On the giant component of random hyperbolic graphs

Michel Bode[1], Nikolaos Fountoulakis[1] and Tobias Müller[2]

1 Introduction

The theory of geometric random graphs was initiated by Gilbert [2] already in 1961 in the context of what is called *continuum percolation*. In 1972, Hafner [4] focused on the typical properties of large but finite random geometric graphs. Here N points are sampled within a certain region of $\mathbb{R}^d$ following a certain distribution and any two of them are joined when their Euclidean distance is smaller than some threshold which, in general, is a function of N. In the last two decades, this class of random graphs has been studied extensively – see the monograph of Penrose [6].

However, what structural characteristics emerge when one considers these points distributed on a curved space where distances are measured through some (non-Euclidean) metric? Such a model was introduced by Krioukov et al. [5] and some typical properties of these random graphs were studied with the use of non-rigorous methods.

1.1 Random geometric graphs on a hyperbolic space

The most common representations of the hyperbolic space is the upper-half plane representation $\{z \ : \ \Im z > 0\}$ as well as the Poincaré unit disc which is simply the open disc of radius one, that is, $\{(u, v) \in \mathbb{R}^2 \ : \ 1 - u^2 - v^2 > 0\}$. Both spaces are equipped with the hyperbolic metric; in the former case this is $\frac{1}{(\zeta y)^2} dy^2$ whereas in the latter this is $\frac{4}{\zeta^2} \frac{du^2 + dv^2}{(1-u^2-v^2)^2}$, where ζ is some positive real number. It can be shown that the (Gaussian) curvature in both cases is equal to $-\zeta^2$.We will denote by $\mathbb{H}^2_\zeta$ the class of these spaces.

[1] School of Mathematics, University of Birmingham, United Kingdom.
Email: michel.bode@gmx.de, n.fountoulakis@bham.ac.uk. This research has been supported by a Marie Curie Career Integration Grant PCIG09-GA2011-293619.

[2] Mathematical Institute, Utrecht University, The Netherlands. Email: t.muller@uu.nl

J. Nešetřil et al. (eds.), *The Seventh European Conference on Combinatorics, Graph Theory and Applications*

In this paper, following the definitions in [5], we shall be using the native representation of $\mathbb{H}^2_\zeta$. Under this representation, the ground space of $\mathbb{H}^2_\zeta$ is $\mathbb{R}^2$ and every point $x \in \mathbb{R}^2$ whose polar coordinates are (r, θ) has hyperbolic distance from the origin O equal to r. Also, a circle of radius r around the origin has length equal to $\frac{2\pi}{\zeta}\sinh(\zeta r)$ and area equal to $\frac{2\pi}{\zeta^2}(\cosh(\zeta r) - 1)$.

Let $N = \nu e^{\zeta R/2}$, where ν is a positive real number that controls the average degree of the random graph. We create a random graph by selecting randomly N points from the disc of radius R centred at the origin O, which we denote by $\mathcal{D}_R$. If such a random point u has polar coordinates (r, θ), then θ is uniformly distributed in $(0, 2\pi]$, whereas the probability density function of r, which we denote by $\rho(r)$, is determined by a parameter $\alpha > 0$ and is equal to

$$\rho(r) = \alpha \frac{\sinh \alpha r}{\cosh \alpha R - 1}. \tag{1.1}$$

When $\alpha = \zeta$, then this is the uniform distribution. This set of points will be the vertex set of the random graph and we denote it by V_N. The random graph $\mathcal{G}(N; \zeta, \alpha)$ is formed when we join two vertices, if they are within (hyperbolic) distance R.

Krioukov et al. [5] focus on the degree distribution of $\mathcal{G}(N; \zeta, \alpha)$, showing that when $0 < \zeta/\alpha < 2$ this follows a power law with exponent $2\alpha/\zeta + 1$. They also discuss clustering on a smooth version of the above model. Their results have been verified rigorously by Gugelmann et al. [3]. When $1 < \zeta/\alpha < 2$, the exponent is between 2 and 3, as is the case in a number of networks that emerge in applications such as computer networks, social networks and biological networks (see for example [1]). Krioukov et al. [5] introduce this model as a geometric framework for the study of complex networks. In fact, they view the degree distribution as well as the existence of clustering at a local level as “natural reflections of the underlying hyperbolic geometry”.

1.2 Component structure of $\mathcal{G}(N; \zeta, \alpha)$

This paper focuses on the component structure of $\mathcal{G}(N; \zeta, \alpha)$ and, in particular, on the size of its largest component. We also denote by $|L_1|$ the size of a largest connected component of $\mathcal{G}(N; \zeta, \alpha)$.

In this contribution, we show that when ζ/α crosses 1 a “phase transition” occurs. More specifically, if $\zeta/\alpha < 1$, then *asymptotically almost surely (a.a.s.)*, that is, with probability $1 - o(1)$ as $N \to \infty$, $|L_1|$ is bounded by a sublinear function, whereas if $\zeta/\alpha > 1$, then $|L_1|$ is linear.

Theorem 1.1. *Let ζ, α be positive real numbers. The following hold:*

- *If $\zeta/\alpha > 1$, then there exists $c = c(\zeta, \alpha, \nu) > 0$ such that a.a.s. $|L_1| > cN$.*
- *If $\zeta/\alpha < 1$, then a.a.s. $|L_1| < CR^2 N^{\zeta/\alpha}$, where $C = C(\zeta, \alpha) > 0$.*

Furthermore, one can show that when $\zeta/\alpha > 2$, then $\mathcal{G}(N; \zeta, \alpha)$ is a.a.s. connected. We now proceed with a brief sketch of the proof of each part of the above theorem.

2 The supercritical regime

For any given point $v \in \mathcal{D}_R$, we let $t_v = R - r_v$, where r_v denotes the radius of v – we call this the *type* of point v. We define a partition of $\mathcal{D}_R$ into homocentric bands $\mathcal{B}_i$, for $i = 1, \dots, T$, where $T = T(N)$ is a suitably defined function of N. More specifically, the central band $\mathcal{B}_0$ consists of all points in $\mathcal{D}_R$ whose type is larger than $R/2$, that is, their radius is less than $R/2$. Note that any two vertices in $\mathcal{B}_0$ are connected by an edge as their hyperbolic distance is less than R. In other words, the subgraph of $\mathcal{G}(N; \zeta, \alpha)$ induced by the vertices in $\mathcal{B}_0$ is the complete graph. To define the remaining bands, we define a decreasing sequence of positive real numbers $t_0 > t_1 > \cdots$, where $t_0 := R/2$ and for $i \geq 1$ we have

$$t_i - \frac{2}{\zeta} \ln \left(\frac{8\pi}{\nu} \ln t_i \right) = \lambda t_{i-1}, \tag{2.1}$$

where $\lambda := \frac{2}{\zeta}\left(\alpha - \frac{\zeta}{2}\right) < 1$ (as $\zeta/\alpha > 1$) and

$$\mathcal{B}_i = \{ v \in \mathcal{D}_R \;:\; t_i \leq t_v < t_{i-1} \}.$$

We define T as the largest i such that $e^{-\alpha(t_{i-1}-t_i)} < 1/2$ and $t_i > e$. Let $\mathcal{N}_0$ denote the set of vertices that belong to the set $\mathcal{B}_0$. In turn, for $i > 0$ we let $\mathcal{N}_i$ denote the set of vertices in $\mathcal{B}_i$ that have at least one neighbour in $\mathcal{N}_{i-1}$. Since $\mathcal{B}_0$ is a clique, the subgraph of $\mathcal{G}(N; \zeta, \alpha)$ that is induced by $\cup_{i=0}^{T} \mathcal{N}_i$ is connected and has size $\sum_{i=0}^{T} |\mathcal{N}_i|$.

We establish bounds on the sizes of the sets $\mathcal{N}_i$, for $i = 1, \dots, T$. In particular, we show that the number of vertices in $\mathcal{N}_i$ stochastically dominates the number of vertices that are contained in a subset of $\mathcal{B}_i$ that has arc Θ_i, where the sequence of Θ_i satisfies for $i \geq 1$

$$\Theta_i \geq \Theta_{i-1} \left(1 - \exp\left(-\frac{\nu}{4\pi} \left(e^{-\alpha t_{i-1}} - e^{-\alpha t_{i-2}} \right) \theta^{(i)} \right) \right), \quad \Theta_0 := 2\pi$$

and $\theta^{(i)} := e^{\frac{\zeta}{2}(t_{i-1}+t_i)}$ (here $t_{-1} := R$). We denote this number by N_i'. A concentration argument shows that a.a.s. $N_i' \geq \frac{1}{2} N \frac{\Theta_i}{2\pi} \left(e^{-\alpha t_i} - e^{-\alpha t_{i-1}} \right)$.

As $e^{-\alpha(t_{i-1}-t_i)} < 1/2$, it follows that

$$\left(e^{-\alpha t_{i-1}} - e^{-\alpha t_{i-2}}\right)\theta^{(i)} = e^{-\alpha t_{i-1}}\left(1 - e^{-\alpha(t_{i-2}-t_{i-1})}\right)\theta^{(i)}$$
$$> \frac{1}{2}\, e^{-\alpha t_{i-1} + \frac{\zeta}{2}(t_{i-1}+t_i)} = \frac{4\pi}{\nu}\ln t_i,$$

whereby

$$\Theta_i \geq \Theta_{i-1}\left(1 - \frac{1}{t_i}\right) \stackrel{(2.1)}{>} \Theta_{i-1}\left(1 - \frac{1}{\lambda^i t_0}\right).$$

It then follows that for some $c = c(\zeta, \alpha, \nu) > 0$ a.a.s.

$$\sum_{i=0}^{T} N_i' \geq N\left(e^{-\alpha t_T} - e^{-\alpha t_0}\right)\prod_{i=0}^{T}\left(1 - \frac{1}{\lambda^i t_0}\right) > cN.$$

The stochastic domination implies this part of the theorem.

3 The subcritical regime

A first moment argument shows that all vertices have type at most $\frac{\zeta}{2\alpha}R + \omega(N)$, where $\omega(N)$ is a function such that $\omega(N) \to \infty$ as $N \to \infty$. Hence, since $\zeta/\alpha < 1$, it follows that all vertices have types which are smaller than and bounded away from $R/2$. We consider a vertex v which has this type and we analyse a *breadth exploration process* through which we bound the *total angle of the component* which v belongs to. We define the *total angle* of the component of v to be the largest relative angle between any two of its vertices – if the component has only one vertex, then this is equal to zero. We denote this angle by $\Theta(v)$. We show that the assumption that the type of v is $\frac{\zeta}{2\alpha}R + \omega(N)$ gives a stochastic upper bound on the total angle of the component of any vertex in V_N. Our bound is as follows.

Lemma 3.1. *Let $v \in \mathrm{V}_N$ be a vertex having $t_v = \frac{\zeta}{2\alpha}R + \omega(N)$. There exists a constant $C' = C'(\zeta, \alpha, \nu) > 0$ such that with probability $1 - o\left(\frac{1}{N^{1-\zeta/\alpha}}\right)$ we have*

$$\Theta(v) \leq C'\frac{R^2 N^{\zeta/\alpha}}{N}.$$

The *breadth exploration process* is a process that is somewhat similar to the breadth-first search algorithm. More specifically, starting from vertex v with type as in the above lemma, we expose the vertex of the largest type among those vertices that are within distance R from v in clockwise direction. Subsequently, we continue this procedure until a vertex of type K is reached, where K is a large constant. We repeat this in anticlockwise direction. This completes the first phase of the process. Thereafter, we bound the contribution that comes from vertices of type smaller than K. If these have also vertices within distance R that have not been covered previously, then we start the first phase again. We show that the number of repetitions of this phase is bounded in probability. A first moment argument shows that the probability that there is a vertex v with $\Theta(v)$ larger than the bound of the above lemma is $o(1)$. The result provides a bound on the total angle of each component. This needs to be complemented by a result which associates the total angle of a component with the number of vertices. We show that a.a.s. there is no component with total angle at most $C'R^2N^{\zeta/\alpha}/N$ that has more than $CR^2N^{\zeta/\alpha}$ vertices, where $C = C(\zeta, \alpha, \nu) > 0$ is another constant. This completes the proof of Theorem 1.1.

4 Conclusions - Further directions

This contribution focuses on the size of the largest component of random geometric graphs on the hyperbolic plane. We show that when the ratio ζ/α crosses 1 a giant component emerges. But is this component unique? What is the size of the largest component in each case? Moreover, our results do not cover the critical case $\zeta/\alpha = 1$, that is, when the points are uniformly distributed on $\mathcal{D}_R$. This is a natural direction that goes along the lines of the theory of random geometric graphs on Euclidean spaces.

References

[1] R. ALBERT and A.-L. BARABÁSI, *Statistical mechanics of complex networks*, *Rev. Mod. Phys.* **74** (2002), 47–97.

[2] E. N. GILBERT, *Random plane networks*, J. Soc. Indust. Appl. Math. **9** (1961), 533–543.

[3] L. GUGELMANN, K. PANAGIOTOU and U. PETER, *Random hyperbolic graphs: degree sequence and clustering*, In: "Proceedings of the 39th International Colloquium on Automata, Languages and Programming", A. Czumaj *et al.* (eds.), Lecture Notes in Computer Science 7392, 2012, 573–585.

[4] R. HAFNER, *The asymptotic distribution of random clumps*, Computing **10** (1972), 335–351.

[5] D. KRIOUKOV, F. PAPADOPOULOS, M. KITSAK, A. VAHDAT and M. BOGUÑÁ, *Hyperbolic geometry of complex networks*, Phys. Rev. E **82** (2010), 036106.

[6] M. PENROSE, "Random Geometric Graphs", Oxford University Press, 2003.