. Here our notation will reflect the fact that in flat superspace world and

tangent-space indices can be identified. The tangent space derivatives are

$$\partial_a = \frac{\partial}{\partial x^a}, \quad D_{\alpha i} = \frac{\partial}{\partial \theta^{\alpha i}} + i(\sigma^a)_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}}_i \frac{\partial}{\partial x^a}, \quad \bar{D}^{\dot{\alpha} i} = \frac{\partial}{\partial \bar{\theta}^{\dot{\alpha} i}} + i(\bar{\sigma}^a)^{\dot{\alpha}\alpha} \theta_\alpha^i \frac{\partial}{\partial x^a}. \quad (2.1)$$

In the context of curved superspace, we will be employing a vector tangent-space derivative ∇_a and spinor tangent-space derivatives $\nabla_{\alpha i}$ and $\bar{\nabla}^{\dot{\alpha} i}$ which are the direct extension of the derivatives in (2.1). We remind the reader that we use two-component spinor notation in the context of superspace where spinor indices are raised and lowered with the antisymmetric epsilon tensor (see appendix A).

Chiral superfields satisfy the differential superspace constraint $\bar{D}^{\dot{\alpha} i} \Phi = 0$. We will denote the components of a general chiral multiplet Φ by [28, 29],

$$\begin{aligned} A &:= \Phi|_{\theta=0}, & \Psi_{\alpha i} &:= D_{\alpha i} \Phi|_{\theta=0}, & B_{ij} &:= -\frac{1}{2} D_{ij} \Phi|_{\theta=0}, \\ F_{ab}^- &:= -\frac{1}{4} (\sigma_{ab})_{\alpha\beta} D_\beta^\alpha \Phi|_{\theta=0}, & \Lambda_{\alpha i} &:= \frac{1}{6} \varepsilon^{jk} D_{\alpha k} D_{ji} \Phi|_{\theta=0}, & C &:= -2D^4 \Phi|_{\theta=0}, \end{aligned} \quad (2.2)$$

where

$$D_{ij} := -D_{\alpha(i} D^{\alpha}_{j)}, \quad D_{\alpha\beta} := -\varepsilon^{ij} D_{(\alpha i} D_{\beta)j}. \quad (2.3)$$

Hence a chiral multiplet comprises a 16 + 16 bosonic and fermionic components, consisting of a complex scalar A , a chiral spinor doublet Ψ_i , a complex symmetric scalar B_{ij} , an anti-selfdual tensor F_{ab}^- , a chiral spinor doublet Λ_i , and a complex scalar C .

Under dilatations and chiral U(1) transformations (with constant parameters Λ_D and Λ_A in flat superspace) the superspace coordinates change according to

$$x' = \exp[-\Lambda_D] x, \quad \theta' = \exp\left[-\frac{1}{2}(\Lambda_D + i\Lambda_A)\right] \theta, \quad \bar{\theta}' = \exp\left[-\frac{1}{2}(\Lambda_D - i\Lambda_A)\right] \bar{\theta}, \quad (2.4)$$

and superfields $\Psi(x, \theta, \bar{\theta})$ are usually assigned to transform as

$$\Psi'(x', \theta', \bar{\theta}') = \exp[w \Lambda_D + ic \Lambda_A] \Psi(x, \theta, \bar{\theta}), \quad (2.5)$$

where w and c are called the Weyl and the chiral weight. For chiral multiplets these weights are related by $c = -w$. In that case the Weyl weight of A equals w and the highest- θ component C has weight $w + 2$. All the components scale homogeneously and since there are no chiral superfield components with Weyl weight less than w it follows that A must be invariant under S-supersymmetry. This implies that it is also invariant under K transformations. Such a chiral superfield is called a *conformal primary* field. All these properties can be derived systematically on the basis of the superconformal algebra using the chiral constraint.

Just as in $N = 1$ superspace one can integrate the product $\Phi' \bar{\Phi}$ of a chiral and an anti-chiral superfield, respectively, to obtain an expression involving *four* space-time derivatives (discarding total derivatives in the equalities),

$$\int d^4\theta d^4\bar{\theta} \Phi' \bar{\Phi} = \int d^4\theta \Phi' (\bar{D}^4 \bar{\Phi}) = A' \square \square \bar{A} + \dots, \quad (2.6)$$

where $\bar{D}^4 = \frac{1}{48}\varepsilon_{ik}\varepsilon_{jl}\bar{D}^{ij}\bar{D}^{kl}$ and A and A' are the lowest- θ components of Φ and Φ' , respectively. Obviously this class of Lagrangians defines a superconformal extension of the second Lagrangian in (1.17). In order for the action to be superconformally invariant, the chiral superfields Φ and Φ' must both have vanishing Weyl weights, implying that A and A' are scale invariant.

The intermediate equality in (2.6) involves the so-called $N = 2$ kinetic multiplet $\mathbb{T}(\bar{\Phi})$ [29], conventionally normalized as $\mathbb{T}(\bar{\Phi}) := -2\bar{D}^4\bar{\Phi}$. When Φ has zero Weyl weight the highest- θ component of the chiral superfield Φ , denoted by C , is S-supersymmetric. Since \bar{C} equals the lowest- θ component of $\mathbb{T}(\bar{\Phi})$, the kinetic multiplet is therefore a conformal primary chiral superfield. The kinetic multiplet itself thus has Weyl weight $w = 2$.⁵ Its flat-space components are

$$\begin{aligned} A|_{\mathbb{T}(\bar{\Phi})} &= \bar{C}, & \Psi_i|_{\mathbb{T}(\bar{\Phi})} &= -2\varepsilon_{ij}\not{\partial}\Lambda^j, \\ B_{ij}|_{\mathbb{T}(\bar{\Phi})} &= -2\varepsilon_{ik}\varepsilon_{jl}\square B^{kl}, & F_{ab}^-|_{\mathbb{T}(\bar{\Phi})} &= -4\left(\delta_a^{[c}\delta_b^{d]} - \frac{1}{2}\varepsilon_{ab}^{cd}\right)\partial_c\partial^e F_{ed}^+, \\ \Lambda_i|_{\mathbb{T}(\bar{\Phi})} &= 2\square\not{\partial}\Psi^j\varepsilon_{ij}, & C|_{\mathbb{T}(\bar{\Phi})} &= 4\square\square\bar{A}. \end{aligned} \tag{2.7}$$

They transform as a chiral multiplet, while depending on the components of the anti-chiral multiplet $\bar{\Phi}$.

An obvious question concerns the derivation of the supersymmetric extension of the first Lagrangian in (1.17), which is only quadratic in space-time derivatives. As it turns out this Lagrangian is associated with a *reduced* chiral superfield \mathcal{X} . Besides the chiral constraint, reduced superfields obey the additional constraint $D_{ij}\mathcal{X} = \varepsilon_{ik}\varepsilon_{jl}\bar{D}^{kl}\bar{\mathcal{X}}$ that halves the number of independent field components by expressing the higher- θ components in terms of space-time derivatives of the lower- θ components. The independent components of the reduced chiral superfield are a complex scalar X , a chiral spinor doublet $\Omega_{\alpha i}$, an anti-selfdual tensor F_{ab}^- and a triplet of auxiliary fields Y_{ij} , conventionally normalized as

$$X := A|_{\mathcal{X}}, \quad \Omega_{\alpha i} := \Psi_{\alpha i}|_{\mathcal{X}}, \quad F_{ab}^- := F_{ab}^-|_{\mathcal{X}}, \quad Y_{ij} := B_{ij}|_{\mathcal{X}}, \tag{2.8}$$

using the definitions for the components of a chiral multiplet. The reducibility constraint on the superfield \mathcal{X} requires that Y_{ij} is real, $(Y_{ij})^* = Y^{ij} = \varepsilon^{ik}\varepsilon^{jl}Y_{kl}$, whereas the tensor obeys a Bianchi identity implying that $F_{ab} = F_{ab}^- + F_{ab}^+$ equals $F_{ab} = 2e_a^\mu e_b^\nu \partial_{[\mu}W_{\nu]}$ where W_μ is a vector gauge field. Therefore this multiplet is known as the *vector multiplet*. It comprises 8 + 8 bosonic and fermionic components. In view of the reducibility constraint the vector multiplet carries Weyl weight $w = 1$. It is now straightforward to verify that

$$\frac{1}{2}\int d^4\theta \mathcal{X}^2 = X\square\bar{X} + \dots, \tag{2.9}$$

where the D'Alembertian arises from the fact that the chiral superfield is reduced. This example demonstrates how supersymmetric versions of actions such as the first one in (1.17) arise in the context of $N = 2$ chiral superspace.

⁵Some of these properties will be more obvious once we present the general transformation rules under Q- and S-supersymmetry for a generic chiral multiplet of arbitrary Weyl weight. Those will be given in (4.1) for a general curved superspace.

Incorporating the third type of Lagrangian (1.20) in the context of chiral multiplets seems rather obvious. Taking $\bar{\Phi}$ to be an anti-chiral multiplet of weight w , we consider the chiral integral

$$\int d^4\theta \Phi' (\bar{D}^4 \ln \bar{\Phi}) = A' \square \square \ln \bar{A} + \dots, \quad (2.10)$$

where Φ' is a $w = 0$ chiral superfield and A' denotes its lowest component. Naively, this resembles the previous action (2.6), but there is a crucial difference: the anti-chiral multiplet $\bar{\Phi}$ has arbitrary Weyl weight w and so $\ln \bar{\Phi}$ transforms non-linearly under dilatations. Remarkably, the corresponding kinetic multiplet $\mathbb{T}(\ln \bar{\Phi}) := -2 \bar{D}^4 \ln \bar{\Phi}$ is nevertheless a conformal primary chiral multiplet in flat superspace.⁶ In other words, it transforms linearly under dilatations with $w = 2$ and its lowest component is invariant under S-supersymmetry.

We should stress that the non-linearities in $\mathbb{T}(\ln \bar{\Phi})$ are of two different types. First of all, the logarithm leads to an anti-chiral superfield that will depend non-linearly on the components of $\bar{\Phi}$. Because of this behaviour, the superconformal transformations will also be realized in a non-linear fashion, and as a result the covariantizations that are required in curved superspace will involve non-linearities depending on the Weyl weight w . In spite of all these complications, there is a rather systematic way of writing the various components of $\mathbb{T}(\ln \bar{\Phi})$, although the various explicit expressions tend to become rather complicated, especially because they involve higher space-time derivatives. These non-linearities are the reason why the kinetic multiplet $\mathbb{T}(\ln \bar{\Phi})$ differs in a crucial way from the original one $\mathbb{T}(\bar{\Phi})$.

As a first step in constructing the components of $\mathbb{T}(\ln \bar{\Phi})$, we must replace the components of $\bar{\Phi}$ in (2.7) with those of $\ln \bar{\Phi}$. This will simply involve replacing $\bar{A} \rightarrow \bar{A}|_{\ln \bar{\Phi}}, \dots, \bar{C} \rightarrow \bar{C}|_{\ln \bar{\Phi}}$, where the components of the multiplet $\ln \bar{\Phi}$ are identified as

$$\begin{aligned} A|_{\ln \bar{\Phi}} &= \ln A, & \Psi_i|_{\ln \bar{\Phi}} &= \frac{\Psi_i}{A}, \\ B_{ij}|_{\ln \bar{\Phi}} &= \frac{B_{ij}}{A} + \frac{1}{2A^2} \bar{\Psi}_{(i} \Psi_{j)}, & F_{ab}^-|_{\ln \bar{\Phi}} &= \frac{F_{ab}^-}{A} + \frac{1}{8A^2} \varepsilon^{ij} \bar{\Psi}_i \gamma_{ab} \Psi_j, \\ \Lambda_i|_{\ln \bar{\Phi}} &= \frac{\Lambda_i}{A} + \frac{1}{2A^2} \left(B_{ij} \varepsilon^{jk} \Psi_k + \frac{1}{2} F_{ab}^- \gamma^{ab} \Psi_i \right) + \frac{1}{24A^3} \gamma^{ab} \Psi_i \varepsilon^{jk} \bar{\Psi}_j \gamma_{ab} \Psi_k, \\ C|_{\ln \bar{\Phi}} &= \frac{C}{A} + \frac{1}{4A^2} (\varepsilon^{ik} \varepsilon^{jl} B_{ij} B_{kl} - 2F^{-ab} F_{ab}^- + 4\varepsilon^{ij} \bar{\Lambda}_i \Psi_j) \\ &\quad + \frac{1}{2A^3} \left(\varepsilon^{ik} \varepsilon^{jl} B_{ij} \bar{\Psi}_k \Psi_l - \frac{1}{2} \varepsilon^{kl} F_{ab}^- \bar{\Psi}_k \gamma^{ab} \Psi_l \right) - \frac{1}{32A^4} \varepsilon^{ij} \bar{\Psi}_i \gamma_{ab} \Psi_j \varepsilon^{kl} \bar{\Psi}_k \gamma^{ab} \Psi_l. \end{aligned} \quad (2.11)$$

When the chiral superfield Φ has zero Weyl weight, the logarithm is merely a field redefinition in superspace, which has no direct consequences. However, in the superconformal setting that we are considering, this is no longer the case for non-zero Weyl weight and the two chiral multiplets Φ and $\ln \Phi$ are very different. In particular $\ln \Phi$ does not satisfy the assignment (2.5) as it transforms *inhomogeneously* under (constant) dilatations and

⁶The multiplet $\mathbb{T}(\ln \bar{\mathcal{X}})/\mathcal{X}^2$ was considered in [9] with \mathcal{X} a reduced chiral superfield, and shown to be a $w = 0$ conformal primary. The extension of that analysis to $\mathbb{T}(\ln \bar{\Phi})$ for an arbitrary anti-chiral multiplet $\bar{\Phi}$ is completely straightforward.

chiral U(1) transformations,

$$\delta A|_{\ln \Phi} = w (\Lambda_D - i\Lambda_A) . \tag{2.12}$$

There are further inhomogeneous transformations, such as S-supersymmetry that acts inhomogeneously on $\Psi_i|_{\ln \Phi}$. However, the higher- θ components all scale consistently as if they belong to a $w = 0$ chiral multiplet. In flat superspace this phenomenon also extends to the Q- and S-supersymmetry transformations, although, as we shall see later, there are some minor exceptions in curved superspace. The explicit components in $\mathbb{T}(\ln \bar{\Phi})$ will take a rather different form than in $\mathbb{T}(\bar{\Phi})$, but much of the global structure of $\mathbb{T}(\ln \bar{\Phi})$ will still match that of $\mathbb{T}(\bar{\Phi})$. In particular, the highest θ -component, $C|_{\ln \Phi}$ will remain *invariant* under S-supersymmetry, irrespective of the value of the Weyl weight of Φ . As explained earlier, the latter implies that the kinetic multiplet $\mathbb{T}(\ln \bar{\Phi})$, defined from a generic chiral multiplet Φ of arbitrary Weyl weight w , will constitute a conformal primary $w = 2$ chiral multiplet. This observation is essential as it forms the basis for the approach followed in this paper. We will be more explicit in section 4.

The last quantity of interest is the Weyl tensor, which turns out to be one of the components of the Weyl multiplet. This multiplet is a reduced chiral *tensor* superfield $W_{\alpha\beta}$, symmetric in $(\alpha\beta)$ with Weyl weight $w = 1$. It obeys the constraint $D^{\alpha\beta}W_{\alpha\beta} = \bar{D}_{\dot{\alpha}\dot{\beta}}\bar{W}^{\dot{\alpha}\dot{\beta}}$, which reduces it to 24 + 24 degrees of freedom. Those are captured by the field strengths for the independent gauge fields, namely the vierbein e_μ^a , the doublet of gravitini ψ_μ^i , the gauge fields of the SU(2) \times U(1) R-symmetry, \mathcal{V}_μ^{ij} and A_μ , as well as three matter fields, an anti-selfdual tensor $T_{ab}{}^{ij}$, a chiral spinor doublet χ^i , and a scalar D . Its lowest independent components are given by

$$\begin{aligned} W_{\alpha\beta}|_{\theta=0} &= -\frac{1}{8}(\sigma^{ab})_{\alpha\beta} T_{ab}{}^{ij}\varepsilon_{ij} , & D_{\gamma j}W_{\alpha\beta}|_{\theta=0} &= -\varepsilon_{jk}(\sigma^{ab})_{\alpha\beta} R(Q)_{ab\gamma}{}^k , \\ D_{ij}W_{\alpha\beta}|_{\theta=0} &= 2\varepsilon_{ik}(\sigma^{ab})_{\alpha\beta} R(\mathcal{V})_{ab}{}^k{}_j , & D_{\gamma\delta}W_{\alpha\beta}|_{\theta=0} &= 2(\sigma^{ab})_{\alpha\beta}(\sigma_{cd})_{\gamma\delta} R(M)_{ab}{}^{cd} , \end{aligned} \tag{2.13}$$

where $R(Q)$, $R(\mathcal{V})$ and $R(M)$ are the (linearized) curvatures of conformal supergravity. The usual Weyl tensor as well as the field D are contained within $R(M)$, while χ^i is contained within $R(Q)^i$. The chiral superspace integral of the square of $W_{\alpha\beta}$ contains therefore the square of the anti-selfdual component of the Weyl tensor. At the linearized level, we can work with flat superspace, and we find

$$\mathcal{L}_W^- = - \int d^4\theta W_{\alpha\beta}W^{\alpha\beta} = C^{abcd}C_{abcd}^- + \dots . \tag{2.14}$$

From these results we can now define characteristic terms of the (linearized and complex) expression for the Gauss-Bonnet density in flat superspace,

$$\begin{aligned} \mathcal{L}_\chi^- &= - \int d^4\theta \{ W_{\alpha\beta}W^{\alpha\beta} + w^{-1} \mathbb{T}(\ln \bar{\Phi}) \} \\ &= \frac{1}{2}C^{abcd}C_{abcd} - \frac{1}{2}C^{abcd}\tilde{C}_{abcd} + 2w^{-1}\square\square \ln \bar{A} + \dots , \end{aligned} \tag{2.15}$$

where the additional terms depend on the remaining components of the linearized Weyl multiplet.

These observations are in principle restricted to flat superspace and to the linearized Weyl multiplet action. Nevertheless, all of the Lagrangians above exist in curved superspace. At the component level, this is due to the existence of an off-shell conformal supergravity multiplet which can be used to extend the global supersymmetry algebra to a local one and impose it on the matter multiplets. Of particular use is the chiral density formula (whose explicit form we give in the next section), which allows the construction of a locally supersymmetric invariant from a generic weight-two chiral multiplet, analogous to chiral superspace integrals. The full Lagrangian corresponding to the Weyl multiplet action (2.14), given long ago in [12], falls into this class, as does the action (2.6) built upon the kinetic multiplet $\mathbb{T}(\bar{\Phi})$, whose locally supersymmetric version was shown to be a conformal primary chiral multiplet in [15]. For the more complicated Lagrangian (2.10), the key property to determine is similarly whether $\mathbb{T}(\ln \bar{\Phi})$ similarly exists as a proper chiral multiplet; once that is established, the locally supersymmetric extension follows. One can then, as a simple application, construct the $N = 2$ Gauss-Bonnet invariant using the non-linear version of (2.15), which we can immediately deduce must look like

$$\begin{aligned}
 e^{-1} \mathcal{L}_\chi^- &= \frac{1}{2} C^{abcd} C_{abcd} - \frac{1}{2} C^{abcd} \tilde{C}_{abcd} + 2 w^{-1} \square_c \square_c \ln \bar{A} + \dots \\
 &= \frac{1}{2} C^{abcd} C_{abcd} - \frac{1}{2} C^{abcd} \tilde{C}_{abcd} - \mathcal{R}^{ab} \mathcal{R}_{ab} + \frac{1}{3} \mathcal{R}^2 + \frac{1}{3} \mathcal{D}^2 \mathcal{R} \\
 &\quad + 2w^{-1} \left\{ (\mathcal{D}^2)^2 \ln \bar{A} + \mathcal{D}^a \left(\frac{2}{3} \mathcal{R} \mathcal{D}_a \ln \bar{A} - 2 \mathcal{R}_{ab} \mathcal{D}^b \ln \bar{A} \right) \right\} + \dots \quad (2.16)
 \end{aligned}$$

where the missing terms depend on the rest of the Weyl and chiral multiplets.

To extend flat superspace to curved superspace has the advantage that local supersymmetry will be manifest from the start. A consistent definition of curved superspace requires a suitable structure group and corresponding constraints on the superspace geometry. Subsequently one can replace the flat spinor derivatives $D_{\alpha i}$ by curved tangent space derivatives $\nabla_{\alpha i}$ in the explicit superspace actions as well as in the definitions of the superfield components and the superfield constraints; these curved derivatives contain the relevant connection fields whereas the gravitino fields are introduced as fermionic components of the superspace vielbein. The superspace formulation that is used here [1] shares the same structure group with the conformal multiplet calculus, encompassing it in a more geometric setting.

3 Curved superspace, chiral superfields and the kinetic multiplet

In this section we first introduce the extension of flat superspace to the $N = 2$ conformal superspace [1], which is closely related to the $N = 2$ superconformal multiplet calculus.⁷

⁷This is not the only way to formulate conformal supergravity in superspace. The most well-known formulations involve either the structure group $\text{SO}(3,1) \times \text{U}(2)$ [30, 31], or the simpler structure group $\text{SO}(3,1) \times \text{SU}(2)$ [5]. Both realize the superconformal symmetries as a super-Weyl transformation, so the connection with superconformal multiplet calculus is less direct. The relation between the two is spelled out in [6], and their relation to conformal superspace is described in [1].

Subsequently, we will discuss the chiral multiplet Lagrangians and the kinetic multiplet in curved superspace.

3.1 Some details of curved superspace

Our starting point is a supermanifold parametrized by local coordinates $z^M = (x^\mu, \theta^{\mathfrak{m}\iota}, \bar{\theta}_{\mathfrak{m}\iota})$.⁸ The coordinates x^μ parametrize the bosonic part of the manifold while the eight Grassmann (anticommuting) coordinates $\theta^{\mathfrak{m}\iota}$ and $\bar{\theta}_{\mathfrak{m}\iota}$, with $\mathfrak{m} = 1, 2$, $\mathfrak{m} = \dot{1}, \dot{2}$ and $\iota = \underline{1}, \underline{2}$, are associated with the eight supersymmetries. In addition to (super)diffeomorphisms, we equip the superspace with the following symmetry generators: Lorentz transformations, M_{ab} ; Weyl dilatations, \mathbb{D} ; chiral U(1) rotations, \mathbb{A} ; SU(2) transformations, I^i_j ; special conformal transformations, K_a ; and the S-supersymmetries, S_α^i and $\bar{S}^{\dot{\alpha}}_i$. We introduce a connection associated with each of these: the spin connection Ω_M^{ab} ; the dilatation connection B_M ; the U(1) and SU(2) connections A_M and $\mathcal{V}_M^i_j$; and the K and S-supersymmetry connections F_M^a , $\Phi_M^{\alpha i}$ and $\bar{\Phi}_{M\dot{\alpha}i}$. In addition, we introduce the superspace vielbein E_M^A , which relates the world index M to the tangent space index A . In terms of the connections, we can construct the covariant derivative $\nabla_A = (\nabla_a, \nabla_{\alpha i}, \bar{\nabla}^{\dot{\alpha}i})$ implicitly via the equation

$$E_M^A \nabla_A = \partial_M - \frac{1}{2} \Omega_M^{ab} M_{ab} - B_M \mathbb{D} - A_M \mathbb{A} - \frac{1}{2} \mathcal{V}_M^i_j I^j_i - \frac{1}{2} \Phi_M^{\alpha i} S_\alpha^i - \frac{1}{2} \bar{\Phi}_{M\dot{\alpha}i} \bar{S}^{\dot{\alpha}}_i - F_M^a K_a, \quad (3.1)$$

from which ∇_A can be solved using the inverse vielbein E_A^M . The supergravity gauge group consists of covariant diffeomorphisms generated by ∇_A and the additional superconformal gauge transformations. A *covariant* (scalar) superfield $\Psi(x, \theta, \bar{\theta})$ transforms as

$$\delta \Psi = \left(\xi^A \nabla_A + \frac{1}{2} \Lambda^{ab} M_{ab} + \Lambda_{\mathbb{D}} \mathbb{D} + \Lambda_{\mathbb{A}} \mathbb{A} + \frac{1}{2} \Lambda^i_j I^j_i + \eta^\alpha_i S_\alpha^i + \bar{\eta}_{\dot{\alpha}i} \bar{S}^{\dot{\alpha}}_i + \Lambda_K^a K_a \right) \Psi, \quad (3.2)$$

without any derivative on the parameters. Ψ has Weyl weight w and chiral weight c if $\mathbb{D}\Psi = w\Psi$ and $\mathbb{A}\Psi = ic\Psi$. Note that the space-time diffeomorphisms and the Q-supersymmetry transformations comprise the superspace diffeomorphisms generated by $\xi^A \nabla_A$.

Just as in flat superspace, invariant actions are constructed in two ways. A *full superspace* integral involves an integral over the eight Grassmann coordinates of some superspace Lagrangian, which we denote using the symbol \mathcal{L} (to distinguish it from a component Lagrangian \mathcal{L}),

$$\int d^4x d^4\theta d^4\bar{\theta} E \mathcal{L}. \quad (3.3)$$

The measure factor $E = \text{Ber}(E_M^A)$ is the Berezinian (or superdeterminant) of the superspace vielbein and plays the same role as the vierbein determinant e on a bosonic manifold. In order for the action to be invariant under the supergravity gauge group, the superspace Lagrangian \mathcal{L} must be a conformal primary scalar with Weyl and chiral weight zero.

⁸The index ι on the Grassmann coordinates is a world index rather than a tangent space SU(2) index.

A *chiral superspace* integral can be written as

$$\int d^4x d^4\theta \mathcal{E} \mathcal{L}_{\text{ch}}, \quad (3.4)$$

where \mathcal{E} is the appropriate chiral measure and the Lagrangian \mathcal{L}_{ch} must be covariantly chiral (i.e. subject to $\bar{\nabla}^{\dot{\alpha}i} \mathcal{L}_{\text{ch}} = 0$) and a conformal primary with Weyl weight 2 and chiral weight -2 . Generally, any integral over the full superspace can be rewritten (up to a total derivative) as an integral over chiral superspace,

$$\int d^4x d^4\theta d^4\bar{\theta} E \mathcal{L} = \int d^4x d^4\theta \mathcal{E} \bar{\nabla}^4 \mathcal{L} \quad (3.5)$$

using the chiral projection operator $\bar{\nabla}^4$,

$$\bar{\nabla}^4 = \frac{1}{48} \varepsilon_{ik} \varepsilon_{jl} \bar{\nabla}^{kl} \bar{\nabla}^{ij}, \quad \bar{\nabla}^{ij} := \bar{\nabla}_{\dot{\alpha}}^{(i} \bar{\nabla}^{\dot{\alpha}j)}. \quad (3.6)$$

This is a non-trivial statement in curved superspace: one must check that $\bar{\nabla}^4 \mathcal{L}$ is indeed chiral and annihilated by S-supersymmetry.

One must have a method to relate superspace integrals to the usual integrals over the bosonic manifold. Performing the θ integrals in (3.4) leads to [1]

$$\int d^4x d^4\theta \mathcal{E} \mathcal{L}_{\text{ch}} = \int d^4x \mathcal{L}_{\text{ch}} \quad (3.7)$$

where, in two-component notation,

$$\begin{aligned} e^{-1} \mathcal{L}_{\text{ch}} = & \left[\nabla^4 \mathcal{L}_{\text{ch}} - \frac{1}{12} i \varepsilon^{ik} \varepsilon^{jl} (\bar{\psi}_{\mu i} \bar{\sigma}^{\mu})^{\alpha} (\nabla_{\alpha j} \nabla_{kl} \mathcal{L}_{\text{ch}}) - \frac{1}{2} i \varepsilon^{ij} \bar{\psi}_{\mu \dot{\gamma} i} \bar{W}^{\dot{\gamma}}_{\dot{\beta}} (\bar{\sigma}^{\mu})^{\dot{\beta} \alpha} (\nabla_{\alpha j} \mathcal{L}_{\text{ch}}) \right. \\ & + \bar{W}^{\dot{\alpha} \dot{\beta}} \bar{W}_{\dot{\alpha} \dot{\beta}} \mathcal{L}_{\text{ch}} + \frac{1}{4} \varepsilon^{ik} \varepsilon^{jl} (\bar{\psi}_{\mu i} \bar{\sigma}^{\mu \nu} \bar{\psi}_{\nu j}) (\nabla_{kl} \mathcal{L}_{\text{ch}}) \\ & + \varepsilon^{ij} (\bar{\psi}_{\mu i} \bar{\psi}_{\nu j}) \left(\frac{1}{8} (\sigma^{\mu \nu})_{\alpha \beta} (\nabla_{\beta}^{\alpha} \mathcal{L}_{\text{ch}}) + (\bar{\sigma}^{\mu \nu})^{\dot{\alpha} \dot{\beta}} \bar{W}_{\dot{\beta}}^{\dot{\alpha}} \mathcal{L}_{\text{ch}} \right) \\ & \left. + \frac{1}{4} e^{-1} \varepsilon^{\mu \nu \rho \tau} \varepsilon^{ij} \varepsilon^{kl} (\bar{\psi}_{\mu i} \bar{\psi}_{\nu j}) \left(i (\bar{\psi}_{\rho k} \bar{\sigma}_{\tau})^{\alpha} (\nabla_{\alpha l} \mathcal{L}_{\text{ch}}) + (\bar{\psi}_{\rho k} \bar{\psi}_{\tau l}) \mathcal{L}_{\text{ch}} \right) \right]_{\theta=0}. \quad (3.8) \end{aligned}$$

Provided one defines the components of the chiral multiplet \mathcal{L}_{ch} as in (2.2), replacing $D_{\alpha i}$ with $\nabla_{\alpha i}$, one recovers the usual chiral density rule of the conformal multiplet calculus [28], but now in four-component form,⁹

$$\begin{aligned} -2 e^{-1} \mathcal{L}_{\text{ch}} = & C - \varepsilon^{ij} \bar{\psi}_{\mu i} \gamma^{\mu} \Lambda_j - \frac{1}{8} \varepsilon^{ij} \varepsilon^{kl} \bar{\psi}_{\mu i} T_{abjk} \gamma^{ab} \gamma^{\mu} \Psi_l - \frac{1}{16} (T_{abij} \varepsilon^{ij})^2 A \\ & - \frac{1}{2} \varepsilon^{ik} \varepsilon^{jl} \bar{\psi}_{\mu i} \gamma^{\mu \nu} \psi_{\nu j} B_{kl} + \varepsilon^{ij} \bar{\psi}_{\mu i} \psi_{\nu j} \left(F^{-\mu \nu} - \frac{1}{2} A T^{\mu \nu}_{kl} \varepsilon^{kl} \right) \\ & - \frac{1}{2} e^{-1} \varepsilon^{\mu \nu \rho \sigma} \varepsilon^{ij} \varepsilon^{kl} \bar{\psi}_{\mu i} \psi_{\nu j} (\bar{\psi}_{\rho k} \gamma_{\sigma} \Psi_l + \bar{\psi}_{\rho k} \psi_{\sigma j} A). \quad (3.9) \end{aligned}$$

For further details of the superspace geometry, we refer to appendix C.

⁹This version of the chiral density formula differs by an overall factor of $-\frac{1}{2}$ from the usual one [28]. This arises as a result of the normalization of the component C of the chiral multiplet, or equivalently, a different definition of the superspace measure.

3.2 Chiral multiplet actions and the kinetic multiplet in curved superspace

In section 2, we discussed four types of actions which could be written down in flat superspace. Each has a straightforward extension to curved superspace. If we restrict ourselves to pure conformal supergravity without additional matter multiplets, there is only a single possible action given by the chiral superspace integral

$$\int d^4x d^4\theta \mathcal{E} W^{\alpha\beta} W_{\alpha\beta}, \tag{3.10}$$

which we have already discussed at the linearized level in (2.14).

The remaining actions that we discussed in section 2 require general chiral multiplets and vector multiplets, which are contained respectively in chiral and reduced chiral superfields. To couple these to conformal supergravity in superspace requires merely the covariantization of the chiral constraint and the reducibility constraint, respectively.

The simplest action we discussed was the vector multiplet action (2.9), whose curved generalization reads simply

$$\frac{1}{2} \int d^4x d^4\theta \mathcal{E} \mathcal{X}^2. \tag{3.11}$$

Its component expression was given in [28] using the superconformal multiplet calculus. However, our focus will be on the curved superspace generalizations of the actions (2.6) and (2.10).

Let us begin with (2.6). It generalizes to curved superspace in a completely straightforward manner:

$$\int d^4x d^4\theta d^4\bar{\theta} E \Phi' \bar{\Phi} = \int d^4x d^4\theta \mathcal{E} \Phi' \bar{\nabla}^4 \bar{\Phi}. \tag{3.12}$$

We have emphasized that the same action can be written using (3.5) as a chiral integral of the product of Φ' and the kinetic multiplet $\mathbb{T}(\bar{\Phi})$. At the component level, the Lagrangian is the supersymmetrization of $A' \square_c \square_c \bar{A}$ and was analyzed in [15]. This class of higher derivative action admits an obvious generalization in the presence of several chiral multiplets Φ^I with weights w_I . Introducing a homogeneous function $\mathcal{H}(\Phi, \bar{\Phi})$ of weight zero,

$$\sum_I w_I \Phi^I \mathcal{H}_I = 0, \tag{3.13}$$

where $\mathcal{H}_I := \partial\mathcal{H}/\partial\Phi^I$, one can construct a higher derivative action by integrating \mathcal{H} over the full superspace,¹⁰

$$\int d^4x d^4\theta d^4\bar{\theta} E \mathcal{H}. \tag{3.14}$$

By virtue of the formula (3.5) and its complex conjugate, one can show that the action is invariant under the Kähler-like transformations

$$\mathcal{H} \rightarrow \mathcal{H} + \Lambda(\Phi) + \bar{\Lambda}(\bar{\Phi}) \tag{3.15}$$

¹⁰Similar structures were considered in the context of low-energy effective actions in flat space [7–11].

where the holomorphic function $\Lambda(\Phi)$ must similarly be homogeneous. It follows that the component action will depend only on the Kähler metric $\mathcal{H}_{I\bar{J}}$, which is subject to the homogeneity condition

$$\sum_I w_I \Phi^I \mathcal{H}_{I\bar{J}} = 0 . \tag{3.16}$$

The locally supersymmetric version was analyzed in [15], with particular attention paid to the special case where the chiral multiplets were vector multiplets \mathcal{X}^I with $w = 1$ or the Weyl-squared chiral multiplet $W^{\alpha\beta}W_{\alpha\beta}$ with $w = 2$. This class can be broadened further while maintaining the Kähler structure by considering the chiral multiplets Φ^I to be themselves composite in various ways.

It was noted in [15] that a broad class of higher derivative chiral superspace integrals lift naturally to full superspace integrals involving functions \mathcal{H} by stripping away an operator $\bar{\nabla}^4$ as in (3.12). However, it turns out that the curved version of the action (2.10),

$$\int d^4x d^4\theta \mathcal{E} \Phi' \bar{\nabla}^4 \ln \bar{\Phi} , \tag{3.17}$$

where Φ' has weight $w' = 0$ and Φ has nonzero weight w , does *not* belong to this class. At first glance, a naive application of (3.5) would seem to indicate

$$\int d^4x d^4\theta \mathcal{E} \Phi' \bar{\nabla}^4 \ln \bar{\Phi} \stackrel{?}{=} \int d^4x d^4\theta d^4\bar{\theta} E \Phi' \ln \bar{\Phi} \tag{3.18}$$

with the full superspace Lagrangian falling into the class of generic function \mathcal{H} already considered. However, the proposed Lagrangian $\mathcal{H} = \Phi' \ln \bar{\Phi}$ transforms inhomogeneously under dilatations and so is *not* permissible; in other words, \mathcal{H} does not obey the homogeneity conditions (3.13) or (3.16).¹¹ Nevertheless, the left-hand side of (3.18) *does* transform appropriately. This is because the kinetic multiplet $\mathbb{T}(\ln \bar{\Phi})$ is a conformal primary chiral multiplet of weight $w = 2$, obeying

$$\bar{\nabla}^{\dot{\alpha}i} \mathbb{T}(\ln \bar{\Phi}) = 0 , \quad S_{\alpha}{}^i \mathbb{T}(\ln \bar{\Phi}) = \bar{S}^{\dot{\alpha}i} \mathbb{T}(\ln \bar{\Phi}) = 0 . \tag{3.19}$$

Both conditions are straightforward enough to check, although they require some SU(2) index gymnastics.¹² Now by comparing to the flat space limit, it is obvious that

$$\int d^4x d^4\theta \mathcal{E} \Phi' \bar{\nabla}^4 \ln \bar{\Phi} = \int d^4x e A' \square_c \square_c \ln \bar{A} + \text{additional terms} . \tag{3.20}$$

The complete expression, which we will present in this paper, corresponds to a *new chiral supersymmetric invariant*.

¹¹This obstruction is specific for curved superspace. For flat superspace, \mathcal{H} must be homogeneous only up to Kähler transformations; see e.g. [9] where such actions were considered.

¹²The key idea for the first condition is that there are only four anti-commuting $\bar{\nabla}^{\dot{\alpha}i}$ derivatives, so a product of five of them must vanish (up to curvatures, which contribute nothing in this case). The next condition, that it is annihilated by $S_{\alpha}{}^i$, is easy enough as that operator anti-commutes with $\bar{\nabla}^{\dot{\alpha}i}$; checking the last condition, that $\bar{S}^{\dot{\alpha}i}$ similarly gives zero, is a minor exercise using the algebra of the operators given in appendix C.

This invariant has already appeared in physical applications. In [18], the 5D mixed gauge-gravitational Chern-Simons invariant [19] was dimensionally reduced, and a characteristic subset of 4D terms was obtained which broke down into three classes. The first class was easily identified as the usual chiral superspace integral of a holomorphic function. Another class seemed to coincide with the full superspace integral of a real function $\mathcal{H} \sim \Phi' \ln \bar{\Phi} + \text{h.c.}$, while the remainder, involving terms of the Gauss-Bonnet variety, could not be identified with any currently known invariant. It is clear to us now that these latter two classes of terms are actually contained within the single invariant (3.20), which is intrinsically chiral and cannot be decomposed further in a manifestly superconformal way.

Before setting out to calculate the expression (3.20) explicitly, we should make an important observation. In the introduction, we noted that the non-linear Lagrangian (1.20) with ϕ' constant, must depend on the field $\ln \bar{\phi}$ only via total derivative terms. We expect the same should hold for its supersymmetrized version, namely that when Φ' is constant in (3.20) the dependence on $\ln \bar{\Phi}$ is only in the form of total-derivative terms. To see this, suppose we have two such kinetic multiplets built out of the logarithm of two different anti-chiral superfields $\bar{\Phi}_1$ and $\bar{\Phi}_2$, taken to have the same weight w for simplicity. The difference is obviously

$$\bar{\nabla}^4 \ln \bar{\Phi}_1 - \bar{\nabla}^4 \ln \bar{\Phi}_2 = \bar{\nabla}^4 \ln(\bar{\Phi}_1/\bar{\Phi}_2), \tag{3.21}$$

and the quantity under the spinor derivatives on the right-hand side is actually a proper weight-zero multiplet. It follows that any chiral integrand involving such a difference can be written as a full superspace integral and then as an anti-chiral superspace integral, discarding total derivatives in the equalities. Hence,

$$\begin{aligned} \int d^4x d^4\theta \mathcal{E} \Phi' \bar{\nabla}^4 \ln(\bar{\Phi}_1/\bar{\Phi}_2) &= \int d^4x d^4\theta d^4\bar{\theta} E \Phi' \ln(\bar{\Phi}_1/\bar{\Phi}_2) \\ &= \int d^4x d^4\bar{\theta} \bar{\mathcal{E}} (\nabla^4 \Phi') \ln(\bar{\Phi}_1/\bar{\Phi}_2). \end{aligned} \tag{3.22}$$

Taking the weight-zero chiral superfield Φ' to be actually constant, it follows that the right-hand side of (3.22) vanishes and therefore

$$\int d^4x d^4\theta \mathcal{E} \bar{\nabla}^4 \ln \bar{\Phi}_1 = \int d^4x d^4\theta \mathcal{E} \bar{\nabla}^4 \ln \bar{\Phi}_2. \tag{3.23}$$

In other words, the integral $\int d^4x d^4\theta \mathcal{E} \bar{\nabla}^4 \ln \bar{\Phi}$ is *independent* of the components of $\ln \bar{\Phi}$ up to total derivatives. This observation will be an important check that we have correctly calculated the additional terms in (3.20). It is to this task which we now turn.

4 The component structure of the kinetic multiplet

In this section, we proceed to construct the kinetic multiplet $\mathbb{T}(\ln \bar{\Phi})$ in supergravity along with the corresponding Lagrangian (2.10). The starting point is the formula for the Q- and S-supersymmetry transformations of a general $N = 2$ chiral multiplet Φ with Weyl

weight w in four-component notation [15, 28, 29],

$$\begin{aligned}
\delta A &= \bar{\epsilon}^i \Psi_i, \\
\delta \Psi_i &= 2 \not{D} A \epsilon_i + B_{ij} \epsilon^j + \frac{1}{2} \gamma^{ab} F_{ab}^- \epsilon_{ij} \epsilon^j + 2 w A \eta_i, \\
\delta B_{ij} &= 2 \bar{\epsilon}_{(i} \not{D} \Psi_{j)} - 2 \bar{\epsilon}^k \Lambda_{(i} \epsilon_{j)k} + 2(1-w) \bar{\eta}_{(i} \Psi_{j)}, \\
\delta F_{ab}^- &= \frac{1}{2} \epsilon^{ij} \bar{\epsilon}_i \not{D} \gamma_{ab} \Psi_j + \frac{1}{2} \bar{\epsilon}^i \gamma_{ab} \Lambda_i - \frac{1}{2} (1+w) \epsilon^{ij} \bar{\eta}_i \gamma_{ab} \Psi_j, \\
\delta \Lambda_i &= -\frac{1}{2} \gamma^{ab} \not{D} F_{ab}^- \epsilon_i - \not{D} B_{ij} \epsilon^{jk} \epsilon_k + C \epsilon_{ij} \epsilon^j + \frac{1}{4} (\not{D} A \gamma^{ab} T_{abij} + w A \not{D} \gamma^{ab} T_{abij}) \epsilon^{jk} \epsilon_k \\
&\quad - 3 \gamma_a \epsilon^{jk} \epsilon_k \bar{\chi}_{[i} \gamma^a \Psi_{j]} - (1+w) B_{ij} \epsilon^{jk} \eta_k + \frac{1}{2} (1-w) \gamma^{ab} F_{ab}^- \eta_i, \\
\delta C &= -2 \epsilon^{ij} \bar{\epsilon}_i \not{D} \Lambda_j - 6 \bar{\epsilon}_i \chi_j \epsilon^{ik} \epsilon^{jl} B_{kl} \\
&\quad - \frac{1}{4} \epsilon^{ij} \epsilon^{kl} ((w-1) \bar{\epsilon}_i \gamma^{ab} \not{D} T_{abjk} \Psi_l + \bar{\epsilon}_i \gamma^{ab} T_{abjk} \not{D} \Psi_l) + 2 w \epsilon^{ij} \bar{\eta}_i \Lambda_j. \tag{4.1}
\end{aligned}$$

In this convention the spinors ϵ^i and η_i are the positive chirality spinorial parameters associated with Q- and S-supersymmetry. The corresponding negative chirality parameters are denoted by ϵ_i and η^i . (In two-component form, the positive chirality spinors would be denoted by ϵ_{α}^i and $\eta_{\alpha i}$, and the negative chirality spinors by $\bar{\epsilon}^{\dot{\alpha} i}$ and $\bar{\eta}^{\dot{\alpha} i}$.)

One can see from (4.1) that the highest component C of a $w = 0$ chiral multiplet is anti-chiral and invariant under S-supersymmetry. This observation allows the construction of the chiral $w = 2$ kinetic multiplet $\mathbb{T}(\bar{\Phi})$ whose lowest component is \bar{C} . Although such an analysis was carried out in components in [15] by consecutively considering supersymmetry transformations and identifying the higher- θ components, it could just as easily be carried through in superspace. The starting point is to take $\mathbb{T}(\bar{\Phi}) := -2\bar{\nabla}^4 \bar{\Phi}$ and to identify its components using the curved superspace version of (2.2),

$$\begin{aligned}
A|_{\mathbb{T}(\bar{\Phi})} &:= -2\bar{\nabla}^4 \bar{\Phi}|_{\theta=0}, & \Psi_{\alpha i}|_{\mathbb{T}(\bar{\Phi})} &:= -2\nabla_{\alpha i} \bar{\nabla}^4 \bar{\Phi}|_{\theta=0}, \\
B_{ij}|_{\mathbb{T}(\bar{\Phi})} &:= \nabla_{ij} \bar{\nabla}^4 \bar{\Phi}|_{\theta=0}, & F_{ab}^-|_{\mathbb{T}(\bar{\Phi})} &:= \frac{1}{2} (\sigma_{ab})_{\alpha}{}^{\beta} \nabla_{\beta}{}^{\alpha} \bar{\nabla}^4 \bar{\Phi}|_{\theta=0}, \\
\Lambda_{\alpha i}|_{\mathbb{T}(\bar{\Phi})} &:= -\frac{1}{3} \epsilon^{jk} \nabla_{\alpha k} \nabla_{ji} \bar{\nabla}^4 \bar{\Phi}|_{\theta=0}, & C|_{\mathbb{T}(\bar{\Phi})} &:= 4\nabla^4 \bar{\nabla}^4 \bar{\Phi}|_{\theta=0}. \tag{4.2}
\end{aligned}$$

Since $\bar{\Phi}$ is anti-chiral, each of these components may be evaluated in the usual superspace fashion: (anti)commute each $\nabla_{\alpha i}$ past the other covariant derivatives until they annihilate $\bar{\Phi}$. The resulting expression should be rearranged (using the superspace commutation relations) so that all the spinor derivatives $\bar{\nabla}^{\dot{\alpha} i}$ act directly upon $\bar{\Phi}$. The calculation is straightforward, although more and more complicated as the number of spinor derivatives increases; for such calculations, the (anti)-commutation relations given in appendix C are necessary.

Now we wish to construct the non-linear version of the kinetic multiplet, $\mathbb{T}(\ln \bar{\Phi})$. As we have already alluded to in section 2, we will choose to define the components of $\ln \bar{\Phi}$ using (2.11), which coincides with using the curved superspace version of (2.2) with $\bar{\Phi}$ replaced by $\ln \bar{\Phi}$. It is straightforward to determine the Q- and S-supersymmetry

transformation rules of these components

$$\begin{aligned}
\delta \hat{A} &= \bar{\epsilon}^i \hat{\Psi}_i, \\
\delta \hat{\Psi}_i &= 2 \not{D} \hat{A} \epsilon_i + \hat{B}_{ij} \epsilon^j + \frac{1}{2} \gamma^{ab} \hat{F}_{ab}^- \epsilon_{ij} \epsilon^j + 2 w \eta_i, \\
\delta \hat{B}_{ij} &= 2 \bar{\epsilon}_{(i} \not{D} \hat{\Psi}_{j)} - 2 \bar{\epsilon}^k \hat{\Lambda}_{(i} \epsilon_{j)k} + 2 \bar{\eta}_{(i} \hat{\Psi}_{j)}, \\
\delta \hat{F}_{ab}^- &= \frac{1}{2} \epsilon^{ij} \bar{\epsilon}_i \not{D} \gamma_{ab} \hat{\Psi}_j + \frac{1}{2} \bar{\epsilon}^i \gamma_{ab} \hat{\Lambda}_i - \frac{1}{2} \epsilon^{ij} \bar{\eta}_i \gamma_{ab} \hat{\Psi}_j, \\
\delta \hat{\Lambda}_i &= -\frac{1}{2} \gamma^{ab} \not{D} \hat{F}_{ab}^- \epsilon_i - \not{D} \hat{B}_{ij} \epsilon^{jk} \epsilon_k + \hat{C} \epsilon_{ij} \epsilon^j + \frac{1}{4} (\not{D} \hat{A} \gamma^{ab} T_{abij} + w \not{D} \gamma^{ab} T_{abij}) \epsilon^{jk} \epsilon_k \\
&\quad - 3 \gamma_a \epsilon^{jk} \epsilon_k \bar{\chi}_{[i} \gamma^a \hat{\Psi}_{j]} - \hat{B}_{ij} \epsilon^{jk} \eta_k + \frac{1}{2} \gamma^{ab} \hat{F}_{ab}^- \eta_i, \\
\delta \hat{C} &= -2 \epsilon^{ij} \bar{\epsilon}_i \not{D} \hat{\Lambda}_j - 6 \bar{\epsilon}_i \chi_j \epsilon^{ik} \epsilon^{jl} \hat{B}_{kl} + \frac{1}{4} \epsilon^{ij} \epsilon^{kl} (\bar{\epsilon}_i \gamma^{ab} \not{D} T_{abjk} \hat{\Psi}_l - \bar{\epsilon}_i \gamma^{ab} T_{abjk} \not{D} \hat{\Psi}_l). \quad (4.3)
\end{aligned}$$

Comparing these transformation laws to those in (4.1), one notes the appearance of non-linearities involving the weight w . Every term is linear in the components of $\hat{\Phi} = \ln \Phi$ except for the terms proportional to w , which are independent of $\ln \Phi$. As discussed earlier, this arises ultimately from the inhomogeneous transformation of $\ln \Phi$ under dilatations. Note, however, that the covariant derivatives in (4.3) do also depend on the Weyl weight and therefore contain similar terms. For instance, consider the transformation (2.12), which obviously requires a term $-w(b_\mu - iA_\mu)$ in the covariant derivative $D_\mu \hat{A}$ which no longer depends on $\ln \Phi$.

As mentioned in section 2, the highest component \hat{C} of $\ln \Phi$ is a weight 2 conformal primary and (anti)chiral under Q-supersymmetry. This means we may use \hat{C} as the lowest component of a chiral multiplet, which will be the kinetic multiplet $\mathbb{T}(\ln \bar{\Phi})$. Within superspace, we can define its components exactly as in (4.2), with $\bar{\Phi}$ replaced by $\ln \bar{\Phi}$, and the subsequent computational steps are as outlined above, except for the generation of terms involving w .

An alternative procedure is to begin with the condition $A|_{\mathbb{T}(\ln \bar{\Phi})} = \hat{C}$ and derive $\Psi_i|_{\mathbb{T}(\ln \bar{\Phi})}$ by applying a Q-supersymmetry transformation to both sides. Continuing in this way, one can build up the entire multiplet. This was the procedure that was originally applied to the linear kinetic multiplet $\mathbb{T}(\bar{\Phi})$ in [15], but which is now considerably more involved. A convenient way of applying the same strategy is to focus only on the w -dependent terms by unpackaging the full covariant derivatives. Although this sacrifices manifest covariance, it exploits the high degree of overlap between $\mathbb{T}(\ln \bar{\Phi})$ and the kinetic multiplet $\mathbb{T}(\bar{\Phi})$ studied in [15].

We have followed both lines of approach and confirmed agreement between them, up to the fermionic terms in $C|_{\mathbb{T}(\ln \bar{\Phi})}$; these have passed other non-trivial checks using S-supersymmetry. The result is (in four component notation),

$$\begin{aligned}
A|_{\mathbb{T}(\ln \bar{\Phi})} &= \hat{C}, \\
\Psi_i|_{\mathbb{T}(\ln \bar{\Phi})} &= -2 \epsilon_{ij} \not{D} \hat{\Lambda}^j - 6 \epsilon_{ik} \epsilon_{jl} \chi^j \hat{B}^{kl} - \frac{1}{4} \epsilon_{ij} \epsilon_{kl} \gamma^{ab} T_{ab}{}^{jk} \overleftrightarrow{\not{D}} \hat{\Psi}^l,
\end{aligned}$$

$$\begin{aligned}
 B_{ij}|_{\mathbb{T}(\ln \bar{\Phi})} &= -2 \varepsilon_{ik} \varepsilon_{jl} (\square_c + 3D) \hat{B}^{kl} - 2 \hat{F}_{ab}^+ R(\mathcal{V})^{abk} \varepsilon_{jk} \\
 &\quad - 6 \varepsilon_{k(i} \bar{\chi}_{j)} \hat{\Lambda}^k + 3 \varepsilon_{ik} \varepsilon_{jl} \hat{\Psi}^{(k} \not{D} \chi^{l)}, \\
 F_{ab}^-|_{\mathbb{T}(\ln \bar{\Phi})} &= - \left(\delta_a^{[c} \delta_b^{d]} - \frac{1}{2} \varepsilon_{ab}^{cd} \right) \\
 &\quad \times [4 D_c D^e \hat{F}_{ed}^+ + (D^e \hat{A} D_c T_{de}^{ij} + D_c \hat{A} D^e T_{ed}^{ij}) \varepsilon_{ij} - w D_c D^e T_{ed}^{ij} \varepsilon_{ij}] \\
 &\quad + \square_c \hat{A} T_{ab}^{ij} \varepsilon_{ij} - R(\mathcal{V})_{ab}^{-i} \hat{B}^{jk} \varepsilon_{ij} + \frac{1}{8} T_{ab}^{ij} T_{cdij} \hat{F}^{+cd} - \varepsilon_{kl} \hat{\Psi}^k \overleftrightarrow{\not{D}} R(Q)_{ab}{}^l \\
 &\quad - \frac{9}{4} \varepsilon_{ij} \hat{\Psi}^i \gamma^c \gamma_{ab} D_c \chi^j + 3 \varepsilon_{ij} \bar{\chi}^i \gamma_{ab} \not{D} \hat{\Psi}^j + \frac{3}{8} T_{ab}^{ij} \varepsilon_{ij} \bar{\chi}_k \hat{\Psi}^k, \\
 \Lambda_i|_{\mathbb{T}(\ln \bar{\Phi})} &= 2 \square_c \not{D} \hat{\Psi}^j \varepsilon_{ij} + \frac{1}{4} \gamma^c \gamma_{ab} (2 D_c T^{ab}{}_{ij} \hat{\Lambda}^j + T^{ab}{}_{ij} D_c \hat{\Lambda}^j) \\
 &\quad - \frac{1}{2} \varepsilon_{ij} (R(\mathcal{V})_{ab}{}^j{}_k + 2i R(A)_{ab} \delta^j{}_k) \gamma^c \gamma^{ab} D_c \hat{\Psi}^k \\
 &\quad + \frac{1}{2} \varepsilon_{ij} \left(3 D_b D^a - 4i D^a R(A)_{ab} + \frac{1}{4} T_{bc}{}^{ij} \overleftrightarrow{D}_a T^{ac}{}_{ij} \right) \gamma^b \hat{\Psi}^j \\
 &\quad - 2 \hat{F}^{+ab} \not{D} R(Q)_{abi} + 6 \varepsilon_{ij} D \not{D} \hat{\Psi}^j \\
 &\quad + 3 \varepsilon_{ij} (\not{D} \chi_k \hat{B}^{kj} + \not{D} \hat{A} \not{D} \chi^j) \\
 &\quad + \frac{3}{2} \left(2 \not{D} \hat{B}^{kj} \varepsilon_{ik} + \not{D} \hat{F}_{ab}^+ \gamma^{ab} \delta_i^j + \frac{1}{4} \varepsilon_{kl} T_{ab}{}^{kl} \gamma^{ab} \not{D} \hat{A} \delta_i^j \right) \chi_j \\
 &\quad + \frac{9}{4} (\bar{\chi}^l \gamma_a \chi^l) \varepsilon_{ij} \gamma^a \hat{\Psi}^j - \frac{9}{2} (\bar{\chi}_i \gamma_a \chi^k) \varepsilon_{kl} \gamma^a \hat{\Psi}^l \\
 &\quad - \frac{3}{2} w \varepsilon_{jk} D^a T_{ab}{}^{jk} \gamma^b \chi_i, \\
 C|_{\mathbb{T}(\ln \bar{\Phi})} &= 4(\square_c + 3D) \square_c \hat{A} + 6(D_a D) D^a \hat{A} - 16 D^a (R(D)_{ab}^+ D^b \hat{A}) \\
 &\quad - D^a (T_{abij} T^{cbij} D_c \hat{A}) - \frac{1}{2} D^a (T_{abij} T^{cbij}) D_c \hat{A} - 9 \bar{\chi}_j \gamma^a \chi^j D_a \hat{A} \\
 &\quad + \frac{1}{2} D_a D^a (T_{bcij} \hat{F}^{bc+}) \varepsilon^{ij} + 4 \varepsilon^{ij} D_a (D^b T_{bcij} \hat{F}^{ac+} + D^b \hat{F}_{bc}^+ T^{ac}{}_{ij}) \\
 &\quad - \frac{9}{2} \varepsilon^{jk} \bar{\chi}_j \gamma^{ab} \chi_k \hat{F}_{ab}^+ + 9 \bar{\chi}_j \chi_k \hat{B}^{jk} + \frac{1}{16} (T_{ab}{}^{ij} \varepsilon_{ij})^2 \hat{C} \\
 &\quad + 6 D^a D_a \bar{\chi}_j \hat{\Psi}^j + 3 \bar{\chi}_j \not{D} \not{D} \hat{\Psi}^j + 3 D_a (\bar{\chi}_j \gamma^a \not{D} \hat{\Psi}^j) + 9 D \bar{\chi}_j \hat{\Psi}^j \\
 &\quad - 8 D^a \bar{R}(Q)_{abj} D^b \hat{\Psi}^j + 6 D_b \bar{\chi}_j \gamma^b \not{D} \hat{\Psi}^j \\
 &\quad + \frac{3}{2} D^a T_{abij} \bar{\chi}^i \gamma^b \hat{\Psi}^j + 3 D^a (T_{abij} \bar{\chi}^i \gamma^b \hat{\Psi}^j) + \frac{3}{2} D^a (T_{abij} \bar{\chi}^i) \gamma^b \hat{\Psi}^j \\
 &\quad + 3 \left(\frac{1}{2} R(\mathcal{V})_{ab}^+{}^i{}_j - R(D)_{ab}^+ \delta^i{}_j \right) \bar{\chi}_i \gamma^{ab} \hat{\Psi}^j - 2 R(\mathcal{V})_{ab}^+{}^i{}_j \bar{R}(Q)^{ab}{}_i \hat{\Psi}^j - \frac{1}{2} T^{ab}{}_{ij} \bar{R}(S)_{ab}^+{}^i \hat{\Psi}^j \\
 &\quad + \frac{1}{8} \varepsilon^{ij} T_{abij} \left(3 \bar{\chi}_k \gamma^{ab} \hat{\Lambda}^k + 2 \bar{R}(Q)_k^{ab} \hat{\Lambda}^k \right) \\
 &\quad + w \left\{ 9 \bar{\chi}_j \not{D} \chi^j - R(\mathcal{V})_{ab}^+{}^i{}_j R(\mathcal{V})^{ab+j}{}_i - 8 R(D)_{ab}^+ R(D)^{ab+} \right. \\
 &\quad \quad \left. - D^a T_{abij} D_c T^{cbij} - D^a (T_{abij} D_c T^{cbij}) \right\}. \tag{4.4}
 \end{aligned}$$

The result agrees with the corresponding expressions for the usual kinetic multiplet discussed in [15] by taking $w = 0$. In this limit, the superfield $\ln \bar{\Phi}$ becomes a normal $w = 0$ anti-chiral multiplet with $\mathbb{T}(\ln \bar{\Phi})$ its associated kinetic multiplet.

Now we can calculate the component Lagrangian \mathcal{L} corresponding to the action

$$-2 \int d^4x d^4\theta \mathcal{E} \Phi' \mathbb{T}(\ln \bar{\Phi}) . \quad (4.5)$$

This is a straightforward application of (3.9) and the product rule, (C.1) of [15], or, equivalently, the direct application of (3.8). We will ignore all fermions, which significantly simplifies the resulting expression. Expanding out the covariant d'Alembertians using, for example, the expression for f_μ^a given in (B.6) leads to

$$\begin{aligned} C|_{\mathbb{T}(\ln \bar{\Phi})} &= \mathcal{D}_a V^a + \frac{1}{16} (T_{abij} \varepsilon^{ij})^2 \hat{C} \\ &+ w \left\{ -2 \mathcal{R}^{ab} \mathcal{R}_{ab} + \frac{2}{3} \mathcal{R}^2 - 6 D^2 + 2 R(A)^{ab} R(A)_{ab} - R(\mathcal{V})_{ab}^{+i} R(\mathcal{V})^{ab+j}_i \right. \\ &\left. + \frac{1}{128} T^{abij} T_{ab}{}^{kl} T^{cd}{}_{ij} T_{cdkl} + T^{acij} \mathcal{D}_a \mathcal{D}^b T_{bcij} \right\} , \end{aligned} \quad (4.6)$$

where V^a is given by

$$\begin{aligned} V^a &= 4 \mathcal{D}^a \mathcal{D}^2 \hat{A} - 8 \mathcal{R}^{ab} \mathcal{D}_b \hat{A} + \frac{8}{3} \mathcal{R} \mathcal{D}^a \hat{A} + 8 D \mathcal{D}^a \hat{A} - 8 i R(A)^{ab} \mathcal{D}_b \hat{A} \\ &- 2 T^{acij} T_{bcij} \mathcal{D}^b \hat{A} + \frac{1}{2} \varepsilon^{ij} \mathcal{D}^a T_{bcij} \hat{F}^{bc+} + 4 \varepsilon^{ij} T^{ac}{}_{ij} \mathcal{D}^b \hat{F}_{bc}^+ \\ &+ w \left\{ \frac{2}{3} \mathcal{D}^a \mathcal{R} - 4 \mathcal{D}^a D - \mathcal{D}^b (T^{acij} T_{bcij}) \right\} . \end{aligned} \quad (4.7)$$

Here the derivatives \mathcal{D}_a are covariant with respect to the linearly acting bosonic transformations. Hence they do not contain the connection field of the conformal boosts f_μ^a . Note that we have kept the K-connection f_μ^a within the fully covariant derivatives in the last term of (4.6) for later convenience, but there is no obstacle in extracting it here as well.

Performing a similar decomposition in $B_{ij}|_{\mathbb{T}(\ln \bar{\Phi})}$ and $F_{ab}^-|_{\mathbb{T}(\ln \bar{\Phi})}$ and dropping a number of total derivatives, we find

$$\begin{aligned} e^{-1} \mathcal{L} &= 4 \mathcal{D}^2 A' \mathcal{D}^2 \hat{A} + 8 \mathcal{D}^a A' \left[\mathcal{R}_{ab} - \frac{1}{3} \mathcal{R} \eta_{ab} \right] \mathcal{D}^b \hat{A} + C' \hat{C} \\ &- \mathcal{D}^\mu B'_{ij} \mathcal{D}_\mu \hat{B}^{ij} + \left(\frac{1}{6} \mathcal{R} + 2 D \right) B'_{ij} \hat{B}^{ij} \\ &- \left[\varepsilon^{ik} B'_{ij} \hat{F}^{+\mu\nu} R(\mathcal{V})_{\mu\nu}{}^j{}_k + \varepsilon_{ik} \hat{B}^{ij} F'^{-\mu\nu} R(\mathcal{V})_{\mu\nu}{}^k{}_j \right] \\ &- 8 D \mathcal{D}^\mu A' \mathcal{D}_\mu \hat{A} + (8 i R(A)_{\mu\nu} + 2 T_\mu{}^{cij} T_{\nu cij}) \mathcal{D}^\mu A' \mathcal{D}^\nu \hat{A} \\ &- \left[\varepsilon^{ij} \mathcal{D}^\mu T_{bcij} \mathcal{D}_\mu A' \hat{F}^{+bc} + \varepsilon_{ij} \mathcal{D}^\mu T_{bc}{}^{ij} \mathcal{D}_\mu \hat{A} F'^{-bc} \right] \\ &- 4 \left[\varepsilon^{ij} T^{\mu b}{}_{ij} \mathcal{D}_\mu A' \mathcal{D}^c \hat{F}_{cb}^+ + \varepsilon_{ij} T^{\mu bij} \mathcal{D}_\mu \hat{A} \mathcal{D}^c F'^{-}_{cb} \right] \\ &+ 8 \mathcal{D}_a F'^{-ab} \mathcal{D}^c \hat{F}_{cb}^+ + 4 F'^{-ac} \hat{F}_{bc}^+ \mathcal{R}_a{}^b + \frac{1}{4} T_{ab}{}^{ij} T_{cdij} F'^{-ab} \hat{F}^{+cd} \\ &+ w \left\{ -\frac{2}{3} \mathcal{D}^a A' \mathcal{D}_a \mathcal{R} + 4 \mathcal{D}^a A' \mathcal{D}_a D - T^{acij} T_{bcij} \mathcal{D}^b \mathcal{D}_a A' \right. \\ &\left. - 2 \mathcal{D}^a F_{ab}{}'^- \mathcal{D}_c T^{cbij} \varepsilon_{ij} + i F'^{-ab} R(A)^-{}_{ad} T_b{}^{dij} \varepsilon_{ij} + F_{ab}^- T^{abij} \varepsilon_{ij} \left(\frac{1}{12} \mathcal{R} - \frac{1}{2} D \right) \right\} \end{aligned}$$

$$\begin{aligned}
& + A' \left[\frac{2}{3} \mathcal{R}^2 - 2 \mathcal{R}^{ab} \mathcal{R}_{ab} - 6 D^2 + 2 R(A)^{ab} R(A)_{ab} - R(\mathcal{V})^{+abi}{}_j R(\mathcal{V})_{ab}{}^{+j}{}_i \right. \\
& \left. + \frac{1}{128} T^{abij} T_{ab}{}^{kl} T^{cd}{}_{ij} T_{cdkl} + T^{acij} D_a D^b T_{bcij} \right] \Big\} . \tag{4.8}
\end{aligned}$$

The above Lagrangian is the central result of this paper and can be used to construct a large variety of invariants in the same way as has been done in [15]. Three brief comments should be made about it. First, in the limit $w = 0$, we recover exactly (4.2) of [15]. Second, the w -terms appear not only explicitly in the final four lines of (4.8) but also implicitly within the covariant derivatives of \hat{A} , as we have already stressed earlier. Finally, we argued in section 3 that if Φ' is set to a constant, then the action cannot actually depend on the components of $\ln \bar{\Phi}$. This is apparent in (4.8) by inspection: only the last two lines survive in this limit and they depend on the conformal supergravity fields alone. We note in particular the appearance of the non-conformal part of the Gauss-Bonnet invariant involving $\frac{2}{3} \mathcal{R}^2 - 2 \mathcal{R}^{ab} \mathcal{R}_{ab}$. This confirms our conjecture that the kinetic multiplet based upon $\ln \bar{\Phi}$ can be used to generate the $N = 2$ Gauss-Bonnet invariant. This will be the topic of the next section.

5 The $N = 2$ Gauss-Bonnet invariant in and out of superspace

We now have all of the building blocks necessary to construct the $N = 2$ Gauss-Bonnet invariant. Based on our discussion in section 2, we were led to postulate the action

$$S_{\chi}^{-} = S_{\text{W}}^{-} + S_{\text{NL}}^{-} = - \int d^4x d^4\theta \mathcal{E} \left(W^{\alpha\beta} W_{\alpha\beta} + w^{-1} \mathbb{T}(\ln \bar{\Phi}) \right) = \int d^4x \left(\mathcal{L}_{\text{W}}^{-} + \mathcal{L}_{\text{NL}}^{-} \right) \tag{5.1}$$

as the $N = 2$ supersymmetric Gauss-Bonnet, based mainly on the form its component action took in the linearized limit. Using the results of section 4, we can verify explicitly that its component Lagrangian contains the combination (1.2) of curvature-squared terms. However, the full $N = 2$ Gauss-Bonnet must not only include this combination, but must also be a topological quantity.

We will establish its topological nature in the next two sections using two complementary methods. First, we will analyze its component structure, keeping only the bosonic terms, and show that it indeed reduces to a topological quantity. In principle, this should be sufficient as it is unlikely that the fermionic terms would not be a topological invariant if the bosonic terms are. However, in order to eliminate this possibility, we will subsequently present a superspace argument which encompasses all terms.

Afterwards, we will comment briefly on an alternative way of formulating the Gauss-Bonnet in superspace which sheds further light on some of its features.

5.1 The $N = 2$ Gauss-Bonnet in components

In section 4, we provided the explicit expressions for the various components of the kinetic multiplet $\mathbb{T}(\ln \bar{\Phi})$. It is straightforward to put them together to construct the component action (5.1). To keep the calculation concise, we will again neglect all fermionic terms.

We begin with the density formula for the kinetic multiplet,

$$-2 \int d^4x d^4\theta \mathcal{E} \mathbb{T}(\ln \bar{\Phi}) = \int d^4x e \left(C|_{\mathbb{T}(\ln \bar{\Phi})} - \frac{1}{16} (T_{abij} \varepsilon^{ij})^2 A|_{\mathbb{T}(\ln \bar{\Phi})} \right) \equiv 2w \int d^4x \mathcal{L}_{\text{NL}}^- \quad (5.2)$$

where $C|_{\mathbb{T}(\ln \bar{\Phi})}$ and $A|_{\mathbb{T}(\ln \bar{\Phi})}$ are given in (4.4). We have already discussed how the dependence on the fields of the anti-chiral multiplet must be limited to total derivative terms, but we would like to explicitly check this. Making use of (4.6), we easily find

$$\begin{aligned} 2w e^{-1} \mathcal{L}_{\text{NL}}^- &= \mathcal{D}_a V^a - 2w \mathcal{R}^{ab} \mathcal{R}_{ab} + \frac{2}{3} w \mathcal{R}^2 - 6w D^2 \\ &\quad + 2w R(A)^{ab} R(A)_{ab} - w R(\mathcal{V})_{ab}^+{}^i R(\mathcal{V})^{ab+j}{}_i \\ &\quad + \frac{1}{128} w T^{abij} T_{ab}{}^{kl} T^{cd}{}_{ij} T_{cdkl} + w T^{acij} D_a D^b T_{bcij} \end{aligned} \quad (5.3)$$

where the components of the multiplet $\ln \bar{\Phi}$ are confined to the covariant term V^a given in (4.7).

The well-known conformal supergravity invariant constructed from the square of the superconformal Weyl tensor is

$$\begin{aligned} e^{-1} \mathcal{L}_{\text{W}}^- &= \frac{1}{2} C^{abcd} C_{abcd} - \frac{1}{2} C^{abcd} \tilde{C}_{abcd} - 2R(A)_{ab}^- R(A)^{ab-} + \frac{1}{2} R(\mathcal{V})_{ab}^-{}^i R(\mathcal{V})^{ab-j}{}_i \\ &\quad + 3D^2 - \frac{1}{2} T^{acij} D_a D^b T_{bcij} - \frac{1}{256} T^{abij} T_{ab}{}^{kl} T^{cd}{}_{ij} T_{cdkl} . \end{aligned} \quad (5.4)$$

Combining the expressions (5.3) and (5.4) with the appropriate coefficients leads to

$$\begin{aligned} e^{-1} \mathcal{L}_{\chi}^- &= e^{-1} \mathcal{L}_{\text{W}}^- + e^{-1} \mathcal{L}_{\text{NL}}^- = \frac{1}{2} C^{abcd} C_{abcd} - \mathcal{R}^{ab} \mathcal{R}_{ab} + \frac{1}{3} \mathcal{R}^2 - \frac{1}{2} C^{abcd} \tilde{C}_{abcd} \\ &\quad + R(A)_{ab} \tilde{R}(A)^{ab} - \frac{1}{2} R(\mathcal{V})_{ab}{}^i R(\mathcal{V})^{abj}{}_i + \frac{1}{2} w^{-1} \mathcal{D}_a V^a . \end{aligned} \quad (5.5)$$

As required, \mathcal{L}_{χ}^- is a topological invariant. It involves respectively the Euler density, the Pontryagin density, the SU(2) and U(1) topological invariants, and an explicit total covariant derivative. It is interesting (although perhaps coincidental) that the specific combination of U(1) and SU(2) curvatures appearing in the above expression can be rewritten purely in terms of the U(2) curvature.

5.2 The $N = 2$ Gauss-Bonnet is topological in superspace

Next, we give a purely superspace argument that the action (5.1) is topological — that is, it is independent (up to a total derivative) of the choice of $\bar{\Phi}$ and of the fields of conformal supergravity. We have already shown it is independent of the components of $\bar{\Phi}$ via a simple argument in section 3. Proving invariance under the supergravity fields is much more involved. In principle, the superspace connections depend in a very complicated way on the $N = 2$ conformal supergravity prepotential, which is a real scalar superfield H .¹³

¹³The references [32, 33] showed that the linearized $\mathcal{N} = 2$ Weyl multiplet can be described by a real unconstrained prepotential H , in agreement with the supercurrent analysis of [34]. The origin of such a prepotential in the harmonic superspace approach to $\mathcal{N} = 2$ supergravity (see [4] and references therein) was revealed in [35] at the linearized level, and in [36] at the fully nonlinear level.

Then applying a small deformation δH to the prepotential, the action shifts to first order, $S \rightarrow S + \delta S$, where

$$\delta S = \int d^4x d^4\theta d^4\bar{\theta} E \delta H \frac{\delta S}{\delta H} . \tag{5.6}$$

The quantity $\delta S/\delta H$ is the supercurrent multiplet provided δH is defined correctly; we will elaborate on this shortly. If the action is topological, then $\delta S/\delta H = 0$. Our goal will be to prove this last condition for the Gauss-Bonnet invariant.

To make these manipulations a bit more concrete, we consider first the second order Weyl action S_{W}^- given by the space-time integral of (2.14) involving the linearized super-Weyl tensor $W_{\alpha\beta}$. This superfield is given in terms of the prepotential H as

$$W_{\alpha\beta} = \bar{D}^4 D_{\alpha\beta} H , \tag{5.7}$$

which satisfies the Bianchi identity $D^{\alpha\beta} W_{\alpha\beta} = \bar{D}_{\dot{\alpha}\dot{\beta}} \bar{W}^{\dot{\alpha}\dot{\beta}}$. The prepotential H contains the linearized connections and covariant fields of the Weyl multiplet. Applying a small deformation $H \rightarrow H + \delta H$, one finds the action changes by

$$\delta S_{\text{W}}^- = -2 \int d^4x d^4\theta d^4\bar{\theta} \delta H D^{\alpha\beta} W_{\alpha\beta} . \tag{5.8}$$

The quantity $\mathcal{J} = -2D^{\alpha\beta} W_{\alpha\beta}$ is the (linearized) $N = 2$ supercurrent for this action. One can check that it satisfies the constraint [34]

$$D_{ij} \mathcal{J} = \bar{D}^{ij} \mathcal{J} = 0 , \tag{5.9}$$

which is a consequence of the fact that H is defined only up to the gauge transformation [36, 37]

$$\delta_{\Omega} H = \frac{1}{12} D_{ij} \Omega^{ij} + \frac{1}{12} \bar{D}^{ij} \bar{\Omega}_{ij} \tag{5.10}$$

for an unconstrained complex superfield Ω^{ij} .

These manipulations were rather simple because of the linearized nature of the super-Weyl tensor. In a generic curved background, there will be some elaborations. For instance, because H is a prepotential, it generically appears non-polynomially in the definitions of the connections and the curvature $W_{\alpha\beta}$, and so there is some ambiguity in how one should define its variation. Nevertheless, one expects that just as one can introduce small covariant deformations to the component fields,

$$\delta e_{\mu}{}^a = e_{\mu}{}^b h_b{}^a , \quad \delta \psi_{\mu}{}^{\alpha i} = e_{\mu}{}^b \varphi_b{}^{\alpha i} , \quad \text{etc.}, \tag{5.11}$$

it should be possible to introduce a similar small covariant deformation \mathcal{H} to the prepotential.¹⁴ Here the key idea is that one is deforming around an *arbitrary curved* background.

¹⁴For an extensive pedagogical discussion of this procedure for $N = 1$ supergravity, we refer the reader to the standard textbook references [38, 39]. The generalization to a manifestly superconformal setting was obtained in [40]. There seems to be no particular obstruction to implementing an analogous procedure for $N = 2$ conformal supergravity, but this has not yet been done.

The corresponding variation of the action would be

$$\delta S = \int d^4x d^4\theta d^4\bar{\theta} E \mathcal{H} \mathcal{J} \tag{5.12}$$

where the deformation \mathcal{H} and the supercurrent \mathcal{J} are both covariant conformal primary superfields, generalizing our previous formula (5.6). At the component level, this formula would simply amount to

$$\delta S = \int d^4x e \left(h^{ba} T_{ba} + \varphi^{b\alpha i} J_{b\alpha i} + \dots + \delta D J_D \right) \tag{5.13}$$

where T_{ba} is the stress-energy tensor, $J_{b\alpha i}$ is the supersymmetry current, and so on up through J_D , which is the variation of the action with respect to the field D . By comparing with the linearized case, we can deduce that \mathcal{H} must have Weyl weight $w = -2$ and so \mathcal{J} must be weight $w = 2$; it follows that the variation δD appears only in the highest component of \mathcal{H} , and so $\mathcal{J}|_{\theta=0} = -\frac{1}{4} J_D$, with the normalization given by matching to the linearized case. A gauge transformation of the component fields corresponds to a superfield gauge transformation¹⁵

$$\delta_\Omega \mathcal{H} = \frac{1}{12} \nabla_{ij} \Omega^{ij} + \frac{1}{12} \bar{\nabla}^{ij} \bar{\Omega}_{ij}, \tag{5.14}$$

which is the curved generalization of (5.10). One can check that this respects the S-supersymmetry invariance of \mathcal{H} provided Ω^{ij} has $w = -3$ and $c = -1$. For this choice, the variation of the action is zero, $\delta S = 0$, so it follows that the supercurrent \mathcal{J} must obey the current conservation equations

$$\nabla_{ij} \mathcal{J} = \bar{\nabla}^{ij} \mathcal{J} = 0, \tag{5.15}$$

which are the curved generalizations of (5.9). These conditions are invariant under S-supersymmetry precisely when \mathcal{J} has $w = 2$.

Now let us return to the case of interest. The naive covariantization of (5.8) is

$$\delta S_{\mathcal{W}}^- = -2 \int d^4x d^4\theta d^4\bar{\theta} E \mathcal{H} \nabla^{\alpha\beta} W_{\alpha\beta} \tag{5.16}$$

and so $\mathcal{J}_{\mathcal{W}} = -2 \nabla^{\alpha\beta} W_{\alpha\beta}$. In principle there could be additional covariant corrections on the right-hand side, but it is easy to see that no such corrections exist. The $N = 2$ supercurrent must be a real conformal primary $w = 2$ superfield, and the *unique* such superfield one may construct in conformal supergravity is $\nabla^{\alpha\beta} W_{\alpha\beta}$.¹⁶ Moreover, $\mathcal{J}_{\mathcal{W}}$ must also obey the constraint (5.15), which one can check is indeed satisfied for $\mathcal{J}_{\mathcal{W}} \propto \nabla^{\alpha\beta} W_{\alpha\beta}$.

¹⁵In the SU(2) superspace formulation of conformal supergravity [5], the gauge transformation (5.14) coincides with the transformation given in [41].

¹⁶This statement is a little too strong. In principle, one could have terms like $(\nabla^{\gamma\delta} W_{\gamma\delta})^2 / |\nabla^{\alpha\beta} W_{\alpha\beta}|$. The correct statement is that so long as our component action has a regular Minkowski limit, we expect that the supercurrent should also have a regular Minkowski limit, and so we may exclude such terms.

Remarkably, we may now apply the same argument to the variation of the kinetic multiplet action S_{NL}^- . Taking

$$\delta S_{\text{NL}}^- = \int d^4x d^4\theta d^4\bar{\theta} E \mathcal{H} \mathcal{J}_{\text{NL}}, \quad (5.17)$$

we observe that \mathcal{J}_{NL} *cannot* depend on $\ln \bar{\Phi}$ since the original action does not actually depend on it. Thus, \mathcal{J}_{NL} can only depend on the conformal supergravity fields. But we have just argued that this leaves only one option: $\mathcal{J}_{\text{NL}} \propto \nabla^{\alpha\beta} W_{\alpha\beta}$. This means that there must be *some* combination of $\mathbb{T}(\ln \bar{\Phi})$ and $W^{\alpha\beta} W_{\alpha\beta}$ that is topological. As we already know that the combination $W^{\alpha\beta} W_{\alpha\beta} + w^{-1} \mathbb{T}(\ln \bar{\Phi})$ yields a topological action if we turn off all fields except the vierbein and the lowest component of $\ln \bar{\Phi}$, we must conclude that

$$\mathcal{J}_{\text{NL}} = 2\nabla^{\alpha\beta} W_{\alpha\beta}. \quad (5.18)$$

It follows that

$$\delta S_{\chi}^- = \delta S_{\text{W}}^- + \delta S_{\text{NL}}^- = 0, \quad (5.19)$$

and therefore this combination is indeed topological for a generic supergravity background.

5.3 The $N = 2$ Gauss-Bonnet in an alternative superspace

We close this section by elaborating upon alternative formulations of the $N = 2$ Gauss-Bonnet in superspace. The formulation in (5.1) is very close in spirit to the component formulation (1.22) constructed from conformal gravity coupled to a scalar field. A natural question to ask is what the superspace analog of (1.23) should be, where the scalar field has been gauge-fixed to unity and conformal gravity reduced to Poincaré gravity.

This question is naturally addressed in the superspace formulation for $N = 2$ conformal supergravity given in [5] where only the Lorentz and $\text{SU}(2)$ transformations are explicitly gauged, while the remaining local superconformal symmetries are realized as super-Weyl transformations.¹⁷ We refer to this conformal supergravity formulation as $\text{SU}(2)$ superspace. The covariant superspace derivatives are given by

$$E_M^A \mathcal{D}_A = \partial_M - \frac{1}{2} \Omega_M^{ab} M_{ab} - \frac{1}{2} \mathcal{V}_M^i{}_j I^j{}_i \quad (5.20)$$

and the algebra of superspace covariant derivatives depends not only on the superfield $W_{\alpha\beta}$, which contains the Weyl multiplet, but also on additional torsion superfields S_{ij} and $Y_{\alpha\beta}$, which are both complex and symmetric in their indices, as well as the real superfield G_a . The latter torsion superfields give direct access to the Ricci tensor (as opposed to merely the Weyl tensor), which is an advantage of using this formulation as opposed to conformal superspace.

It turns out there is a straightforward mapping between conformal and $\text{SU}(2)$ superspace, which can be accomplished by adopting the K- and S-gauge $B_M = 0$ and

¹⁷This formulation makes use of the superspace geometry originally proposed in [42] without any connection with conformal supergravity.

extracting the U(1), K- and S-connections from the covariant derivative.¹⁸ These turn out to contain the multiplet associated with the Ricci tensor. Just as adopting the gauge $b_\mu = 0$ in conformal gravity allows the decomposition

$$\begin{aligned} \square_c \square_c \ln \phi &= \mathcal{D}^2 \mathcal{D}^2 \ln \phi + \mathcal{D}^a \left(\frac{2}{3} \mathcal{R} \mathcal{D}_a \ln \phi - 2 \mathcal{R}_{ab} \mathcal{D}^b \ln \phi \right) \\ &\quad - \frac{1}{2} w \mathcal{R}^{ab} \mathcal{R}_{ab} + \frac{1}{6} w \mathcal{R}^2 + \frac{1}{6} w \mathcal{D}^2 \mathcal{R}, \end{aligned} \quad (5.21)$$

performing the same procedure in superspace allows

$$\bar{\nabla}^4 \ln \bar{\Phi} = \bar{\Delta} \ln \bar{\Phi} - \frac{1}{2} w \mathbb{T}_0 \quad (5.22)$$

where

$$\bar{\Delta} \ln \bar{\Phi} := \frac{1}{96} \varepsilon_{ik} \varepsilon_{jl} \bar{\mathcal{D}}^{ij} \bar{\mathcal{D}}^{kl} \ln \bar{\Phi} - \frac{1}{96} \bar{\mathcal{D}}_{\dot{\alpha}\dot{\beta}} \bar{\mathcal{D}}^{\dot{\alpha}\dot{\beta}} \ln \bar{\Phi} + \frac{1}{6} \varepsilon_{ik} \varepsilon_{jl} \bar{\mathcal{S}}^{ij} \bar{\mathcal{D}}^{kl} \ln \bar{\Phi} + \frac{1}{6} \bar{Y}_{\dot{\alpha}\dot{\beta}} \bar{\mathcal{D}}^{\dot{\alpha}\dot{\beta}} \ln \bar{\Phi} \quad (5.23)$$

is the chiral projection operator of SU(2) superspace [43, 44] and

$$\mathbb{T}_0 := -\bar{Y}_{\dot{\alpha}\dot{\beta}} \bar{Y}^{\dot{\alpha}\dot{\beta}} - \varepsilon_{ik} \varepsilon_{jl} \bar{\mathcal{S}}^{ij} \bar{\mathcal{S}}^{kl} - \frac{1}{6} \varepsilon_{ik} \varepsilon_{jl} \bar{\mathcal{D}}^{ij} \bar{\mathcal{S}}^{kl} \quad (5.24)$$

is a combination of torsion superfields which is independent of $\bar{\Phi}$. The term $\bar{\Delta} \ln \bar{\Phi}$ of (5.22) corresponds to the first line of (5.21) while the second term involving \mathbb{T}_0 corresponds to the three w -dependent curvature terms. Moreover, the combination \mathbb{T}_0 is actually chiral since both $\bar{\nabla}^4 \ln \bar{\Phi}$ and $\bar{\Delta} \ln \bar{\Phi}$ are chiral in SU(2) superspace. This is quite a non-trivial statement since none of the individual terms are chiral, nor can the expression be written as the chiral projection of some covariant term. In other words, \mathbb{T}_0 is an *additional* non-trivial chiral invariant in SU(2) superspace, which contains the second line of (5.21) as its highest component.

The analogy we have drawn between (5.21) and (5.22) is not superficial. In the component expression (5.21), the first line possesses an inhomogeneous contribution under a Weyl transformation which is precisely balanced by the second line. The same property holds for (5.22). Using the super-Weyl transformation introduced in [5], one finds

$$\delta_\Sigma \bar{\Delta} \ln \bar{\Phi} = 2 \Sigma \bar{\Delta} \ln \bar{\Phi} + w \bar{\Delta} \bar{\Sigma}, \quad \delta_\Sigma \mathbb{T}_0 = 2 \Sigma \mathbb{T}_0 + 2 \bar{\Delta} \bar{\Sigma}, \quad (5.25)$$

with chiral parameter Σ . It follows that $\delta_\Sigma \bar{\nabla}^4 \ln \bar{\Phi} = 2 \Sigma \bar{\nabla}^4 \ln \bar{\Phi}$.

Let us now consider the action for the kinetic multiplet in SU(2) superspace, where it becomes

$$\begin{aligned} \int d^4 x d^4 \theta \mathcal{E} \Phi' \mathbb{T}(\ln \bar{\Phi}) &= -2 \int d^4 x d^4 \theta \mathcal{E} \Phi' \bar{\nabla}^4 \ln \bar{\Phi} \\ &= -2 \int d^4 x d^4 \theta d^4 \bar{\theta} E \Phi' \ln \bar{\Phi} + w \int d^4 x d^4 \theta \mathcal{E} \Phi' \mathbb{T}_0 \end{aligned} \quad (5.26)$$

¹⁸More precisely, one recovers U(2) superspace [30, 31] (see also [6]) in this manner, which can be further reduced to SU(2) superspace by an additional super-Weyl gauge-fixing [6].

after using the chiral projection operator $\bar{\Delta}$ to rewrite a chiral superspace integral in terms of a full superspace integral. It is easy to see that the part of the action involving $\bar{\Phi}$ vanishes when Φ' is a constant. The pure curvature contributions to the Gauss-Bonnet are isolated in the remaining term \mathbb{T}_0 , which is the *intrinsic* part of the kinetic multiplet and is explicitly independent of the components of $\bar{\Phi}$. In fact, there is no obstruction to performing a super-Weyl transformation to explicitly fix $\bar{\Phi}$ to a constant; then its contribution to the action vanishes completely.

Just as we proposed the action

$$\begin{aligned}
 S_{\chi}^{-} &= \int d^4x d^4\theta \mathcal{E} \left(-W^{\alpha\beta}W_{\alpha\beta} + 2w^{-1}\bar{\nabla}^4 \ln \bar{\Phi} \right) \\
 &= \int d^4x e \left(\frac{1}{2}C^{abcd}C_{abcd} - \frac{1}{2}C^{abcd}\tilde{C}_{abcd} + 2w^{-1}\square_c\square_c\bar{A} + \dots \right) \quad (5.27)
 \end{aligned}$$

as the Gauss-Bonnet in conformal superspace, we can similarly now exhibit the Gauss-Bonnet in SU(2) superspace as

$$\begin{aligned}
 S_{\chi}^{-} &= - \int d^4x d^4\theta \mathcal{E} \left(W^{\alpha\beta}W_{\alpha\beta} + \mathbb{T}_0 \right) \\
 &= \int d^4x d^4\theta \mathcal{E} \left(-W^{\alpha\beta}W_{\alpha\beta} + \bar{Y}_{\dot{\alpha}\dot{\beta}}\bar{Y}^{\dot{\alpha}\dot{\beta}} + \varepsilon_{ik}\varepsilon_{jl}\bar{S}^{ij}\bar{S}^{kl} + \frac{1}{6}\varepsilon_{ik}\varepsilon_{jl}\bar{D}^{ij}\bar{S}^{kl} \right) \\
 &= \int d^4x e \left(\frac{1}{2}C^{abcd}C_{abcd} - \frac{1}{2}C^{abcd}\tilde{C}_{abcd} - \mathcal{R}^{ab}\mathcal{R}_{ab} + \frac{1}{3}\mathcal{R}^2 + \frac{1}{3}\mathcal{D}^2\mathcal{R} + \dots \right). \quad (5.28)
 \end{aligned}$$

These two actions correspond respectively to the supersymmetric versions of the actions (1.22) and (1.23). In both cases, the details of the elided terms can be reconstructed using the explicit results for the kinetic multiplet. We further observe that because the imaginary part of the chiral superspace integral of $W^{\alpha\beta}W_{\alpha\beta}$ is a total derivative, the supersymmetric Pontryagin term, the same must hold for \mathbb{T}_0 .

It is actually possible to cast the conformal superspace action into the same form as (5.26). Suppose we have some chiral multiplet Φ_0 which is nowhere vanishing. For simplicity, let us take its weight to be $w_0 = 1$, although any nonzero weight will suffice. It is trivial to rewrite

$$\mathbb{T}(\ln \bar{\Phi}) = \mathbb{T}(\ln(\bar{\Phi}/(\bar{\Phi}_0)^w)) + w\mathbb{T}(\ln \bar{\Phi}_0) \quad (5.29)$$

where $\mathbb{T}(\ln(\bar{\Phi}/(\bar{\Phi}_0)^w))$ is a usual kinetic multiplet of a weight zero anti-chiral superfield and the non-linear behavior has been isolated within $\mathbb{T}_0 := \mathbb{T}(\ln \bar{\Phi}_0)$. It follows that

$$\int d^4x d^4\theta \mathcal{E} \Phi' \mathbb{T}(\ln \bar{\Phi}) = -2 \int d^4x d^4\theta d^4\bar{\theta} E \Phi' \ln(\bar{\Phi}/(\bar{\Phi}_0)^w) + w \int d^4x d^4\theta \mathcal{E} \Phi' \mathbb{T}_0. \quad (5.30)$$

If one then chooses to work in the gauge where $\bar{\Phi}_0 = 1$, one recovers (5.26). Note however that the action is actually independent of the choice of $\bar{\Phi}_0$.

6 Summary and conclusions

The main goal of this paper was to establish the existence of a new class of higher-derivative $N = 2$ supersymmetric invariants based on the non-linear extension of the

kinetic multiplet. Now that we have obtained its explicit form, we can address in more detail the two issues mentioned in the introduction.

In the recent paper [18], the 5D mixed gauge-gravitational Chern-Simons term constructed originally in [19] was reduced to four dimensions. The resulting 4D Lagrangian, denoted \mathcal{L}_{vww} , could not be completely classified in terms of known supersymmetric invariants. In particular, there appeared to be three sets of terms. The first set was easily identified as arising from a known chiral invariant based on a holomorphic function; the second and third sets were more puzzling. One seemed to belong to the class based on the kinetic multiplet that had already been constructed in [15], corresponding, as discussed in section 3, to a full superspace integral of a Kähler potential

$$\mathcal{H} \propto i c_A (t^A \ln \bar{\mathcal{X}}^0 - \bar{t}^A \ln \mathcal{X}^0), \tag{6.1}$$

where the coefficients c_A were real constants, the field \mathcal{X}^0 was the Kaluza-Klein vector multiplet, and the fields $t^A = \mathcal{X}^A / \mathcal{X}^0$ were the ratio of vector multiplets. The other set of terms involved curvature bilinears such as $c_A t^A \mathcal{R}^{ab} \mathcal{R}_{ab}$ and $c_A t^A R(\mathcal{V})_{ab}^{+i} R(\mathcal{V})^{+abj}_i$.

Based on the results of this paper, it has become clear to us that the second and third sets of terms actually arise from a single invariant based on the non-linear version of the kinetic multiplet. The key point is that the higher-derivative Lagrangian constructed in [15] depended on a Kähler metric $\mathcal{H}_{I\bar{J}}$ with the additional homogeneity condition $\mathcal{X}^I \mathcal{H}_{I\bar{J}} = 0$. The proposed function (6.1) does not obey this condition; however, it seems that one can relax slightly the homogeneity condition and “patch up” the component Lagrangian by including certain curvature-squared combinations, such as $c_A t^A \mathcal{R}^{ab} \mathcal{R}_{ab}$, exactly of the sort found in [18]. The resulting higher-derivative action arises not by using the full superspace action of [15], but rather the non-linear kinetic multiplet action

$$i c_A \int d^4x d^4\theta \mathcal{E} t^A \mathbb{T}(\ln \bar{\mathcal{X}}^0) + \text{h.c.} \tag{6.2}$$

constructed in this paper. This single action appears to contain the second and third sets of terms identified in [18]. In other words, it seems that the 4D Lagrangian \mathcal{L}_{vww} contains only two supersymmetric invariants: one based on a holomorphic function of chiral multiplets and the other based on (6.2). At the present time, we cannot be more definitive, as the analysis of [18] was based on a few characteristic terms only, with the goal of reconstructing what the 4D invariant should be. Now that we are confident in our identification of these terms, we plan to revisit the analysis of [18] to ensure full equality.

It is an interesting question whether the new 4D invariants we have constructed also arise from reduction of other 5D invariants. Recently, the dilaton-Weyl formulation of 5D conformal supergravity has been used to construct all of the 5D R^2 invariants [45], in addition to the Gauss-Bonnet combination [46], building on the work of [47]. It is probable that an off-shell dimensional reduction of these actions would produce 4D invariants equivalent to the ones under consideration, but such explicit reductions have not yet been undertaken.

The second question has to do with black hole entropy. Originally the first calculation of the entropy of BPS black holes involving higher-derivative couplings was based on the supersymmetric extension of the square of the Weyl tensor [21, 22]. More precisely (2.14)

was generalized to a holomorphic and homogeneous function F of weight two, depending on $W^2 = W_{\alpha\beta}W^{\alpha\beta}$ and the vector multiplets \mathcal{X}^I , i.e.

$$S \propto \int d^4x d^4\theta \mathcal{E} F(\mathcal{X}^I, W^2) + \text{h.c.} \tag{6.3}$$

A somewhat surprising result was that the actual contribution from the higher-derivative terms did not originate from the square of the Weyl tensor, but from the terms $T^{acij}D_aD^bT_{bcij}$ required by supersymmetry. Some time later, in a specific model [20], the entropy was calculated by replacing the square of the Weyl tensor by the Gauss-Bonnet combination

$$C^{abcd}C_{abcd} \implies C^{abcd}C_{abcd} - 2\mathcal{R}^{ab}\mathcal{R}_{ab} + \frac{2}{3}\mathcal{R}^2, \tag{6.4}$$

keeping the same coefficient in front of C^2 term. Since the supersymmetrization of the Gauss-Bonnet term was not known, no additional terms were included. The surprising result was that this pure Gauss-Bonnet coupling gave rise, at least in this model, to the same result as [21, 22].

With the results of this paper it is now straightforward to analyze the reasons behind this unexpected match, which holds even when including all the terms required by supersymmetry. The relevant terms in the supersymmetrization (6.3) of the Weyl tensor squared are

$$e^{-1}A' \mathcal{L}_W^- \sim A' \left\{ \frac{1}{2}C^{abcd}C_{abcd} - \frac{1}{2}C^{abcd}\tilde{C}_{abcd} - \frac{1}{2}T^{acij}D_aD^bT_{bcij} - \frac{1}{256}T^{abij}T_{ab}{}^{kl}T^{cd}{}_{ij}T_{cdkl} \right\}, \tag{6.5}$$

where A' denotes the scalar associated with the ratio of two vector multiplets. As already mentioned, the sole contribution to the BPS black hole entropy in the original calculation came from the third term above. The reason is that the Wald entropy follows in this particular case from varying the action with respect to $\mathcal{R}_{ab}{}^{cd}$ and subsequently restricting the background to ensure that the near-horizon horizon is fully supersymmetric (for further details we refer to [21, 22]). In this near-horizon background both the Weyl tensor and the Ricci scalar vanish, so that the term quadratic in the Weyl tensor cannot give a contribution to the entropy. However, it turns out that the square of the (conformally) covariant derivatives acting on T_{bcij} involve terms linear in the Ricci tensor, while the tensor fields T are non-vanishing so that this term determines the entropy.

Let us now give the relevant terms in the non-linear kinetic multiplet, which can be added to (6.5) to carry out the replacement (6.4) in the fully supersymmetric context,

$$e^{-1}A' \mathcal{L}_{NL}^- \sim A' \left\{ -\mathcal{R}^{ab}\mathcal{R}_{ab} + \frac{1}{3}\mathcal{R}^2 + \frac{1}{2}T^{acij}D_aD^bT_{bcij} + \frac{1}{256}T^{abij}T_{ab}{}^{kl}T^{cd}{}_{ij}T_{cdkl} \right\}. \tag{6.6}$$

Here the first and the third term do both contribute to the entropy, but as it turns out their contribution cancels in the near-horizon geometry by virtue of the relation

$\mathcal{R}_{ab} = -\frac{1}{8}T_a^{cij}T_{bcij}$. Hence it follows that the replacement (6.4) at the fully supersymmetric level does not affect the result for the BPS black hole entropy.¹⁹ Moreover, the terms depending on the tensor fields cancel in the sum of (6.5) and (6.6), so that in the calculation based on the Gauss-Bonnet term the supersymmetric completion will not contribute, just as indicated by the result of [20].

In addition one may also consider the actual value of the two invariants in the supersymmetric near-horizon background. This is the reason why we also included the T^4 terms in (6.5) and (6.6), as they are the only other terms that can generate additional contributions to the action in the near-horizon geometry. Working out this particular contribution, we find that (6.6) vanishes, and furthermore that the T -dependent terms vanish in the sum of (6.5) and (6.6). Hence the supersymmetric completion does not contribute to the Gauss-Bonnet coupling, and the value of the action will not change under the replacement (6.4) at the fully supersymmetric level. We should add that this last result has a bearing on the evaluation of the logarithmic corrections to the BPS entropy in [48]. There the square of the Weyl tensor and the Gauss-Bonnet invariant were equated and their contributions summed without further information of the possible supersymmetric completion of the coupling to a Gauss-Bonnet term. This was necessary in order to obtain quantitative agreement when comparing two methods for calculating the logarithmic corrections. Our above analysis thus confirms and clarifies the earlier observations in [20, 48].

We have showed for this case that the non-linear version of the kinetic multiplet vanishes at supersymmetric field configurations and it does not contribute to the entropy of a BPS black hole. A more complete analysis, establishing the existence of a BPS non-renormalization theorem in a more general Lagrangian, would proceed along the same lines as in [15], which established that Lagrangians involving the usual kinetic multiplet $\mathbb{T}(\bar{\Phi})$ will vanish for a supersymmetric background and also their first derivative with respect to fields or parameters will vanish in such a background. The latter would imply in particular that they cannot contribute to the BPS black hole entropy or to the electric charges. The proof was based on the fact that weight-zero chiral superfields must be proportional to a constant in the supersymmetric limit. For the non-linear version of the kinetic multiplet $\mathbb{T}(\ln \bar{\Phi})$ considered here, there is a marked difference because $\bar{\Phi}$ is a chiral multiplet of non-zero weight. Its supersymmetric value is therefore not necessarily proportional to a constant, which makes the corresponding BPS analysis significantly more involved, with constraints imposed on the supergravity background as well as the chiral multiplet itself. We intend to give a more thorough analysis of these features in the near future.

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¹⁹Similarly, $\mathcal{L}_{\text{NL}}^-$ contributes nothing to the electric charges of BPS black holes.

A Notations and conventions

In this paper, we have used in parallel both superspace, which is conventionally written in two-component notation, and multiplet calculus, which is usually carried out in four-component notation. To aid the reader in translating any given formula between the two notations, we summarize our conventions for both.

Space-time indices are denoted μ, ν, \dots , Lorentz indices are denoted a, b, \dots , and SU(2) indices are denoted i, j, \dots . The Lorentz metric is $\eta_{ab} = \text{diag}(-1, 1, 1, 1)$ and the antisymmetric tensor ε_{abcd} is imaginary, with $\varepsilon_{0123} = -i$. Our two-component conventions follow mainly [49] with the following modification: the spinor matrices are given by $\sigma^a = (-\mathbf{1}, -\tau^i)$ with τ^i the Pauli matrices, so that the matrix in the Dirac conjugate can be written $i\gamma^0$ as in [15]. A generic four-component Dirac fermion Ψ decomposes into spinors ψ_α and $\bar{\chi}^{\dot{\alpha}}$, which are respectively left-handed and right-handed two-component spinors. The Dirac conjugate $\bar{\Psi} = i\Psi^\dagger\gamma^0$ has components $\chi^\alpha = (\bar{\chi}^{\dot{\alpha}})^*$ and $\bar{\psi}_{\dot{\alpha}} = (\psi_\alpha)^*$. Spinor indices can be raised and lowered using the antisymmetric tensor $\epsilon_{\alpha\beta}$:

$$\psi^\beta = \epsilon^{\beta\alpha}\psi_\alpha, \quad \psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta, \quad \epsilon_{\alpha\beta}\epsilon^{\beta\gamma} = \delta_\alpha^\gamma, \quad \epsilon^{12} = \epsilon_{21} = 1. \quad (\text{A.1})$$

Similar equations pertain for $\epsilon_{\dot{\alpha}\dot{\beta}}$ and dotted spinors. We define

$$(\bar{\sigma}^a)^{\dot{\alpha}\alpha} := \epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{\alpha\beta}(\sigma^a)_{\beta\dot{\beta}}, \quad \bar{\sigma}^a = (\sigma^0, -\sigma^i) \quad (\text{A.2})$$

so that

$$(\sigma^a\bar{\sigma}^b + \sigma^b\bar{\sigma}^a)_{\alpha\beta} = -2\eta^{ab}\delta_{\alpha\beta}, \quad (\bar{\sigma}^a\sigma^b + \bar{\sigma}^b\sigma^a)^{\dot{\alpha}\dot{\beta}} = -2\eta^{ab}\delta^{\dot{\alpha}\dot{\beta}}. \quad (\text{A.3})$$

The four-component γ matrices, which differ from those of [49], are built out of the σ matrices and obey

$$\begin{aligned} \gamma^a &= \begin{pmatrix} 0 & i(\sigma^a)_{\alpha\dot{\beta}} \\ i(\bar{\sigma}^a)^{\dot{\alpha}\beta} & 0 \end{pmatrix}, \quad (\gamma^a)^\dagger = \gamma_a, \quad \{\gamma^a, \gamma^b\} = 2\eta^{ab}, \\ \gamma_5 &= -i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} \delta_{\alpha\beta} & 0 \\ 0 & -\delta^{\dot{\alpha}\dot{\beta}} \end{pmatrix}. \end{aligned} \quad (\text{A.4})$$

We define antisymmetric combinations of γ and σ matrices as

$$\begin{aligned} (\sigma^{ab})_{\alpha\beta} &:= \frac{1}{4}(\sigma^a\bar{\sigma}^b - \sigma^b\bar{\sigma}^a)_{\alpha\beta}, \quad (\bar{\sigma}^{ab})^{\dot{\alpha}\dot{\beta}} := \frac{1}{4}(\bar{\sigma}^a\sigma^b - \bar{\sigma}^b\sigma^a)^{\dot{\alpha}\dot{\beta}}, \\ \gamma^{ab} &:= \frac{1}{2}[\gamma^a, \gamma^b] = \begin{pmatrix} -2(\sigma^{ab})_{\alpha\beta} & 0 \\ 0 & -2(\bar{\sigma}^{ab})^{\dot{\alpha}\dot{\beta}} \end{pmatrix}. \end{aligned} \quad (\text{A.5})$$

One can check that $(\sigma^{ab})_{\alpha\beta} = \epsilon_{\beta\gamma}(\sigma^{ab})_{\alpha}{}^\gamma$ is *symmetric* in its spinor indices and similarly for $(\bar{\sigma}^{ab})^{\dot{\alpha}\dot{\beta}} = \epsilon^{\dot{\alpha}\dot{\gamma}}(\bar{\sigma}^{ab})^{\dot{\gamma}}{}_{\dot{\beta}}$. These obey the duality properties

$$\frac{1}{2}\varepsilon_{abcd}\sigma^{cd} = -\sigma_{ab}, \quad \frac{1}{2}\varepsilon_{abcd}\bar{\sigma}^{cd} = +\bar{\sigma}_{ab}, \quad \frac{1}{2}\varepsilon_{abcd}\gamma^{cd} = -\gamma_5\gamma_{ab}. \quad (\text{A.6})$$

The main difference between four-component and two-component notation (aside from the use of γ - versus σ -matrices) is that the latter usually yields more direct information about the Lorentz group representation of the field in question. For example, in four-component calculations, one must remember the chirality of all spinor quantities. This is accomplished in $N = 2$ multiplet calculus by using the location of the $SU(2)$ index to distinguish between the left-handed and right-handed fields; for example, ψ_μ^i and $\psi_{\mu i}$ are always, respectively, the left-handed and right-handed gravitinos while ϕ_μ^i and $\phi_{\mu i}$ are always, respectively, the right-handed and left-handed S-supersymmetry connections. In two-component notation, the first pair are written as $\psi_{\mu\alpha}^i$ and $\bar{\psi}_{\dot{\mu}}^{\dot{\alpha}i}$ and the second pair by $\bar{\phi}_{\mu}^{\dot{\alpha}i}$ and $\phi_{\mu\alpha i}$ with the explicit spinor index denoting the chirality, so one can in principle raise or lower the $SU(2)$ index using the antisymmetric tensor ε_{ij} . However, we will avoid doing this to maintain maximum compatibility with four-component notation.

Similarly, vectors and tensors can be written with spinor indices to explicitly indicate their properties under the Lorentz group. A vector V^a is associated with a field $V_{\alpha\dot{\alpha}}$ with one dotted and one undotted index via

$$V_{\alpha\dot{\alpha}} = (\sigma^a)_{\alpha\dot{\alpha}} V^a, \quad V_a = -2(\bar{\sigma}_a)^{\dot{\alpha}\alpha} V_{\alpha\dot{\alpha}}. \quad (\text{A.7})$$

Similarly, an antisymmetric two-form F_{ab} is associated with symmetric bi-spinors $F_{\alpha\beta}$ and $F_{\dot{\alpha}\dot{\beta}}$ corresponding to its anti-selfdual and selfdual parts,

$$\begin{aligned} F_{ab}^- &= (\sigma_{ab})_{\alpha\beta} F_{\alpha\beta}, & F_{ab}^+ &= (\bar{\sigma}_{ab})^{\dot{\alpha}\dot{\beta}} F_{\dot{\alpha}\dot{\beta}}, \\ F_{ab}^\pm &= \frac{1}{2}(F_{ab} \pm \tilde{F}_{ab}), & \tilde{F}_{ab} &= \frac{1}{2}\varepsilon_{abcd} F^{cd}, & \tilde{F}_{ab}^\pm &= \pm F_{ab}^\pm. \end{aligned} \quad (\text{A.8})$$

If F_{ab} is real, then $(F_{\alpha\beta})^* = -F_{\dot{\alpha}\dot{\beta}}$. We always apply symmetrization and antisymmetrization with unit strength, so that $F_{[ab]} = F_{ab}$ and $F_{(\alpha\beta)} = F_{\alpha\beta}$.

Finally, we remind the reader that $SU(2)$ indices are swapped by complex conjugation, $(T_{abij})^* = T_{ab}^{ij}$, and we make use of the invariant $SU(2)$ tensor ε^{ij} and ε_{ij} defined as $\varepsilon^{12} = \varepsilon_{12} = 1$ with $\varepsilon^{ij}\varepsilon_{kj} = \delta_k^i$. As already stated, unlike in the superspace approaches [1, 5], we do *not* raise or lower $SU(2)$ indices with the ε_{ij} tensor.

B Multiplet calculus formulation of $N = 2$ conformal supergravity

In this appendix, we present the transformation rules for the $N = 2$ conformal supergravity (or Weyl) multiplet and their relation to the superconformal algebra. Recall that the superconformal algebra comprises the generators of the general-coordinate, local Lorentz, dilatation, special conformal, chiral $SU(2)$ and $U(1)$, supersymmetry (Q) and special supersymmetry (S) transformations. The gauge fields associated with general-coordinate transformations (e_μ^a), dilatations (b_μ), R-symmetry ($\mathcal{V}_\mu^i{}_j$ and A_μ) and Q-supersymmetry (ψ_μ^i) are independent fields. The remaining gauge fields associated with the Lorentz (ω_μ^{ab}), special conformal (f_μ^a) and S-supersymmetry transformations (ϕ_μ^i) are composite objects [2, 3, 29]. The multiplet also contains three other fields: a Majorana spinor doublet χ^i , a scalar D , and a selfdual Lorentz tensor T_{abij} , which is anti-symmetric in $[ab]$ and $[ij]$. The Weyl and chiral weights have been collected in table 1.

	Weyl multiplet										parameters		
field	e_μ^a	ψ_μ^i	b_μ	A_μ	$\mathcal{V}_\mu^i{}_j$	$T_{ab}{}^{ij}$	χ^i	D	$\omega_\mu{}^{ab}$	f_μ^a	ϕ_μ^i	ϵ^i	η^i
w	-1	$-\frac{1}{2}$	0	0	0	1	$\frac{3}{2}$	2	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
c	0	$-\frac{1}{2}$	0	0	0	-1	$-\frac{1}{2}$	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
γ_5		+					+				-	+	-

Table 1. Weyl and chiral weights (w and c) and fermion chirality (γ_5) of the Weyl multiplet component fields and the supersymmetry transformation parameters.

Under Q-supersymmetry, S-supersymmetry and special conformal transformations the Weyl multiplet fields transform as

$$\begin{aligned}
 \delta e_\mu^a &= \bar{\epsilon}^i \gamma^a \psi_{\mu i} + \bar{\epsilon}_i \gamma^a \psi_\mu^i, \\
 \delta \psi_\mu^i &= 2 \mathcal{D}_\mu \epsilon^i - \frac{1}{8} T_{ab}{}^{ij} \gamma^{ab} \gamma_\mu \epsilon_j - \gamma_\mu \eta^i \\
 \delta b_\mu &= \frac{1}{2} \bar{\epsilon}^i \phi_{\mu i} - \frac{3}{4} \bar{\epsilon}^i \gamma_\mu \chi_i - \frac{1}{2} \bar{\eta}^i \psi_{\mu i} + \text{h.c.} + \Lambda_K^a e_{\mu a}, \\
 \delta A_\mu &= \frac{1}{2} i \bar{\epsilon}^i \phi_{\mu i} + \frac{3}{4} i \bar{\epsilon}^i \gamma_\mu \chi_i + \frac{1}{2} i \bar{\eta}^i \psi_{\mu i} + \text{h.c.}, \\
 \delta \mathcal{V}_\mu^i{}_j &= 2 \bar{\epsilon}_j \phi_\mu^i - 3 \bar{\epsilon}_j \gamma_\mu \chi^i + 2 \bar{\eta}_j \psi_\mu^i - (\text{h.c.}; \text{traceless}), \\
 \delta T_{ab}{}^{ij} &= 8 \bar{\epsilon}^{[i} R(Q)_{ab}{}^{j]}, \\
 \delta \chi^i &= -\frac{1}{12} \gamma^{ab} \not{D} T_{ab}{}^{ij} \epsilon_j + \frac{1}{6} R(\mathcal{V})_{\mu\nu}{}^i{}_j \gamma^{\mu\nu} \epsilon^j - \frac{1}{3} i R_{\mu\nu}(A) \gamma^{\mu\nu} \epsilon^i + D \epsilon^i + \frac{1}{12} \gamma_{ab} T^{abij} \eta_j, \\
 \delta D &= \bar{\epsilon}^i \not{D} \chi_i + \bar{\epsilon}_i \not{D} \chi^i.
 \end{aligned} \tag{B.1}$$

Here ϵ^i and ϵ_i denote the spinorial parameters of Q-supersymmetry, η^i and η_i those of S-supersymmetry, and Λ_K^a is the transformation parameter for special conformal boosts. The full superconformally covariant derivative is denoted by D_μ , while \mathcal{D}_μ denotes a covariant derivative with respect to Lorentz, dilatation, chiral U(1), and SU(2) transformations,

$$\mathcal{D}_\mu \epsilon^i = \left(\partial_\mu - \frac{1}{4} \omega_\mu{}^{cd} \gamma_{cd} + \frac{1}{2} b_\mu + \frac{1}{2} i A_\mu \right) \epsilon^i + \frac{1}{2} \mathcal{V}_\mu^i{}_j \epsilon^j. \tag{B.2}$$

The covariant curvatures are given by

$$\begin{aligned}
 R(P)_{\mu\nu}{}^a &= 2 \partial_{[\mu} e_{\nu]}^a + 2 b_{[\mu} e_{\nu]}^a - 2 \omega_{[\mu}{}^{ab} e_{\nu]b} - \frac{1}{2} (\bar{\psi}_{[\mu}{}^i \gamma^a \psi_{\nu]i} + \text{h.c.}), \\
 R(Q)_{\mu\nu}{}^i &= 2 \mathcal{D}_{[\mu} \psi_{\nu]}^i - \gamma_{[\mu} \phi_{\nu]}^i - \frac{1}{8} T^{abij} \gamma_{ab} \gamma_{[\mu} \psi_{\nu]j}, \\
 R(A)_{\mu\nu} &= 2 \partial_{[\mu} A_{\nu]} - i \left(\frac{1}{2} \bar{\psi}_{[\mu}{}^i \phi_{\nu]i} + \frac{3}{4} \bar{\psi}_{[\mu}{}^i \gamma_\nu \chi_i - \text{h.c.} \right), \\
 R(\mathcal{V})_{\mu\nu}{}^i{}_j &= 2 \partial_{[\mu} \mathcal{V}_{\nu]}^i{}_j + \mathcal{V}_{[\mu}{}^i{}_k \mathcal{V}_{\nu]}^k{}_j + 2 (\bar{\psi}_{[\mu}{}^i \phi_{\nu]j} - \bar{\psi}_{[\mu j} \phi_{\nu]}^i) - 3 (\bar{\psi}_{[\mu}{}^i \gamma_\nu \chi_j - \bar{\psi}_{[\mu j} \gamma_\nu \chi^i) \\
 &\quad - \delta_j^i (\bar{\psi}_{[\mu}{}^k \phi_{\nu]k} - \bar{\psi}_{[\mu k} \phi_{\nu]}^k) + \frac{3}{2} \delta_j^i (\bar{\psi}_{[\mu}{}^k \gamma_\nu \chi_k - \bar{\psi}_{[\mu k} \gamma_\nu \chi^k), \\
 R(M)_{\mu\nu}{}^{ab} &= 2 \partial_{[\mu} \omega_{\nu]}{}^{ab} - 2 \omega_{[\mu}{}^{ac} \omega_{\nu]c}{}^b - 4 f_{[\mu}{}^{[a} e_{\nu]}{}^{b]} + \frac{1}{2} (\bar{\psi}_{[\mu}{}^i \gamma^{ab} \phi_{\nu]i} + \text{h.c.})
 \end{aligned}$$

$$\begin{aligned}
 & + \left(\frac{1}{4} \bar{\psi}_\mu^i \psi_\nu^j T^{ab}{}_{ij} - \frac{3}{4} \bar{\psi}_{[\mu}^i \gamma_{\nu]} \gamma^{ab} \chi_i - \bar{\psi}_{[\mu}^i \gamma_{\nu]} R(Q)^{ab}{}_i + \text{h.c.} \right), \\
 R(D)_{\mu\nu} &= 2 \partial_{[\mu} b_{\nu]} - 2 f_{[\mu}{}^a e_{\nu]a} - \frac{1}{2} \bar{\psi}_{[\mu}^i \phi_{\nu]i} + \frac{3}{4} \bar{\psi}_{[\mu}^i \gamma_{\nu]} \chi_i - \frac{1}{2} \bar{\psi}_{[\mu i} \phi_{\nu]}^i + \frac{3}{4} \bar{\psi}_{[\mu i} \gamma_{\nu]} \chi^i, \\
 R(S)_{\mu\nu}{}^i &= 2 \mathcal{D}_{[\mu} \phi_{\nu]}^i - 2 f_{[\mu}{}^a \gamma_a \psi_{\nu]}^i - \frac{1}{8} \not{D} T_{ab}{}^{ij} \gamma^{ab} \gamma_{[\mu} \psi_{\nu]}^j - \frac{3}{2} \gamma_a \psi_{[\mu}^i \bar{\psi}_{\nu]}^j \gamma^a \chi_j \\
 & + \frac{1}{4} R(\mathcal{V})_{ab}{}^i \gamma^{ab} \gamma_{[\mu} \psi_{\nu]}^j + \frac{1}{2} i R(A)_{ab} \gamma^{ab} \gamma_{[\mu} \psi_{\nu]}^i, \\
 R(K)_{\mu\nu}{}^a &= 2 \mathcal{D}_{[\mu} f_{\nu]}^a - \frac{1}{4} (\bar{\phi}_{[\mu}^i \gamma^a \phi_{\nu]}^i + \bar{\phi}_{[\mu i} \gamma^a \phi_{\nu]}^i) \\
 & + \frac{1}{4} (\bar{\psi}_\mu^i D_b T^{ba}{}_{ij} \psi_\nu^j - 3 e_{[\mu}{}^a \psi_{\nu]}^i \not{D} \chi_i + \frac{3}{2} D \bar{\psi}_{[\mu}^i \gamma^a \psi_{\nu]}^j - 4 \bar{\psi}_{[\mu}^i \gamma_{\nu]} D_b R(Q)^{ba}{}_i + \text{h.c.}). \quad (\text{B.3})
 \end{aligned}$$

The connections $\omega_\mu{}^{ab}$, ϕ_μ^i and f_μ^a are algebraically determined by imposing the conventional constraints

$$\begin{aligned}
 R(P)_{\mu\nu}{}^a &= 0, \quad \gamma^\mu R(Q)_{\mu\nu}{}^i + \frac{3}{2} \gamma_\nu \chi^i = 0, \\
 e^\nu{}_b R(M)_{\mu\nu}{}^b - i \tilde{R}(A)_{\mu a} + \frac{1}{8} T_{abij} T_\mu{}^{bij} - \frac{3}{2} D e_{\mu a} &= 0. \quad (\text{B.4})
 \end{aligned}$$

Their solution is given by

$$\begin{aligned}
 \omega_\mu{}^{ab} &= -2 e^{\nu[a} \partial_{[\mu} e_{\nu]}^{b]} - e^{\nu[a} e^{b]\sigma} e_{\mu c} \partial_\sigma e_\nu{}^c - 2 e_\mu{}^{[a} e^{b]\nu} b_\nu \\
 & - \frac{1}{4} (2 \bar{\psi}_\mu^i \gamma^{[a} \psi_i^{b]} + \bar{\psi}^{ai} \gamma_\mu \psi_i^b + \text{h.c.}), \\
 \phi_\mu^i &= \frac{1}{2} \left(\gamma^{\rho\sigma} \gamma_\mu - \frac{1}{3} \gamma_\mu \gamma^{\rho\sigma} \right) \left(\mathcal{D}_\rho \psi_\sigma^i - \frac{1}{16} T^{abij} \gamma_{ab} \gamma_\rho \psi_{\sigma j} + \frac{1}{4} \gamma_{\rho\sigma} \chi^i \right), \\
 f_\mu{}^a &= \frac{1}{6} R(\omega, e) - D - \left(\frac{1}{12} e^{-1} \varepsilon^{\mu\nu\rho\sigma} \bar{\psi}_\mu^i \gamma_\nu \mathcal{D}_\rho \psi_{\sigma i} - \frac{1}{12} \bar{\psi}_\mu^i \psi_\nu^j T^{\mu\nu}{}_{ij} - \frac{1}{4} \bar{\psi}_\mu^i \gamma^\mu \chi_i + \text{h.c.} \right). \quad (\text{B.5})
 \end{aligned}$$

We will also need the bosonic part of the expression for the uncontracted connection $f_\mu{}^a$,

$$f_\mu{}^a = \frac{1}{2} R(\omega, e)_{\mu}{}^a - \frac{1}{4} \left(D + \frac{1}{3} R(\omega, e) \right) e_\mu{}^a - \frac{1}{2} i \tilde{R}(A)_{\mu}{}^a + \frac{1}{16} T_{\mu b}{}^{ij} T^{ab}{}_{ij}, \quad (\text{B.6})$$

where $R(\omega, e)_{\mu}{}^a = R(\omega)_{\mu\nu}{}^{ab} e_b{}^\nu$ is the non-symmetric Ricci tensor, and $R(\omega, e)$ the corresponding Ricci scalar. The curvature $R(\omega)_{\mu\nu}{}^{ab}$ is associated with the spin connection field $\omega_\mu{}^{ab}$.

C Superspace formulation of $N = 2$ conformal supergravity

We summarize in this appendix the structure of conformal superspace, whose component reduction reproduces the superconformal multiplet calculus. Relative to [1], we have made several changes of normalization of various operators, connections and curvatures so that the matching with tensor calculus is as transparent as possible. With the explicit results given here, one can (with some effort) reproduce the component results of section 4.

Recall that $\mathcal{N} = 2$ superspace is a supermanifold parametrized by local coordinates $z^M = (x^\mu, \theta^{m\dot{a}}, \bar{\theta}_{\dot{m}a})$. Together with superdiffeomorphisms, we equip the superspace

with additional symmetry generators — the Lorentz transformations (M_{ab}), Weyl dilatations (\mathbb{D}), chiral U(1) rotations (\mathbb{A}), SU(2) transformations (I^i_j), special conformal transformations (K_a), and S-supersymmetry (S_α^i and $\bar{S}^{\dot{\alpha}}_i$). One introduces connection one-forms associated with each of these generators, including a vielbein E_M^A associated with covariant diffeomorphisms, and defines the covariant derivative ∇_A as in (3.1). It transforms under Lorentz, dilatation and $SU(2) \times U(1)$ transformations as

$$\begin{aligned}
 [M_{ab}, \nabla_c] &= -\eta_{bc} \nabla_a + \eta_{ac} \nabla_b, & [M_{ab}, \nabla_{\gamma i}] &= -(\sigma_{ab})_{\gamma}{}^{\beta} \nabla_{\beta i}, & [M_{ab}, \bar{\nabla}^{\dot{\gamma} i}] &= -(\bar{\sigma}_{ab})^{\dot{\gamma}}{}_{\dot{\beta}} \bar{\nabla}^{\dot{\beta} i}, \\
 [\mathbb{D}, \nabla_a] &= \nabla_a, & [\mathbb{D}, \nabla_{\alpha i}] &= \frac{1}{2} \nabla_{\alpha i}, & [\mathbb{D}, \bar{\nabla}^{\dot{\alpha} i}] &= \frac{1}{2} \bar{\nabla}^{\dot{\alpha} i}, \\
 [\mathbb{A}, \nabla_{\alpha i}] &= \frac{1}{2} i \nabla_{\alpha i}, & [\mathbb{A}, \bar{\nabla}^{\dot{\alpha} i}] &= -\frac{1}{2} i \bar{\nabla}^{\dot{\alpha} i}, \\
 [I^j_i, \nabla_{\alpha k}] &= \delta_k^j \nabla_{\alpha i} - \frac{1}{2} \delta_i^j \nabla_{\alpha k}, & [I^j_i, \bar{\nabla}^{\dot{\alpha} k}] &= -\delta_k^j \bar{\nabla}^{\dot{\alpha} j} + \frac{1}{2} \delta_j^i \bar{\nabla}^{\dot{\alpha} k}.
 \end{aligned} \tag{C.1}$$

The non-trivial algebraic relations involving S , \bar{S} and K are

$$\begin{aligned}
 \{S_\alpha^i, \bar{S}^{\dot{\alpha} j}\} &= -i \delta_j^i (\sigma^a)_{\alpha\dot{\alpha}} K_a, & [K_a, \nabla_b] &= -\eta_{ab} \mathbb{D} - M_{ab}, \\
 \{S_\alpha^i, \nabla_{\beta j}\} &= -\delta_j^i \epsilon_{\alpha\beta} \mathbb{D} + 2\delta_j^i M_{\alpha\beta} - i \delta_j^i \epsilon_{\alpha\beta} \mathbb{A} + 2\epsilon_{\alpha\beta} I^i_j, \\
 \{\bar{S}^{\dot{\alpha} i}, \bar{\nabla}^{\dot{\beta} j}\} &= -\delta_j^i \epsilon^{\dot{\alpha}\dot{\beta}} \mathbb{D} + 2\delta_j^i \bar{M}^{\dot{\alpha}\dot{\beta}} + i \delta_j^i \epsilon^{\dot{\alpha}\dot{\beta}} \mathbb{A} - 2\epsilon^{\dot{\alpha}\dot{\beta}} I^j_i, \\
 [K_a, \nabla_{\alpha i}] &= i (\sigma_a)_{\alpha\dot{\beta}} \bar{S}^{\dot{\beta}}_i, & [K_a, \bar{\nabla}^{\dot{\alpha} i}] &= i (\bar{\sigma}_a)^{\dot{\alpha}\beta} S_{\beta}^i, \\
 [S_\alpha^i, \nabla_a] &= -\frac{1}{2} i (\sigma_a)_{\alpha\dot{\beta}} \bar{\nabla}^{\dot{\beta} i}, & [\bar{S}^{\dot{\alpha} i}, \nabla_a] &= -\frac{1}{2} i (\bar{\sigma}_a)^{\dot{\alpha}\beta} \nabla_{\beta i}.
 \end{aligned} \tag{C.2}$$

Above we have used $M_{\alpha\beta}$ and $\bar{M}^{\dot{\beta}\dot{\alpha}}$ as the anti-selfdual and selfdual parts of M_{ab} .

We have not yet specified the (anti-)commutation relations of the covariant derivatives. These involve non-vanishing torsion and curvature tensors, but they are all built out of the covariant Weyl superfield $W_{\alpha\beta}$, which is a chiral primary superfield obeying the Bianchi identity $\nabla^{\alpha\beta} W_{\alpha\beta} = \bar{\nabla}^{\dot{\alpha}\dot{\beta}} \bar{W}_{\dot{\alpha}\dot{\beta}}$. The algebra of the spinor derivatives is the simplest:

$$\begin{aligned}
 \{\nabla_{\alpha i}, \bar{\nabla}^{\dot{\beta} j}\} &= -2i \delta_i^j (\sigma^c)_{\alpha\dot{\beta}} \nabla_c = -2i \delta_i^j \nabla_{\alpha\dot{\beta}}, \\
 \{\nabla_{\alpha i}, \nabla_{\beta j}\} &= -2 \epsilon_{ij} \epsilon_{\alpha\beta} \bar{\mathcal{W}}, & \{\bar{\nabla}^{\dot{\alpha} i}, \bar{\nabla}^{\dot{\beta} j}\} &= -2 \epsilon^{ij} \epsilon^{\dot{\alpha}\dot{\beta}} \mathcal{W} \\
 \mathcal{W} &:= W^{\alpha\beta} M_{\alpha\beta} - \frac{1}{2} \nabla^\beta_j W_{\beta}{}^\alpha S_\alpha^j - \frac{1}{2} \nabla^{\dot{\alpha}\beta} W_{\beta}{}^\alpha K_{\alpha\dot{\alpha}}, & \bar{\mathcal{W}} &= (\mathcal{W})^*.
 \end{aligned} \tag{C.3}$$

The commutator of the spinor and vector derivatives is

$$[\nabla_{\alpha i}, \nabla_{\beta\dot{\beta}}] = -2 \epsilon_{\alpha\beta} \bar{\mathcal{W}}_{\dot{\beta} i}, \quad [\bar{\nabla}^{\dot{\alpha} i}, \nabla_{\beta\dot{\beta}}] = -2 \epsilon_{\dot{\alpha}\dot{\beta}} \mathcal{W}_{\beta}^i \tag{C.4}$$

where

$$\begin{aligned}
 \epsilon_{ij} \mathcal{W}_\alpha^j &= \frac{1}{2} i W_\alpha{}^\gamma \nabla_{\gamma i} - \frac{1}{4} i \nabla^\delta_i W_{\delta\alpha} \mathbb{D} + \frac{1}{4} \nabla^\delta_i W_{\delta\alpha} \mathbb{A} + \frac{1}{2} i \nabla^\delta_j W_{\delta\alpha} I^j_i - \frac{1}{2} i \nabla^\beta_i W^\gamma{}_\alpha M_{\beta\gamma} \\
 &+ \frac{1}{4} i \nabla_{\alpha i} \nabla^\delta_k W_{\delta}{}^\gamma S_\gamma^k - \frac{1}{2} \nabla_{\dot{\beta}}^\delta W_{\delta\alpha} \bar{S}^{\dot{\beta}}_i + \frac{1}{4} i \nabla_{\alpha i} \nabla_{\dot{\beta}}^\delta W_{\delta\beta} K^{\dot{\beta}\beta},
 \end{aligned} \tag{C.5}$$

and $\bar{W}_{\dot{\alpha}j} = (W_{\alpha^j})^*$. Finally, the commutator of the vector derivatives can be written

$$\begin{aligned}
 [\nabla_a, \nabla_b] &= -T_{ab}{}^c \nabla_c - T_{ab}{}^{\gamma j} \nabla_{\gamma j} - T_{ab\dot{\gamma}j} \bar{\nabla}^{\dot{\gamma}j} \\
 &\quad - \frac{1}{2} \hat{R}(M)_{ab}{}^{cd} M_{cd} - \frac{1}{2} R(\mathcal{V})_{ab}{}^i{}_j I^j{}_i - R(D)_{ab} \mathbb{D} - R(A)_{ab} \mathbb{A} \\
 &\quad - \frac{1}{2} R(S)_{ab}{}^{\gamma j} S_{\gamma^j} - \frac{1}{2} R(S)_{ab\dot{\gamma}j} \bar{S}^{\dot{\gamma}j} - \hat{R}(K)_{ab}{}^c K_c .
 \end{aligned} \tag{C.6}$$

We have placed circumflexes on the Lorentz curvature and K-curvature because their lowest components will differ from the corresponding curvatures in tensor calculus in a way we will soon describe. The anti-selfdual parts of the torsion and curvature tensors are

$$\begin{aligned}
 T_{\alpha\beta}{}^c &= 0, \quad T_{\alpha\beta}{}^{\gamma i} = -\frac{1}{4} \varepsilon^{ij} \nabla_j^\gamma W_{\alpha\beta}, \quad T_{\alpha\beta\dot{\gamma}i} = 0, \\
 R(\mathcal{V})_{\alpha\beta}{}^i{}_j &= \frac{1}{4} \varepsilon^{ik} \nabla_{jk} W_{\alpha\beta}, \quad R(D)_{\alpha\beta} = -iR(A)_{\alpha\beta} = \frac{1}{16} \nabla_\alpha{}^\gamma W_{\gamma\beta} + \frac{1}{16} \nabla_\beta{}^\gamma W_{\gamma\alpha}, \\
 \hat{R}(M)_{\alpha\beta}{}^{cd} M_{cd} &= \frac{1}{4} \nabla^{\gamma\delta} W_{\alpha\beta} M_{\delta\gamma} - \frac{1}{4} \nabla_{\delta\gamma} W^{\delta\gamma} M_{\alpha\beta} - W_{\alpha\beta} \bar{W}_{\dot{\gamma}\delta} \bar{M}^{\dot{\delta}\gamma}, \\
 R(S)_{\alpha\beta}{}^{\gamma i} &= \frac{1}{24} \varepsilon^{jk} \nabla_{ij} \nabla_{\beta k} W_\alpha{}^\gamma + \frac{1}{24} \varepsilon^{jk} \nabla_{ij} \nabla_{\alpha k} W_\beta{}^\gamma, \\
 R(S)_{\alpha\beta\dot{\gamma}i} &= -\frac{1}{4} i \varepsilon^{ij} \nabla_{\beta j} \nabla_{\dot{\gamma}}{}^\gamma W_{\gamma\alpha} - \frac{1}{4} i \varepsilon^{ij} \nabla_{\alpha j} \nabla_{\dot{\gamma}}{}^\gamma W_{\gamma\beta} + \frac{1}{2} W_{\alpha\beta} \bar{\nabla}_{\dot{\delta}}{}^i \bar{W}^{\dot{\delta}\gamma}, \\
 \hat{R}(K)_{\alpha\beta}{}^c &= -\frac{1}{16} \nabla_{\alpha\beta} \nabla^{\gamma\delta} W_\delta{}^\gamma (\sigma^c)_{\gamma\dot{\gamma}} + \frac{1}{4} W_{\alpha\beta} \nabla_{\dot{\gamma}\delta} \bar{W}^{\dot{\delta}\gamma} (\bar{\sigma}^c)^{\dot{\gamma}\gamma}.
 \end{aligned} \tag{C.7}$$

The selfdual parts can be found by complex conjugation. These algebraic relations completely determine the superspace geometry.

The component structure of any superspace theory can be found by identifying the independent components of the superfields and taking the $\theta = \bar{\theta} = 0$ limit, which we denote by $|\theta=0$. For the connections, we identify

$$\begin{aligned}
 e_\mu{}^a &\equiv E_\mu{}^a|_{\theta=0}, & \psi_\mu{}^{\alpha i} &\equiv 2E_\mu{}^{\alpha i}|_{\theta=0}, & \bar{\psi}_{\mu\dot{\alpha}i} &\equiv 2E_{\mu\dot{\alpha}i}|_{\theta=0} \\
 A_\mu &\equiv A_\mu|_{\theta=0}, & b_\mu &\equiv B_\mu|_{\theta=0}, & \omega_\mu{}^{ab} &\equiv \Omega_\mu{}^{ab}|_{\theta=0}, & \mathcal{V}_\mu{}^i{}_j &\equiv \mathcal{V}_\mu{}^i{}_j|_{\theta=0} \\
 \hat{f}_\mu{}^a &\equiv F_\mu{}^a|_{\theta=0}, & \phi_\mu{}^{\alpha i} &\equiv \Phi_\mu{}^{\alpha i}|_{\theta=0}, & \bar{\phi}_{\mu\dot{\alpha}i} &\equiv \bar{\Phi}_{\mu\dot{\alpha}i}|_{\theta=0}.
 \end{aligned} \tag{C.8}$$

The covariant components of the Weyl multiplet are found within the superfield $W_{\alpha\beta}$. The tensor $T_{ab}{}^{ij}$, spinor $\chi_\alpha{}^i$ and scalar D are given by

$$T_{ab}{}^{ij} := 2\varepsilon^{ij} (\sigma_{ab})_\beta{}^\alpha W_\alpha{}^\beta|_{\theta=0}, \quad \chi_\alpha{}^i := -\frac{1}{3} \nabla^{\beta i} W_{\beta\alpha}|_{\theta=0}, \quad D := \frac{1}{12} \nabla^{\alpha\beta} W_{\beta\alpha}|_{\theta=0}. \tag{C.9}$$

One can define the component covariant derivative by $\hat{D}_a = \nabla_a|_{\theta=0}$, leading to

$$\begin{aligned}
 e_\mu{}^a \hat{D}_a &= \partial_\mu - \frac{1}{2} \psi_\mu{}^{\alpha i} Q_{\alpha i} - \frac{1}{2} \bar{\psi}_{\mu\dot{\alpha}i} \bar{Q}^{\dot{\alpha}i} - \frac{1}{2} \omega_\mu{}^{ab} M_{ab} - b_\mu \mathbb{D} - A_\mu \mathbb{A} - \frac{1}{2} \mathcal{V}_\mu{}^i{}_j I^j{}_i \\
 &\quad - \frac{1}{2} \phi_\mu{}^{\alpha i} S_\alpha{}^i - \frac{1}{2} \bar{\phi}_{\mu\dot{\alpha}i} \bar{S}^{\dot{\alpha}i} - \hat{f}_\mu{}^a K_a,
 \end{aligned} \tag{C.10}$$

where we identify $Q_{\alpha i} := \nabla_{\alpha i}|_{\theta=0}$ as the supersymmetry transformation on a component field.

The covariant derivative D_a used in multiplet calculus differs slightly from \hat{D}_a . They are related by a redefinition of the K-connection,²⁰

$$D_a = \hat{D}_a + \frac{3}{4}D K_a, \quad f_\mu^a = \hat{f}_\mu^a - \frac{3}{4}e_\mu^a. \quad (\text{C.11})$$

The component curvatures are

$$\begin{aligned} [D_a, D_b] = & -\frac{1}{2}R(Q)_{ab}{}^{\gamma j} Q_{\gamma j} - \frac{1}{2}R(Q)_{ab\dot{\gamma}j} \bar{Q}^{\dot{\gamma}j} \\ & - \frac{1}{2}R(M)_{ab}{}^{cd} M_{cd} - \frac{1}{2}R(\mathcal{V})_{ab}{}^i{}_j I^i{}_j - R(D)_{ab} \mathbb{D} - R(A)_{ab} \mathbb{A} \\ & - \frac{1}{2}R(S)_{ab}{}^{\gamma j} S_{\gamma j} - \frac{1}{2}R(S)_{ab\dot{\gamma}j} \bar{S}^{\dot{\gamma}j} - R(K)_{ab}{}^c K_c. \end{aligned} \quad (\text{C.12})$$

These are related to the superspace curvatures by

$$\begin{aligned} R(Q)_{ab}{}^{\alpha i} &= 2T_{ab}{}^{\alpha i}|_{\theta=0}, & R(M)_{ab}{}^{cd} &= \hat{R}(M)_{ab}{}^{cd}|_{\theta=0} + 3D \delta_a^{[c} \delta_b^{d]}, \\ R(K)_{ab}{}^c &= \hat{R}(K)_{ab}{}^c|_{\theta=0} - \frac{3}{2}D_{[a} D \delta_{b]}^c, \end{aligned} \quad (\text{C.13})$$

while the other curvatures in (C.12) are the $\theta = 0$ projections of the corresponding superspace curvatures.

Component gauge transformations can be derived directly from how their corresponding superfields transform. One may explicitly rederive (B.1), for example, by taking the transformations with Q-supersymmetry parameters $\xi^{\alpha i} = \epsilon^{\alpha i}$ and $\bar{\xi}_{\dot{\alpha} i} = \bar{\epsilon}_{\dot{\alpha} i}$, S-supersymmetry parameters $\eta_\alpha{}^i$ and $\bar{\eta}_{\dot{\alpha} i}$, and special conformal parameter Λ_K^a . For example, if Ψ is some covariant superfield (e.g. Φ , $\nabla_{\alpha i} \Phi$, etc.)

$$\delta\Psi|_{\theta=0} = \left(\epsilon^{\alpha i} \nabla_{\alpha i} \Psi + \bar{\epsilon}_{\dot{\alpha} i} \bar{\nabla}^{\dot{\alpha} i} \Psi + \eta_\alpha{}^i S_\alpha{}^i \Psi + \bar{\eta}_{\dot{\alpha} i} \bar{S}^{\dot{\alpha} i} \Psi + \Lambda_K^a K_a \Psi \right)|_{\theta=0}. \quad (\text{C.14})$$

As a simple example, let us consider $T_{ab}{}^{ij}$:

$$\delta T_{ab}{}^{ij} = -2\varepsilon^{ij} (\sigma_{ab})^{\beta\alpha} \epsilon^{\gamma k} \nabla_{\gamma k} W_{\alpha\beta}|_{\theta=0} = 8\epsilon^{\gamma[i} R(Q)_{ab}{}^{\gamma j]}, \quad (\text{C.15})$$

using (C.7) and (C.13). The transformation rules for $\chi_\alpha{}^i$ and D can be derived similarly. For the connections (C.8), the transformation rules follow from covariant diffeomorphisms and gauge transformations for the superspace connections.²¹

D Gauss-Bonnet invariant in $N = 1$ conformal supergravity

In the main body of the paper, we have constructed the $N = 2$ Gauss-Bonnet using conformal supergravity, corresponding to the approach taken in the introduction for the non-supersymmetric case. Because the $N = 1$ Gauss-Bonnet is not usually described in this way, it is reasonable to give a brief discussion showing how the same construction

²⁰The difference in the K-connection corresponds to a slight modification of the third conventional constraint (B.4).

²¹See, for example, the recent discussion in [50].

proceeds in that case. As it is somewhat out of the main line of presentation of the paper, we have placed the discussion in this brief appendix.

Recall that all invariants in $N = 1$ superspace can be written either as integrals over the full superspace or over chiral superspace,

$$\int d^4x d^2\theta d^2\bar{\theta} E \mathcal{L}, \quad \int d^4x d^2\theta \mathcal{E} \mathcal{L}_{\text{ch}}. \quad (\text{D.1})$$

In the superspace associated with conformal supergravity, the covariant derivative ∇_A is constructed with a connection associated with each generator in the $N = 1$ superconformal algebra. The algebra of covariant derivatives is given in [51] and is constrained so that all curvatures depend only on the weight 3/2 chiral superfield $W_{\alpha\beta\gamma}$, which contains the $N = 1$ Weyl multiplet.²² The single invariant action one can construct in pure $N = 1$ conformal supergravity involves the chiral superspace integral

$$\int d^4x d^2\theta \mathcal{E} W^{\alpha\beta\gamma} W_{\alpha\beta\gamma} = \int d^4x e \left(\frac{1}{4} C_{abcd} C^{abcd} - \frac{1}{4} C_{abcd} \tilde{C}^{abcd} + \text{additional terms} \right). \quad (\text{D.2})$$

To construct the additional terms in (1.22) requires a compensator field. The simplest possibility is a chiral superfield Φ of weight w . It is easy to see that in flat superspace

$$-\frac{1}{64} \int d^2\theta \bar{D}^2 D^2 \bar{D}^2 \ln \bar{\Phi} = -\frac{1}{4} \int d^2\theta \bar{D}^2 \square \ln \bar{\Phi} = \square \square \ln \bar{A}. \quad (\text{D.3})$$

The generalization of this chiral integrand to conformal superspace turns out to be its naive covariantization: $\mathbb{S}(\ln \bar{\Phi}) = -\frac{1}{64} \bar{\nabla}^2 \nabla^2 \bar{\nabla}^2 \ln \bar{\Phi}$. One can check that \mathbb{S} is a covariant conformal primary chiral multiplet of weight 3. The proposed chiral invariant corresponding to the $N = 1$ Gauss-Bonnet is

$$\Gamma := W^{\alpha\beta\gamma} W_{\alpha\beta\gamma} + w^{-1} \mathbb{S}(\ln \bar{\Phi}),$$

$$\int d^4x d^2\theta \mathcal{E} \Gamma = \int d^4x e \left(\frac{1}{4} C^{abcd} C_{abcd} - \frac{1}{4} C^{abcd} \tilde{C}_{abcd} + w^{-1} \square_c \square_c \ln \bar{A} + \dots \right) \quad (\text{D.4})$$

where $\square_c := D^a D_a$ for $D_a = \nabla_a|_{\theta=0}$, the supercovariant derivative of $N = 1$ conformal supergravity, and we have kept only the relevant terms.

The $N = 1$ Gauss-Bonnet is usually formulated in Wess-Zumino superspace.²³ To compare our expression to the usual one, we must rewrite the conformally covariant derivatives ∇_A in terms of the Wess-Zumino covariant derivatives \mathcal{D}_A . The result of the degauging process is

$$\mathbb{S}(\ln \bar{\Phi}) = \Delta \ln \bar{\Phi} + w \mathbb{S}_0,$$

$$\Delta \ln \bar{\Phi} := -\frac{1}{64} (\bar{D}^2 - 4R) \left(\mathcal{D}^2 \bar{D}^2 \ln \bar{\Phi} + 8\mathcal{D}^\alpha (G_{\alpha\dot{\alpha}} \bar{D}^{\dot{\alpha}} \ln \bar{\Phi}) \right),$$

²²The normalization conventions in [51] were originally chosen to coincide with [49], but in this appendix we follow the normalization conventions of [39]. This requires that we rescale the supersymmetric Weyl tensor as $W_{\alpha\beta\gamma} \rightarrow 2W_{\alpha\beta\gamma}$.

²³The details of Wess-Zumino superspace are covered in the standard references [38, 39, 49], and its auxiliary field structure corresponds to old minimal supergravity. The Gauss-Bonnet invariant may be equally well constructed in new minimal [52] or U(1) supergravity [53].

$$\mathbb{S}_0 := -\frac{1}{4}(\bar{\mathcal{D}}^2 - 4R)\left(2R\bar{R} + G^a G_a - \frac{1}{4}\mathcal{D}^2 R\right), \quad (\text{D.5})$$

where here and below we use the usual $N = 1$ abbreviation $\mathcal{D}^2 = \mathcal{D}^\alpha \mathcal{D}_\alpha$ and $\bar{\mathcal{D}}^2 = \bar{\mathcal{D}}_{\dot{\alpha}} \bar{\mathcal{D}}^{\dot{\alpha}}$. In this expression, the additional torsion superfields R and G_a of Wess-Zumino superspace appear; these contain respectively the Ricci scalar and the Einstein tensor. The equation (D.5) may be compared both to the analogous $N = 0$ result (5.21) and to the $N = 2$ result (5.22). Under a super-Weyl transformation [54, 55] involving a chiral parameter Σ , the spinor covariant derivative and the curvature superfields transform as

$$\begin{aligned} \delta_\Sigma \mathcal{D}_\alpha &= \left(\bar{\Sigma} - \frac{1}{2}\Sigma\right)\mathcal{D}_\alpha + \mathcal{D}^\beta \Sigma M_{\beta\alpha}, & \delta_\Sigma W_{\alpha\beta\gamma} &= \frac{3}{2}\Sigma W_{\alpha\beta\gamma}, \\ \delta_\Sigma R &= \frac{1}{4}\bar{\mathcal{D}}^2 \bar{\Sigma} + (2\Sigma - \bar{\Sigma})R, & \delta_\Sigma G_{\alpha\dot{\alpha}} &= \frac{1}{2}(\Sigma + \bar{\Sigma})G_{\alpha\dot{\alpha}} + i\mathcal{D}_{\alpha\dot{\alpha}}(\Sigma - \bar{\Sigma}), \end{aligned} \quad (\text{D.6})$$

while Φ transforms as $\delta\Phi = w\Sigma\Phi$. One can check that

$$\delta_\Sigma \Delta \ln \bar{\Phi} = 3\Sigma \Delta \ln \bar{\Phi} + w\Delta \bar{\Sigma}, \quad \delta_\Sigma \mathbb{S}_0 = 3\Sigma \mathbb{S}_0 - \Delta \bar{\Sigma}, \quad (\text{D.7})$$

which ensures that $\mathbb{S}(\ln \bar{\Phi})$ transforms homogeneously, $\delta_\Sigma \mathbb{S}(\ln \bar{\Phi}) = 3\Sigma \mathbb{S}(\ln \bar{\Phi})$.

In the form (D.5), it is easy to see that the chiral superspace integral of $\mathbb{S}(\ln \bar{\Phi})$ depends on the superfield $\bar{\Phi}$ only via a total covariant derivative; discarding any explicit total derivatives, one finds

$$\begin{aligned} \int d^4x d^2\theta \mathcal{E} \Gamma &= \int d^4x d^2\theta \mathcal{E} W^{\alpha\beta\gamma} W_{\alpha\beta\gamma} + \int d^4x d^2\theta d^2\bar{\theta} E \left(2R\bar{R} + G^a G_a\right) \\ &= \int d^4x \left(\frac{1}{4}\mathcal{L}_\chi - \frac{1}{4}\mathcal{L}_P + \text{total derivative}\right). \end{aligned} \quad (\text{D.8})$$

This is the $N = 1$ Gauss-Bonnet in old minimal supergravity [56, 57], whose real and imaginary parts correspond respectively to the usual Gauss-Bonnet invariant and the Pontryagin term. Its topological nature in superspace was first demonstrated in [58] (see also [39]). The full component expression appeared first in [59].

There is a curious feature of the operator Δ which deserves comment. Taking two chiral multiplets Φ and Φ' , both now of weight zero, it is natural to define

$$\mathbb{S}(\bar{\Phi}) := -\frac{1}{64}\bar{\nabla}^2 \nabla^2 \bar{\nabla}^2 \bar{\Phi} \equiv \Delta \bar{\Phi}. \quad (\text{D.9})$$

One can check that

$$\delta_\Sigma \Delta \bar{\Phi} = 3\Sigma \Delta \bar{\Phi} \quad (\text{D.10})$$

and so Δ can be viewed as a super-Weyl covariant mapping from a weight-zero anti-chiral multiplet to a weight-3 chiral multiplet. This is the $N = 1$ generalization of the Fradkin-Tseytlin operator discussed in section 1. One finds²⁴

$$\begin{aligned} \int d^4x d^2\theta \mathcal{E} \Phi' \mathbb{S}(\bar{\Phi}) &= \int d^4x d^2\theta \mathcal{E} \Phi' \Delta \bar{\Phi} = \int d^4x e A' \square_c \square_c \bar{A} + \dots \\ &= \int d^4x e \left(\mathcal{D}^a \mathcal{D}_a A' \mathcal{D}^b \mathcal{D}_b \bar{A} + \mathcal{D}^a A' \left(2\mathcal{R}_{ab} - \frac{2}{3}\eta_{ab}\mathcal{R} \right) \mathcal{D}^b \bar{A} + \dots \right). \end{aligned} \quad (\text{D.11})$$

²⁴More details regarding the component expression can be found in [60].

One may equally write

$$\int d^4x d^2\theta \mathcal{E} \Phi' \Delta \bar{\Phi} = \frac{1}{16} \int d^4x d^4\theta E \left(\mathcal{D}^2 \Phi' \bar{\mathcal{D}}^2 \bar{\Phi} - 8 \mathcal{D}^\alpha \Phi' G_{\alpha\dot{\alpha}} \bar{\mathcal{D}}^{\dot{\alpha}} \bar{\Phi} \right). \quad (\text{D.12})$$

The expression on the left is super-Weyl invariant as a consequence of (D.10), while this property is obscured for the expression on the right. However, one can check that it does transform into

$$\begin{aligned} & -\frac{1}{8} \int d^4x d^4\theta E \left(\mathcal{D}^\alpha \Sigma \mathcal{D}_\alpha \Phi \bar{\mathcal{D}}^2 \bar{\Phi} + 4i \mathcal{D}_{\alpha\dot{\alpha}} \Sigma \mathcal{D}^\alpha \Phi \bar{\mathcal{D}}^{\dot{\alpha}} \bar{\Phi} + \text{c.c.} \right) \\ & = -\frac{1}{8} \int d^4x d^4\theta E \left(\bar{\Phi} \bar{\mathcal{D}}^2 (\mathcal{D}^\alpha \Sigma \mathcal{D}_\alpha \Phi) + 4i \bar{\Phi} \bar{\mathcal{D}}^{\dot{\alpha}} (\mathcal{D}_{\alpha\dot{\alpha}} \Sigma \mathcal{D}^\alpha \Phi) + \text{c.c.} \right) = 0. \end{aligned} \quad (\text{D.13})$$

One might wonder whether a simpler version of this operator could be constructed. The obvious proposal of $\Phi \nabla^2 \bar{\nabla}^2 \bar{\Phi}$ integrated over the full superspace is unfortunately not a conformal primary; equivalently, there is no conformally covariant anti-chiral (or chiral) d'Alembertian in Wess-Zumino superspace. This is in agreement with the discussion in [61] that the analysis of [26] is not directly applicable in superspace. Rather, one requires the (higher dimension) operator Δ . The $N = 1$ supersymmetric generalization of the construction of [26] is given in [62].

We should also mention that the $N = 2$ version of the Fradkin-Tseytlin operator can be constructed in the context of the $SU(2)$ superspace discussed in section 5.3. It is simply the covariant chiral projector $\bar{\Delta}$, and the actions analogous to (D.12) are

$$\int d^4x d^4\theta \mathcal{E} \Phi' \bar{\Delta} \bar{\Phi} = \int d^4x d^4\theta d^4\bar{\theta} E \Phi' \bar{\Phi}, \quad (\text{D.14})$$

where Φ and Φ' are weight-zero chiral superfields and both integrands are manifestly super-Weyl covariant. This is exactly the action considered in [15].

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