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Twisted Heisenberg chain and the six-vertex model with DWBC

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Abstract

In this work we establish a relation between the six-vertex model with domain wall boundaries and the XXZ spin chain with non-diagonal twisted boundary condition. In particular, we demonstrate a formal relation between the zeroes of the partition function of the six-vertex model with Domain Wall Boundary Conditions (DWBC) and the zeroes of the twisted six-vertex model eigenvalues.

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1 Introduction

Integrable systems have played a remarkable role for the discovery of connections between seemingly unrelated topics. For instance, although the relation between one-dimensional quantum spin chains and two-dimensional classical vertex models is nowadays clear, this notable relation only emerged after Lieb's observation that the ice model transfer matrix and the XXX spin chain hamiltonian share the same eigenvectors [1]. As a matter of fact, this relation only became clear with the work [2] where the one-dimensional spin chain hamiltonian was shown to correspond to the logarithmic derivative of a two-dimensional vertex model transfer matrix. This correspondence between quantum spin chains and classical vertex models is well established for lattice systems but we also have interesting relations emerging in the continuum limit. For instance, it is believed that the massless regimes of vertex models in the continuum are described by the critical properties of Wess-Zumino-Witten field theories [3].

As far as vertex models with domain wall boundary conditions [4] are concerned, we can not immediately associate an one-dimensional spin chain within the lines of [2]. However, there exist some interesting connections between the six-vertex model with domain wall boundaries and the theory of classical integrable systems [5], special functions [6, 7] and enumerative combinatorics [8]. Moreover, in the recent paper [9] we have shown that the partition function of the six-vertex model with domain wall boundaries corresponds to the null eigenvalue wave-function of a certain many-body hamiltonian operator.

On the other hand, the XXZ spin chain with non-diagonal twisted boundary conditions can be embedded in the transfer matrix of a $U_q[\mathfrak{sl}(2)]$ invariant six-vertex model with generalized toroidal boundary conditions in the same lines of [2, 10]. This particular spin chain has also been studied in [11–13], and it was the first system approached through the algebraic-functional method employed in [14–16] for partition functions with domain wall boundaries. This method has been refined in a series of papers and here, due to this approach, we intend to report a connection found between the twisted Heisenberg chain and the six-vertex model with DWBC.

This paper is organized as follows. In Section 2 we briefly present the transfer matrix embedding the XXZ chain with twisted boundary conditions and introduce the notation we shall use throughout this paper. In Section 3 we explore the Yang-Baxter algebra in the lines of [9] to derive a functional equation relating the transfer matrix eigenvalues and the partition function of the six-vertex model with DWBC. The implications of this functional equation is discussed in Section 4, and in particular we show how our results simplify when the model anisotropy parameter is a root of unity. Concluding remarks are then presented in Section 5.

2 Heisenberg chain and the DWBC partition function

In this section we shall give a brief description of the anisotropic Heisenberg chain with non-diagonal twisted boundary conditions. This model consists of a spin-$\frac{1}{2}$ system and here we shall mainly adopt the conventions of [12]. The hamiltonian $\mathcal{H}$ of the system acts on the tensor product space $\mathbb{V}_{\mathcal{Q}} \cong (\mathbb{C}^2)^{\otimes L}$ and it reads

$$\mathcal{H} = \sum_{i=1}^{L} \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + 2\cosh(\gamma) \sigma_i^z \sigma_{i+1}^z \qquad \in \quad \mathrm{End}(\mathbb{V}_{\mathcal{Q}}) \; . \tag{2.1}$$

In (2.1) we have employed the notation $\sigma_i = \mathbb{1}^{\otimes(i-1)} \otimes \sigma \otimes \mathbb{1}^{\otimes(L-i)}$ for elements $\sigma \in \{\sigma^x, \sigma^y, \sigma^z\}$ characterizing the standard Pauli matrices. As far as the boundary terms are concerned, here we have the following twisted conditions

$$\sigma_{L+1}^x = \sigma_1^x \qquad \sigma_{L+1}^y = -\sigma_1^y \qquad \sigma_{L+1}^z = -\sigma_1^z \; . \tag{2.2}$$

Transfer matrix. The hamiltonian (2.1) descends from a commuting transfer matrix as its logarithmic derivative. Let $\mathbb{V}_{\mathcal{A}} \cong \mathbb{V}_j \cong \mathbb{C}^2$ and consider elements $\mathcal{R}_{\mathcal{A}j} \in \mathrm{End}(\mathbb{V}_{\mathcal{A}} \otimes \mathbb{V}_j)$ and $G_{\mathcal{A}} \in \mathrm{End}(\mathbb{V}_{\mathcal{A}})$. Then we define the transfer matrix

$$T(\lambda) = \mathrm{Tr}_{\mathcal{A}}[G_{\mathcal{A}} \prod_{1 \leq j \leq L}^{\longrightarrow} \mathcal{R}_{\mathcal{A}j}(\lambda - \mu_j)] \quad \in \quad \mathrm{End}(\mathbb{V}_{\mathcal{Q}}) \; , \tag{2.3}$$

where $\lambda, \mu_j \in \mathbb{C}$ and the matrix $G_{\mathcal{A}}$ is explicitly given by $G_{\mathcal{A}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The trace in (2.3) is taken only over the space $\mathbb{V}_{\mathcal{A}}$ while the matrix $\mathcal{R}$ reads

$$\mathcal{R}(\lambda) = \begin{pmatrix} a(\lambda) & 0 & 0 & 0 \\ 0 & b(\lambda) & c(\lambda) & 0 \\ 0 & c(\lambda) & b(\lambda) & 0 \\ 0 & 0 & 0 & a(\lambda) \end{pmatrix} \; . \tag{2.4}$$

The non-null entries of (2.4) are the following functions, $a(\lambda) = \sinh(\lambda + \gamma)$, $b(\lambda) = \sinh(\lambda)$ and $c(\lambda) = \sinh(\gamma)$ for parameters $\lambda, \gamma \in \mathbb{C}$. In this way the hamiltonian (2.1) is obtained from the relation $\mathcal{H} \sim \frac{d}{d\lambda} \ln T(\lambda)\big|_{\substack{\lambda=0 \\ \mu_j=0}}$. Here it is worth mentioning that the $\mathcal{R}$-matrix (2.4) also satisfies the standard Yang-Baxter equation [17]. In addition to that we also have the property $[\mathcal{R}, G_{\mathcal{A}} \otimes G_{\mathcal{A}}] = 0$ ensuring that the transfer matrix (2.3) forms a commutative family.

Monodromy matrix. Let $\mathcal{T}$ be the following operator

$$\mathcal{T}_{\mathcal{A}}(\lambda) = \overrightarrow{\prod_{1 \leq j \leq L}} \mathcal{R}_{\mathcal{A}j}(\lambda - \mu_j) \qquad \in \quad \mathrm{End}(\mathbb{V}_{\mathcal{A}} \otimes \mathbb{V}_{\mathcal{Q}}) \; , \tag{2.5}$$

which we shall refer to as monodromy matrix. As the $\mathcal{R}$-matrix (2.4) satisfies the Yang-Baxter equation, one can show that the monodromy matrix (2.5) satisfies the following quadratic identity

$$\mathcal{R}_{12}(\lambda_1 - \lambda_2)\mathcal{T}_1(\lambda_1)\mathcal{T}_2(\lambda_2) = \mathcal{T}_2(\lambda_2)\mathcal{T}_1(\lambda_1)\mathcal{R}_{12}(\lambda_1 - \lambda_2) \; . \tag{2.6}$$

The relation (2.6) is usually referred to as Yang-Baxter algebra and since $\mathbb{V}_{\mathcal{A}} \cong \mathbb{C}^2$, the monodromy matrix $\mathcal{T}$ can be recasted as

$$\mathcal{T}_{\mathcal{A}}(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} \; . \tag{2.7}$$

In terms of the operators $A, B, C, D \in \mathrm{End}(\mathbb{V}_{\mathcal{Q}})$, we find that the transfer matrix (2.3) simply reads $T(\lambda) = B(\lambda) + C(\lambda)$.

Domain wall boundaries. The $\mathcal{R}$-matrix (2.4) encodes the statistical weights of a six-vertex model as discussed in [17]. However, one still needs to consider appropriate boundary conditions in order to having a non-trivial partition function for the model. The case of domain wall boundary conditions for the six-vertex model was then introduced in [4] considering a square lattice with dimensions $L \times L$. In that case the model partition function reads

$$Z(\lambda_1, \dots, \lambda_L) = \langle \bar{0} | \overrightarrow{\prod_{1 \leq j \leq L}} B(\lambda_j) \, | 0 \rangle \; , \tag{2.8}$$

where the vectors $|0\rangle$ and $|\bar{0}\rangle$ are defined as

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}^{\otimes L} \qquad \text{and} \qquad |\bar{0}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^{\otimes L} \; . \tag{2.9}$$

Highest/lowest weight vectors. The vectors $|0\rangle$ and $|\bar{0}\rangle$ defined in (2.9) are respectively $\mathfrak{sl}(2)$ highest and lowest weight vectors. The action of the entries of the monodromy matrix (2.7) on those vectors are given as follows:

$$\begin{aligned}
A(\lambda)\,|0\rangle &= \prod_{j=1}^{L} a(\lambda-\mu_j)\,|0\rangle & D(\lambda)\,|0\rangle &= \prod_{j=1}^{L} b(\lambda-\mu_j)\,|0\rangle \\
A(\lambda)\,|\bar{0}\rangle &= \prod_{j=1}^{L} b(\lambda-\mu_j)\,|\bar{0}\rangle & D(\lambda)\,|\bar{0}\rangle &= \prod_{j=1}^{L} a(\lambda-\mu_j)\,|\bar{0}\rangle \\
B(\lambda)\,|\bar{0}\rangle &= 0 & C(\lambda)\,|0\rangle &= 0\;.
\end{aligned} \tag{2.10}$$

3 Functional equations

The spectrum of the twisted Heisenberg chain hamiltonian (2.1) can be obtained directly from the spectrum of the transfer matrix (2.3). This is due to the fact that the hamiltonian $\mathcal{H}$ is given by the logarithmic derivative of the transfer matrix T, in addition to the property $[T(\lambda), T(\mu)] = 0$ which follows from the relation (2.6). The eigenvalue problem for the transfer matrix T can be tackled through the method introduced in [12] and subsequently extended in [14–16, 18]. In order to do so we first need to introduce some extra definitions and conventions.

Definition 1. *Let $B(\lambda_i) \in \mathbb{V}_{\mathcal{Q}}$ be an off-diagonal element of the Yang-Baxter algebra as defined in (2.6). In this way we introduce the following notation for the product of n generators B,*

$$[\lambda_1, \dots, \lambda_n] = \prod_{1\leq i\leq n}^{\longrightarrow} B(\lambda_i)\;. \tag{3.1}$$

Remark 1. *The property $B(\lambda)B(\mu) = B(\mu)B(\lambda)$ encoded within the relation (2.6) ensures that $[\lambda_1, \dots, \lambda_n]$ is symmetric under the permutation of variables $\lambda_i \leftrightarrow \lambda_j$. Thus we shall also employ the simplified notation $[X^{1,n}] = [\lambda_1, \dots, \lambda_n]$ where $X^{i,j} = \{\lambda_k \mid i \leq k \leq j\}$.*

Now we recall that $T(\lambda) = B(\lambda) + C(\lambda)$ and consider the action of $T(\lambda_0)$ over the element $[X^{1,n}]$. For that the most lengthy computation is the term $C(\lambda_0)\,[X^{1,n}]$ which can be evaluated with the help of the commutation relations contained in (2.6). This computation has been performed in [4, 14] and here we restrict ourselves to presenting only the final results. In this way we have the following expression,

$$\begin{aligned}
T(\lambda_0)\left[X^{1,n}\right] \;=\;& \left[X^{0,n}\right] + \sum_{1\leq i\leq n} \left[X_i^{1,n}\right]\left(\mathit{\Gamma}_{0,i}^{i} A(\lambda_0)D(\lambda_i) + \mathit{\Gamma}_{i,0}^{i} A(\lambda_i)D(\lambda_0)\right) \\
&+ \sum_{1\leq i<j\leq n} \left[X_{i,j}^{0,n}\right]\left(\mathit{\Omega}_{i,j} A(\lambda_i)D(\lambda_j) + \mathit{\Omega}_{j,i} A(\lambda_j)D(\lambda_i)\right)\,,
\end{aligned} \tag{3.2}$$

where $X_i^{1,n} = X^{1,n} \backslash \{\lambda_i\}$ and $X_{i,j}^{0,n} = X^{0,n} \backslash \{\lambda_i, \lambda_j\}$. The coefficients $\Gamma_{j,k}^i$ and $\Omega_{i,j}$ in (3.2) explicitly read

$$\begin{aligned} \Gamma_{j,k}^i &= \frac{c(\lambda_k - \lambda_j)}{b(\lambda_k - \lambda_j)} \prod_{\lambda \in X_i^{1,n}} \frac{a(\lambda_k - \lambda)}{b(\lambda_k - \lambda)} \frac{a(\lambda - \lambda_j)}{b(\lambda - \lambda_j)} \\ \Omega_{i,j} &= \frac{c(\lambda_j - \lambda_0)}{a(\lambda_j - \lambda_0)} \frac{c(\lambda_0 - \lambda_i)}{a(\lambda_0 - \lambda_i)} \frac{a(\lambda_j - \lambda_i)}{b(\lambda_j - \lambda_i)} \prod_{\lambda \in X_{i,j}^{0,n}} \frac{a(\lambda_j - \lambda)}{b(\lambda_j - \lambda)} \frac{a(\lambda - \lambda_i)}{b(\lambda - \lambda_i)} . \end{aligned} \tag{3.3}$$

Next we take into account the recent discussion of [9] and immediately recognize (3.2) as a Yang-Baxter relation of order $n+1$. This relation, in addition to some extra properties, allows us to derive a functional equation describing the spectrum of the transfer matrix T. For that we shall also employ the following definition.

Definition 2. *Let $n \in \mathbb{N}$ be an index and $\mathcal{M}(\lambda) = \{A, B, C, D\}(\lambda)$. Then we define π_n as the following n-additive continuous map*

$$\pi_n \quad : \quad \mathcal{M}(\lambda_1) \times \mathcal{M}(\lambda_2) \times \cdots \times \mathcal{M}(\lambda_n) \mapsto \mathbb{C}[\lambda_1^{\pm 1}, \lambda_2^{\pm 1}, \ldots, \lambda_n^{\pm 1}] . \tag{3.4}$$

The n-tuple $(\xi_1, \xi_2, \ldots, \xi_n) \; : \; \xi_i \in \mathcal{M}(\lambda_i)$ is understood as the non-commutative product $\overrightarrow{\prod\limits_{1 \leq i \leq n}} \xi_i$. The map π_n thus associates a multivariate complex function to any product of n generators of the Yang-Baxter algebra.

The next step in our approach is to find a suitable realization of the map π_n which is able to convert (3.2) into appropriate functional equations. As discussed in [9], the simplest realization of π_n seems to be the scalar product with particular vectors, and that is precisely the realization we shall adopt here.

Realization of $\boldsymbol{\pi_n}$. Let $|\Psi\rangle \in \mathrm{span}(\mathbb{V}_{\mathcal{Q}})$ be an eigenvector of the transfer matrix (2.3), i.e. $T(\lambda)\,|\Psi\rangle = \Lambda(\lambda)\,|\Psi\rangle$, and let $|0\rangle \in \mathbb{V}_{\mathcal{Q}}$ be the $\mathfrak{sl}(2)$ highest weight vector defined in (2.9). Then we consider

$$\pi_{n+1}(\mathcal{A}) = \langle \Psi | \, \mathcal{A} \, |0\rangle \qquad\qquad \forall \, \mathcal{A} \, \in \, \mathcal{K}_{n+1} \; , \tag{3.5}$$

where $\mathcal{K}_{n+1} = \mathcal{M}(\lambda_0) \times \mathcal{M}(\lambda_1) \times \cdots \times \mathcal{M}(\lambda_n)$.

Now we have gathered all the ingredients required to convert the Yang-Baxter algebra relation (3.2) into a functional equation for the eigenvalues of T. For that we then apply the map (3.4) to the relation (3.2) and consider the realization (3.5). In this way we find the following property for the term in the LHS of (3.2),

$$\pi_{n+1}(T(\lambda_0) \left[X^{1,n}\right]) = \Lambda(\lambda_0) \pi_n(\left[X^{1,n}\right]) . \tag{3.6}$$

On the other hand, the terms in the RHS of (3.2) are all of the form $\pi_{n+1}([X^{0,n}])$ and $\pi_{n+1}([Y^{1,n-1}]\, A(z_i) D(z_j))$ where $z_k \in \mathbb{C}$ and $Y^{1,n-1} = \{z_k \mid 1 \leq k \leq n-1 \; ; \; z_k \neq z_i, z_j\}$. The term $\pi_{n+1}([X^{0,n}])$ can not be significantly simplified but using (2.10) we find that

$$\pi_{n+1}(\left[Y^{1,n-1}\right] A(z_i) D(z_j)) = \prod_{k=1}^{L} a(z_i - \mu_k) b(z_j - \mu_k) \; \pi_{n-1}(\left[Y^{1,n-1}\right]) . \tag{3.7}$$

Our results so far can be written in a more convenient form with the help of the notation $\pi_n([X]) = \mathcal{F}_n(X)$ for a given set $X = \{\lambda_k\}$ of cardinality n. In this way, the relations (3.2), (3.6) and (3.7) yields the following set of functional equations,

$$\begin{aligned} \Lambda(\lambda_0)\mathcal{F}_n(X^{1,n}) &= \mathcal{F}_{n+1}(X^{0,n}) + \sum_{1 \leq i \leq n} M_i^{(n)} \mathcal{F}_{n-1}(X_i^{1,n}) \\ &+ \sum_{1 \leq i < j \leq n} N_{j,i}^{(n)} \mathcal{F}_{n-1}(X_{i,j}^{0,n}) \;, \end{aligned} \tag{3.8}$$

with coefficients $M_i^{(n)} = M_i^{(n)}(X^{0,n})$ and $N_{j,i}^{(n)} = N_{j,i}^{(n)}(X^{0,n})$ given by

$$\begin{aligned} M_i^{(n)} &= \Gamma_{0,i}^i \prod_{k=1}^{L} a(\lambda_0 - \mu_k) b(\lambda_i - \mu_k) + \Gamma_{i,0}^i \prod_{k=1}^{L} a(\lambda_i - \mu_k) b(\lambda_0 - \mu_k) \\ N_{j,i}^{(n)} &= \Omega_{i,j} \prod_{k=1}^{L} a(\lambda_i - \mu_k) b(\lambda_j - \mu_k) + \Omega_{j,i} \prod_{k=1}^{L} a(\lambda_j - \mu_k) b(\lambda_i - \mu_k) \;. \end{aligned} \tag{3.9}$$

Some remarks are required at this stage. For instance, the functional equation (3.8) consists of an extension of the equation obtained in [12] and it has also been recently described in [13]. Moreover, at algebraic level there is no upper limit for the discrete index n in equation (3.8), however, we shall see that the $\mathfrak{sl}(2)$ highest weight representation theory imposes an upper bound for its possible values.

4 The eigenvalues Λ and the partition function Z

In the previous section we have derived a functional equation involving the eigenvalues Λ of the twisted transfer matrix (2.3) and a certain set of functions $\mathcal{F}_n$. Here we intend to show that the functions $\mathcal{F}_n$ can be eliminated from the system of Eqs. (3.8) yielding a single equation for the eigenvalue Λ. Furthermore, the equation obtained for the eigenvalues will depend explicitly on the partition function of the six-vertex model with domain wall boundaries (2.8).

Highest weight and domain walls. As discussed in [4, 14], the highest weight representation theory of the $\mathfrak{sl}(2)$ algebra gives an upper bound for the number of operators B entering the product (3.1). Moreover, taking into account the *Remark 1* we have the following property,

$$[X^{1,L}] \, |0\rangle = Z(X^{1,L}) \, |\bar{0}\rangle \;. \tag{4.1}$$

In this way we find from (4.1) and (3.5) that $\mathcal{F}_L(X^{1,L}) = Z(X^{1,L})\bar{\mathcal{F}}_0$ where $\bar{\mathcal{F}}_0 = \langle\Psi| \, \bar{0}\rangle$. Besides that, the relations (2.10), (3.5) and (4.1) imply that the function $\mathcal{F}_n$ vanishes for $n > L$. Also, in writing (3.8) we have assumed that $\mathcal{F}_n$ vanishes for $n < 0$.

In order to illustrate our procedure, let us first have a closer look at Eq. (3.8) for the case $L = 2$. In that case we can set $n = 0, 1, 2, 3$ which correspond to the following relations,

$$
\begin{aligned}
\Lambda(\lambda_0)\mathcal{F}_0 &= \mathcal{F}_1(X^{0,0}) \\
\Lambda(\lambda_0)\mathcal{F}_1(X^{1,1}) &= Z(X^{0,1})\bar{\mathcal{F}}_0 + M_1^{(1)}\mathcal{F}_0 \\
\Lambda(\lambda_0)Z(X^{1,2})\bar{\mathcal{F}}_0 &= \sum_{1\leq i\leq 2} M_i^{(2)}\mathcal{F}_1(X_i^{1,2}) + N_{2,1}^{(2)}\mathcal{F}_1(X_{1,2}^{0,2}) \\
0 &= \sum_{1\leq i\leq 3} M_i^{(3)} Z(X_i^{1,3}) + \sum_{1\leq i<j\leq 3} N_{j,i}^{(3)} Z(X_{i,j}^{0,3}) \, .
\end{aligned} \tag{4.2}
$$

The last equation in (4.2) is a functional relation involving solely the partition function Z which had been previously described in [14]. This equation alone is fully able to determine the function Z while the remaining equations relate both Λ and $\mathcal{F}_1$. Next we use the first equation of (4.2) to eliminate $\mathcal{F}_1$ from the second and third equations. By doing so we are then left with the relations,

$$
\begin{aligned}
\Lambda(\lambda_0)\Lambda(\lambda_1) &= Z(X^{0,1})k_0 + M_1^{(1)} \\
\Lambda(\lambda_0)\left[Z(X^{1,2})k_0 - N_{2,1}^{(2)}\right] &= M_1^{(2)}\Lambda(\lambda_2) + M_2^{(2)}\Lambda(\lambda_1) \, ,
\end{aligned} \tag{4.3}
$$

where $k_0 = \bar{\mathcal{F}}_0/\mathcal{F}_0$. Here we shall assume that the partition function Z is already determined and the only unknown factors in (4.3) are the coefficients k_0 and the function Λ. Both equations (4.3) are able to determine those terms, however, the first Eq. in (4.3) is non-linear in Λ while the second is linear. In fact, the direct inspection of (4.3) reveals that $k_0 = \Lambda(\mu_1)\Lambda(\mu_2)[1-(\mu_1-\mu_2)^2]^{-1}$.

Next we consider Eq. (3.8) with $L = 3$. In that case we can set $n = 0, 1, 2, 3, 4$ which yields the following system of equations

$$
\begin{aligned}
\Lambda(\lambda_0)\mathcal{F}_0 &= \mathcal{F}_1(X^{0,0}) \\
\Lambda(\lambda_0)\mathcal{F}_1(X^{1,1}) &= \mathcal{F}_2(X^{0,1}) + M_1^{(1)}\mathcal{F}_0 \\
\Lambda(\lambda_0)\mathcal{F}_2(X^{1,2}) &= Z(X^{0,2})\bar{\mathcal{F}}_0 + \sum_{1\leq i\leq 2} M_i^{(2)}\mathcal{F}_1(X_i^{1,2}) + N_{2,1}^{(2)}\mathcal{F}_1(X_{1,2}^{0,2}) \\
\Lambda(\lambda_0)Z(X^{1,3})\bar{\mathcal{F}}_0 &= \sum_{1\leq i\leq 3} M_i^{(3)}\mathcal{F}_2(X_i^{1,3}) + \sum_{1\leq i<j\leq 3} N_{j,i}^{(3)}\mathcal{F}_2(X_{i,j}^{0,3}) \\
0 &= \sum_{1\leq i\leq 4} M_i^{(4)} Z(X_i^{1,4}) + \sum_{1\leq i<j\leq 4} N_{j,i}^{(4)} Z(X_{i,j}^{0,4}) \, .
\end{aligned} \tag{4.4}
$$

Similarly to the case $L = 2$, the system of Eqs. (4.4) can be solved recursively for the functions $\mathcal{F}_i$. By carrying out this procedure we find that the third and fourth equations in (4.4) can be rewritten only in terms of the unknown function Λ. Those equations then read

$$
\begin{aligned}
\Lambda(\lambda_0)\Lambda(\lambda_1)\Lambda(\lambda_2) &= Z(X^{0,2})k_0 + \Lambda(\lambda_0)[M_1^{(1)}(X^{1,2}) + N_{2,1}^{(2)}(X^{0,2})] \\
&\quad + \Lambda(\lambda_1)M_2^{(2)}(X^{0,2}) + \Lambda(\lambda_2)M_1^{(2)}(X^{0,2}) \\
\Lambda(\lambda_0)Z(X^{1,3})k_0 &= \sum_{1\leq i\leq 3} M_i^{(3)}(X^{0,3}) \prod_{\substack{k=1 \\ k\neq i}}^{3} \Lambda(\lambda_k) + \sum_{1\leq i<j\leq 3} N_{j,i}^{(3)}(X^{0,3})\Lambda(\lambda_0) \prod_{\substack{k=1 \\ k\neq i,j}}^{3} \Lambda(\lambda_k)
\end{aligned}
$$

$$- \sum_{1\leq i\leq 3} M_i^{(3)}(X^{0,3}) M_1^{(1)}(X_i^{1,3}) - \sum_{1\leq i<j\leq 3} N_{j,i}^{(3)}(X^{0,3}) M_1^{(1)}(X_{i,j}^{0,3}) \; . \tag{4.5}$$

In contrast to the case $L = 2$, none of the Eqs. in (4.5) is linear and this is the general behavior for arbitrary L.

At this stage it is important to remark that for both cases $L = 2$ and $L = 3$ we have explicitly written two sets of two equations each, namely (4.3) and (4.5), relating the spectrum Λ and the partition function Z. However, there is a dramatic difference between the first and the second equations of (4.3) and (4.5). For instance, while the first equation runs over the set of variables $\{\lambda_k \mid 0 \leq k \leq L-1\}$, we can see that the second equation is defined over the set $\{\lambda_k \mid 0 \leq k \leq L\}$. A similar issue had appeared previously in the literature when we compare the functional equations of [14] and [16] describing the partition function of the six-vertex model with DWBC. In what follows we shall focus on the functional equation relating Λ and Z which generalizes the first equation of (4.3) and (4.5) for arbitrary values of L.

The procedure above described holds for any value of L and it yields the following equation,

$$Z(X^{0,L-1}) k_0 = \left\{ \sum_{m=0}^{[L]/2} \sum_{0\leq i_1<\cdots<i_{2m}\leq L-1} V_{i_{2m},\dots,i_1}^{(2m)} \widehat{\prod}_{\lambda}^{i_1,\dots,i_{2m}} \right\} \Lambda(\lambda) \; , \tag{4.6}$$

where $V^{(0)} = 1$ and the symbol $[x]$ is defined as

$$[x] = \begin{cases} x & \text{if } x \in 2\mathbb{Z} \\ x-1 & \text{if } x \in 2\mathbb{Z}+1 \end{cases} \; . \tag{4.7}$$

In order to clarify our notation we remark that $\widehat{\prod}_{\lambda}^{i_1,\dots,i_{2m}}$ appearing in the RHS of (4.6) should be regarded as the following product operator.

Definition 3. *Let $\Lambda(\lambda) \in \mathbb{C}[\lambda]$ and $\widehat{\prod}_{\lambda}^{i_1,\dots,i_{2m}} : \mathbb{C}[\lambda] \mapsto \mathbb{C}[\lambda_{v_1}] \times \cdots \times \mathbb{C}[\lambda_{v_{L-2m}}]$ for $\lambda_{v_j} \in X_{i_1,\dots,i_{2m}}^{0,L-1}$ such that $\lambda_{v_j} \neq \lambda_{v_k}$ if $j \neq k$. The relation $X_{i_1,\dots,i_{n+1}}^{l,m} = X_{i_1,\dots,i_n}^{l,m} \backslash \{\lambda_{i_{n+1}}\}$ generalizes recursively our previous definition and we have*

$$\widehat{\prod}_{\lambda}^{i_1,\dots,i_{2m}} \Lambda(\lambda) \;\equiv\; \prod_{\lambda\in X_{i_1,\dots,i_{2m}}^{0,L-1}} \Lambda(\lambda) = \prod_{\substack{k=0 \\ k\neq i_1,\dots,i_{2m}}}^{L-1} \Lambda(\lambda_k) \; . \tag{4.8}$$

Example 1. *The RHS of (4.6) for $L = 2$ reads*

$$\Lambda(\lambda_0)\Lambda(\lambda_1) + V_{1,0}^{(2)} \; , \tag{4.9}$$

while for $L = 3$ we have

$$\Lambda(\lambda_0)\Lambda(\lambda_1)\Lambda(\lambda_2) + \sum_{0\leq i_1<i_2\leq 2} V_{i_2,i_1}^{(2)} \prod_{\substack{k=0 \\ k\neq i_1,i_2}}^{2} \Lambda(\lambda_k) \; . \tag{4.10}$$

Now the only factors in (4.6) that we still need to define are the functions $V^{(2m)}_{i_{2m},\dots,i_1}$ for $m \geq 1$. Those functions are essentially the result of particular combinations of $M^{(n)}_i$ and $N^{(n)}_{ji}$ which have been previously defined in (3.9). In this way we find that the functions $V^{(2m)}_{i_{2m},\dots,i_1}$ are explicitly given by

$$\begin{aligned} V^{(2m)}_{i_{2m},\dots,i_1} &= (-1)^m \sum_J \prod_{l=1}^{m} \prod_{n=1}^{L} a(\lambda_{j_l} - \mu_n) \prod_{\lambda \in X^{0,L-1}_{i_1,\dots,i_{2m}}} \frac{a(\lambda - \lambda_{j_l})}{b(\lambda - \lambda_{j_l})} \\ &\quad \times \sum_{K(J)} \prod_{l=1}^{m} \prod_{n=1}^{L} b(\lambda_{k_l} - \mu_n) \frac{c(\lambda_{j_l} - \lambda_{k_l})}{b(\lambda_{j_l} - \lambda_{k_l})} \prod_{\lambda \in X^{0,L-1}_{i_1,\dots,i_{2m}}} \frac{a(\lambda_{k_l} - \lambda)}{b(\lambda_{k_l} - \lambda)} \\ &\quad \times \prod_{1 \leq r < s \leq m} \frac{a(\lambda_{k_r} - \lambda_{k_s})}{b(\lambda_{k_r} - \lambda_{k_s})} \frac{a(\lambda_{k_r} - \lambda_{j_s})}{b(\lambda_{k_r} - \lambda_{j_s})} \frac{a(\lambda_{k_s} - \lambda_{j_r} + \gamma)}{b(\lambda_{k_s} - \lambda_{j_r})} , \end{aligned} \tag{4.11}$$

with summation symbols defined as $\sum_J = \sum_{\substack{j_1,\dots,j_m \in X^{i_1,i_{2m}} \\ j_1 < j_2 < \dots < j_m}}$ and $\sum_{K(J)} = \sum_{k_1,\dots,k_m \in X^{i_1,i_{2m}}_{j_1,\dots,j_m}}$.

Concerning the general structure of Eq. (4.6) some remarks are in order. For instance, Eq. (4.6) expresses the partition function of the six-vertex model with DWBC in terms of the eigenvalues of the six-vertex model transfer matrix with twisted boundary conditions. This is analogous to the case with periodic boundary conditions where the partition function of the six-vertex model on a torus is given in terms of the eigenvalues of a standard six-vertex model transfer matrix [17, 19]. Moreover, if one assumes the eigenvalues Λ is given in terms of Bethe ansatz like roots as obtained in [11], then Eq. (4.6) would allow us to extract thermodynamic properties of the six-vertex model with DWBC in the same fashion as for the case with toroidal boundary conditions.

4.1 The zeroes w_j

As demonstrated in [12] the eigenvalues $\Lambda(\lambda)$ are essentially a polynomial of order $L-1$ in the variable $x = e^{2\lambda}$. Thus it can be written in terms of its zeroes in the following way,

$$\Lambda(\lambda) = \Lambda(0) \prod_{j=1}^{L-1} \frac{\sinh(w_j - \lambda)}{\sinh(w_j)} . \tag{4.12}$$

Next we assume that the zeroes w_j are all distinct and also consider the functional equation (4.6) under the specialization $\lambda_j = w_j$ for $1 \leq j \leq L-1$. By doing so we are left with the following relations,

- $L \in 2\mathbb{Z}$:

$$\frac{Z(\lambda_0, w_1, \dots, w_{L-1})}{V^{(L)}_{L-1,\dots,0}(\lambda_0, w_1, \dots, w_{L-1})} k_0 = (-1)^{\frac{L}{2}} \tag{4.13}$$

- $L \in 2\mathbb{Z}+1$:
$$\frac{Z(\lambda_0, w_1, \dots, w_{L-1})}{V^{(L-1)}_{L-1,\dots,0}(\lambda_0, w_1, \dots, w_{L-1})} k_0 = \Lambda(\lambda_0) \; . \tag{4.14}$$

The variable λ_0 is still an arbitrary complex parameter in both relations (4.13) and (4.14), and this fact will pave the way to determine the set $\{w_j\}$ from the analytic properties of the functions Z and $V^{([L])}_{L-1,\dots,0}$.

The relations (4.13) and (4.14) are given in terms of the partition function Z which is a symmetric multivariate polynomial [4,15]. On the other hand, the function $V^{([L])}_{L-1,\dots,0}$ also appearing in (4.13) and (4.14) consists of a polynomial of order $L-1$ in the variable $x_0 = e^{2\lambda_0}$ only for the case L even. For odd values of L we have

$$V^{(L-1)}_{L-1,\dots,0}(\lambda_0, w_1, \dots, w_{L-1}) = \frac{\tilde{V}^{(L-1)}_{L-1,\dots,0}(\lambda_0, w_1, \dots, w_{L-1})}{\prod_{j=1}^{L-1} b(\lambda_0 - w_j)} \; , \tag{4.15}$$

where the function $\tilde{V}^{(L-1)}_{L-1,\dots,0}(\lambda_0, w_1, \dots, w_{L-1})$ is then a polynomial of order $L-1$ in the variable x_0.

The case $L \in 2\mathbb{Z}$. The set of variables $\{w_j\}$ for L even can be characterized by the analytical properties of (4.13). In order to carry out this analysis we first need to remind ourselves that both functions Z and $V^{(L)}_{L-1,\dots,0}$ in the LHS of (4.13) are polynomials of the same order in the variable x_0. Now, since the RHS of (4.13) is a constant, we can conclude that the residues of the LHS must vanish at the zeroes of $V^{(L)}_{L-1,\dots,0}$. In other words, the zeroes of Z and $V^{(L)}_{L-1,\dots,0}$ must coincide which leads us to the following formal condition. Let $\lambda^Z_k \in \{\lambda \in \mathbb{C} \mid Z(\lambda, w_1, \dots, w_{L-1}) = 0\}$ and $\lambda^V_k \in \{\lambda \in \mathbb{C} \mid V^{(L)}_{L-1,\dots,0}(\lambda, w_1, \dots, w_{L-1}) = 0\}$. The elements λ^Z_k and λ^V_k shall depend on the set of parameters $\{w_j\}$ and we can conclude that

$$\lambda^Z_k(\{w_j\}) = \lambda^V_k(\{w_j\}) \qquad\qquad k = 1, \dots, L-1 \; . \tag{4.16}$$

The direct inspection of (4.16) for small values of L reveals that the variables w_j are then completely fixed by the aforementioned constraints.

The case $L \in 2\mathbb{Z}+1$. The situation for L odd requires a slightly more elaborated analysis due to the presence of the eigenvalue Λ in the RHS of (4.14). However, in that case we also need to consider (4.15) and it turns out that (4.14) simplifies to

$$\frac{\Lambda(0)}{k_0} \prod_{j=1}^{L-1} b(-w_j)^{-1} = \frac{Z(\lambda_0, w_1, \dots, w_{L-1})}{\tilde{V}^{(L-1)}_{L-1,\dots,0}(\lambda_0, w_1, \dots, w_{L-1})} \; . \tag{4.17}$$

The LHS of (4.17) is a constant with respect to the variable λ_0 while the RHS is given by a ratio of two polynomials. Thus the polynomials Z and $\tilde{V}^{(L-1)}_{L-1,\dots,0}$ must share the same zeroes. Similarly to the case with L even we can formulate this statement as follows. Let $\lambda_k^Z \in \{\lambda \in \mathbb{C} \mid Z(\lambda, w_1, \dots, w_{L-1}) = 0\}$ as previously defined and let $\tilde{\lambda}_k^V \in \{\lambda \in \mathbb{C} \mid \tilde{V}^{(L-1)}_{L-1,\dots,0}(\lambda, w_1, \dots, w_{L-1}) = 0\}$. In this way we have the condition

$$\lambda_k^Z(\{w_j\}) = \tilde{\lambda}_k^V(\{w_j\}) \qquad k = 1, \dots, L-1 \; , \tag{4.18}$$

determining the set of variables $\{w_j\}$.

Both relations (4.16) and (4.18) state that the zeroes of the partition function Z with respect to one of its variables, when properly evaluated at the zeroes of the transfer matrix eigenvalues Λ, coincide with the zeroes of the function $V^{([L])}_{L-1,\dots,0}$.

Wronskian condition. The relations (4.16) and (4.18) are constraints involving the zeroes of certain polynomials. However, the explicit evaluation of those zeroes might be a very non-trivial problem. Alternatively, we can also obtain equations determining the set of zeroes $\{w_j\}$ in terms of the coefficients of the polynomial parts of Z and $V^{([L])}_{L-1,\dots,0}$. In order to perform this analysis in an unified manner for both even and odd values of L, we then define the function F as

$$F = \begin{cases} V^{(L)}_{L-1,\dots,0} & \text{if } L \in 2\mathbb{Z} \\ \tilde{V}^{(L-1)}_{L-1,\dots,0} & \text{if } L \in 2\mathbb{Z}+1 \end{cases} \tag{4.19}$$

where $\tilde{V}^{(L-1)}_{L-1,\dots,0}$ has been previously defined in (4.15). The function $F(\lambda_0, w_1, \dots, w_{L-1})$ is thus a polynomial of order $L-1$ in the variable x_0 for any value of L.

The condition on the coefficients of Z and F fixing the zeroes w_j could have been read directly from the relations (4.13) and (4.17). However, one would have to deal with an overall constant factor which can be avoided by simply demanding that the Wronskian determinant between Z and F vanishes. This is justified by the fact that (4.13) and (4.17) state that Z and F and two linearly dependent polynomials. Thus we have the following condition

$$P(x_0) \equiv Z(x_0, \{w_j\}) F'(x_0, \{w_j\}) - F(x_0, \{w_j\}) Z'(x_0, \{w_j\}) = 0 \; , \tag{4.20}$$

where the symbol $(')$ denotes differentiation with respect to the variable x_0. The function P is a polynomial of order $[L]$ in the variable x_0 which should vanish in the entire complex domain according to the Wronskian condition (4.20). The coefficients of P are given by

$$\mathcal{C}_k = \frac{1}{k!} \left. \frac{\partial^k P}{\partial x_0^k} \right|_{x_0=0} \; , \tag{4.21}$$

and we can demand them to vanish in order to satisfy (4.20). Thus we end up with the following formal condition fixing the zeroes w_j,

$$\mathcal{C}_k(\{w_j\}) = 0 \qquad 0 \leq k \leq [L] \; . \tag{4.22}$$

The relation (4.22) contains one or two more equations than variables w_j to be determined depending if L is even or odd. However, each one of the equations is a non-linear algebraic relation and due to that we actually have a large number of solutions, as it also happens with usual Bethe ansatz equations. In fact, the direct inspection of the solutions of (4.22) for small values of L reveals that the extra equations act as a filter keeping only the solutions which are actually contained in the spectrum of the transfer matrix (2.3).

4.2 Truncation at roots of unity

Vertex models based on solutions of the Yang-Baxter equation can exhibit special properties when its anisotropy parameter satisfies certain root of unity condition. For instance, it was shown by Tarasov in [20] that for the $U_q[\mathfrak{sl}(2)]$ invariant six-vertex model we have the property

$$\overrightarrow{\prod_{0\leq k\leq l-1}} B(\lambda - k\gamma) = 0 \tag{4.23}$$

when the anisotropy parameter γ obeys the condition $e^{2l\gamma} = 1$. The case $l = 1$ is not illuminating as we can see from the definitions (2.5) and (2.7) that both operators $B(\lambda)$ and $C(\lambda)$ are proportional to the factor $(e^{2\gamma} - 1)$. Consequently the transfer matrix (2.3) is also proportional to that same quantity which implies that its eigenvalues trivially vanishes when we set $l = 1$. A similar argument can be used for the case $l = 2$ and in order to illustrate how the property (4.23) can be explored we shall initially consider the cases $l = 3, 4$ before discussing the general case.

***l* = 3.** In that case the property (4.23) explicitly reads $B(\lambda)B(\lambda-\gamma)B(\lambda-2\gamma) = 0$ which can be readily substituted in Eqs. (4.4) and (4.5) under the specialization $\lambda_j = \lambda - j\gamma$. Notice however that this is valid for arbitrary chain length L. By doing so we are left with the following relation,

$$\begin{aligned} \Lambda(\lambda)\Lambda(\lambda-\gamma)\Lambda(\lambda-2\gamma) &= \Lambda(\lambda)[M_1^{(1)}(\lambda-\gamma,\lambda-2\gamma) + N_{2,1}^{(2)}(\lambda,\lambda-\gamma,\lambda-2\gamma)] \\ &\quad + \Lambda(\lambda-\gamma)M_2^{(2)}(\lambda,\lambda-\gamma,\lambda-2\gamma) \\ &\quad + \Lambda(\lambda-2\gamma)M_1^{(2)}(\lambda,\lambda-\gamma,\lambda-2\gamma)\ , \end{aligned} \tag{4.24}$$

which simplifies to

$$\begin{aligned} \Lambda(\lambda)\Lambda(\lambda-\gamma)\Lambda(\lambda-2\gamma) &= -\Lambda(\lambda)\prod_{j=1}^{L}\sinh(\lambda-\mu_j)\sinh(\lambda-\mu_j-2\gamma) \\ &\quad + \Lambda(\lambda-\gamma)2\cosh(\gamma)\prod_{j=1}^{L}\sinh(\lambda-\mu_j)\sinh(\lambda-\mu_j-\gamma) \\ &\quad - \Lambda(\lambda-2\gamma)\prod_{j=1}^{L}\sinh(\lambda-\mu_j+\gamma)\sinh(\lambda-\mu_j-\gamma)\ . \end{aligned} \tag{4.25}$$

Now we can substitute the representation (4.12) in (4.25) and, for instance, set $\lambda = w_i+\gamma$. For this particular specialization the term $\Lambda(\lambda-\gamma)|_{\lambda=w_i+\gamma}$ vanishes and we find the following equation determining the zeroes w_j,

$$\prod_{k=1}^{L}\frac{\sinh(w_i-\mu_k+\gamma)\sinh(w_i-\mu_k-\gamma)}{\sinh(w_i-\mu_k+2\gamma)\sinh(w_i-\mu_k)} = -\prod_{j=1}^{L-1}\frac{\sinh(w_j-w_i+\gamma)}{\sinh(w_j-w_i-\gamma)}\ . \tag{4.26}$$

$l = 4$. For this particular root of unity condition the system of equations (3.8) also truncates leaving us with the following relation,

$$\left\{ \sum_{m=0}^{2} \sum_{0 \leq i_1 < \dots < i_{2m} \leq 3} V^{(2m)}_{i_{2m}, \dots, i_1} \widehat{\prod}_{\lambda}^{i_1, \dots, i_{2m}} \right\} \Lambda(\lambda)|_{\lambda_j = \lambda - j\gamma} = 0 \; . \qquad (4.27)$$

Using the explicit form of the functions $V^{(2m)}_{i_{2m}, \dots, i_1}$ given in (4.11), Eq. (4.27) then explicitly reads

$$\begin{aligned}
&\Lambda(\lambda)\Lambda(\lambda-\gamma)\Lambda(\lambda-2\gamma)\Lambda(\lambda-3\gamma) = \\
&+ \Lambda(\lambda-\gamma)\Lambda(\lambda-2\gamma)\frac{\sinh(3\gamma)}{\sinh(\gamma)}\prod_{k=1}^{L}\sinh(\lambda-\mu_k)\sinh(\lambda-\mu_k-2\gamma) \\
&- \Lambda(\lambda-2\gamma)\Lambda(\lambda-3\gamma)\prod_{k=1}^{L}\sinh(\lambda-\mu_k+\gamma)\sinh(\lambda-\mu_k-\gamma) \\
&- \Lambda(\lambda)\Lambda(\lambda-\gamma)\prod_{k=1}^{L}\sinh(\lambda-\mu_k-\gamma)\sinh(\lambda-\mu_k-3\gamma) \\
&- \Lambda(\lambda)\Lambda(\lambda-3\gamma)\prod_{k=1}^{L}\sinh(\lambda-\mu_k)\sinh(\lambda-\mu_k-2\gamma) + \mathcal{Q}(\lambda) \; ,
\end{aligned} \qquad (4.28)$$

where the function $\mathcal{Q}(\lambda)$ is given by

$$\begin{aligned}
\mathcal{Q}(\lambda) \;=\; & \frac{\sinh(3\gamma)}{\sinh(\gamma)}\prod_{k=1}^{L}\sinh(\lambda-\mu_k)^2\sinh(\lambda-\mu_k-2\gamma)^2 \\
& - \prod_{k=1}^{L}\sinh(\lambda-\mu_k+\gamma)\sinh(\lambda-\mu_k-3\gamma)\sinh(\lambda-\mu_k-\gamma)^2 \\
& - 2\cosh(2\gamma)\prod_{k=1}^{L}\sinh(\lambda-\mu_k)\sinh(\lambda-\mu_k-2\gamma)\sinh(\lambda-\mu_k-\gamma)^2 \; .
\end{aligned} \qquad (4.29)$$

Next we consider the representation (4.12) and firstly set $\lambda = w_i + \gamma$ in (4.28). Under this specialization we have that $\Lambda(\lambda-\gamma)|_{\lambda=w_i+\gamma}$ vanishes, and we are left with a relation depending on the function $\mathcal{Q}(w_i+\gamma)$. Then we also consider the specialization $\lambda = w_i + 2\gamma$ where $\Lambda(\lambda-2\gamma)|_{\lambda=w_i+2\gamma}$ is null. In this way we obtain two equations which involves both the functions $\mathcal{Q}(w_i+\gamma)$ and $\mathcal{Q}(w_i+2\gamma)$. However, under the root of unity condition $l = 4$ we have the property $\mathcal{Q}(\lambda) = \mathcal{Q}(\lambda+\gamma)$ which allows us to eliminate those functions from our equations. By doing so we obtain the relation

$$\prod_{k=1}^{L}\frac{\sinh(w_i-\mu_k+\gamma)\sinh(w_i-\mu_k-\gamma)}{\sinh(w_i-\mu_k+2\gamma)\sinh(w_i-\mu_k)} = -\frac{\Lambda(w_i-\gamma)}{\Lambda(w_i+\gamma)} \; , \qquad (4.30)$$

which is precisely the same equation found for the case $l = 3$ (4.26). The free-fermion point $\gamma = \mathrm{i}\pi/2$ fits in the case $l = 4$ and at that point Eq. (4.30) reduces to

$$\prod_{k=1}^{L} \coth\left(w_i - \mu_k\right)^2 = 1 \; , \tag{4.31}$$

which generalizes the proposal of [13] in the presence of inhomogeneities μ_k.

General case. For arbitrary values of l the property (4.23) truncates the system of functional relations (3.8) and we only need to consider the following equation,

$$\left\{ \sum_{m=0}^{[l]/2} \sum_{0 \leq i_1 < \cdots < i_{2m} \leq l-1} V^{(2m)}_{i_{2m}, \dots, i_1} \widehat{\prod_{\lambda}}^{i_1, \dots, i_{2m}} \right\} \Lambda(\lambda)|_{\lambda_j = \lambda - j\gamma} = 0 \; . \tag{4.32}$$

Then we assume the representation (4.12) and consider the sequence of specializations $\lambda = w_i + p\gamma$ for $1 \leq p \leq p-2$. This procedure yields one equation at each p-level of specialization, and similarly to the case $l = 4$, we can manipulate the system of equations to find a compact equation determining the roots w_j. It turns out that for general values of l we find the same equation obtained for the particular cases $l = 3, 4$. Therefore, we find no need to rewrite (4.26) as it is also valid for general root of unity cases.

5 Concluding remarks

The main result of this work is the functional equation (4.6) which states an interesting relation between the six-vertex model with DWBC and the twisted anisotropic Heisenberg chain. The derivation of (4.6) follows the approach introduced in [12] and refined through the works [14–16]. The main ingredient of this derivation is the Yang-Baxter algebra and, in particular, the functional equation (4.6) allows us to show a very non-trivial relation between the zeroes of the DWBC partition function and the zeroes of the twisted transfer matrix eigenvalues. This relation is expressed in (4.16) and (4.18).

The case where the anisotropy parameter satisfies a root of unity condition has also been considered here. In that case we have found a compact set of equations, namely (4.26), describing the zeroes of the eigenvalues Λ.

The six-vertex model is not the only integrable system which admits boundary conditions of domain wall type. For instance, domain wall boundaries can also be formulated for the so called $8V$-SOS model which has been tackled through this algebraic-functional method in [15, 16]. The latter is an elliptic model and in this way one might wonder if there is a twisted transfer matrix such that an analogous relation to (4.6) holds. This problem has eluded us so far and its investigation would probably bring further insights into the structure of elliptic integrable systems.

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References

[1] E. H. Lieb. Residual entropy of square lattice. *Phys. Rev.*, 162(1):162, 1967.

[2] R. J. Baxter. Three-colorings of the square lattice: A hard squares model. *J. Math. Phys.*, 11(10):3116, 1970.

[3] V. G. Knizhnik and A. B. Zamolodchikov. Current algebra and Wess-Zumino model in two dimensions. *Nucl. Phys. B*, 247(1):83–103, 1984.

[4] V. E. Korepin. Calculation of norms of Bethe wave functions. *Commun. Math. Phys.*, 86:391–418, 1982.

[5] O. Foda, M. Wheeler, and M. Zuparic. Domain wall partition functions and KP. *J. Stat. Mech.*, 2009.

[6] A. Lascoux. The 6 Vertex Model and Schubert Polynomials. *SIGMA*, 3:029, 2007.

[7] H. Rosengren. The three-colour model with domain wall boundary conditions. *Adv. Appl. Math.*, 46:481, 2011.

[8] G. Kuperberg. Another proof of the alternating sign matrix conjecture. *Inter. Math. Res. Notes*, (3):139–150, 1996.

[9] W. Galleas. Functional relations and the Yang-Baxter algebra. *Journal of Physics: Conference Series*, 474:012020, 2013.

[10] H. J. de Vega. Families of commuting transfer matrices and integrable models with disorder. *Nucl. Phys. B*, 240(4):495–513, 1984.

[11] M. T. Batchelor, R. J. Baxter, M. J. Orourke, and C. M. Yung. Exact solution and interfacial tension of the 6-vertex model with antiperiodic boundary conditions. *J. Phys. A - Math. and Gen.*, 28(10):2759–2770, 1995.

[12] W. Galleas. Functional relations from the Yang-Baxter algebra: Eigenvalues of the XXZ model with non-diagonal twisted and open boundary conditions. *Nucl. Phys. B*, 790(3):524–542, 2008.

[13] J. Cao, W. L. Yang, K. Shi, and Y. Wang. Off-diagonal Bethe ansatz and exact solution of a topological spin ring. *Phys. Rev. Lett.*, 111:137201, 2013.

[14] W. Galleas. Functional relations for the six-vertex model with domain wall boundary conditions. *J. Stat. Mech.*, (06):P06008, 2010.

[15] W. Galleas. Multiple integral representation for the trigonometric SOS model with domain wall boundaries. *Nucl. Phys. B*, 858(1):117–141, 2012, math-ph/1111.6683.

[16] W. Galleas. Refined functional relations for the elliptic SOS model. *Nucl. Phys. B*, 867:855–871, 2013.

[17] R. J. Baxter. *Exactly Solved Models in Statistical Mechanics.* Dover Publications, Inc., Mineola, New York, 2007.

[18] W. Galleas. Scalar product of Bethe vectors from functional equations. to appear in Comm. Math. Phys., 2013.

[19] J. de Gier, W. Galleas, and M. Sorrell. Multiple integral formula for the off-shell six vertex scalar product. 2011, hep-th/1111.3712.

[20] V. O. Tarasov. On Bethe vectors for the XXZ model at roots of unity. *Journal of Mathematical Sciences*, 125(2):242–248, 2005.