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Functional relations and the Yang-Baxter algebra

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Abstract. Functional equations methods are a fundamental part of the theory of Exactly Solvable Models in Statistical Mechanics and they are intimately connected with Baxter's concept of commuting transfer matrices. This concept has culminated in the celebrated Yang-Baxter equation which plays a fundamental role for the construction of quantum integrable systems and also for obtaining their exact solution. Here I shall discuss a proposal that has been put forward in the past years, in which the Yang-Baxter algebra is viewed as a source of functional equations describing quantities of physical interest. For instance, this method has been successfully applied for the description of the spectrum of open spin chains, partition functions of elliptic models with domain wall boundaries and scalar product of Bethe vectors. Further applications of this method are also discussed.

1. Introduction

Exact solutions have played an important role for the development of physical theories and assumptions and their contributions can be seen in a variety of contexts. For instance, Onsager's solution of the two-dimensional Ising model [1] not only showed that the formalism of Statistical Mechanics was indeed able to describe phase transitions, but also unveiled that the critical behavior of the Ising model specific heat was not included in Landau's theory of critical exponents [2]. In a different context, the exact solution of the one-dimensional Heisenberg spin chain [3] was also of fundamental importance for elucidating the value of the spin carried by a spin wave [4].

Bethe's celebrated solution of the isotropic Heisenberg chain [3] consists of a fundamental stone of the modern theory of quantum integrable systems and its influence can be seen in several areas ranging from Quantum Field Theory [5, 6] to Combinatorics [7]. The hypothesis employed by Bethe for the model wave function is known nowadays as 'Bethe ansatz' and it became a standard tool in the theory of quantum integrable systems. On the other hand, Onsager's solution of the two-dimensional Ising model was based on Kramers and Wannier transfer matrix technique [8, 9] which did not have any previous connection with Bethe ansatz. Nevertheless, the transfer matrix technique was later on recognized as a fundamental ingredient for the establishment of integrability in the sense of Baxter [10].

Within this scenario Baxter's concept of integrability appeared as an analogous of Liouville's classical concept where now the transfer matrix was playing the role of generating function of quantities in involution [10]. More precisely, in Baxter's framework a family of mutually commuting operators is obtained as a consequence of transfer matrices which commute for different values of their parameters. In their turn, these commutative transfer matrices are built directly from solutions of the Yang-Baxter equation.

The importance of the Yang-Baxter equation was only better understood with the proposal of the Quantum Inverse Scattering Method [11, 12]. This method unified the transfer matrix approach, the Yang-Baxter equation and the Bethe ansatz employed to solve a variety of one-dimensional quantum many-body systems. Besides that, the Quantum Inverse Scattering Method, or QISM for short, put in evidence the so called Yang-Baxter algebra which latter on led to the notion of Quantum Groups [13].

The Yang-Baxter algebra plays a fundamental role within the QISM and it is one of the main ingredients for the construction of exact eigenvectors of transfer matrices of two-dimensional lattice systems and hamiltonians of one-dimensional quantum many-body systems. However, the applications of the Yang-Baxter algebra are not limited to that and alternative ways of exploring the Yang-Baxter algebra are also known in the literature. For instance, it can be used to build solutions of the Knizhnik-Zamolodchikov equation [14] in the sense of [15, 16].

More recently, the Yang-Baxter algebra was also shown to be capable of rendering functional equations describing quantities such as the spectrum of spin chains and partition functions of vertex models [17–22]. Here we aim to discuss this latter possibility.

This article is organized as follows. In Section 2 we introduce definitions which will be relevant throughout this paper and also present the lattice systems we shall consider by means of this algebraic-functional approach. In Section 3 we illustrate how the Yang-Baxter algebra can be converted into functional equations and, in particular, we derive functional relations describing the partition function of the elliptic Eight-Vertex-SOS model with domain wall boundaries and scalar products of Bethe vectors. The solutions of the aforementioned functional equations are also presented in Section 3, and in Section 4 we unveil a family of partial differential equations underlying our functional relations. Concluding remarks are then discussed in Section 5.

2. Yang-Baxter relations and lattice systems

Lattice systems of Statistical Mechanics have a long history and remarkable examples share the property of being exactly solvable [23]. The most prominent examples, such as the Ising model and Eight-Vertex model, have been solved in two-dimensions and this choice of dimensionality undoubtedly grants them special properties. For instance, in [24, 25] Smirnov proved that the scaling limit of the critical site percolation on a two-dimensional triangular lattice is conformally invariant. The importance of this proof can be seen in two ways: from the mathematical perspective Smirnov's proof introduced the concept of 'discrete' harmonic functions. On the other hand, this proof provides a solid ground for the CFT (Conformal Field Theory) methods employed by Cardy in [26].

The concept of exact solvability seems to be intrinsically dependent on the method we are employing. However, it is nowadays well accepted that Baxter's commuting transfer matrices approach [10] plays a fundamental role for two-dimensional lattice systems within a variety of methods. For instance, the requirement of commuting transfer matrices leads us to the Yang-Baxter equation/algebra [11, 12] and also their dynamical counterparts [27, 28]. Those algebraic relations constitute the foundations of the algebraic Bethe ansatz [12] and, as firstly demonstrated in [17], they are also able to describe spectral problems in terms of functional equations.

Vertex and Solid-on-Solid models are some important examples of exactly solvable lattice systems and both of them admit an operatorial description in terms of generators of the Yang-Baxter algebra and its dynamical version. Here we will be mainly interested in the so called Eight-Vertex-SOS and Six-Vertex models, and for that it is enough to present only the dynamical Yang-Baxter equation/algebra while the standard Yang-Baxter relations will be obtained as a particular limit.

Dynamical Yang-Baxter equation. Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\mathbb{C}$ and $\mathfrak{h} \subset \mathfrak{g}$ be an abelian Lie subalgebra. Also, let $\mathbb{V} = \bigoplus_{\phi \in \mathfrak{h}^*} \mathbb{V}[\phi]$ with $\mathbb{V}[\phi] = \{ v \in \mathbb{V} \mid hv = \phi(h) v \text{ for } h \in \mathfrak{h} \}$ be a diagonalizable $\mathfrak{h}$-module. Then for $\lambda_j, \gamma, \theta \in \mathbb{C}$ the dynamical Yang-Baxter equation reads

$$\begin{aligned} \mathcal{R}_{12}(\lambda_1 - \lambda_2, \theta - \gamma h_3) \mathcal{R}_{13}(\lambda_1 - \lambda_3, \theta) \mathcal{R}_{23}(\lambda_2 - \lambda_3, \theta - \gamma h_1) = \\ \mathcal{R}_{23}(\lambda_2 - \lambda_3, \theta) \mathcal{R}_{13}(\lambda_1 - \lambda_3, \theta - \gamma h_2) \mathcal{R}_{12}(\lambda_1 - \lambda_2, \theta) . \end{aligned} \tag{2.1}$$

Eq. (2.1) is a relation for an operator $\mathcal{R}_{ij} : \mathbb{C} \times \mathfrak{h}^* \mapsto \mathrm{End}(\mathbb{V}_i \otimes \mathbb{V}_j)$ where $\mathbb{V}_i$ $(i = 1, 2, 3)$ are finite dimensional diagonalizable $\mathfrak{h}$-modules. In this way we have (2.1) defined in $\mathrm{End}(\mathbb{V}_1 \otimes \mathbb{V}_2 \otimes \mathbb{V}_3)$ with tensor products being understood as

$$\mathcal{R}_{12}(\lambda, \theta - \gamma h_3)(v_1 \otimes v_2 \otimes v_3) = (\mathcal{R}_{12}(\lambda, \theta - \gamma \phi)(v_1 \otimes v_2)) \otimes v_3 . \tag{2.2}$$

The term ϕ in (2.2) corresponds to the weight of v_3 while the remaining elements $\mathcal{R}_{13}$ and $\mathcal{R}_{23}$ are computed by analogy. As far as the solutions of (2.1) are concerned, it is currently well understood the importance of the elliptic quantum groups $E_{p,q}[\mathfrak{g}]$ for their characterization. Here we shall restrict ourselves to the case $\mathfrak{g} \simeq \mathfrak{sl}(2)$, and in that case we consider $\mathfrak{h}$ as the $\mathfrak{sl}(2)$ Cartan subalgebra while $\mathbb{V} \cong \mathbb{C}^2$. Thus $h = \mathrm{diag}(1, -1)$ and we have the explicit solution

$$\mathcal{R}(\lambda, \theta) = \begin{pmatrix} a_+(\lambda,\theta) & 0 & 0 & 0 \\ 0 & b_+(\lambda,\theta) & c_+(\lambda,\theta) & 0 \\ 0 & c_-(\lambda,\theta) & b_-(\lambda,\theta) & 0 \\ 0 & 0 & 0 & a_-(\lambda,\theta) \end{pmatrix} \qquad \begin{aligned} a_\pm(\lambda,\theta) &= f(\lambda + \gamma) \\ b_\pm(\lambda,\theta) &= f(\lambda) \tfrac{f(\theta \mp \gamma)}{f(\theta)} \\ c_\pm(\lambda,\theta) &= f(\gamma) \tfrac{f(\theta \mp \lambda)}{f(\theta)} \end{aligned} . \tag{2.3}$$

The function f in (2.3) is essentially a Jacobi Theta-function. More precisely we have $f(\lambda) = \Theta_1(\mathrm{i}\lambda, \tau)/2$, according to the conventions of [29], and the dependence of f with the elliptic nome τ is omitted for convenience.

Dynamical Yang-Baxter algebra. Let $\mathbb{V}_a \cong \mathbb{V}$, $\mathbb{V}_{\mathcal{Q}} \cong \mathbb{V}^{\otimes L}$ and consider an operator $\mathcal{T}_a \in \mathrm{End}(\mathbb{V}_a \otimes \mathbb{V}_{\mathcal{Q}})$ for an arbitrary integer L. Then the dynamical Yang-Baxter equation (2.1) ensures the associativity of the relation

$$\begin{aligned} \mathcal{R}_{ab}(\lambda_1 - \lambda_2, \theta - \gamma \mathrm{H}) \mathcal{T}_a(\lambda_1, \theta) \mathcal{T}_b(\lambda_2, \theta - \gamma \hat{h}_a) = \\ \mathcal{T}_b(\lambda_2, \theta) \mathcal{T}_a(\lambda_1, \theta - \gamma \hat{h}_b) \mathcal{R}_{ab}(\lambda_1 - \lambda_2, \theta) , \end{aligned} \tag{2.4}$$

for $\mathcal{R}$-matrices satisfying the weight zero condition, i.e. $[\mathcal{R}, h \otimes 1 + 1 \otimes h] = 0$. As usual we define $h_i \in \mathbb{V}_{\mathcal{Q}}$ as h acting on the i-th node of the tensor product space $\mathbb{V}_{\mathcal{Q}}$ while $\mathrm{H} = \sum_{i=1}^{L} h_i$. The generator H can be identified with the $\mathfrak{sl}(2)$ Cartan generator acting on $\mathbb{V}_{\mathcal{Q}}$ and one can also show that

$$\mathcal{T}_a(\lambda, \theta) = \overrightarrow{\prod_{1 \leq i \leq L}} \mathcal{R}_{ai}(\lambda - \mu_i, \hat{\theta}_i) \tag{2.5}$$

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is a representation of (2.4) with $\hat{\theta}_i = \theta - \gamma \sum_{k=i+1}^{L} h_k$ and arbitrary parameters $\mu_i \in \mathbb{C}$. The operator $\mathcal{T}$ is usually denominated dynamical monodromy matrix, or simply monodromy matrix, and it consists of a matrix in the space $\mathbb{V}_a$ whose entries are operators in the space $\mathbb{V}_{\mathcal{Q}}$. Thus for the $E_{p,q}[\mathfrak{sl}(2)]$ solution (2.3), our monodromy matrix can be recasted as

$$\mathcal{T}(\lambda, \theta) = \begin{pmatrix} \mathcal{A}(\lambda, \theta) & \mathcal{B}(\lambda, \theta) \\ \mathcal{C}(\lambda, \theta) & \mathcal{D}(\lambda, \theta) \end{pmatrix} \tag{2.6}$$

where $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D} \in \mathrm{End}((\mathbb{C}^2)^{\otimes L})$.

Remark 1. *An important limit of (2.3) is the limit $p = e^{i\pi\tau} \to 0$ where the elliptic Theta-functions degenerate into hyperbolic ones. Furthermore, if we also consider the limit $\theta \to \infty$ we are left with the standard $\mathcal{R}$-matrix and Yang-Baxter relations of the six-vertex model [23].*

Integrable lattice systems. Exact solvability of two-dimensional lattice systems can be achieved from certain conditions of integrability in analogy to the theory of integrable differential equations. This condition is fulfilled by commuting transfer matrices which is assured by local equivalence transformations satisfied by the model statistical weights [23]. For vertex models we can encode the statistical weight of a given vertex configuration as the entry of a certain matrix $\mathcal{R}$. In this way the aforementioned equivalence transformation requires this $\mathcal{R}$-matrix to satisfy the celebrated Yang-Baxter equation. For Solid-on-Solid models, or SOS for short, the condition of integrability in the sense of Baxter requires the model statistical weights to satisfy the so called Hexagon identity [30]. This condition was shown in [27] to be directly related to the dynamical Yang-Baxter equation (2.1), and in what follows we shall briefly describe some important examples of integrable SOS and vertex lattice systems.

(i) *Eight-Vertex-SOS model with domain wall boundaries:* This model consists of a two-dimensional lattice system defined on a square lattice. It is built from the juxtaposition of plaquettes where we associate a set of state variables to the corners of each plaquette in order to characterize its allowed configurations. As far as boundary conditions are concerned, the case of domain wall boundaries consists of the assumption that the plaquettes at the border are fixed at a particular configuration. This model has been previously considered in [31–34] and here we shall adopt the conventions of [20, 21]. In this way the partition function of the Eight-Vertex-SOS model with domain wall boundaries can be written as

$$\mathcal{Z}_\theta = \langle \bar{0} | \prod_{1 \leq j \leq L}^{\longrightarrow} \mathcal{B}(\lambda_j, \theta + j\gamma) | 0 \rangle \ , \tag{2.7}$$

where the operators $\mathcal{B}(\lambda, \theta)$ are defined through the relations (2.6), (2.3) and (2.5). In their turn the vectors $|0\rangle$ and $|\bar{0}\rangle$ are explicitly given by

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}^{\otimes L} \qquad\qquad |\bar{0}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^{\otimes L} \ . \tag{2.8}$$

(ii) *Scalar product of Bethe vectors:* The evaluation of partition functions of lattice systems such as vertex models with periodic boundary conditions can be conveniently translated into an eigenvalue problem for an operator usually denominated transfer matrix [8, 9]. The diagonalization of transfer matrices for integrable vertex models can be performed, for instance, through the Quantum Inverse Scattering Method [12]. Within that method, Bethe vectors arise as an ansatz capable of determining transfer matrices exact eigenvectors. In a slightly different context, scalar products of Bethe vectors can also

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be regarded as the partition function of a vertex model with special boundary conditions [35,36], and this is the interpretation we shall pursue here. As it was stressed out in Remark 1, the standard six-vertex model relations can be obtained from (2.3) in a particular limit. In that limit we have $\mathcal{A}(\lambda,\theta) \rightarrow A(\lambda)$, $\mathcal{B}(\lambda,\theta) \rightarrow B(\lambda)$, $\mathcal{C}(\lambda,\theta) \rightarrow C(\lambda)$, $\mathcal{D}(\lambda,\theta) \rightarrow D(\lambda)$ and the scalar product of Bethe vectors S_n then reads

$$S_n = \langle 0| \overleftarrow{\prod_{1\leq i\leq n}} C(\lambda_i^C) \overrightarrow{\prod_{1\leq i\leq n}} B(\lambda_i^B) |0\rangle \ . \tag{2.9}$$

In formulae (2.9) we have considered the description employed in [36] while the vector $|0\rangle$ has been defined in (2.8).

From (2.7) and (2.9) we can see that the above defined partition functions are given as the expected value of a product of generators of the Yang-Baxter algebra and its dynamical counterpart. In what follows we shall demonstrate how the algebraic relations (2.4) can be explored yielding functional equations determining the aforementioned quantities.

3. Yang-Baxter algebra and functional relations

In order to illustrate how the Yang-Baxter algebra can be employed to derive functional relations, we shall consider the six-vertex model limit of (2.1), (2.3) and (2.4) for the sake of simplicity. In particular, the relation (2.4) then encodes a total of sixteen commutation rules involving the set of generators $\mathcal{M}(\lambda) = \{A, B, C, D\}(\lambda)$ evaluated at different values of the spectral parameter λ.

From the perspective of Quantum Field Theory one can regard the operators $A(\lambda)$ and $D(\lambda)$ as diagonal fields while $B(\lambda)$ and $C(\lambda)$ plays the role of creation and annihilation fields. As far as we are concerned with an eigenvalue problem involving the set of generators $\mathcal{M}(\lambda)$, i.e. the diagonalization of a transfer matrix [37, 12], the framework of Quantum Field Theory is quite appealing and it seems natural to build the corresponding eigenvectors as elements of a Fock space. In particular, this approach is encouraged by the structure of the commutation relations in (2.4). Those commutation rules constitute one of the corner stones of the algebraic Bethe ansatz [12] and for instance let us single out the following one

$$A(\lambda_1)B(\lambda_2) = \frac{a(\lambda_2-\lambda_1)}{b(\lambda_2-\lambda_1)}B(\lambda_2)A(\lambda_1) - \frac{c(\lambda_2-\lambda_1)}{b(\lambda_2-\lambda_1)}B(\lambda_1)A(\lambda_2) \ , \tag{3.1}$$

where $a(\lambda) = \sinh(\lambda+\gamma)$, $b(\lambda) = \sinh(\lambda)$ and $c(\lambda) = \sinh(\gamma)$. Within the framework of the algebraic Bethe ansatz we usually regard (3.1) as a relation between a diagonal field and a creation field. Nevertheless, this is not the only way one can explore relations of type (3.1), and in what follows we shall see they can also be projected as a functional relation.

Definition 1. *Let $\pi : End(\mathbb{V}_{\mathcal{Q}}) \times End(\mathbb{V}_{\mathcal{Q}}) \mapsto \mathbb{C}$ be a continuous and bi-additive map. Due to (3.1), or more generally (2.4), it is convenient to specialize the map π to π_2 defined as*

$$\pi_2 : \qquad \mathcal{M}(\lambda) \times \mathcal{M}(\mu) \mapsto \mathbb{C}[\lambda^{\pm 1}, \mu^{\pm 1}] \ . \tag{3.2}$$

The 2-tuple $(\xi_1, \xi_2) : \xi_1 \in \mathcal{M}(\lambda),\ \xi_2 \in \mathcal{M}(\mu)$ originated from the Cartesian product $\mathcal{M}(\lambda) \times \mathcal{M}(\mu)$ is then simply understood as the matrix product $\xi_1\xi_2$. In other words the map π_2 associates a two-variable complex function to any quadratic form in (2.4).

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The map π_2 defined in (3.2) is able to associate a functional relation to any commutation rule contained in (2.4). For instance, the map (3.2) applied on (3.1) yields the relation

$$b(\lambda_2 - \lambda_1) f(\lambda_1, \lambda_2) = a(\lambda_2 - \lambda_1) \bar{f}(\lambda_2, \lambda_1) - c(\lambda_2 - \lambda_1) \bar{f}(\lambda_1, \lambda_2) \tag{3.3}$$

where $f(\lambda_1, \lambda_2) = \pi_2(A(\lambda_1)B(\lambda_2))$ and $\bar{f}(\lambda_1, \lambda_2) = \pi_2(B(\lambda_1)A(\lambda_2))$.

The study of functional equations has a long history, see for instance the monograph [38], and they play a remarkable role in Statistical Mechanics [23] and Conformal Field Theory [39]. Within those contexts they appear intimately related to Baxter's concept of commuting transfer matrices [10] and among prominent examples we have Baxter's $T-Q$ relation [40], inversion relation [41], analytical Bethe ansatz [42] and Y-system [43,44]. Also, it is worth mentioning the quantum Knizhnik-Zamolodchikov equation [45] which describes form factors and correlation functions in integrable field theories [46]. Here we intend to demonstrate that partition functions of integrable lattice models, such as (2.7) and (2.9), can also be described by functional equations. Interestingly, the functional equations describing those partition functions follow directly from the Yang-Baxter algebra within the lines above discussed. In order to show that we first need to generalize the Definition 1 in the following way.

Definition 2. *Let n be an integer. Then we define the n-additive continuous map π_n as*

$$\pi_n : \qquad \mathcal{M}(\lambda_1) \times \mathcal{M}(\lambda_2) \times \cdots \times \mathcal{M}(\lambda_n) \mapsto \mathbb{C}[\lambda_1^{\pm 1}, \lambda_2^{\pm 1}, \ldots, \lambda_n^{\pm 1}] \, . \tag{3.4}$$

Similarly to (3.2) the n-tuple $(\xi_1, \xi_2, \ldots, \xi_n) : \xi_i \in \mathcal{M}(\lambda_i)$ is understood as the non-commutative product $\overrightarrow{\prod\limits_{1 \leq i \leq n}} \xi_i$. In other words, the map π_n associates a n-variable complex function to a product of n generators of the Yang-Baxter algebra.

Realization of π_n. A simple choice of realization of π_n is the scalar product with vectors $|\psi\rangle \, , |\psi'\rangle \in \mathbb{V}_{\mathcal{Q}}$. More precisely, we can readily see that

$$\pi_n(\mathcal{F}) = \langle \psi' | \, \mathcal{F} \, |\psi\rangle \tag{3.5}$$

is a realization of π_n for any element $\mathcal{F} \in \mathcal{M}(\lambda_1) \times \mathcal{M}(\lambda_2) \times \cdots \times \mathcal{M}(\lambda_n)$. At this stage $|\psi\rangle$ and $|\psi'\rangle$ are arbitrary vectors but it will become clear that particular choices can render interesting functional equations describing quantities such as (2.7) and (2.9).

3.1. Functional equation for $\mathcal{Z}_\theta$

In this section we aim to show how a functional equation for the partition function $\mathcal{Z}_\theta$ can be derived within the lines discussed in Section 3. For that the first step is to find appropriate vectors $|\psi\rangle$ and $|\psi'\rangle$ in order to employ the realization (3.5). In addition to that it is also important to consider suitable elements $\mathcal{F}$ such that we end up with an equation capable of determining the desired partition function. Although there is no precise recipe for selecting $|\psi\rangle$, $|\psi'\rangle$ and $\mathcal{F}$, we shall see that some properties of the elements entering in the definitions (2.7) and (2.8) can help us to sort that out.

Highest weight vectors. The vectors $|0\rangle$ and $|\bar{0}\rangle$ defined in (2.8) are $\mathfrak{sl}(2)$ highest and lowest weight vectors. They satisfy the following properties:

$$
\begin{aligned}
&\mathcal{A}(\lambda,\theta)\,|\bar{0}\rangle = \frac{f(\theta-\gamma)}{f(\theta+(L-1)\gamma)}\prod_{j=1}^{L} f(\lambda-\mu_j)\,|\bar{0}\rangle \qquad && \mathcal{A}(\lambda,\theta)\,|0\rangle = \prod_{j=1}^{L} f(\lambda-\mu_j+\gamma)\,|0\rangle \\
&\mathcal{D}(\lambda,\theta)\,|0\rangle = \frac{f(\theta+\gamma)}{f(\theta-(L-1)\gamma)}\prod_{j=1}^{L} f(\lambda-\mu_j)\,|0\rangle && \mathcal{D}(\lambda,\theta)\,|\bar{0}\rangle = \prod_{j=1}^{L} f(\lambda-\mu_j+\gamma)\,|\bar{0}\rangle \\
&\mathcal{C}(\lambda,\theta)\,|0\rangle = 0 && \mathcal{B}(\lambda,\theta)\,|\bar{0}\rangle = 0
\end{aligned} \tag{3.6}
$$

$$
\begin{aligned}
&\langle\bar{0}|\,\mathcal{A}(\lambda,\theta) = \frac{f(\theta-\gamma)}{f(\theta+(L-1)\gamma)}\prod_{j=1}^{L} f(\lambda-\mu_j)\,\langle\bar{0}| \qquad && \langle 0|\,\mathcal{A}(\lambda,\theta) = \prod_{j=1}^{L} f(\lambda-\mu_j+\gamma)\,\langle 0| \\
&\langle 0|\,\mathcal{D}(\lambda,\theta) = \frac{f(\theta+\gamma)}{f(\theta-(L-1)\gamma)}\prod_{j=1}^{L} f(\lambda-\mu_j)\,\langle 0| && \langle\bar{0}|\,\mathcal{D}(\lambda,\theta) = \prod_{j=1}^{L} f(\lambda-\mu_j+\gamma)\,\langle\bar{0}| \\
&\langle\bar{0}|\,\mathcal{C}(\lambda,\theta) = 0 && \langle 0|\,\mathcal{B}(\lambda,\theta) = 0
\end{aligned} \tag{3.7}
$$

The expressions (3.6) and (3.7) follow from the definitions (2.5), (2.3) and the highest/lowest weight property of $|0\rangle$ and $|\bar{0}\rangle$.

Yang-Baxter relations of degree n. The relations arising from the dynamical Yang-Baxter algebra (2.4) involve the set of generators $\mathcal{M}(\lambda,\theta) = \{\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}\}(\lambda,\theta)$ and H. Although some entries of (2.4) contain products of the form $\mathcal{M}(\lambda_1,\theta_1)\times\mathcal{M}(\lambda_2,\theta_2)\times f(\mathrm{H})$, there are still commutation rules with all terms in $\mathcal{M}(\lambda_1,\theta_1)\times\mathcal{M}(\lambda_2,\theta_2)$. Those are the relations that will be explored here and among them we have the following ones

$$
\begin{aligned}
\mathcal{B}(\lambda_1,\theta)\mathcal{B}(\lambda_2,\theta+\gamma) &= \mathcal{B}(\lambda_2,\theta)\mathcal{B}(\lambda_1,\theta+\gamma) \\
\mathcal{A}(\lambda_1,\theta+\gamma)\mathcal{B}(\lambda_2,\theta) &= \frac{f(\lambda_2-\lambda_1+\gamma)}{f(\lambda_2-\lambda_1)}\frac{f(\theta+\gamma)}{f(\theta+2\gamma)}\mathcal{B}(\lambda_2,\theta+\gamma)\mathcal{A}(\lambda_1,\theta+2\gamma) \\
&- \frac{f(\theta+\gamma-\lambda_2+\lambda_1)}{f(\lambda_2-\lambda_1)}\frac{f(\gamma)}{f(\theta+2\gamma)}\mathcal{B}(\lambda_1,\theta+\gamma)\mathcal{A}(\lambda_2,\theta+2\gamma)\ .
\end{aligned} \tag{3.8}
$$

Both expressions in (3.8) are quadratic and their repeated use is able to provide relations of degree n for a subset of elements $\mathcal{F}\subset\mathcal{W}_n = \mathcal{M}(\lambda_0,\theta_0)\times\mathcal{M}(\lambda_1,\theta_1)\times\cdots\times\mathcal{M}(\lambda_{n-1},\theta_{n-1})$. More precisely, the iteration of (3.8) yields the following relation of degree $n+1$,

$$
\begin{aligned}
&\mathcal{A}(\lambda_0,\theta+\gamma)Y_{\theta-\gamma}(\lambda_1,\dots,\lambda_n) = \\
&\frac{f(\theta+\gamma)}{f(\theta+(n+1)\gamma)}\prod_{j=1}^{n}\frac{f(\lambda_j-\lambda_0+\gamma)}{f(\lambda_j-\lambda_0)}Y_\theta(\lambda_1,\dots,\lambda_n)\mathcal{A}(\lambda_0,\theta+(n+1)\gamma) \\
&-\sum_{i=1}^{n}\frac{f(\theta+\gamma-\lambda_i+\lambda_0)}{f(\theta+(n+1)\gamma)}\frac{f(\gamma)}{f(\lambda_i-\lambda_0)}\prod_{\substack{j=1\\ j\neq i}}^{n}\frac{f(\lambda_j-\lambda_i+\gamma)}{f(\lambda_j-\lambda_i)}\times \\
&\qquad Y_\theta(\lambda_0,\lambda_1,\dots,\lambda_{i-1},\lambda_{i+1},\dots,\lambda_n)\mathcal{A}(\lambda_i,\theta+(n+1)\gamma)\ ,
\end{aligned} \tag{3.9}
$$

where $Y_\theta(\lambda_1,\dots,\lambda_n) = \overrightarrow{\prod\limits_{1\leq j\leq n}} B(\lambda_j,\theta+j\gamma)$.

Building π_n for $\mathcal{Z}_\theta$. The relations (3.6) and (3.7) suggest the prescription $|\psi\rangle = |0\rangle$ and $|\psi'\rangle = |\bar{0}\rangle$. As we shall see this particular choice of π_n is able to generate a functional equation for the partition function $\mathcal{Z}_\theta$ from the algebraic relation (3.9).

Taking into account the discussion of Section 3 we then set $n = L$ in the relation (3.9). Next we consider the action of the map π_{L+1} on (3.9) and by doing so we find only terms of the form

$$\pi_{L+1}(\mathcal{A}(\lambda_0, \theta+\gamma) Y_{\theta-\gamma}(\lambda_1, \dots, \lambda_L)) \tag{3.10}$$

and

$$\pi_{L+1}(Y_\theta(\lambda_0, \lambda_1, \dots, \lambda_{i-1}, \lambda_{i+1}, \dots, \lambda_L) \mathcal{A}(\lambda_i, \theta + (L+1)\gamma)) \ . \tag{3.11}$$

Interestingly, the relations (3.6) and (3.7) obtained as a consequence of the highest/lowest weight property of $|0\rangle$ and $|\bar{0}\rangle$ give rise to a map $\pi_{L+1} \mapsto \pi_L$. More precisely we have

$$\pi_{L+1}(\mathcal{A}(\lambda_0, \theta+\gamma) Y_{\theta-\gamma}(\lambda_1, \dots, \lambda_L)) = \frac{f(\theta)}{f(\theta+L\gamma)} \prod_{j=1}^{L} f(\lambda_0 - \mu_j) \ \pi_L(Y_{\theta-\gamma}(\lambda_1, \dots, \lambda_L)) \tag{3.12}$$

and

$$\pi_{L+1}(Y_\theta(\lambda_0, \lambda_1, \dots, \lambda_{i-1}, \lambda_{i+1}, \dots, \lambda_L) \mathcal{A}(\lambda_i, \theta + (L+1)\gamma)) = \prod_{j=1}^{L} f(\lambda_i - \mu_j) \ \pi_L(Y_\theta(\lambda_0, \lambda_1, \dots, \lambda_{i-1}, \lambda_{i+1}, \dots, \lambda_L)) \ . \tag{3.13}$$

The partition function (2.7) can now be promptly identified with $\pi_L(Y_\theta(\lambda_1, \dots, \lambda_L))$. In this way the action of π_{L+1} on (3.9), in addition to the relations (3.12) and (3.13), leaves us with the following functional equation for $\mathcal{Z}_\theta$,

$$M_0 \ \mathcal{Z}_{\theta-\gamma}(\lambda_1, \dots, \lambda_L) + \sum_{i=0}^{L} N_i \ \mathcal{Z}_\theta(\lambda_0, \dots, \lambda_{i-1}, \lambda_{i+1}, \dots, \lambda_L) = 0 \ . \tag{3.14}$$

The structure of the coefficients M_0 and N_i is a direct consequence of the dynamical Yang-Baxter algebra relations (2.4) and the highest weight properties (3.6) and (3.7). For convenience we shall postpone presenting their explicit form. It is also worth remarking that (3.14) made its first appearance in [21] and due to the fact that the operators $\mathcal{B}$ satisfy the relation $\mathcal{B}(\lambda_1, \theta)\mathcal{B}(\lambda_2, \theta+\gamma) = \mathcal{B}(\lambda_2, \theta)\mathcal{B}(\lambda_1, \theta+\gamma)$, as given by (3.8), the ordering of the arguments of $\mathcal{Z}_\theta$ in (3.14) is indeed arbitrary. This property leads us to the following lemma.

Lemma 1 (Symmetric function)**.** *The partition function $\mathcal{Z}_\theta$ is symmetric under the permutation of its variables, i.e.*

$$\mathcal{Z}_\theta(\dots, \lambda_i, \dots, \lambda_j, \dots) = \mathcal{Z}_\theta(\dots, \lambda_j, \dots, \lambda_i, \dots) \ . \tag{3.15}$$

Proof. This property follows directly from the commutation relation (3.8) or from the functional equation (3.14) as demonstrated in [21]. □

Remark 2. *Due to (3.15) we can safely employ the notation $\mathcal{Z}_\theta(\lambda_1, \dots, \lambda_L) = \mathcal{Z}_\theta(X^{1,L})$ where $X^{i,j} = \{\lambda_k \ : \ i \leq k \leq j\}$.*

Taking into account Remark 2, it is also convenient to introduce the set $X_k^{i,j} = X^{i,j} \backslash \{\lambda_k\}$ in such a way that (3.14) can be simply recasted as

$$M_0 \; \mathcal{Z}_{\theta-\gamma}(X^{1,L}) + \sum_{i=0}^{L} N_i \; \mathcal{Z}_{\theta}(X_i^{0,L}) = 0 \; . \tag{3.16}$$

In their turn the coefficients M_0 and N_i explicitly read

$$\begin{aligned} M_0 &= \frac{f(\theta)}{f(\theta + L\gamma)} \prod_{j=1}^{L} f(\lambda_0 - \mu_j) \\ N_i &= -\frac{f(\theta + \gamma + \lambda_0 - \lambda_i)}{f(\theta + (L+1)\gamma)} \frac{f(\gamma)}{f(\lambda_0 - \lambda_i + \gamma)} \prod_{j=1}^{L} f(\lambda_i - \mu_j + \gamma) \prod_{\lambda \in X_i^{0,L}} \frac{f(\lambda - \lambda_i + \gamma)}{f(\lambda - \lambda_i)} \; . \end{aligned} \tag{3.17}$$

Some further remarks are important at this stage. The reader familiar with the theory of Knizhnik-Zamolodchikov (KZ) equations can notice that (3.16) exhibits a structure which resembles that of the classical KZ equation [14]. For instance, the first term of (3.16) consists of the partition function with shifted variable θ which could be thought of as an analogous of the derivative. The second term of (3.16) consists of a linear combination of partition functions with a given variable λ_i in the argument being replaced by a variable λ_0. This variable replacement can be regarded as the action of an operator which can be considered as a sort of 'hamiltonian'. Although KZ equations are vector equations while here we are dealing with a scalar equation, we can see that both terms of (3.16) have a counterpart in the KZ theory. Furthermore, we shall find that the solution of (3.16) also resembles solutions of KZ-like equations [47].

Solution. The functional relation (3.16) consists of an equation for the partition function $\mathcal{Z}_\theta(X^{1,L})$ over the set of variables $X^{0,L}$. Thus we have one more variable than is required to describe the partition function itself. This feature is typical of functional equations such as the d'Alembert equation [38], but had not appeared previously in the functional relations describing Exactly Solvable Models to the best of our knowledge. This extra variable can be set at will in order to help us with the resolution of (3.16), and this approach is the basis of the method considered in [20] and [21]. Moreover, Eq. (3.16) also exhibits some special properties providing some guidance through the steps required to obtain its solution. These properties are as follows:

- *Scale invariance*: Eq. (3.16) is invariant under the symmetry transformation $\mathcal{Z}_\theta(X^{1,L}) \mapsto \alpha \mathcal{Z}_\theta(X^{1,L})$ where $\alpha \in \mathbb{C}$ is independent of θ and λ_j. This property tells us that (3.16) is only able to determine the partition function up to an overall multiplicative factor independent of the variables θ and λ_j. In this way the full determination of the partition function will require we know the precise value of $\mathcal{Z}_\theta$ for a particular choice of the aforementioned variables.
- *Linearity*: Eq. (3.16) is linear and as usual this implies that if $\mathcal{Z}_\theta^{(1)}$ and $\mathcal{Z}_\theta^{(2)}$ are solutions, then the linear combination $\mathcal{Z}_\theta = \mathcal{Z}_\theta^{(1)} + \mathcal{Z}_\theta^{(2)}$ is also a solution. As far as the determination of (2.7) is concerned, this property is telling us we need to establish an uniqueness criterium in order to characterize the desired partition function.

Here we do not intend to give a detailed description of the method developed to solve (3.16). Nevertheless, we can still comment on how the properties of scale invariance and linearity have been employed. In [21] we have shown that the partition function (2.7) can be explicitly computed in the limit $(\theta, \lambda_j) \to \infty$. This result can then be used to completely fix the overall multiplicative factor which is not constrained by (3.16). Concerning the issue raised by the linearity of (3.16), we have also shown in [21] that the desired solution consists of a higher order Theta-function [48]. As such, it is uniquely characterized by its zeroes up to an overall factor.

As a matter of fact, unveiling special zeroes of $\mathcal{Z}_\theta$ plays an important role for the resolution of (3.16) and we have found the following solution in [21],

$$\mathcal{Z}_\theta(X^{1,L}) = [f'(0)f(\gamma)]^L$$
$$\oint \cdots \oint \prod_{j=1}^{L} \frac{\mathrm{d}w_j}{2\mathrm{i}\pi} \frac{\prod_{j>i}^{L} f(w_j - w_i + \gamma) f(w_j - w_i)}{\prod_{i,j=1}^{L} f(w_i - \lambda_j)} \prod_{j=1}^{L} \frac{f(\theta + j\gamma - w_j + \mu_j)}{f(\theta + j\gamma)}$$
$$\times \prod_{j<i}^{L} f(\mu_j - w_i) \prod_{j>i}^{L} f(w_i - \mu_j + \gamma) \; . \qquad (3.18)$$

Formulae (3.18) is given in terms of a multiple contour integral whose integration contour for each variable w_j encloses all variables in the set $X^{1,L}$. It is also worth remarking that similar multiple contour integrals also emerge as solutions of the KZ equation [47] and its q-deformed version [49].

3.2. *Functional equation for S_n*

The partition function (2.7) is not the only quantity which satisfy a functional equation such as the one described in Section 3.1. Similar equations can also be derived for scalar products of Bethe vectors as we shall demonstrate. Although the case of domain wall boundary conditions was introduced in [35] as a building block of scalar products, here we shall not follow that approach but instead consider scalar products defined by (2.9) as an independent quantity.

The derivation of (3.14) required two main ingredients: the construction of a suitable realization of π_n and the derivation of appropriate Yang-Baxter relations of degree n. In order to apply the same methodology for scalar products S_n we then first need to consider the commutation relations from (2.4) in the six-vertex model limit as discussed in Remark 1. In what follows we present the ones that will be required,

$$\begin{aligned} A(\lambda_1)B(\lambda_2) &= \frac{a(\lambda_2 - \lambda_1)}{b(\lambda_2 - \lambda_1)} B(\lambda_2)A(\lambda_1) - \frac{c(\lambda_2 - \lambda_1)}{b(\lambda_2 - \lambda_1)} B(\lambda_1)A(\lambda_2) \\ C(\lambda_1)A(\lambda_2) &= \frac{a(\lambda_1 - \lambda_2)}{b(\lambda_1 - \lambda_2)} A(\lambda_2)C(\lambda_1) - \frac{c(\lambda_1 - \lambda_2)}{b(\lambda_1 - \lambda_2)} A(\lambda_1)C(\lambda_2) \end{aligned} \qquad (3.19)$$

$$\begin{aligned} D(\lambda_1)B(\lambda_2) &= \frac{a(\lambda_1 - \lambda_2)}{b(\lambda_1 - \lambda_2)} B(\lambda_2)D(\lambda_1) - \frac{c(\lambda_1 - \lambda_2)}{b(\lambda_1 - \lambda_2)} B(\lambda_1)D(\lambda_2) \\ C(\lambda_1)D(\lambda_2) &= \frac{a(\lambda_2 - \lambda_1)}{b(\lambda_2 - \lambda_1)} D(\lambda_2)C(\lambda_1) - \frac{c(\lambda_2 - \lambda_1)}{b(\lambda_2 - \lambda_1)} D(\lambda_1)C(\lambda_2) \end{aligned} \qquad (3.20)$$

$$\begin{aligned} B(\lambda)B(\mu) &= B(\mu)B(\lambda) \\ C(\lambda)C(\mu) &= C(\mu)C(\lambda) \; . \end{aligned} \qquad (3.21)$$

Yang-Baxter relation of degree n. The relations (3.19), (3.20) and (3.21) are a subset of the commutation rules contained in (2.4) in the six-vertex model limit and, as such, they are relations of degree 2 according to the discussion of Section 3.1. In order to describe the scalar product S_n we need an appropriate relation of degree n which can be obtained by the repeated use of (3.19)-(3.21). The direct inspection of the commutation relations (3.19)-(3.21) suggests us to consider the quantity

$$T_A = \overleftarrow{\prod_{1\leq i\leq n}} C(\lambda_i^C)\, A(\lambda_0) \overrightarrow{\prod_{1\leq i\leq n}} B(\lambda_i^B)\, , \tag{3.22}$$

which can be analyzed in at least two different ways through the relations (3.19) and (3.21). Here we shall restrict our discussion to the following ways of evaluating T_A. Firstly, we can move the operator $A(\lambda_0)$ in (3.22) all the way to the right through all the string of operators B with the help of the first relation in (3.19). Alternatively, we can also move the operator $A(\lambda_0)$ to the left by making use of the second relation in (3.19). These two different ways of evaluating the same quantity yields the following Yang-Baxter relation of order $2n+1$,

$$\begin{aligned}
&\prod_{i=1}^{n} \frac{a(\lambda_i^C - \lambda_0)}{b(\lambda_i^C - \lambda_0)} A(\lambda_0) \overleftarrow{\prod_{1\leq i\leq n}} C(\lambda_i^C) \overrightarrow{\prod_{1\leq i\leq n}} B(\lambda_i^B) \\
&- \sum_{i=1}^{n} \frac{c(\lambda_i^C - \lambda_0)}{b(\lambda_i^C - \lambda_0)} \prod_{\substack{j=1\\ j\neq i}}^{n} \frac{a(\lambda_j^C - \lambda_i^C)}{b(\lambda_j^C - \lambda_i^C)} A(\lambda_i^C) \overleftarrow{\prod_{\substack{0\leq j\leq n\\ j\neq i}}} C(\lambda_j^C) \overrightarrow{\prod_{1\leq j\leq n}} B(\lambda_j^B) = \\
&\prod_{i=1}^{n} \frac{a(\lambda_i^B - \lambda_0)}{b(\lambda_i^B - \lambda_0)} \overleftarrow{\prod_{1\leq i\leq n}} C(\lambda_i^C) \overrightarrow{\prod_{1\leq i\leq n}} B(\lambda_i^B)\, A(\lambda_0) \\
&- \sum_{i=1}^{n} \frac{c(\lambda_i^B - \lambda_0)}{b(\lambda_i^B - \lambda_0)} \prod_{\substack{j=1\\ j\neq i}}^{n} \frac{a(\lambda_j^B - \lambda_i^B)}{b(\lambda_j^B - \lambda_i^B)} \overleftarrow{\prod_{1\leq j\leq n}} C(\lambda_j^C) \overrightarrow{\prod_{\substack{0\leq j\leq n\\ j\neq i}}} B(\lambda_j^B)\, A(\lambda_i^B)\, .
\end{aligned} \tag{3.23}$$

It is important to stress here that the derivation of (3.23) also makes explicit use of the relations (3.21). The expression (3.23) will be left at rest for a while and we shall focus on another quantity. For instance, instead of (3.22) we could have performed the same analysis starting with

$$T_D = \overleftarrow{\prod_{1\leq i\leq n}} C(\lambda_i^C)\, D(\lambda_0) \overrightarrow{\prod_{1\leq i\leq n}} B(\lambda_i^B)\, . \tag{3.24}$$

In that case we need to consider the relations (3.20) and (3.21), and we end up with the following identity,

$$\begin{aligned}
&\prod_{i=1}^{n} \frac{a(\lambda_0 - \lambda_i^C)}{b(\lambda_0 - \lambda_i^C)} D(\lambda_0) \overleftarrow{\prod_{1\leq i\leq n}} C(\lambda_i^C) \overrightarrow{\prod_{1\leq i\leq n}} B(\lambda_i^B) \\
&- \sum_{i=1}^{n} \frac{c(\lambda_0 - \lambda_i^C)}{b(\lambda_0 - \lambda_i^C)} \prod_{\substack{j=1\\ j\neq i}}^{n} \frac{a(\lambda_i^C - \lambda_j^C)}{b(\lambda_i^C - \lambda_j^C)} D(\lambda_i^C) \overleftarrow{\prod_{\substack{0\leq j\leq n\\ j\neq i}}} C(\lambda_j^C) \overrightarrow{\prod_{1\leq j\leq n}} B(\lambda_j^B) = \\
&\prod_{i=1}^{n} \frac{a(\lambda_0 - \lambda_i^B)}{b(\lambda_0 - \lambda_i^B)} \overleftarrow{\prod_{1\leq i\leq n}} C(\lambda_i^C) \overrightarrow{\prod_{1\leq i\leq n}} B(\lambda_i^B)\, D(\lambda_0) \\
&- \sum_{i=1}^{n} \frac{c(\lambda_0 - \lambda_i^B)}{b(\lambda_0 - \lambda_i^B)} \prod_{\substack{j=1\\ j\neq i}}^{n} \frac{a(\lambda_i^B - \lambda_j^B)}{b(\lambda_i^B - \lambda_j^B)} \overleftarrow{\prod_{1\leq j\leq n}} C(\lambda_j^C) \overrightarrow{\prod_{\substack{0\leq j\leq n\\ j\neq i}}} B(\lambda_j^B)\, D(\lambda_i^B)\, .
\end{aligned} \tag{3.25}$$

Both expressions (3.23) and (3.25) consist of Yang-Baxter relations of order $2n+1$ and they can be converted into functional equations for S_n with a proper choice of π_{2n+1}.

The map π_m for S_n. Taking into account the relations (3.6), (3.7), (3.23) and (3.25) we can readily see that the choice $|\psi\rangle = |\psi'\rangle = |0\rangle$ for the realization (3.5) gives rise to functional relations for the scalar product S_n. In order to see that we apply the map π_{2n+1} on (3.23) and (3.25). By doing so we only find terms of the form

$$\begin{aligned}
&\pi_{2n+1}(A(v_0^A) \overleftarrow{\prod_{1\leq i\leq n}} C(v_i^C) \overrightarrow{\prod_{1\leq i\leq n}} B(v_i^B)) \quad , \quad \pi_{2n+1}(\overleftarrow{\prod_{1\leq i\leq n}} C(v_i^C) \overrightarrow{\prod_{1\leq i\leq n}} B(v_i^B)\, A(\bar{v}_0^A)) \\
&\pi_{2n+1}(D(v_0^D) \overleftarrow{\prod_{1\leq i\leq n}} C(v_i^C) \overrightarrow{\prod_{1\leq i\leq n}} B(v_i^B)) \quad , \quad \pi_{2n+1}(\overleftarrow{\prod_{1\leq i\leq n}} C(v_i^C) \overrightarrow{\prod_{1\leq i\leq n}} B(v_i^B)\, D(\bar{v}_0^D))
\end{aligned} \tag{3.26}$$

with parameters $v_0^A, v_0^D, v_i^C \in \{\lambda_0, \lambda_1^C, \dots, \lambda_n^C\}$ and $\bar{v}_0^A, \bar{v}_0^D, v_i^B \in \{\lambda_0, \lambda_1^B, \dots, \lambda_n^B\}$.

Similarly to the case discussed in Section 3.1, here we also have a map $\pi_{2n+1} \mapsto \pi_{2n}$ induced by the highest weight property of $|0\rangle$. More precisely, from (3.6) we obtain

$$\begin{aligned}
\pi_{2n+1}(\overleftarrow{\prod_{1\leq i\leq n}} C(v_i^C) \overrightarrow{\prod_{1\leq i\leq n}} B(v_i^B)\, A(\bar{v}_0^A)) &= \\
\prod_{j=1}^{L} a(\bar{v}_0^A - \mu_j) &\quad \pi_{2n}(\overleftarrow{\prod_{1\leq i\leq n}} C(v_i^C) \overrightarrow{\prod_{1\leq i\leq n}} B(v_i^B)) \\
\pi_{2n+1}(\overleftarrow{\prod_{1\leq i\leq n}} C(v_i^C) \overrightarrow{\prod_{1\leq i\leq n}} B(v_i^B)\, D(\bar{v}_0^A)) &= \\
\prod_{j=1}^{L} b(\bar{v}_0^A - \mu_j) &\quad \pi_{2n}(\overleftarrow{\prod_{1\leq i\leq n}} C(v_i^C) \overrightarrow{\prod_{1\leq i\leq n}} B(v_i^B)) \, .
\end{aligned} \tag{3.27}$$

On the other hand, the property (3.7) yields

$$\begin{aligned}
\pi_{2n+1}(A(v_0^A) \overleftarrow{\prod_{1\leq i\leq n}} C(v_i^C) \overrightarrow{\prod_{1\leq i\leq n}} B(v_i^B)) &= \\
\prod_{j=1}^{L} a(v_0^A - \mu_j) &\quad \pi_{2n}(\overleftarrow{\prod_{1\leq i\leq n}} C(v_i^C) \overrightarrow{\prod_{1\leq i\leq n}} B(v_i^B)) \\
\pi_{2n+1}(D(v_0^D) \overleftarrow{\prod_{1\leq i\leq n}} C(v_i^C) \overrightarrow{\prod_{1\leq i\leq n}} B(v_i^B)) &= \\
\prod_{j=1}^{L} b(v_0^D - \mu_j) &\quad \pi_{2n}(\overleftarrow{\prod_{1\leq i\leq n}} C(v_i^C) \overrightarrow{\prod_{1\leq i\leq n}} B(v_i^B)) \, .
\end{aligned} \tag{3.28}$$

The terms $\pi_{2n}(\overleftarrow{\prod_{1\leq i\leq n}} C(v_i^C) \overrightarrow{\prod_{1\leq i\leq n}} B(v_i^B))$ can now be identified with the scalar products S_n as defined in (2.9). In this way the map π_m above discussed, together with the relations (3.23), (3.25), (3.27) and (3.28), leaves us with the following functional equations,

$$\begin{aligned} J_0 S_n(X^{1,n}|Y^{1,n}) &+ \sum_{i=1}^{n} K_i^{(B)} S_n(X_i^{0,n}|Y^{1,n}) + \sum_{i=1}^{n} K_i^{(C)} S_n(X^{1,n}|Y_i^{0,n}) = 0 \\ \tilde{J}_0 S_n(X^{1,n}|Y^{1,n}) &+ \sum_{i=1}^{n} \tilde{K}_i^{(B)} S_n(X_i^{0,n}|Y^{1,n}) + \sum_{i=1}^{n} \tilde{K}_i^{(C)} S_n(X^{1,n}|Y_i^{0,n}) = 0 \; . \end{aligned} \tag{3.29}$$

In their turn the coefficients appearing in (3.29) can be conveniently written as

$$\begin{aligned} J_0 &= \prod_{j=1}^{L} a(\lambda_0 - \mu_j) \left[\prod_{i=1}^{n} \frac{a(\lambda_i^C - \lambda_0)}{b(\lambda_i^C - \lambda_0)} - \prod_{i=1}^{n} \frac{a(\lambda_i^B - \lambda_0)}{b(\lambda_i^B - \lambda_0)} \right] \\ K_i^{(B,C)} &= \alpha_{B,C} \frac{c(\lambda_i^{B,C} - \lambda_0)}{b(\lambda_i^{B,C} - \lambda_0)} \prod_{j=1}^{L} a(\lambda_i^{B,C} - \mu_j) \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{a(\lambda_j^{B,C} - \lambda_i^{B,C})}{b(\lambda_j^{B,C} - \lambda_i^{B,C})} \; , \end{aligned} \tag{3.30}$$

and

$$\begin{aligned} \widetilde{J}_0 &= \prod_{j=1}^{L} b(\lambda_0 - \mu_j) \left[\prod_{i=1}^{n} \frac{a(\lambda_0 - \lambda_i^C)}{b(\lambda_0 - \lambda_i^C)} - \prod_{i=1}^{n} \frac{a(\lambda_0 - \lambda_i^B)}{b(\lambda_0 - \lambda_i^B)} \right] \\ \widetilde{K}_i^{(B,C)} &= \alpha_{B,C} \frac{c(\lambda_0 - \lambda_i^{B,C})}{b(\lambda_0 - \lambda_i^{B,C})} \prod_{j=1}^{L} b(\lambda_i^{B,C} - \mu_j) \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{a(\lambda_i^{B,C} - \lambda_j^{B,C})}{b(\lambda_i^{B,C} - \lambda_j^{B,C})} \; . \end{aligned} \tag{3.31}$$

where $\alpha_B = 1$ and $\alpha_C = -1$.

The Remark 2 of Section 3.1 can be immediately extended to the case of scalar products, and in (3.29) we have employed the notation

$$S_n(\lambda_1^B, \dots, \lambda_n^B | \lambda_1^C, \dots, \lambda_n^C) = S_n(X^{1,n}|Y^{1,n}) \tag{3.32}$$

where $X^{i,j} = \{\lambda_k^B \; : \; i \leq k \leq j\}$ and $Y^{i,j} = \{\lambda_k^C \; : \; i \leq k \leq j\}$. Moreover, we have also considered the definitions $X_k^{i,j} = X^{i,j} \backslash \{\lambda_k^B\}$ and $Y_k^{i,j} = Y^{i,j} \backslash \{\lambda_k^C\}$. This possibility is granted by the following lemma.

Lemma 2 (Doubly symmetric function)**.** *The scalar product* $S_n(\lambda_1^B, \dots, \lambda_n^B | \lambda_1^C, \dots, \lambda_n^C)$ *is symmetric under the permutation of variables* $\lambda_i^B \leftrightarrow \lambda_j^B$ *and* $\lambda_i^C \leftrightarrow \lambda_j^C$ *performed independently. More precisely we have*

$$S_n(\dots, \lambda_i^B, \dots, \lambda_j^B, \dots | \lambda_1^C, \dots, \lambda_n^C) = S_n(\dots, \lambda_j^B, \dots, \lambda_i^B, \dots | \lambda_1^C, \dots, \lambda_n^C) \tag{3.33}$$

and

$$S_n(\lambda_1^C, \dots, \lambda_n^C | \dots, \lambda_i^B, \dots, \lambda_j^B, \dots) = S_n(\lambda_1^C, \dots, \lambda_n^C | \dots, \lambda_j^B, \dots, \lambda_i^B, \dots) \; . \tag{3.34}$$

Proof. The proof follows directly from the commutation rules (3.21). Alternatively, one can demonstrate it from the analysis of (3.29) as performed in [22]. □

Solution. The same discussion of Section 3.1 concerning the resolution of the functional equation (3.14) is also valid for the set of equations (3.29). For instance, we can readily see that each equation in (3.29) is scale invariant and linear. Nevertheless, there is one main difference concerning (3.29) which is the fact that here we have two equations instead of only one. This might suggest that one of the equations is redundant but the direct inspection of our system of equations reveals that this is not the case. In fact, the scalar products we are interested consist of certain polynomials and the use of a polynomial ansatz for solving (3.29) shows that only one equation is not able to fix all the coefficients. On the other hand, the simultaneous resolution of both equations indeed fix the coefficients up to an overall multiplicative factor.

Although the process of solving the system of equations (3.29) is more involving than that for the single equation (3.14), the same methodology still applies. The solution of (3.29) was firstly obtained in [22] and here we restrict ourselves to presenting only the final expression. The scalar product S_n is then given by,

$$S_n(X^{1,n}|Y^{1,n}) = \oint \dots \oint \prod_{i=1}^{n} \frac{\mathrm{d}w_i}{2\mathrm{i}\pi} \frac{\mathrm{d}\bar{w}_i}{2\mathrm{i}\pi} \frac{H(w_1, \dots, w_n|\bar{w}_1, \dots, \bar{w}_n)}{\prod_{i,j=1}^{n} b(w_i - \lambda_j^C) b(\bar{w}_i - \lambda_j^B)} , \tag{3.35}$$

where the function H explicitly reads,

$$H(w_1, \dots, w_n|\bar{w}_1, \dots, \bar{w}_n) = (-1)^{Ln + \frac{n(n+1)}{2}} c^{2n} \frac{\prod_{j>i}^{n} b(w_i - w_j)^2 b(\bar{w}_i - \bar{w}_j)^2 a(w_j - \mu_i) a(\bar{w}_j - \mu_i)}{\prod_{i=1}^{n} b(w_i - \mu_i) b(\bar{w}_i - \mu_i)} \prod_{i=1}^{n} R_i^{-1} \Lambda_i , \tag{3.36}$$

with functions R_i and Λ_i given by

$$\begin{aligned} R_i &= \prod_{k=i}^{n} \frac{a(w_k - \mu_i)}{b(w_k - \mu_i)} - \prod_{k=i}^{n} \frac{a(\bar{w}_k - \mu_i)}{b(\bar{w}_k - \mu_i)} \\ \Lambda_i &= \prod_{k=i}^{L} a(\bar{w}_i - \mu_k) b(\mu_k - w_i) \prod_{k=i+1}^{n} \frac{a(w_i - w_k)}{b(w_i - w_k)} \frac{a(\bar{w}_k - \bar{w}_i)}{b(\bar{w}_k - \bar{w}_i)} \\ &\quad - \prod_{k=i}^{L} a(w_i - \mu_k) b(\mu_k - \bar{w}_i) \prod_{k=i+1}^{n} \frac{a(w_k - w_i)}{b(w_k - w_i)} \frac{a(\bar{w}_i - \bar{w}_k)}{b(\bar{w}_i - \bar{w}_k)} . \end{aligned} \tag{3.37}$$

Formulae (3.35) is commonly denominated off-shell scalar product as it is valid for arbitrary complex parameters λ_i^B and λ_i^C. In its turn, when the variables λ_i^B are constrained by Bethe ansatz equations, see [37] for instance, the function S_n receives the name on-shell scalar product and the analysis of (3.29) in that case has also been performed in [22].

4. Partial differential equations

The functional equations (3.14) and (3.29) contain a very rich structure which is not apparent at first sight. In order to illustrate how these hidden structures emerge let us consider Eq. (3.14) in the standard six-vertex model limit. In that case we have $\mathcal{Z}_\theta \to Z$ and are left with the following equation,

$$\bar{M}_0 \, Z(X^{1,L}) + \sum_{i=1}^{L} \bar{N}_i \, Z(X_i^{0,L}) = 0 , \tag{4.1}$$

with coefficients $\bar{M}_0$ and $\bar{N}_i$ given by

$$\begin{aligned}
\bar{M}_0 &= \prod_{j=1}^{L} b(\lambda_0 - \mu_j) - \prod_{j=1}^{L} a(\lambda_0 - \mu_j) \prod_{j=1}^{L} \frac{a(\lambda_j - \lambda_0)}{b(\lambda_j - \lambda_0)} \\
\bar{N}_i &= \frac{c(\lambda_i - \lambda_0)}{b(\lambda_i - \lambda_0)} \prod_{j=1}^{L} a(\lambda_i - \mu_j) \prod_{\substack{j=1 \\ j\neq i}}^{L} \frac{a(\lambda_j - \lambda_i)}{b(\lambda_j - \lambda_i)} \; .
\end{aligned} \tag{4.2}$$

In what follows we intend to demonstrate that (4.1) encodes a family of partial differential equations and for that we need to introduce some extra definitions and lemmas.

Definition 3. *Let f be a complex valued function $f(z) \in \mathbb{C}[z]$ and $z = (z_1, \dots, z_n) \in \mathbb{C}^n$. Then for $\alpha \notin \{1, 2, \dots, n\}$ we define the operator D_i^α as*

$$D_i^\alpha \; : \qquad f(z_1, \dots, z_i, \dots, z_n) \mapsto f(z_1, \dots, z_\alpha, \dots, z_n) \; . \tag{4.3}$$

The operator D_i^α essentially replaces the variable z_i by z_α. It is worth mentioning that D_i^α had been previously introduced in [19].

Lemma 3 (Differential realization)**.** *The operator D_i^α admits the realization*

$$D_i^\alpha = \sum_{k=0}^{m} \frac{(z_\alpha - z_i)^k}{k!} \frac{\partial^k}{\partial z_i^k} \tag{4.4}$$

when its action is restricted to the ring of multivariate polynomials of degree m.

Proof. Let $f = f(z_1, z_2, \dots, z_n)$ and $\mathbb{K}^m[z_1, z_2, \dots, z_n]$ be the ring of polynomials in $z_1, \dots, z_n$ with degree m. The ring $\mathbb{K}^m[z_1, z_2, \dots, z_n]$ shall be simply denoted as $\mathbb{K}^m[z]$ and the condition $f \in \mathbb{K}^m[z]$ implies

$$\frac{\partial^k f}{\partial z_i^k} = 0 \qquad \text{if} \quad k > m \; . \tag{4.5}$$

Next we consider the Taylor expansion of f in the variable z_i around the point z_α. Due to (4.5) the expansion is truncated and convergent. Thus we have,

$$\begin{aligned}
f &= f(\dots, z_{i-1}, z_\alpha, z_{i+1}, \dots) + \left.\frac{\partial f}{\partial z_i}\right|_{i=\alpha} (z_i - z_\alpha) \\
&\quad + \frac{1}{2} \left.\frac{\partial^2 f}{\partial z_i^2}\right|_{i=\alpha} (z_i - z_\alpha)^2 + \dots + \frac{1}{m!} \left.\frac{\partial^m f}{\partial z_i^m}\right|_{i=\alpha} (z_i - z_\alpha)^m \; .
\end{aligned} \tag{4.6}$$

The expression (4.6) holds for indexes $i \in \{1, 2, \dots, n\}$, and as long as $\alpha \notin \{1, 2, \dots, n\}$ we can write

$$\left.\frac{\partial^k f}{\partial z_i^k}\right|_{i=\alpha} = \frac{\partial^k}{\partial z_\alpha^k} f(\dots, z_{i-1}, z_\alpha, z_{i+1}, \dots) \; . \tag{4.7}$$

In this way formulas (4.6) and (4.7) yields the relation

$$f(\dots, z_{i-1}, z_i, z_{i+1}, \dots) = \left[\sum_{k=0}^{m} \frac{(z_i - z_\alpha)^k}{k!} \frac{\partial^k}{\partial z_\alpha^k} \right] f(\dots, z_{i-1}, z_\alpha, z_{i+1}, \dots) \; . \tag{4.8}$$

The term inside the brackets in (4.8) performs the operation (4.3) and we thus obtain the differential realization (4.4). □

Next we notice the functional equation (4.1) can be written in terms of operators D_i^α. For that we only need to consider $n = L$ and $z_i = \lambda_i$. By doing so (4.1) becomes $\mathfrak{L}(\lambda_0) Z(X^{1,L}) = 0$ where

$$\mathfrak{L}(\lambda_0) = \bar{M}_0 + \sum_{i=1}^{L} \bar{N}_i \, D_i^0 \; . \tag{4.9}$$

Some remarks are required at this stage. For instance, although the functional equation (4.1) depends on the set of variables $X^{0,L}$, the use of Definition 3 localizes the whole dependence with the variable λ_0 in the operator $\mathfrak{L}$. It is also important to stress here that we can not immediately use the differential realization (4.4) in (4.9) since it is valid only for functions in $\mathbb{K}^m[z]$. Nevertheless, in what follows we shall discuss how (4.4) can be adjusted for $Z(X^{1,L})$.

Lemma 4 (Polynomial structure)**.** *In terms of variables $x_i = e^{2\lambda_i}$ the partition function $Z(X^{1,L})$ is of the form*

$$Z(X^{1,L}) = \prod_{j=1}^{L} x_j^{\frac{1-L}{2}} \bar{Z}(x_1, \dots, x_L) \; , \tag{4.10}$$

where $\bar{Z}(x_1, \dots, x_L)$ is a polynomial of degree $L-1$ in each variable x_i separately.

Proof. A detailed proof can be found in [35] and [18]. □

Lemma 4 is telling us that $Z(X^{1,L})$ consists of a multivariate polynomial up to an overall multiplicative factor when the appropriate variable is considered. As a matter of fact we have $\bar{Z}(x_1, \dots, x_L) \in \mathbb{K}^{L-1}[x]$ and therefore the realization (4.4) can be employed for $\bar{Z}$. Due to that it is convenient to define the functions

$$\check{M}_0 = \bar{M}_0 \prod_{j=1}^{L} x_j^{\frac{1-L}{2}} \qquad \text{and} \qquad \check{N}_i = \bar{N}_i \prod_{\substack{j=0 \\ j \neq i}}^{L} x_j^{\frac{1-L}{2}} \tag{4.11}$$

in such a way that (4.1) becomes $\bar{\mathfrak{L}}(x_0) \bar{Z}(x_1, \dots, x_L) = 0$ with

$$\bar{\mathfrak{L}}(x_0) = \check{M}_0 + \sum_{i=1}^{L} \check{N}_i \, D_i^0 \; . \tag{4.12}$$

Now the formulae (4.4) can be substituted into (4.12) [1] leaving us with the expression

$$\bar{\mathfrak{L}}(x_0) = \sum_{k=0}^{L-2} x_0^k \, \Omega_k \quad . \tag{4.13}$$

The coefficients Ω_k are differential operators whilst $\bar{\mathfrak{L}}(x_0)$ is a polynomial of degree $L-2$ in the variable x_0. In this way the equation $\bar{\mathfrak{L}}(x_0) \bar{Z}(x_1, \dots, x_L) = 0$ needs to be independently satisfied for each power in x_0 which leaves us with the following family of partial differential equations,

$$\Omega_k \, \bar{Z}(x_1, \dots, x_L) = 0 \qquad 0 \leq k \leq L-2 \; . \tag{4.14}$$

Although the explicit form of the operators Ω_k can be written down for any length L, it is usually given by cumbersome expressions for most of the indexes k. Fortunately the situation

[1] Here we need to set $z_i = x_i$ due to the change of variables discussed in Lemma 4.

is different for $k = L-2$ and the leading term coefficient Ω_{L-2} exhibits a compact expression given by

$$\Omega_{L-2} = \sum_{i=1}^{L} \bar{a}(x_i, y_i) - \frac{q^{2(1-L)}}{(L-1)!} \sum_{i=1}^{L} \prod_{j=1}^{L} \bar{a}(x_i, y_j) \prod_{\substack{j=1 \\ j \neq i}}^{L} \frac{\bar{a}(x_j, x_i)}{\bar{b}(x_j, x_i)} \frac{\partial^{L-1}}{\partial x_i^{L-1}} . \tag{4.15}$$

The expression (4.15) takes into account the further conventions $q = e^{\gamma}$, $y_i = e^{2\mu_i}$ and the remaining functions are then defined as $\bar{a}(x,y) = xq^2 - y$ and $\bar{b}(x,y) = x - y$.

From (4.15) we can see that the operator Ω_{L-2} exhibits some very interesting characteristics. For instance, it naturally decomposes into two kinds of terms and it is tempting to interpret it as the hamiltonian of a many-body system. Although Ω_{L-2} contains higher order derivatives, the first term in the RHS of (4.15) could be thought of as 'potential energy' while the second term can be regarded as 'kinetic energy'. The most obvious problem with this interpretation is that the interaction factor $\prod_{j\neq i}^{L} \frac{\bar{a}(x_j,x_i)}{\bar{b}(x_j,x_i)}$ appears in the 'kinetic energy' term and it is not clear if a change of variables could have this issue properly fixed. Nevertheless, taking into account this analogy it is sensible to consider the eigenvalue problem for the operator (4.15), i.e. $\Omega_{L-2}\Psi = \Lambda\Psi$. In this way the partition function $\bar{Z}$ can be regarded as the null eigenvalue wave function associated with Ω_{L-2}.

5. Concluding remarks

In this article we have described a mechanism allowing to extract functional equations satisfied by certain partition functions of two-dimensional lattice models directly from the Yang-Baxter algebra. More precisely, we have applied this method for the elliptic Eight-Vertex-SOS model with domain wall boundaries [21] and for scalar products of Bethe vectors [22]. For those systems we have obtained functional relations satisfied by their partition functions whose solution are then given in terms of multiple contour integrals.

The class of functional equations we describe here share some similarities, as far as their structure is concerned, with the classical Knizhnik-Zamolodchikov equation [14] as discussed in Section 3.1. This similarity seems to extend to their solutions as multiple contour integrals are also convenient to describe solutions of KZ equations [47].

Although classical KZ equations consist of a system of partial differential equations, whilst here we are dealing with functional equations, in Section 4 we have also demonstrated that there is a family of partial differential equations underlying our functional relations originated from the Yang-Baxter algebra. Interestingly, one member of this family exhibits a structure which resembles that of a generalized Schrödinger equation for a quantum many-body hamiltonian.

Concerning this algebraic-functional approach, it is fair to say that this method is still under development as there are still many opened questions. For instance, motivated by the similarities shared with the classical KZ equation, one could ask if there is an analogous of the whole theory of KZ equations [47, 49] for the functional equations presented here. Moreover, as far as the computation of physical quantities are concerned, one important problem is the evaluation of the model free-energy per site in the thermodynamical limit from the multiple contour integrals we have obtained. This analysis would not only give us access to the model physical properties but also help us to understand the influence of boundary conditions for vertex and SOS models [50, 51].

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References

[1] Onsager, L 1944 *Phys. Rev.* **65** 117–149
[2] Ma S 2000 *Modern Theory of Critical Phenomena* Advanced book classics (Perseus)
[3] Bethe H 1931 *Zeitschrift für Physik* 225–226
[4] Faddeev, L D and Takhtajan, L A 1981 *Phys. Lett. A* **85** 375–377
[5] Zamolodchikov, A B 1990 *Nucl. Phys. B* **342** 695–720
[6] Minahan J A and Zarembo K 2003 *JHEP*
[7] Brak R and Galleas W 2013 *Lett. Math. Phys.* **103** 1261–1272
[8] Kramers H A and Wannier G H 1941 *Phys. Rev.* **60** 252
[9] Kramers H A and Wannier G H 1941 *Phys. Rev.* **60** 263
[10] Baxter R J 1971 *Phys. Rev. Lett.* **26** 832
[11] Sklyanin E K, Takhtadzhyan L A and Faddeev L D 1979 *Theor. Math. Phys.* **40** 688
[12] Takhtadzhyan L A and Faddeev L D 1979 *Russ. Math. Surv.* **34** 11
[13] Chari V and Pressley A 1995 *A Guide to Quantum Groups* (Cambridge University Press)
[14] Knizhnik, V G and Zamolodchikov, A B 1984 *Nucl. Phys. B* **247** 83–103
[15] Babujian, H M 1993 *J. Phys. A: Math. and Gen.* **26** 6981–6990
[16] Babujian, H M and Flume, R 1994 *Mod. Phys. Lett. A* **9** 2029–2039
[17] Galleas W 2008 *Nucl. Phys. B* **790** 524–542
[18] Galleas W 2010 *J. Stat. Mech.* P06008
[19] Galleas W 2011 *J. Stat. Mech.* P01013
[20] Galleas W 2012 *Nucl. Phys. B* **858** 117–141 (*Preprint* `math-ph/1111.6683`)
[21] Galleas W 2013 *Nucl. Phys. B* **867** 855–871
[22] Galleas, W (*Preprint* `arXiv: 1211.7342`)
[23] Baxter R J 2007 *Exactly Solved Models in Statistical Mechanics* (Mineola, New York: Dover Publications, Inc.) ISBN 978-0-486-46271-4; 0-486-46271-4
[24] Smirnov S 2001 *Comptes rendus de L'academie des sciences serie I - Mathematique* **333** 239–244
[25] Smirnov S 2001 *Pre-print* (*Preprint* `arXiv: math.PR/1211.3968`)
[26] Cardy J L 1992 *J. Phys. A - Math. and Gen.* **25** L201–L206
[27] Felder G 1995 *Proceedings of the International Congress of Mathematicians* **1** 1247
[28] Felder G 1996 *Nucl. Phys. B* **480** 485
[29] Whittaker E T and Watson G N 1927 *A Course of Modern Analysis* 4th ed (Cambridge University Press)
[30] D'Ariano G M, Montorsi A and Rasetti M G 1985 *Integrable Systems in Statistical Mechanics* (World Scientific)
[31] Rosengren H 2009 *Adv. Appl. Math.* **43** 137
[32] Pakuliak S, Rubtsov V and Silantyev A 2008 *J. Phys. A* **41** 295204
[33] Yang W L and Zhang Y Z 2009 *J. Math. Phys.* **50** 083518
[34] Razumov A G and Stroganov Y G 2009 *Theor. Math. Phys.* **161** 1325
[35] Korepin V E 1982 *Commun. Math. Phys.* **86**(3) 391–418
[36] de Gier J, Galleas W and Sorrell M 2011 (*Preprint* `hep-th/1111.3712`)
[37] Korepin V E, Bogoliubov N M and Izergin A G 1993 *Quantum inverse scattering method and correlation functions* (Cambridge University Press)
[38] Aczél, J 1984 *Functional equations: History, Applications and Theory* (D. Reidel Publishing Company)
[39] Bazhanov V V, Lukyanov S L and Zamolodchikov A B 1997 *Comm. Math. Phys.* **190** 247–278
[40] Baxter R J 1972 *Ann. Phys.* **70** 193
[41] Stroganov Y G 1982 *Unpublished thesis*
[42] Reshetikhin N Y 1987 *Lett. Math. Phys.* **14** 235–246
[43] Kuniba A, Nakanishi T and Suzuki J 1994 *Int. J. Mod. Phys. A* **9** 5215–5266
[44] Kuniba A, Nakanishi T and Suzuki J 2011 *J. Phys. A - Math. and Theor.* **44**
[45] Frenkel I B and Reshetikhin N Y 1992 *Comm. Math. Phys.* **146** 1–60
[46] Smirnov, F A 1992 *Form factors in completely integrable models of quantum field theory* (Advanced Series in Mathematical Physics. 14. Singapore: World Scientific,. xi, 208 p.)

[47] Varchenko, A N 2003 *Special Functions, KZ Type Equations, and Representation Theory* (American Mathematical Society)

[48] Weber H 1891 *Elliptische Functionen und algebraische Zahlen* 4th ed (Braunschweig: Friedrich Vieweg und Sohn)

[49] Etingof, P I and Frenkel, I B and Kirillov, A A 1998 *Lectures on Representation Theory and Knizhnik-Zamolodchikov Equations* (American Mathematical Society)

[50] Rosengren H 2011 *Adv. Appl. Math.* **46** 481

[51] Korepin V E and Zinn-Justin P 2000 *J. Phys. A: Math. Gen.* **33** 7053