FROM WZW MODELS TO MODULAR FUNCTORS

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ABSTRACT. In this survey paper we give a relatively simple and coordinate free description of the WZW model as a local system whose base is the $\mathbb{G}_m$-bundle associated to the determinant bundle on the moduli stack of pointed curves. We derive its main properties and show how it leads to a modular functor in the spirit of Segal. The approach presented here is almost purely algebro-geometric in character; it avoids the Boson-Fermion correspondence, operator product expansions as well as Teichmüller theory.

The tumultuous interaction between mathematicians and theoretical physicists that began more than two decades ago left some of us hardly time to take stock. It is telling for this era that it took physicists (Witten, mainly) to point out in the late eighties that there must exist a bridge between two, at the time hardly connected, mathematical land masses, *viz.* algebraic geometry and knot theory, and it is equally telling that it was only recently that this was materialized with mathematically rigorous underpinnings (and strictly speaking not even in the desired form yet). We are here referring on the algebro-geometric side to a subject that has its place in the present handbook, namely moduli spaces of vector bundles over curves, and on the other side to the knot invariants (like the Jones polynomial) that are furnished by Chern-Simons theory. The bridge metaphor is actually a bit misleading, because on either side the roads leading to it had yet to be constructed. Let us use the remainder of this introduction to survey very briefly the part this route that involves algebraic geometry, stopping short at the point were the crossing is made, then say which segment is covered by this paper and conclude in the customary manner by commenting on the various sections.

To set the stage, let C be a compact Riemann surface and G a (say, simply connected) complex algebraic group with simple Lie algebra $\mathfrak{g}$. Then there is a moduli stack $\mathrm{M}(\mathsf{C},\mathsf{G})$ of G-principal bundles over C. This stack carries a natural ample line bundle $\Theta(\mathsf{C},\mathsf{G})$, which in fact generates its Picard group, and for which the vector space of sections of $\Theta(\mathsf{C},\mathsf{G})^{\otimes\ell}$, the so-called *Verlinde space of level* ℓ and here denoted by $\mathbb{H}_\ell(\mathsf{G})_{\mathsf{C}}$, is finite dimensional for all ℓ. Its dimension is independent of C and indeed, if we vary C over a base S, then we get a vector bundle $\mathcal{H}_\ell(\mathsf{G})_{\mathcal{C}/\mathsf{S}}$ over that base. Although we required G to be simply connected, one can make sense of this for reductive groups as well, although some care is needed. For instance, for $\mathsf{G}=\mathbb{C}^\times$, we let $\mathrm{M}(\mathsf{C},\mathbb{C}^\times)$ not be the full Picard variety $\mathrm{Pic}(\mathsf{C})$ of C, but pick the component $\mathrm{Pic}(\mathsf{C})^{g-1}$

parameterizing line bundles of degree $(g-1)$, as this is the one which carries a natural line bundle that can play the role of $\Theta(C, \mathbb{C}^\times)$ (and which is indeed known as the theta bundle). In that case $\mathbb{H}_\ell(G)_C$ is just the space of theta functions of degree ℓ. These theta functions satisfy a heat equation and it is our understanding that Mumford was the first to observe that this property may be interpreted as defining a flat connection for the associated projective space bundle. Hitchin [8] proved that this is also true for the case considered here: the projectivized Verlinde bundles come naturally with a flat connection. But if one aims for flat connections on the bundles themselves, then one should work on the total space of a $\mathbb{C}^\times$-bundle over S (which allows for nontrivial monodromy in a fiber). For the line bundle attached to this $\mathbb{C}^\times$-bundle we can take the determinant bundle of the direct image of the sheaf of relative differentials on $\mathcal{C}/S$. For many purposes—certainly for topological applications—it is desirable to allow for certain 'impurities' of the principal bundle, in the form of a parabolic structure. Such a structure is specified by giving on C a finite set of points $(x_i \in C)_{i \in I}$, and for each such point a finite dimensional irreducible representation V_i of G. It was shown by Scheinost-Schottenloher [15] that in this setting there are still corresponding Verlinde bundles that come with a flat connection after a pull-back to a $\mathbb{C}^\times$-bundle. There is an infinitesimal counterpart of the above construction via holomorphic conformal field theory where the group G enters only via its Lie algebra $\mathfrak{g}$, known as the *Wess-Zumino-Witten model*. This centers on the affine Lie algebra associated to $\mathfrak{g}$ and its representation theory and leads to similar constructs such as the Verlinde bundles with a projectively flat connection. Its mathematically rigorous treatment began with the fundamental paper by Tsuchiya-Ueno-Yamada [20] with subsequent extensions and refinements, mainly by Andersen-Ueno [1], [2]. It was however not a priori clear that this led to the same local system as the global approach. Indeed, this turned out to be not trivial at all: after partial results by Beauville-Laszlo and others, Laszlo-Sorger [13] proved that the Verlinde bundles can be identified and Laszlo [12] showed that via this identification the two connections are the same as well, at least when no parabolic structure is present.

The bridge is now crossed as follows: a nonzero point of the determinant line over C can be topologically specified by means of the choice of Lagrangian sublattice in $H_1(C; \mathbb{Z})$. This enables us to understand the existence of the flat connection on the Verlinde bundles as telling us that these spaces only depend on the isotopy class of the complex structure of C. In particular, they naturally receive the structure of a projective representation of the mapping class group of the pointed surface. This puts these spaces into the topological realm and we thus arrive at an example of a topological quantum field theory, more precisely, at one of Segal's modular functors [16].

Let us now turn to the central goal of this paper, which is to define the Wess-Zumino-Witten connection and to derive its principal properties, to wit its flatness, factorization, the relation with the KZ-system, ..., in short, to recover all the properties needed for defining the underlying (topological) modular functor as found in the papers above mentioned by Tsuchiya-Ueno-Yamada and Andersen-Ueno. For an audience of algebraic geometers knowing, or willing to accept, some rather basic facts about affine Lie algebras, our presentation is essentially self-contained. It is also shorter and possibly at several points more transparent than the literature we are aware of. This is to a large extent due to our consistent coordinate free approach, which not only has the advantage of making it unnecessary to constantly check for gauge invariance, but is also conceptually more satisfying. Cases in point are our definition of the WZW-connection and our treatment of the Fock representation (leading up to Corollary 8) which enables us to avoid resorting to the infinite wedge representation and allied techniques.

Let us take the occasion to point out that what makes the WZW-story still incomplete is an explanation of the duality property and the unitary structure that the associated modular functor should possess.

As promised, we finish with brief comments on the contents of the separate sections. The rather short Section 1 essentially elaborates on the notion of a projectively flat connection. Logically, this material should have its place later in the paper, but as it has some motivating content for what comes right after it, we felt it best to put it there. Section 2 introduces in a canonical way the Virasoro algebra and its Fock representation and the associated Segal-Suguwara construction in a relative setting. New is the last subsection about symplectic local systems, where we see the determinant bundle appear in a canonical fashion. The Lie algebra $\mathfrak{g}$ enters in Section 3. We found it helpful to present this material in an abstract algebraic setting, replacing for instance the ring of complex Laurent polynomials by a complete local field containing $\mathbb{Q}$ (or rather a direct sum of these), which is then also allowed to 'depend on parameters'. Our extension 13 of the Sugawara representation to a relative situation involving a Leibniz rule in the horizontal direction serves here as the origin of WZW-connection and its projective flatness. We keep that setting in Section 4, where the connection itself is defined. In the subsequent section we derive the coherence of the Verlinde sheaf and establish what is called the propagation of covacua. Special attention is paid to the genus zero case and it shown how the WZW-connection is then related to the one of Knizhnik-Zamolodchikov. Section 6 is devoted to the basic results associated to a double point degeneration such as local freeness, factorization and monodromy. Finally, in Section 7, we establish the conversion into a modular functor. Notice that the approach described here is elementary and does not resort to Teichmüller theory.

This paper is based on (but substantially supersedes) our arXiv preprint `math.AG/0507086`.

The author is grateful to two referees for helpful comments.

We find it convenient to work over an arbitrary algebraically closed field k of characteristic zero, but in Section 7, where we discuss the link with topological quantum field theory, we assume $k = \mathbb{C}$. As an intermediate base we use a regular k-algebra, denoted R.

1. Flat and projectively flat connections

A central notion of this article is that of a flat projective connection. Although it enters the scene much later in the paper, some of the work done in the first part is motivated by the particular way this notion appears here. So we start with a brief section discussing it.

We begin with recalling some basic facts. Let $\mathcal{H}$ be a rank r vector bundle over a smooth base S (in other words, is a locally a free $\mathcal{O}_S$-module of rank r). Then the Lie algebra $\mathcal{D}_1(\mathcal{H})$ of first order differential operators $\mathcal{H} \to \mathcal{H}$ fits in an exact sequence of coherent sheaves of Lie algebras

$$0 \to \mathcal{E}nd(\mathcal{H}) \to \mathcal{D}_1(\mathcal{H}) \xrightarrow{sb} \theta_S \otimes_{\mathcal{O}_S} \mathcal{E}nd(\mathcal{H}) \to 0,$$

where sb is the symbol map which assigns to $D \in \mathcal{D}_1(\mathcal{H})$ the k-derivation $\phi \in \mathcal{O}_S \mapsto [D, \phi] \in \mathcal{O}_S$. The local sections of $\mathcal{D}_1(\mathcal{H})$ whose symbol land in $\theta_S \otimes 1_{\mathcal{H}} \cong \theta_S$ make up a coherent subsheaf of Lie subalgebras $sb^{-1}(\theta_S) \subset \mathcal{D}_1(\mathcal{H})$ so that we have an exact sequence of coherent sheaves of Lie algebras

$$0 \to \mathcal{E}nd(\mathcal{H}) \to sb^{-1}(\theta_S) \to \theta_S \to 0.$$

A connection on $\mathcal{H}$ is then simply a section $X \in \theta_S \mapsto \nabla_X \in sb^{-1}(\theta_S)$ of $sb^{-1}(\theta_S) \to \theta_S$ and it is flat precisely if this section is a Lie homomorphism. This suggests that we define a flat connection on the associated projective space bundle $\mathbb{P}_S(\mathcal{H})$ as a Lie subalgebra $\hat{\mathcal{D}} \subset \mathcal{D}_1(\mathcal{H})$ with $sb(\hat{\mathcal{D}}) = \theta_S$ and $\hat{\mathcal{D}} \cap \mathcal{E}nd(\mathcal{H}) = \mathcal{O}_S \otimes 1_{\mathcal{H}} \cong \mathcal{O}_S$ so that we have an exact subsequence

$$0 \to \mathcal{O}_S \to \hat{\mathcal{D}} \to \theta_S \to 0$$

of the one displayed above. That this is a sensible definition follows from the observation that an $\mathcal{O}_S$-linear section ∇ of $\hat{\mathcal{D}} \to \theta_S$ defines a connection on $\mathcal{H}$ whose curvature form $R(\nabla)$ is a closed 2-form on S. Any other section ∇' differs from ∇ by an $\mathcal{O}_S$-linear map $\theta_S \to \mathcal{O}_S$, in other words, by a differential ω, and we have $R(\nabla') = R(\nabla) + d\omega$. So this indeed gives rise to a flat connection in $\mathbb{P}_S(\mathcal{H})$ and it is easily seen that this connection is independent of the choice of the section. Locally on S, $R(\nabla)$ is exact, and so we can always choose ∇ to be flat as a connection. Any other such local section that is flat is necessarily of the form $\nabla + d\phi$ with $\phi \in \mathcal{O}$ and conversely, any such local section has that property. The Lie algebra sheaf $\hat{\mathcal{D}}$ itself does not determine a connection on $\mathcal{H}$; this is most evident when $\mathcal{H}$ is a line bundle, for then we must have $\hat{\mathcal{D}}(\mathcal{H}) = \mathcal{D}_1(\mathcal{H})$.

In the above situation we let $\hat{\mathcal{D}}$ act on the determinant bundle $\det(\mathcal{H}) = \wedge^r_{\mathcal{O}_S}\mathcal{H}$ by means of the formula

$$\hat{D}(e_1 \wedge \cdots \wedge e_r) := \sum_{i=1}^{r} e_1 \wedge \cdots \wedge \hat{D}(e_i) \wedge \cdots \wedge e_r.$$

This is indeed well-defined, and identifies $\hat{\mathcal{D}}$ as a Lie algebra with the Lie algebra of first order differential operators $\mathcal{D}_1(\det(\mathcal{H}))$. Notice however that this identification makes $f \in \mathcal{O}_S \subset \hat{\mathcal{D}}$ act on $\det(\mathcal{H})$ as multiplication by rf.

Let us next observe that if λ is a line bundle on S and N is a positive integer, then a similar formula identifies $\mathcal{D}_1(\lambda)$ with $\mathcal{D}_1(\lambda^{\otimes N})$ (both as $\mathcal{O}_S$-modules and as k-Lie algebras), but induces multiplication by N on $\mathcal{O}_S$. This leads us to make the following

Definition 1. Let be given a smooth base variety S over which we are given a line bundle λ and a locally free $\mathcal{O}_S$-module $\mathcal{H}$ of finite rank. A λ-*flat connection* on $\mathcal{H}$ is homomorphism of $\mathcal{O}_S$-modules $u : \mathcal{D}_1(\lambda) \to \mathcal{D}_1(\mathcal{H})$ that is also a Lie homomorphism over k, commutes with the symbol maps (so these must land in θ_S) and takes scalars to scalars: $\mathcal{O}_S \subset \mathcal{D}_1(\lambda)$ is mapped to $\mathcal{O}_S \subset \mathcal{D}_1(\mathcal{H})$.

It follows from the preceding that such a homomorphism u determines a flat connection on the projectivization of $\mathcal{H}$. The map u preserves $\mathcal{O}_S$ and since this restriction is $\mathcal{O}_S$-linear, it is given by multiplication by some regular function w on S. If $D \in \theta_S$ is lifted to $\hat{D} \in \mathcal{D}_1(\lambda)$, then $u(\hat{D}) \in \mathcal{D}_1(\mathcal{H})$ is also a lift of D and so $D(w) = [u(\hat{D}), u(1)] = u([\hat{D}, 1]) = 0$. This shows that w must be locally constant; we call this the *weight* of u. So in the above discussion, $\hat{\mathcal{D}}$ comes with $\det(\mathcal{H})$-flat connection of weight r^{-1}.

It is clear that if the weight of u is constant zero, then u factors through θ_S, so that we get a flat connection in $\mathcal{H}$. This is also the case when $\lambda = \mathcal{O}_S$, for then $\mathcal{D}_1(\mathcal{O}_S)$ contains θ_S canonically as a direct summand (both as $\mathcal{O}_S$-module and as a sheaf of k-Lie algebras) and the flat connection is then given by the action of θ_S. This has an interesting consequence: if $\pi : \Lambda^\times \to S$ is the geometric realization of the $\mathbb{G}_m$-bundle defined by λ, then $\pi^*\lambda$ has a 'tautological' generating section and thus gets identified with $\mathcal{O}_{\Lambda^\times}$. Hence a λ-flat connection on $\mathcal{H}$ defines an ordinary flat connection on $\pi^*\mathcal{H}$. One checks that if w is the weight of u, then the connection is homogeneous of degree w along the fibers. So in case $k = \mathbb{C}$, $s \in S$ and $\tilde{s} \in \Lambda^\times$ lies over $s \in S$, then the multivalued map $(z, h) \in \mathbb{C}^\times \times H_s \mapsto (z\tilde{s}, z^w h) \in \Lambda^\times_s \times H_s$ is flat, and so the monodromy of the connection in $\Lambda^\times_s$ is scalar multiplication by $e^{2\pi\sqrt{-1}w}$.

We will also encounter a logarithmic version. Here we are given a closed subvariety $\Delta \subset S$ of lower dimension (usually a normal crossing hypersurface). Then the θ_S-stabilizer of the ideal defining Δ, denoted $\theta_S(\log\Delta)$, is a coherent $\mathcal{O}_S$-submodule of θ_S closed under the Lie bracket. If in Definition 1 we have u only defined on the preimage of $\theta_S(\log\Delta) \subset \theta_S$ in $\mathcal{D}_1(\lambda)$ (which we denote here by $\mathcal{D}_1(\lambda)(\log\Delta))$, then we say that we have a *logarithmic λ-flat connection relative to* Δ on $\mathcal{H}$.

2. The Virasoro algebra and its basic representation

Much of the material exposed in this section is a conversion of certain standard constructions (as can be found for instance in [10]) into a coordinate invariant and relative setting. But the way we introduce the Virasoro algebra is less standard and may be even new. A similar remark applies to part of the last subsection (in particular, Corollary 8), which is devoted to the Fock module attached to a symplectic local system.

In this section we fix an R-algebra $\mathcal{O}$ isomorphic to the formal power series ring $R[[t]]$. In other words, $\mathcal{O}$ comes with a principal ideal $\mathfrak{m}$ so that $\mathcal{O}$ is complete for the $\mathfrak{m}$-adic topology and the associated graded R-algebra $\oplus_{j=0}^{\infty}\mathfrak{m}^j/\mathfrak{m}^{j+1}$ is a polynomial ring over R in one variable. The choice of a generator t of the ideal $\mathfrak{m}$ identifies $\mathcal{O}$ with $R[[t]]$. We denote by L the localization of $\mathcal{O}$ obtained by inverting a generator of $\mathfrak{m}$. For $N\in\mathbb{Z}$, $\mathfrak{m}^N$ has the obvious meaning as an $\mathcal{O}$-submodule of L. The $\mathfrak{m}$-*adic topology* on L is the topology that has the collection of cosets $\{f+\mathfrak{m}^N\}_{f\in L, N\in\mathbb{Z}}$ as a basis of open subsets. We sometimes write $F^N L$ for $\mathfrak{m}^N$. We further denote by θ the L-module of continuous R-derivations from L into L and by ω the L-dual of θ. These L-modules come with filtrations (making them principal filtered L-modules): $F^N\theta$ consists of the derivations that take $\mathfrak{m}$ to $\mathfrak{m}^{N+1}$ and $F^N\omega$ consists of the L-homomorphisms $\theta\to L$ that take $F^0\theta$ to $\mathfrak{m}^N$. So in terms of the generator t above, $L=R((t))$, $\theta=R((t))\frac{d}{dt}$, $F^N\theta=R[[t]]t^{N+1}\frac{d}{dt}$, $\omega=R((t))dt$ and $F^N\omega=R[[t]]t^{N-1}dt$.

The residue map $\mathrm{Res}:\omega\to R$ which assigns to an element of $R((t))dt$ the coefficient of $t^{-1}dt$ is canonical, i.e., is independent of the choice of t. The R-bilinear map

$$r: L\times\omega\to R,\quad (f,\alpha)\mapsto \mathrm{Res}(f\alpha)$$

is a topologically perfect pairing of filtered R-modules: we have $r(t^k, t^{-l-1}dt)=\delta_{k,l}$ and so any R-linear $\phi: L\to R$ which is continuous (i.e., ϕ zero on $\mathfrak{m}^N$ for some N) is definable by an element of ω (namely by $\sum_{k>N}\phi(t^{-k})t^{k-1}dt$) and likewise for an R-linear continuous map $\omega\to R$.

A trivial Lie algebra. If we think of of the multiplicative group $L^\times$ of L as an algebraic group over R (or rather, as a group object in a category of ind schemes over R), then its Lie algebra, denoted here by $\mathfrak{l}$, is L, regarded as a R-module with trivial Lie bracket. It comes with a decreasing filtration $F^\bullet\mathfrak{l}$ (as a Lie algebra) defined by the valuation. The universal enveloping algebra $U\mathfrak{l}$ is clearly the symmetric algebra of $\mathfrak{l}$ as an R-module, $\mathrm{Sym}^\bullet_R(\mathfrak{l})$. The ideal $U_+\mathfrak{l}\subset U\mathfrak{l}$ generated by $\mathfrak{l}$ is also a right $\mathcal{O}$-module (since $\mathfrak{l}$ is). We complete it $\mathfrak{m}$-adically: given an integer $N\geq 0$, then an R-basis of the truncation

$U_+\mathfrak{l}/(U\mathfrak{l}\circ F^N\mathfrak{l})$ is the collection $t^{k_1}\circ\cdots\circ t^{k_r}$ with $k_1\le k_2\le\cdots\le k_r<N$. So elements of the completion

$$U_+\mathfrak{l}\to \overline{U}_+\mathfrak{l}:=\varprojlim_N U_+\mathfrak{l}/U\mathfrak{l}\circ F^N\mathfrak{l}$$

are series of the form $\sum_{i=1}^{\infty} r_i t^{k_{i,1}}\circ\cdots\circ t^{k_{i,r_i}}$ with $r_i\in R$, $c\le k_{1,i}\le k_{2,i}\le\cdots\le k_{i,r_i}$ for some constant c. We put $\overline{U}\mathfrak{l}:=R\oplus\overline{U}_+\mathfrak{l}$, which we could of course have defined just as well directly as

$$U\mathfrak{l}\to\overline{U}\mathfrak{l}:=\varprojlim_N U\mathfrak{l}/U\mathfrak{l}\circ F^N\mathfrak{l}.$$

We will refer to this construction as the $\mathfrak{m}$-*adic completion on the right*, although in the present case there is no difference with the analogously defined $\mathfrak{m}$-adic completion on the left, as $\mathfrak{l}$ is commutative.

Any continuous derivation $D\in\theta$ defines an R-linear map $\omega\to L$ which is self-adjoint relative the residue pairing: $r(\langle D,\alpha\rangle,\beta)=r(\alpha,\langle D,\beta\rangle)$. We use that pairing to identify D with an element of the closure of $\mathrm{Sym}^2\mathfrak{l}$ in $\overline{U}\mathfrak{l}$. Let $C(D)$ be half this element, so that in terms of the above topological basis,

$$C(D)=\tfrac{1}{2}\sum_{i,j\in\mathbb{Z}} r(\langle D,t^{-i-1}dt\rangle,t^{-j-1}dt)t^i\circ t^j.$$

In particular for $D=D_k=t^{k+1}\frac{d}{dt}$, $C(D_k)=\frac{1}{2}\sum_{i+j=k}t^i\circ t^j$. Observe that the map $C:\theta\to\overline{U}\mathfrak{l}$ is continuous.

Oscillator and Virasoro algebra. The residue map defines a central extension of $\mathfrak{l}$, the *oscillator algebra* $\hat{\mathfrak{l}}$, which as an R-module is simply $\mathfrak{l}\oplus R$. If we denote the generator of the second summand by $\hbar$, then the Lie bracket is given by

$$[f+\hbar r,g+\hbar s]:=\mathrm{Res}(g\,df)\hbar.$$

So $[t^k,t^{-l}]=k\delta_{k,l}\hbar$ and the center of $\hat{\mathfrak{l}}$ is $Re\oplus R\hbar$, where $e=t^0$ denotes the unit element of L viewed as an element of $\mathfrak{l}$. It follows that $U\hat{\mathfrak{l}}$ is an $R[e,\hbar]$-algebra. As an $R[\hbar]$-algebra it is obtained as follows: take the tensor algebra of $\mathfrak{l}$ (over R) tensored with $R[\hbar]$, $\otimes_R^\bullet\mathfrak{l}\otimes_R R[\hbar]$, and divide that out by the two-sided ideal generated by the elements $f\otimes g-g\otimes f-\mathrm{Res}(gdf)\hbar$. The obvious surjection $\pi:U\hat{\mathfrak{l}}\to U\mathfrak{l}=\mathrm{Sym}_R^\bullet(\mathfrak{l})$ is the reduction modulo $\hbar$.

We filter $\hat{\mathfrak{l}}$ by letting $F^N\hat{\mathfrak{l}}$ be $F^N\mathfrak{l}$ for $N>0$ and $F^N\mathfrak{l}+R\hbar$ for $N\le 0$. This filtration is used to complete $U\hat{\mathfrak{l}}$ $\mathfrak{m}$-adically on the right:

$$U\hat{\mathfrak{l}}\to\overline{U}\hat{\mathfrak{l}}:=\varprojlim_N U\hat{\mathfrak{l}}/U\hat{\mathfrak{l}}\circ F^N\mathfrak{l}.$$

Notice that this completion has the collection $t^{k_1}\circ\cdots\circ t^{k_r}$ with $r\ge 0$, $k_1\le k_2\le\cdots\le k_r$, as topological $R[\hbar]$-basis. Since $\hat{\mathfrak{l}}$ is not abelian, the left and right $\mathfrak{m}$-adic topologies now differ. For instance, $\sum_{k\ge1}t^k\circ t^{-k}$ does not converge in $\overline{U}\hat{\mathfrak{l}}$, whereas $\sum_{k\ge1}t^{-k}\circ t^k$ does. The obvious surjection $\pi:\overline{U}\hat{\mathfrak{l}}\to\overline{U}\mathfrak{l}$ is still given by reduction modulo $\hbar$. We also observe that the filtrations of $\mathfrak{l}$ and $\hat{\mathfrak{l}}$ determine decreasing filtrations of their (completed) universal enveloping algebras, e.g., $F^N U\hat{\mathfrak{l}} = \sum_{r\geq 0}\sum_{n_1+\cdots+n_r\geq N} F^{n_1}\hat{\mathfrak{l}}\circ\cdots\circ F^{n_r}\hat{\mathfrak{l}}$.

Let us denote by $\mathfrak{l}_2$ the image of $\mathfrak{l}\otimes_R \mathfrak{l}\subset \hat{\mathfrak{l}}\otimes_R\hat{\mathfrak{l}}\to U\hat{\mathfrak{l}}$. Under the reduction modulo $\hbar$, $\mathfrak{l}_2$ maps onto $\mathrm{Sym}^2_R(\mathfrak{l})\subset U\mathfrak{l}$ with kernel $R\hbar$. Its closure $\bar{\mathfrak{l}}_2$ in $\overline{U}\hat{\mathfrak{l}}$ maps onto the closure of $\mathrm{Sym}^2_R(\mathfrak{l})$ in $\overline{U}\mathfrak{l}$ with the same kernel.

The generator t defines a continuous R-linear map $D\in\theta\mapsto \hat{C}(D)\in\bar{\mathfrak{l}}_2$ characterized by

$$\hat{C}(D_k) := \tfrac12 \sum_{i+j=k} : t^i\circ t^j : \ .$$

We here adhered to the *normal ordering convention*, which prescribes that the factor with the highest index comes last and hence acts first (here the exponent serves as index). So $: t^i\circ t^j :$ equals $t^i\circ t^j$ if $i\leq j$ and $t^j\circ t^i$ if $i>j$. This map is clearly a lift of $C:\theta\to \mathrm{Sym}^2\,\mathfrak{l}$, but is otherwise non-canonical.

Lemma 2. *We have*

(i) $[\hat{C}(D), f] = -\hbar D(f)$ *as an identity in* $\overline{U}\hat{\mathfrak{l}}$ *(where* $f\in\mathfrak{l}\subset\hat{\mathfrak{l}}$*) and*
(ii) $[\hat{C}(D_k),\hat{C}(D_l)] = -\hbar(l-k)\hat{C}(D_{k+l}) + \hbar^2\frac{1}{12}(k^3-k)\delta_{k+l,0}$.

Proof. For the first statement we compute $[\hat{C}(D_k), t^l]$. If we substitute $\hat{C}(D_k) = \frac12\sum_{i+j=k} : t^i\circ t^j :$, then we see that only terms of the form $[t^{k+l}\circ t^{-l}, t^l]$ or $[t^{-l}\circ t^{k+l}, t^l]$ (depending on whether $k+2l\leq 0$ or $k+2l\geq 0$) can make a contribution and then have coefficient $\frac12$ if $k+2l=0$ and 1 otherwise. In all cases the result is $-\hbar l t^{k+l} = -\hbar D_k(t^l)$.

Formula (i) implies that

$$\begin{aligned}[\hat{C}(D_k),\hat{C}(D_l)] &= \lim_{N\to\infty}\sum_{|i|\leq N}\tfrac12\Big(D_k(t^i)\circ t^{l-i} + t^i\circ D_k(t^{l-i})\Big)\\ &= -\hbar\lim_{N\to\infty}\sum_{|i|\leq N}\Big(it^{k+i}\circ t^{l-i} + t^i\circ (l-i)t^{k+l-i}\Big).\end{aligned}$$

This is up to a reordering equal to $-\hbar(l-k)\hat{C}(D_{k+l})$. The terms which do not commute and are in the wrong order are those for which $0<k+i=-(l-i)$ (with coefficient i) and for which $0<i=-(k+l-i)$ (with coefficient $(l-i)$). This accounts for the extra term $\hbar^2\frac{1}{12}(k^3-k)\delta_{k+l,0}$. □

This lemma shows that $-\hbar^{-1}\hat{C}$ behaves better than $\hat{C}$ (but requires us of course to assume that $\hbar$ be invertible). In fact, it suggests to consider the set $\hat{\theta}$ of pairs $(D,u)\in\theta\times\hbar^{-1}\bar{\mathfrak{l}}_2$ for which $C(D)\in\mathrm{Sym}^2\,\mathfrak{l}$ is the mod $\hbar$ reduction of $-\hbar u$, so that we have an exact sequence

$$0\to R\to\hat{\theta}\to\theta\to 0$$

of R-modules. Then a non-canonical section of $\hat{\theta}\to\theta$ is given by $D\mapsto\hat{D} := (D, -\hbar^{-1}\hat{C}(D))$. In order to avoid confusion, we denote the generator of the copy of R by c_0.

Corollary-Definition 3. *This defines a central extension of Lie algebras, called the* Virasoro algebra *(of the* R*-algebra* L*). Precisely, if* $T : \hat{\theta} \to \overline{U}\hat{l}[\frac{1}{\hbar}]$ *is given by the second component, then* T *is injective, maps* $\hat{\theta}$ *onto a Lie subalgebra of* $\overline{U}\hat{l}[\frac{1}{\hbar}]$ *and sends* c_0 *to* 1*. If we transfer the Lie bracket on* $\overline{U}\hat{l}[\frac{1}{\hbar}]$ *to* $\hat{\theta}$*, then in terms of our non-canonical section,*

$$[\hat{D}_k, \hat{D}_l] = (l-k)\hat{D}_{k+l} + \frac{k^3-k}{12}\delta_{k+l,0}c_0.$$

Moreover, $\mathrm{ad}_{T(\hat{D})}$ *leaves* l *invariant (as a subspace of* $\overline{U}\hat{l}$*) and acts on that subspace by derivation with respect to* $D \in \theta$*.*

Remark 4. An alternative coordinate free definition of the Virasoro algebra, based on the algebra of pseudo-differential operators on L, can be found in [5].

Fock representation. It is clear that $F^0\hat{l} = R\hbar \oplus \mathcal{O}$ is an abelian subalgebra of $\hat{l}$. We let $F^0\hat{l} = \mathcal{O} \oplus R\hbar$ act on a free rank one module Rv_o by letting $\mathcal{O}$ act trivially and $\hbar$ as the identity. The induced representation of $\hat{l}$ over R,

$$\mathbb{F} := U\hat{l} \otimes_{U\hat{l} \circ F^0\hat{l}} Rv_o,$$

will be regarded as a $Ul[\hbar^{-1}]$-module. It comes with an increasing PBW (Poincaré-Birkhoff-Witt) filtration $W_\bullet\mathbb{F}$ by R-submodules, with $W_r\mathbb{F}$ being the image of $\oplus_{s \le r}\hat{l}^{\otimes s} \otimes Rv_o$. Since the scalars $R \subset l$ are central in $\hat{l}$ and kill $\mathbb{F}$ (because $R \subset \mathcal{O}$), they act trivially in all of $\mathbb{F}$. As an R-module, $\mathbb{F}$ is free with basis the collection $t^{-k_r} \circ \cdots \circ t^{-k_1} \otimes v_o$, where $r \ge 0$ and $1 \le k_1 \le k_2 \le \cdots \le k_r$ (for $r = 0$, read v_o). (In fact, $\mathrm{Gr}^W_\bullet \mathbb{F}$ can be identified as a graded R-module with the symmetric algebra $\mathrm{Sym}^\bullet(l/F^0l)$.) This also shows that $\mathbb{F}$ is even a $\overline{U}\hat{l}[\hbar^{-1}]$-module. Thus $\mathbb{F}$ affords a representation of $\hat{\theta}$ over R, called its *Fock representation.*

It follows from Lemma 2 that for any $D \in \theta$ with lift $\hat{D} \in \hat{\theta}$,

$$T(\hat{D})t^{-k_r} \circ \cdots \circ t^{-k_1} \otimes v_o =$$
$$= \left(\sum_{i=1}^{r} t^{-k_r} \circ \cdots \circ D(t^{-k_i}) \circ \cdots \circ t^{-k_1}\right) \otimes v_o + t^{-k_r} \circ \cdots \circ t^{-k_1} \circ T(\hat{D})v_o.$$

Since $T(\hat{D})v_o = 0$ when $D \in F^0\theta$, it follows that $F^0\theta$ acts on $\mathbb{F}$ by coefficient-wise derivation. This observation has an interesting consequence. Consider the module of k-derivations $R \to R$ (denoted here simply by θ_R instead of the more accurate $\theta_{R/k}$) and the module $\theta_{L,R}$ of k-derivations of L that are continuous for the $\mathfrak{m}$-adic topology and preserve $R \subset L$. Since $L \cong R((t))$ as an R-algebra, every k-derivation $R \to R$ extends to one from L to L. So we have an exact sequence

$$0 \to \theta \to \theta_{L,R} \to \theta_R \to 0.$$

The following corollary essentially says that we have defined in the L-module $\mathbb{F}$ a Lie algebra $\hat{\theta}_{L,R}$ of first order (k-linear) differential operators which contains R as the degree zero operators and for which the symbol map (which is just the formation of the degree one quotient) has image $\theta_{L,R}$.

Corollary 5. *The actions on* $\mathbb{F}$ *of* $F^0\theta_{L,R} = \mathfrak{m}\theta_{\mathcal{O},R} \subset \theta_{L,R}$ *(given by coefficient-wise derivation, killing the generator* ν_o*) and* $\hat{\theta}$ *coincide on* $F^0\theta$ *and generate a central extension of Lie algebras* $\hat{\theta}_{L,R} \to \theta_{L,R}$ *by* Rc_o*. Its defining representation on* $\mathbb{F}$ *(still denoted* T*) is faithful and has the property that for every lift* $\hat{D} \in \hat{\theta}_{L,R}$ *of* $D \in \theta_{L,R}$ *and* $f \in \mathfrak{l}$ *we have* $[T(\hat{D}), f] = Df$ *(in particular, it preserves every* $U\hat{\mathfrak{l}}$*-submodule of* $\mathbb{F}$*).*

Proof. The generator t can be used to define a section of $\theta_{L,R} \to \theta_R$: the set of elements of $\theta_{L,R}$ which kill t is a k-Lie subalgebra of $\theta_{L,R}$ which projects isomorphically onto θ_R. Now if $D \in \theta_{L,R}$, write $D = D_{\mathrm{vert}} + D_{\mathrm{hor}}$ with $D_{\mathrm{vert}} \in \theta$ and $D_{\mathrm{hor}}(t) = 0$ and define an R-linear operator $\hat{D}$ in $\mathbb{F}$ as the sum of $T(\hat{D}_{\mathrm{vert}})$ and coefficient-wise derivation by D_{hor}. This map clearly has the properties mentioned.

As to its dependence on t: another choice yields a decomposition of the form $D = (D_{\mathrm{hor}} + D_0) + (D_{\mathrm{vert}} - D_0)$ with $D_0 \in F^0\theta$ and in view of the above $\hat{D}_0$ acts in $\mathbb{F}$ by coefficient-wise derivation. □

The Fock representation for a symplectic local system. In Section 4 we shall run into a particular type of finite rank subquotient of the Fock representation and it seems best to discuss the resulting structure here. We start out from the following data:

(i) a free R-module H of finite rank endowed with a symplectic form $\langle\ ,\ \rangle : H \otimes_R H \to R$, which is nondegenerate in the sense that the induced map $H \to H^*$, $a \mapsto \langle\ , a\rangle$ is an isomorphism of R-modules,

(ii) an R-submodule $\mathfrak{D} \subset \theta_R$ closed under the Lie bracket for which the inclusion is an equality over the generic point and a Lie action $D \mapsto \nabla_D$ of $\mathfrak{D}$ on H by k-derivations which preserves the symplectic form,

(iii) a Lagrangian R-submodule $F \subset H$.

Property (ii) means that $D \in \mathfrak{D} \mapsto \nabla_D \in \mathrm{End}_k(H)$ is R-linear, obeys the Leibniz rule: $\nabla_D(ra) = r\nabla_D(a) + D(r)a$ and satisfies $\langle\nabla_D a, b\rangle + \langle a, \nabla_D b\rangle = D\langle a, b\rangle$. In the cases of interest, $\mathfrak{D}$ will be the θ_R-stabilizer of a principal ideal in R (and often be all of θ_R). One might think of ∇ as a flat meromorphic connection on the symplectic bundle represented by H.

In this setting, a Heisenberg algebra is defined in an obvious manner: it is $\hat{H} := H \oplus R\hbar$ endowed with the bracket $[a + R\hbar, b + R\hbar] = \langle a, b\rangle\hbar$. We also have defined a Fock representation $\mathbb{F}(H, F)$ of $\hat{H}$ as the induced module of the rank one representation of $\hat{F} = F + R\hbar$ on R given by the coefficient of $\hbar$. Notice that if we grade $\mathbb{F}(H, F)$ with respect to the PBW filtration, we get a copy of the symmetric algebra of H/F over R. We aim to define a projective Lie action of $\mathfrak{D}$ on $\mathbb{F}(H, F)$.

We begin with extending the $\mathfrak{D}$-action to $\hat{H}$ by stipulating that it kills $\hbar$. This action clearly preserves the Lie bracket and hence determines one of $\mathfrak{D}$

on the universal enveloping algebra $U\hat{H}$. This does not however induce one in $\mathbb{F}(H,F)$, as ∇_D will not respect the right ideal in $U\hat{H}$ generated by $\hbar - 1$ and F. We will remedy this by means of a 'twist'.

We shall use the isomorphism $\sigma : H \otimes_R H \cong \mathrm{End}_R(H)$ of R-modules defined by associating to $a \otimes b$ the endomorphism $\sigma(a \otimes b) : x \in H \mapsto a\langle b, x\rangle \in H$. If we agree to identify an element in the tensor algebra of H, in particular, an element of H, as the operator in $U\hat{H}$ or $\mathbb{F}(H,F)$ given by left multiplication, then it is ready checked that for $x \in H$,

$$[a \circ b, x] = \sigma(a \otimes b + b \otimes a)(x).$$

We choose a Lagrangian supplement of F in H, i.e., a Lagrangian R-submodule $F' \subset H$ that is also a section of $H \to H/F$. Since F' is an abelian Lie subalgebra of $\hat{H}$, we have a natural map $\mathrm{Sym}^\bullet_R(F') \to \mathbb{F}(H,F)$. It is clearly an isomorphism of $\mathrm{Sym}^\bullet_R(F')$-modules. Now write ∇_D according to the Lagrangian decomposition $H = F' \oplus F$:

$$\nabla_D = \begin{pmatrix} \nabla_D^{F'} & \sigma'_D \\ \sigma_D & \nabla_D^F \end{pmatrix}.$$

Here the diagonal entries represent the induced connections on F' and F, whereas $\sigma_D \in \mathrm{Hom}_R(F', F)$ and $\sigma'_D \in \mathrm{Hom}_R(F, F')$. Since σ identifies $F \otimes_R F$ resp. $F' \otimes_R F'$ with $\mathrm{Hom}_R(F', F)$ resp. $\mathrm{Hom}_R(F, F')$, we can write $\sigma_D = \sigma(s_D)$ with $s_D \in F \otimes_R F$ and $\sigma'_D = \sigma(s'_D)$ and $s'_D \in F' \otimes_R F'$. These tensors are symmetric and represent the second fundamental form of $F' \subset H$ resp. $F \subset H$. Notice that if $a \in F$, then

$$[\nabla_D, a] = \nabla_D(a) = \nabla_D^F(a) + \sigma_{F'}^F(a) = \nabla_D^F(a) + \tfrac{1}{2}[s'_D, a]$$

and similarly, if $a' \in F'$, then $[\nabla_D, a'] = \nabla_D^{F'}(a') + \frac{1}{2}[s_D, a']$. This suggests we should assign to $D \in \mathfrak{D}$ the first order differential operator $T_{F'}(D)$ in $\mathbb{F}(H,F) \cong \mathrm{Sym}^\bullet F'$ defined by

$$T_{F'}(D) := \nabla_D^{F'} + \tfrac{1}{2}s_D + \tfrac{1}{2}s'_D.$$

Proposition 6. *The map* $T_{F'} : \mathfrak{D} \to \mathrm{End}_k(\mathrm{Sym}^\bullet F')$ *is* R*-linear and has the property that* $[T_{F'}(D), a] = \nabla_D(a)$ *for every* $D \in \mathfrak{D}$ *and* $a \in \hat{H}$. *Any other map* $\mathfrak{D} \to \mathrm{End}_k(\mathrm{Sym}^\bullet F')$ *enjoying these properties differs from* $T_{F'}$ *by a multiple of the identity operator, in other words, is of the form* $D \mapsto T_{F'}(D) + \eta(D)$ *for some* $\eta \in \mathrm{Hom}_R(\mathfrak{D}, R)$.

Proof. That $T_{F'}(D)$ has the stated property follows from the preceding. Let $\eta : \mathfrak{D} \to \mathrm{End}_k(\mathrm{Sym}^\bullet F')$ be the difference of two such maps. Then for every $D \in \mathfrak{D}$, $\eta(D) \in \mathrm{End}_R(\mathbb{F}(H,F))$ commutes with all elements of $\hat{H}$. Since $\mathbb{F}(H,F)$ is irreducible as a representation of $\hat{H}$, it follows that $\eta(D)$ is a scalar in R. □

Notice that if $u_1, \dots, u_r \in \hat{H}$, then

$$T_{F'}(D)(u_r \circ \dots \circ u_1 \otimes v_o) =$$
$$= \left(\sum_{i=1}^{r} u_r \circ \dots \circ \nabla_D(u_i) \circ \dots \circ u_1 + u_r \circ \dots \circ u_1 \circ \tfrac{1}{2}s'_D\right) \otimes v_o.$$

So this looks like the operator $T_{\hat{D}}$ acting in $\mathbb{F}$ with s'_D playing the role of $-\hat{C}(D)$. Here is the key result about the 'curvature' of $T_{F'}$.

Lemma 7. *Given* $D, E \in \mathfrak{D}$*, then* $[T_{F'}(D), T_{F'}(E)] - T_{F'}([D,E])$ *is scalar multiplication by* $1/2$ *times the value on of the* ∇^F*-curvature on* $\det(F)$ *on the pair* (D, E).

Proof. The fact that ∇ preserves the Lie bracket is expressed by the following identities:

$$\nabla^F_D\nabla^F_E - \nabla^F_E\nabla^F_D - \nabla^F_{[D,E]} = \sigma_E\sigma'_D - \sigma_D\sigma'_E,$$
$$\nabla^{F'}_D\nabla^{F'}_E - \nabla^{F'}_E\nabla^{F'}_D - \nabla^{F'}_{[D,E]} = \sigma'_E\sigma_D - \sigma'_D\sigma_E,$$
$$\nabla^{\mathrm{Hom}(F',F)}_D(\sigma_E) - \nabla^{\mathrm{Hom}(F',F)}_E(\sigma_D) = \sigma_{[D,E]},$$
$$\nabla^{\mathrm{Hom}(F,F')}_D(\sigma'_E) - \nabla^{\mathrm{Hom}(F,F')}_E(\sigma'_D) = \sigma'_{[D,E]}.$$

The first two give the curvature of ∇^F and $\nabla^{F'}$ on the pair (D, E). The last two can also be written as operator identities in $\mathrm{Sym}^\bullet F'$:

$$[\nabla^{F'}_D, s_E] - [\nabla^{F'}_E, s_D] = s_{[D,E]},$$
$$[\nabla^{F'}_D, s'_E] - [\nabla^{F'}_E, s'_D] = s'_{[D,E]}.$$

If we feed these identities in:

$$[T_{F'}(D), T_{F'}(E)] - T_{F'}([D,E]) =$$
$$= [\nabla^{F'}_D + \tfrac{1}{2}s_D + \tfrac{1}{2}s'_D, \nabla^{F'}_E + \tfrac{1}{2}s_E + \tfrac{1}{2}s'_E] - (\nabla^{F'}_{[D,E]} + \tfrac{1}{2}s_{[D,E]} + \tfrac{1}{2}s'_{[D,E]}) =$$
$$= \big([\nabla^{F'}_D, \nabla^{F'}_E] - \nabla^{F'}_{[D,E]}\big) + \tfrac{1}{2}\big([\nabla^{F'}_D, s_E] - [\nabla^{F'}_E, s_D] - s_{[D,E]}\big)\big)$$
$$+ \tfrac{1}{2}\big([\nabla^{F'}_D, s'_E] - [\nabla^{F'}_E, s'_D] - s'_{[D,E]}\big)\big) + \tfrac{1}{4}\big([s_D, s'_E] - [s_E, s'_D]\big)$$

(where we identified $\mathbb{F}(H, F)$ with $\mathrm{Sym}^\bullet F'$), we obtain

$$[T_{F'}(D), T_{F'}(E)] - T_{F'}([D,E]) = \big(\sigma'_E\sigma_D + \tfrac{1}{4}[s_D, s'_E]\big) - \big(\sigma'_D\sigma_E + \tfrac{1}{4}[s_{D'}, s_E]\big).$$

We must show that the right hand side is equal to $\frac{1}{2}\,\mathrm{Tr}(\sigma_E\sigma'_D - \sigma_D\sigma'_E)$, or perhaps more specifically, that $\sigma'_E\sigma_D + \frac{1}{4}[s_D, s'_E] = -\frac{1}{2}\,\mathrm{Tr}(\sigma_D\sigma'_E)$ (and similarly if we exchange D and E). This reduces to the following identity in linear algebra: if $a \in F$ and $\beta \in F'$, then in $\mathrm{Sym}^\bullet F'$ we have

$$\sigma(\beta \otimes \beta)\sigma(a \otimes a) + \tfrac{1}{4}[a \circ a, \beta \circ \beta] = -\tfrac{1}{2}\,\mathrm{Tr}_{F'}(\sigma_{a\otimes a}\sigma_{\beta\otimes\beta}),$$

Indeed, a straightforward computation shows that

$$[a \circ a, \beta \circ \beta] = 2\langle a, \beta\rangle(a \circ \beta + \beta \circ a) = 4\langle a, \beta\rangle\beta \circ a + 2\langle a, \beta\rangle^2.$$

If we interpret $\langle a, \beta\rangle\beta \circ a$ as an operator in $\mathrm{Sym}^\bullet F'$, then applying it to $x \in F'$ yields $\langle a, \beta\rangle\beta\langle a, x\rangle = -\sigma(\beta \otimes \beta)\sigma(a \otimes a)(x)$. We also find that $\langle a, \beta\rangle^2 = -\operatorname{Tr}_{F'}(\sigma(a \otimes a)\sigma(\beta \otimes \beta)$. □

If N is a free R-module of rank one, then by a *square root of* N we mean a free R-module Θ of rank one together with an isomorphism of $\Theta \otimes_R \Theta$ onto N.

Corollary 8. *Let* Θ *be a square root of* $\det_R(F)$. *Then the twisted Fock module* $\operatorname{Hom}_R(\Theta, \mathbb{F}(H, F))$ *comes with a natural action of* $\mathfrak{D}$ *by derivations.*

Proof. Given the Lagrangian supplement F' of F in H, then endow Θ with the unique $\mathfrak{D}$-module structure that makes the given isomorphism $\Theta \otimes_R \Theta \cong \det_R(F)$ one of $\mathfrak{D}$-modules: if $w \in \Theta$ is a generator and $\nabla_D^{\det F}(w \otimes w) = rw \otimes w$, then $\nabla_D^{\Theta}(w) = \frac{1}{2} rw$. This ensures that the $\mathfrak{D}$-action on $\operatorname{Hom}_R(\Theta, \mathrm{Sym}^\bullet F')$ preserves the Lie bracket. It remains to show that this action is independent of F'. This can be verified by a computation, but rather than carrying this out, we give an abstract argument that avoids this. It is based on the well-known fact that if H_o is a fixed symplectic k-vector space of finite dimension $2g$, and $F_o \subset H_o$ is Lagrangian, then the set of Lagrangian supplements of F_o in H_o form in the Grassmannian of H_o an affine space over $\mathrm{Sym}^2_k F_o$ (and hence is simply connected). Now by doing the preceding construction universally over the corresponding affine space over $\mathrm{Sym}^2_R F$, we see that the flatness on the universal example immediately gives the independence. □

Remark 9. We will use this corollary mainly via the following reformulation. First we observe that the Lie algebra of first order k-linear differential operators $\Theta \to \Theta$ projects to θ_R (this is the symbol map) with kernel the scalars R. Denote by $\mathfrak{D}(\Theta)$ the preimage of $\mathfrak{D}$. This is clearly a Lie subalgebra. Then the above corollary can be understood as saying that there is a natural Lie action of $\mathfrak{D}(\Theta)$ on $\mathbb{F}(H, F)$ by first order differential operators, acting, in the terminology of Section 1, with weight 1. The image in $\operatorname{End}_k(\mathbb{F}(H, F))$ is the R-submodule of $\operatorname{End}_k(\mathbb{F}(H, F))$ generated by the $T_{F'}(D)$ and the identity operator. We may also use $\mathfrak{D}(\det_R(F))$ instead, although then the weight will be $\frac{1}{2}$. Note that our discussion of projectively flat connections in Section 1 now suggests a formulation in more geometric terms, namely that the pull-back of $\mathbb{F}(H, F)$ to the geometric realization of the $\mathbb{G}_m$-bundle over $\operatorname{Spec}(R)$ defined by $\det_R(F)$ acquires a flat meromorphic connection with fiber monodromy minus the identity.

Remark 10. The preceding follows the presentation of Boer-Looijenga [6] rather closely. The quadratic terms that enter in the definition of $T_{F'}$ are in a way a relict of the heat operator of which the theta functions associated to this symplectic local system are solutions (flat sections are expansions of theta functions relative to an unspecified lattice).

3. The Sugawara construction

In this section we show how the Virasoro algebra acts in the standard representions of a centrally extended loop algebra. This construction goes back to the physicist H. Sugawara (in 1968), but it was probably Graeme Segal who first noticed its relevance for the present context.

Most of the material below can for instance be found in [10] (Lecture 10) and [9] (Ch. 12), but our presentation slightly deviates from the standard sources in substance as well in form: we approach the Sugawara construction via the construction discussed in the previous section and put it in the (coordinate free) setting that makes it appropriate for the application we have in mind.

In this section, we fix a simple Lie algebra $\mathfrak{g}$ over k of finite dimension. We retain the data and the notation of Section 2.

Loop algebras. We identify $\mathfrak{g}\otimes\mathfrak{g}$ with the space of bilinear forms $\mathfrak{g}^*\times\mathfrak{g}^*\to k$, where $\mathfrak{g}^*$ denotes the k-dual of $\mathfrak{g}$, as usual. We form its space of $\mathfrak{g}$-covariants (relative to the adjoint action on both factors):

$$q:\mathfrak{g}\otimes\mathfrak{g}\to(\mathfrak{g}\otimes\mathfrak{g})_{\mathfrak{g}}=:\mathfrak{c}.$$

This space is known to be of dimension one and to consist of symmetric tensors. It has a canonical generator which is characterized by the property that it is represented by a $\mathfrak{g}$-invariant symmetric tensor $c\in\mathfrak{g}\otimes\mathfrak{g}$ with the property that $c(\theta\otimes\theta)=2$ if $\theta\in\mathfrak{g}^*$ is a long root (relative to a choice of Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$; the roots then lie in the zero eigenspace of $\mathfrak{h}$ in $\mathfrak{g}^*$). This element is in fact invariant under the full automorphism group of the Lie algebra $\mathfrak{g}$, not just the inner ones. It is nondegenerate when viewed as a symmetric bilinear form on $\mathfrak{g}^*$ and so the inverse form on $\mathfrak{g}$ is defined. If we denote the latter by $\check{c}$, then the equivariant projection $q:\mathfrak{g}\otimes\mathfrak{g}\to\mathfrak{c}$ is given by $X\otimes Y\mapsto\check{c}(X,Y)\mathfrak{c}$.

It is well-known and easy to prove that c maps to the center of $U\mathfrak{g}$. This implies that c acts in any irreducible representation of $\mathfrak{g}$ by a scalar. In the case of the adjoint representation half this scalar is called the *dual Coxeter number* of $\mathfrak{g}$ and is denoted by $\check{h}$. So if we choose an orthonormal basis $\{X_\kappa\}_\kappa$ of $\mathfrak{g}$ relative to $\check{c}$ so that c takes the form $\sum_\kappa X_\kappa\otimes X_\kappa$, then

$$\sum_\kappa[X_\kappa,[X_\kappa,Y]]=2\check{h}Y\quad\text{for all } Y\in\mathfrak{g}.$$

Let $L\mathfrak{g}$ stand for $\mathfrak{g}\otimes_k L$, but considered as a filtered R-Lie algebra (so we restrict the scalars to R) with $F^N L\mathfrak{g}=\mathfrak{g}\otimes_k\mathfrak{m}^N$. An argument similar to the one we used to prove that the pairing r is topologically perfect shows that the pairing

$$r_{\mathfrak{g}}:(\mathfrak{g}\otimes_k L)\times(\mathfrak{g}\otimes_k\omega)\to\mathfrak{c}\otimes_k R=:\mathfrak{c}_R$$

which sends $(Xf,Y\alpha)$ to $q(X\otimes Y)\operatorname{Res}(f\alpha)$ is topologically perfect (the basis dual to $(X_\kappa t^l)_{\kappa,l}$ is $(X_\kappa t^{-l-1}dt\otimes c)_{\kappa,l}$).

For an integer $N \geq 0$, the quotient $U L\mathfrak{g}/U L\mathfrak{g} \circ F^N L\mathfrak{g}$ is a free R-module (a set of generators is $X_{\kappa_1} t^{k_1} \circ \cdots \circ X_{\kappa_r} t^{k_r}$, $k_1 \leq \cdots \leq k_r < N$). We complete $U L\mathfrak{g}$ $\mathfrak{m}$-adically on the right:

$$\overline{U} L\mathfrak{g} := \varprojlim_N U L\mathfrak{g}/U L\mathfrak{g} \circ F^N L\mathfrak{g}.$$

A central extension of Lie algebras

$$0 \to \mathfrak{c}_R \to \widehat{L\mathfrak{g}} \to L\mathfrak{g} \to 0$$

is defined by endowing the sum $L\mathfrak{g} \oplus \mathfrak{c}_R$ with the Lie bracket

$$[Xf + cr, Yg + cs] := [X, Y] fg + r_{\mathfrak{g}}(Yg, Xdf).$$

Since the residue is zero on $\mathcal{O}$, the inclusion of $\mathcal{O}\mathfrak{g}$ in $\widehat{L\mathfrak{g}}$ is a homomorphism of Lie algebras. In fact, this is a canonical (and even unique) Lie section of the central extension over $\mathcal{O}\mathfrak{g}$, for it is just the derived Lie algebra of the preimage of $\mathcal{O}\mathfrak{g}$ in $\widehat{L\mathfrak{g}}$. The Aut$(\mathfrak{g})$-invariance of c implies that the tautological action of Aut$(\mathfrak{g})$ on $\mathfrak{g}$ extends to $\widehat{L\mathfrak{g}}$.

We filter $\widehat{L\mathfrak{g}}$ by setting $F^N \widehat{L\mathfrak{g}} = F^N L\mathfrak{g}$ for $N > 0$ and $F^N \widehat{L\mathfrak{g}} = F^N L\mathfrak{g} + \mathfrak{c}_R$ for $N \leq 0$. Then $U\widehat{L\mathfrak{g}}$ is a filtered $R[c]$-algebra whose reduction modulo c is $U L\mathfrak{g}$. The $\mathfrak{m}$-adic completion on the right

$$\overline{U}\widehat{L\mathfrak{g}} := \varprojlim_N U\widehat{L\mathfrak{g}}/(U\widehat{L\mathfrak{g}} \circ F^N L\mathfrak{g})$$

is still an $R[c]$-algebra and the obvious surjection $\overline{U}\widehat{L\mathfrak{g}} \to \overline{U} L\mathfrak{g}$ is the reduction modulo c. These (completed) enveloping algebras not only come with the (increasing) Poincaré-Birkhoff-Witt filtration, but also inherit a (decreasing) filtration from L.

Segal-Sugawara representation. Tensoring with $c \in \mathfrak{g} \otimes_k \mathfrak{g}$ defines the R-linear map

$$\mathfrak{l} \otimes_R \mathfrak{l} \to L\mathfrak{g} \otimes_R L\mathfrak{g}, \quad f \otimes g \mapsto c \cdot f \otimes g = \sum_\kappa X_\kappa f \otimes X_\kappa g,$$

which, when composed with $L\mathfrak{g} \otimes_R L\mathfrak{g} \subset \widehat{L\mathfrak{g}} \otimes_R \widehat{L\mathfrak{g}} \to U\widehat{L\mathfrak{g}}$, yields a map $\gamma : \mathfrak{l} \otimes_R \mathfrak{l} \to U\widehat{L\mathfrak{g}}$. Since $\gamma(f \otimes g - g \otimes f) = \sum_\kappa [X_\kappa f, X_\kappa g] = c \dim \mathfrak{g} \operatorname{Res}(gdf)$, γ drops and extends naturally to an R-module homomorphism $\hat{\gamma} : \mathfrak{l}_2 \to U\widehat{L\mathfrak{g}}$ which sends $\hbar$ to $c \dim \mathfrak{g}$. This, in turn, extends continuously to a map from the closure $\bar{\mathfrak{l}}_2$ of $\mathfrak{l}_2$ in $\overline{U}\hat{\mathfrak{l}}$ to $\overline{U}\widehat{L\mathfrak{g}}$. As $\bar{\mathfrak{l}}_2$ contains the image of $\hat{C} : \theta \to \overline{U}\hat{\mathfrak{l}}$, and since c is Aut$(\mathfrak{g})$-invariant, we get a R-homomorphism

$$\hat{C}_{\mathfrak{g}} := \hat{\gamma}\hat{C} : \theta \to (\overline{U}\widehat{L\mathfrak{g}})^{\mathrm{Aut}(\mathfrak{g})}.$$

We may also describe $\hat{C}_{\mathfrak{g}}$ in the spirit of Section 2: given $D \in \theta$, then the R-linear map

$$1 \otimes D : \mathfrak{g} \otimes_k \omega \to \mathfrak{g} \otimes_k L$$

is continuous and self-adjoint relative to $r_{\mathfrak{g}}$ and the perfect pairing $r_{\mathfrak{g}}$ allows us to identify it with an element of $\overline{U}L\mathfrak{g}$; this element produces our $\hat{C}_{\mathfrak{g}}(D)$. Thus the choice of the parameter t yields

$$\hat{C}_{\mathfrak{g}}(D_k) = \tfrac{1}{2}\sum_{\kappa,l} : X_\kappa t^{k-l} \circ X_\kappa t^l : \quad .$$

This formula can be used to define $\hat{C}_{\mathfrak{g}}$, but this approach does not exhibit its naturality.

Lemma 11. *For $X \in \mathfrak{g}$ and $f \in L$ we have*

$$[\hat{C}_{\mathfrak{g}}(D_k), Xf] = -(c + \check{h})XD_k(f)$$

(an identity in $\overline{U}\widehat{L\mathfrak{g}}$) and upon a choice of a parameter t, then with the preceding notation

$$[\hat{C}_{\mathfrak{g}}(D_k), \hat{C}_{\mathfrak{g}}(D_l)] = (c + \check{h})(k - l)\hat{C}_{\mathfrak{g}}(D_{k+l}) + c(c + \check{h})\delta_{k+l,0}\frac{k^3 - k}{12}\dim \mathfrak{g}.$$

For the proof (which is a bit tricky, but not very deep), we refer to Lecture 10 of [10] (our $C_{\mathfrak{g}}(\hat{D}_k)$ is their T_k). This formula suggests that we make the central element $c + \check{h}$ of $\overline{U}\widehat{L\mathfrak{g}}$ invertible (its inverse might be viewed as a rational function on $\mathfrak{c}^*$), so that we can state this lemma in a more natural manner as follows.

Corollary 12 (Sugawara representation)**.** *The map $\hat{D}_k \mapsto \frac{-1}{c+\check{h}}\hat{C}_{\mathfrak{g}}(D_k)$ induces a natural homomorphism of R-Lie algebras*

$$T_{\mathfrak{g}} : \hat{\theta} \to \big(\overline{U}\widehat{L\mathfrak{g}}[\tfrac{1}{c+\check{h}}]\big)^{\mathrm{Aut}(\mathfrak{g})}$$

which sends the central element $c_0 \in \hat{\theta}$ to $c(c + \check{h})^{-1} \dim \mathfrak{g}$. Moreover, if $\hat{D} \in \hat{\theta}$, then $\mathrm{ad}_{T_{\mathfrak{g}}(\hat{D})}$ leaves $L\mathfrak{g}$ invariant (as a subspace of $\overline{U}\widehat{L\mathfrak{g}}$) and acts on that subspace by derivation with respect to the image of $\hat{D}$ in θ.

A representation for $\widehat{L\mathfrak{g}}$. We fix $\ell \in k$ with $\ell \neq -\check{h}$. Let $F^1L\mathfrak{g} \oplus \mathfrak{c}_R$ act on the free R-module of rank one $R\nu_\ell$ via the projection onto the second factor $\mathfrak{c}_R = Rc$ with c acting as multiplication by ℓ. We regard $F^1L\mathfrak{g} \oplus \mathfrak{c}_R$ as a subalgebra of $U\widehat{L\mathfrak{g}}$ so that we can form the induced module

$$\mathbb{F}_\ell(\mathfrak{g}, L) := U\widehat{L\mathfrak{g}} \otimes_{U(F^1L\mathfrak{g}\oplus\mathfrak{c}_R)} R\nu_\ell,$$

which we often simply denote by $\mathbb{F}_\ell(\mathfrak{g})$. We use ν_ℓ also to denote its image in this module. As an R-module $\mathbb{F}_\ell(\mathfrak{g})$ is generated by $X_{\kappa_r}t^{-k_r}\circ\cdots\circ X_{\kappa_1}t^{-k_1}\otimes\nu_\ell$, where $r \geq 0$, $0 \leq k_1 \leq k_2 \leq \cdots \leq k_r$ and where $(X_\kappa)_\kappa$ is a given k-basis of $\mathfrak{g}$. If we let $\hat{\theta}$ act on $\mathbb{F}_\ell(\mathfrak{g})$ via $T_{\mathfrak{g}}$, then it follows from Corollary 12 that if

$\hat{D} \in \hat{\theta}$ lifts $D \in \theta$, then

$$\begin{aligned} T_{\mathfrak{g}}(\hat{D})X_{\kappa_r}t^{-k_r} \circ \cdots \circ X_{\kappa_1}t^{-k_1} \otimes v_\ell = \\ = \sum_{i=1}^{r} X_{\kappa_r}t^{-k_r} \circ \cdots X_{\kappa_i}D(t^{-k_i}) \circ \cdots \circ X_{\kappa_1}t^{-k_1} \otimes v_\ell + \\ + X_{\kappa_r}t^{-k_r} \circ \cdots \circ X_{\kappa_1}t^{-k_1} \circ T_{\mathfrak{g}}(\hat{D})v_\ell. \end{aligned}$$

Thus $\hat{\theta}$ is faithfully represented as a Lie algebra of R-linear endomorphisms of $\mathbb{F}_\ell(\mathfrak{g})$. If $D \in F^0\theta$, then clearly $T_{\mathfrak{g}}(\hat{D})v_\ell = 0$ and hence we have the following counterpart of Corollary 5 (with the same proof). It tells us that $\hat{\theta}_{L,R}$ acts in $\mathbb{F}_\ell(\mathfrak{g})$ as a Lie algebra of first order differential operators, but with its degree zero part R acting with weight $(c + \check{h})^{-1} c \dim \mathfrak{g}$:

Corollary 13. *The Sugawara representation* $T_{\mathfrak{g}}$ *of* $\hat{\theta}$ *on* $\mathbb{F}_\ell(\mathfrak{g})$ *extends to* $\hat{\theta}_{L,R}$ *in such a manner that* $F^0\theta_{L,R}$ *acts by coefficientwise derivation (killing the generator* v_ℓ*),* $[T_{\mathfrak{g}}(\hat{D}), Xf] = X(Df)$ *for* $X \in \mathfrak{g}$*,* $f \in L$ *and* $T_{\mathfrak{g}}(\hat{D})$ *is* Aut$(\mathfrak{g})$*-invariant. In particular, this action preserves every* $U\widehat{L\mathfrak{g}}$*-submodule of* $\mathbb{F}_\ell(\mathfrak{g})$.

Semi-local case. This refers to the situation where we allow the R-algebra L to be a finite direct sum of R-algebras isomorphic to $R((t))$: $L = \oplus_{i \in I} L_i$, where I is a nonempty finite index set and L_i as before. We then extend the notation employed earlier in the most natural fashion. For instance, $\mathcal{O}$, $\mathfrak{m}$, ω, $\mathfrak{l}$ are now the direct sums over I (as filtered objects) of the items suggested by the notation. If $r : L \times \omega \to R$ denotes the sum of the residue pairings of the summands, then r is still topologically perfect. However, we take for the oscillator algebra $\hat{\mathfrak{l}}$ not the direct sum of the $\hat{\mathfrak{l}}_i$, but rather the quotient of $\oplus_i \hat{\mathfrak{l}}_i$ that identifies the central generators of the summands with a single $\hbar$. We thus get a Virasoro extension $\hat{\theta}$ of θ by $c_0 R$ and a (faithful) oscillator representation of $\hat{\theta}$ in $\overline{U\hat{\mathfrak{l}}}$. The decreasing filtrations are the obvious ones. We shall denote by $\mathbb{F}$ the Fock representation $\mathbb{F}$ of $\hat{\mathfrak{l}}$ that ensures that the unit of every summand $\mathcal{O}_i$ acts the identity; it is then the induced representation of the rank one representation of $F^0\hat{\mathfrak{l}} = \mathcal{O} \oplus R\hbar$ in Rv_o.

In likewise manner we define $\widehat{L\mathfrak{g}}$ (a central extension of $\oplus_{i \in I} L\mathfrak{g}_i$ by c_R) and construct the associated Sugawara representation. The representation $\mathbb{F}_\ell(\mathfrak{g})$ of $\widehat{L\mathfrak{g}}$ is as before. We have defined $\hat{\theta}_{L,R}$ and Corollaries 12 and 13 continue to hold.

4. The WZW connection: algebraic aspects

From now on we place ourselves in the semi-local case, so $L = \oplus_{i \in I} L_i$ with I nonempty and finite and $L_i \cong R((t))$. For the sake of transparency, we begin with an abstract discussion that will lead us to the Fock representation of a symplectic local system.

Abstract spaces of covacua I. Let A be a R-subalgebra of L and let $\theta_{A/R}$ have the usual meaning as the Lie algebra of R-derivations $A \to A$. We denote by $A^\perp \subset L$ the annihilator of A relative to the residue pairing. We assume that:

- (A_1) as an R-algebra, A is flat and of finite type and $A \cap \mathcal{O} = R$,
- (A_2) the R-modules $L/(A+\mathcal{O})$ and $F := A^\perp \cap \mathcal{O}$ are free of finite rank and the residue pairing induces a perfect pairing $L/(A+\mathcal{O}) \otimes_R F \to R$.
- (A_3) the universal continuous R-derivation $d : L \to \omega$ maps A to $A^\perp$ and the A-dual of the resulting A-homomorphism $\Omega_{A/R} \to A^\perp$ is an R-isomorphism $\mathrm{Hom}_A(A^\perp, A) \cong \theta_{A/R}$.

Remark 14. The example to keep in mind is the following. Since R is regular local k-algebra, it represents a smooth germ (S, o). Suppose we are given a family $\pi : \mathcal{C} \to S$ of smooth projective curves of genus g over this germ, endowed with pairwise disjoint sections $(x_i)_{i\in I}$. We let $\mathcal{O}_i$ be is the formal completion of $\mathcal{O}_\mathcal{C}$ along x_i, let L_i be obtained from $\mathcal{O}_i$ by inverting a generator for the ideal defining $x_i(S)$, and take for A the R-algebra of regular functions on $\mathcal{C}^\circ := \mathcal{C} - \cup_i x_i(S)$ (or rather its isomorphic image in $L = \oplus_i L_i$). It is a classical fact that the three properties A_1, A_2, A_3 are then satisfied. For instance, $L/(A+\mathcal{O})$ has according to Weil the interpretation of $R^1\pi_*\mathcal{O}_\mathcal{C}$ and hence is free of rank g. It is also classical that the annihilator of A in ω is precisely the image of the space relative rational differentials on $\mathcal{C}/S$ that are regular on $\mathcal{C}^\circ$ (so in this case $\Omega_{A/R} \to A^\perp$ is already an isomorphism before dualizing).

We put $H := A^\perp/A$. It follows from properties (A_1) and (A_2), that the natural map $F \to H$ is an embedding with image a Lagrangian subspace. Recall that $\theta_{A,R}$ denotes the Lie algebra of k-derivations $A \to A$ which preserve R. The kernel of the natural map $\theta_{A,R} \to \theta_R$ is $\theta_{A/R}$ and its image, is by definition the R-submodule of k-derivations $R \to R$ that extend to one of A. We denote this image by $\theta_R^A \subset \theta_R$ and refer to it as the module of *liftable derivations*. This module is clearly closed under the Lie bracket. We shall assume that we have equality in the generic point, so that θ_R^A is as our $\mathfrak{D}$. According to (A_3) any element of $\theta_{A/R}$ induces the zero map in H and so $\theta_{A,R}$ acts in H (as a k-Lie algebra) through $\theta_{A,R}$. It is clear that $\theta_{A,R} \subset \theta_{L,R}$.

(In the above example, H would represent the first De Rham cohomology module of $\mathcal{C}/S$, F the module of relative regular differentials, and we would have $\theta_R^A = \theta_R$, as every vector field germ on (S, o) lifts to rational vector field on $\mathcal{C}$ that is regular on $\mathcal{C}^\circ$. The Lie action is then that of covariant derivation of relative cohomology classes. The reason for us to allow $\theta_R^A \neq \theta_R$ is because we want to admit the central fibers of $\mathcal{C} \to S$ to have modest singularities; in that case θ_R^A is the θ_R-stabilizer of a principal ideal in R, the *discriminant* ideal of π.)

We write $\hat{\theta}_{A,R}$ for the preimage of $\theta_{A,R}$ in $\hat{\theta}_{L,R}$ and by $\hat{\theta}^A_R$ the quotient $\hat{\theta}_{A,R}/\theta_{A/R}$. These are extensions of $\theta_{A,R}$ resp. θ^A_R by c_0R. They can be split, but not canonically so.

Since $\mathrm{Ad}(A) \subset A^\perp$, the residue pairing vanishes on $A \times \mathrm{Ad}(A)$ and hence A is contained in $\hat{\mathfrak{l}}$ as an abelian Lie subalgebra. Let $\mathbb{F}_A := \mathbb{F}/A\mathbb{F}$ denote the space of A-covariants.

Theorem 15. *The following properties hold:*

(i) *The space of covariants $\mathbb{F}_A$ is naturally identified with the Fock representation $\mathbb{F}(H, F)$,*

(ii) *for every $D \in \theta_{A/R}$ there exists a lift $\hat{D} \in \hat{\theta}_{A/R}$ such that $T(\hat{D})$ lies in the closure of $A \circ \hat{\mathfrak{l}}$ in $\overline{U\hat{\mathfrak{l}}}$,*

(iii) *the representation of the Lie algebra $\hat{\theta}_{A,R}$ on $\mathbb{F}$ preserves the submodule $A\mathbb{F}$ and $\hat{\theta}_{A,R}$ acts in $\mathbb{F}_A$ through $\hat{\theta}^A_R$ by differential operators of degree ≤ 1 (with c_0 acting as the identity),*

(iv) *if Θ is a square root of $\det_R(F)$, then the image of this action on $\mathbb{F}_A$ is equal to the image of the Lie algebra of first order differential operators $\theta^A_R(\Theta)$ (as described in Remark 9).*

Proof. The proof of the first assertion is straightforward and left to the reader.

Since $L/(A+\mathcal{O})$ is finitely generated as a R-module, we can choose a finite subset $M \subset L$ such that $L = A + \sum_{f\in M} Rf + \mathcal{O}$.

Now let $D \in \theta_{A/R}$. According to (A_3), we may view D as a L-linear map $\omega \to L$ which maps $A^\perp$ to A. This implies that $\hat{C}(\hat{D})$ lies in the closure of the image of $A \otimes_R \hat{\mathfrak{l}} + \hat{\mathfrak{l}} \otimes_R A$ in $\overline{U\hat{\mathfrak{l}}}$. It follows that $\hat{C}(\hat{D})$ has the form $\hbar r + \sum_{n\geq 1} f_n \circ g_n$ with $r \in R$, one of $f_n, g_n \in L$ being in A and the order of f_n smaller than that of g_n for almost all n. In view of the fact that the nonzero elements of A are of lower order than those of $\mathcal{O}$ and $f_n \circ g_n \equiv g_n \circ f_n$ (mod $\hbar R$), we can assume that all f_n lie in A and so we can arrange that $\hat{C}(\hat{D})$ lies in the closure of $A \circ \hat{\mathfrak{l}}$.

For (iii) we observe that if $D \in \theta_{A,R}$ and $f \in A$, then $[D, f] = Df$ lies in A. This shows that $T(\hat{D})$ preserves $A\mathbb{F}$ and hence acts in $\mathbb{F}_A$. When $D \in \theta_{A/R}$ and if we choose $\hat{D} \in \hat{\theta}_{A/R}$ as in (ii), then $T(\hat{D})$ is clearly zero in $\mathbb{F}_A$. Thus $\hat{\theta}_{A,R}$ acts in $\mathbb{F}_A$ through $\hat{\theta}^A_R$.

Property (iv) follows from the observation that the action of $\hat{\theta}_{A,R}$ on $\mathbb{F}_A \cong \mathbb{F}(H, F)$ evidently has the properties described in Proposition-Definition 6. □

Abstract spaces of covacua II. We continue with the setting of the previous subsection. With $\mathfrak{g}$ as before we have defined $\mathbb{F}_\ell(\mathfrak{g})$. We first consider the space of $A\mathfrak{g}$-covariants in $\mathbb{F}_\ell(\mathfrak{g})$,

$$\mathbb{F}_\ell(\mathfrak{g})_{A\mathfrak{g}} := \mathbb{F}_\ell(\mathfrak{g})/A\mathfrak{g}\mathbb{F}_\ell(\mathfrak{g}).$$

Proposition 16. *For* $\hat{D} \in \hat{\theta}_{A/R}$, $T_{\mathfrak{g}}(\hat{D})$ *lies in the closure of* $A\mathfrak{g} \circ \widehat{L\mathfrak{g}}$ *in* $\overline{U\widehat{L\mathfrak{g}}}$. *The Sugawara representation of the Lie algebra* $\hat{\theta}_{A,R}$ *on* $\mathbb{F}_\ell(\mathfrak{g})$ *preserves the submodule* $A\mathfrak{g}\mathbb{F}_\ell(\mathfrak{g}) \subset \mathbb{F}_\ell(\mathfrak{g})$ *and acts in the space of* $A\mathfrak{g}$*-covariants in* $\mathbb{F}_\ell(\mathfrak{g})$, $\mathbb{F}_\ell(\mathfrak{g})_{A\mathfrak{g}}$, *via* $\hat{\theta}^A_R$*; this representation is one by differential operators of degree* ≤ 1 *(with* c_0 *acting as multiplication by* $(c+\check{h})^{-1}c\dim\mathfrak{g}$*).*

Proof. The proof is similar to the arguments used to prove Theorem 15. Since D maps $A^\perp$ to $A \subset L$, $1 \otimes D$ maps the submodule $\mathfrak{g}\otimes A^\perp$ of $\mathfrak{g}\otimes\omega$ to the submodule $\mathfrak{g}\otimes A = A\mathfrak{g}$ of $\mathfrak{g}\otimes L = L\mathfrak{g}$. It is clear that $\mathfrak{g}\otimes A^\perp$ and $A\mathfrak{g}$ are each others annihilator relative to the pairing $r_{\mathfrak{g}}$. This implies that $\hat{C}(\hat{D})$ lies in the closure of the image of $A\mathfrak{g}\otimes_k L\mathfrak{g} + L\mathfrak{g}\otimes_k A\mathfrak{g}$ in $\overline{U\widehat{L\mathfrak{g}}}$. It follows that $\hat{C}(\hat{D})$ has the form $cr + \sum_\kappa \sum_{n\geq 1} X_\kappa f_{\kappa,n} \circ X_\kappa g_{\kappa,n}$ with $r \in R$, one of $f_{\kappa_n}, g_{\kappa,n} \in L$ being in A and the order of f_{κ_n} smaller than that of $g_{\kappa,n}$ for almost all κ, n. Since the elements of A have order ≤ 0 and $X_\kappa f_{\kappa,n} \circ X_\kappa g_{\kappa,n} \equiv X_\kappa g_{\kappa,n} \circ X_\kappa f_{\kappa,n}$ (mod cR), we can assume that all $f_{\kappa,n}$ lie in A and so the first assertion follows.

If $D \in \theta_{A,R}$, then for $X \in \mathfrak{g}$ and $f \in A$, we have $[D, Xf] = X(Df)$, which is an element of $A\mathfrak{g}$ (since $Df \in A$). This shows that $T_{\mathfrak{g}}(\hat{D})$ preserves $A\mathfrak{g}\mathbb{F}_\ell(\mathfrak{g})$. If $D \in \theta_{A/R}$, then it follows from the proven part that $T_{\mathfrak{g}}(\hat{D})$ maps $\mathbb{F}_\ell(\mathfrak{g})$ to $A\mathfrak{g}\mathbb{F}_\ell(\mathfrak{g})$ and hence induces the zero map in $\mathbb{F}_\ell(\mathfrak{g})_{A\mathfrak{g}}$. So $\hat{\theta}_{A,R}$ acts on $\mathbb{F}_\ell(\mathfrak{g})_{A\mathfrak{g}}$ via $\hat{\theta}^A_R$. □

For what follows we need to briefly review from [9] the theory of highest weight representations of a loop algebra such as $\widehat{L\mathfrak{g}}$. According to that theory, the natural analogues for $\widehat{L\mathfrak{g}}$ of the finite dimensional irreducible representations of the finite dimensional semi-simple Lie algebras are obtained as follows, assuming that I is a singleton. Fix an integer $\ell \geq 0$ and let V be a finite dimensional irreducible representation of $\mathfrak{g}$. Make V a k-representation of $F^0 L\mathfrak{g}$ by letting c act as multiplication by ℓ and by letting $\mathfrak{g}\otimes_k \mathcal{O}$ act via its projection onto $\mathfrak{g}$. If we induce this up to $\widehat{L\mathfrak{g}}$ we get a representation $\widetilde{\mathbb{H}}_\ell(V)$ of $\widehat{L\mathfrak{g}}$ which clearly is a quotient of $\mathbb{F}_\ell(\mathfrak{g})$. Its irreducible quotient is denoted by $\mathbb{H}_\ell(V)$. This is integrable as an $\widehat{L\mathfrak{g}}$-module: if $Y \in \mathfrak{g}$ is nilpotent and $f \in L$, then Yf acts locally nilpotently in $\mathbb{H}_\ell(V)$ (which means that the latter is a union of finite dimensional Yf-invariant subspaces in which Yf acts nilpotently). We can be more precise if we fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and a system of positive roots $(\alpha_1, \dots, \alpha_r)$ in $\mathfrak{h}^*$. Let $\theta \in \mathfrak{h}^*$ the highest root, $\check{\theta} \in \mathfrak{h}$ the corresponding coroot and $X \in \mathfrak{g}$ a generator of the root space $\mathfrak{g}_\theta$.

Lemma 17. *If* $\lambda \in \mathfrak{h}^*$ *be the highest weight of* V*, then* $\mathbb{H}_\ell(V)$ *is zero unless* $\lambda(\check{\theta}) \leq \ell$*. Assuming this inequality, then* $\mathbb{H}_\ell(V)$ *can be obtained as the quotient of* $U\widehat{L\mathfrak{g}}$ *by the left ideal generated by* $\mathfrak{g}\otimes_k \mathfrak{m}$, $c - \ell$ *and* $(Xf)^{1+\ell-\lambda(\check{\theta})}$, *where we can take for* f *any* $\mathcal{O}$*-generator of* $F^{-1}\mathfrak{l}$*. In fact, the image of* V *in* $\mathbb{H}_\ell(V)$ *(which generates* $\mathbb{H}_\ell(V)$ *as a* $\widehat{L\mathfrak{g}}$*-representation) is annihilated by all expressions of the form* $Xf_N \circ \cdots \circ Xf_1$ *with* $f_k \in F^{-1}\mathfrak{l}$ *and* $N > \ell - \lambda(\check{\theta})$.

Proof. The first assertion is in the literature in the form of an Exercise (12.12 of [9]). As to the second statement: choose variables $u_1, \dots, u_N$ and observe that $f_u := f + \sum_k u_k f_k$ is an $\mathcal{O}$-generator of $F^{-1}\mathfrak{l}$ for generic u. So V is killed by $(Xf_u)^N$ for generic u and hence for all u. By taking the coefficient of $u_1 \cdots u_N$ (and using that the Xf_k's commute with each other), we find that $Xf_N \circ \cdots \circ Xf_1$ annihilates V. □

Let us call the k-span of an X as above a *highest root line*. Since the Cartan subalgebras of $\mathfrak{g}$ are all conjugate under the adjoint representation, the same is true for the highest root lines.

Definition 18. The *level* of a finite dimensional representation V of $\mathfrak{g}$ is the smallest integer ℓ for which some (or equivalently, any) highest root line $\mathfrak{n}$ has the property that $\mathfrak{n}^{\ell+1} \subset U\mathfrak{g}$ kills V. We denote it by $\ell(V)$.

It is clear that in terms of the above root data, the set P_ℓ of equivalence classes of irreducible representations of level $\leq \ell$ can be identified with the set of integral weights in a simplex, hence is finite. Notice that P_ℓ is invariant under dualization and more generally, under all outer automorphisms of $\mathfrak{g}$.

Returning to the general case in which I need not be a singleton, we put $\mathbb{H}_\ell(V) := \otimes_{i\in I}\mathbb{H}_\ell(V_i)$. So this is zero unless every V_i is of level $\leq \ell$. Inspired by the physicists terminology, the R-module $\mathbb{H}_\ell(V)_{A\mathfrak{g}}$ is called the space of *covacua* attached to A. The following proposition says that it is of finite rank and describes the WZW-connection.

Proposition 19 (Finiteness)**.** *The space $\mathbb{H}_\ell(V)$ is finitely generated as a $UA\mathfrak{g}$-module (so that $\mathbb{H}_\ell(V)_{A\mathfrak{g}}$ is a finitely generated R-module). The Lie algebra $\hat{\theta}^A_R$ acts on $\mathbb{H}_\ell(V)_{A\mathfrak{g}}$ via the Sugawara representation with c_0 acting as multiplication by $\frac{\ell}{\ell+\check{h}} \dim \mathfrak{g}$.*

Proof. Choose a generator t_i of $\mathfrak{m}_i$. The issue being local on Spec(R), we may assume that after localizing R, there exists a finite set Φ of *negative* powers of these generators mapping to an R-basis set of $L/(\mathcal{O} + A)$. The nilpotent elements of $\mathfrak{g}$ span a nontrivial subspace that is invariant under the adjoint action and hence span all of $\mathfrak{g}$. Let $\Xi \subset \mathfrak{g}$ be a k-basis of $\mathfrak{g}$ consisting of nilpotent elements. Then for a pair $(X, f) \in \Xi \times \Phi$, Xf acts locally nilpotently in $\mathbb{H}_\ell(V)$ and so there exists a positive integer N such that the Nth power of any such element kills the image of $\otimes_{i\in I}V_i$ in $\mathbb{H}_\ell(V)$.

A PBW type of argument then shows that $\mathbb{H}_\ell(V)$ is the sum of the subspaces

$$A\mathfrak{g} \circ (X_r f_r)^{\circ n_r} \circ \cdots \circ (X_1 f_1)^{\circ n_1} \otimes (\otimes_{i\in I}V_i) \subset \mathbb{H}_\ell(V)$$

with $(X_i, f_i) \in \Xi\times\Phi$ pairwise distinct for $i = 1, \dots, r$, and $n_1 \geq \cdots \geq n_r \geq 0$. Since we get a nonzero element only when $n_1 < N$, we thus obtain a finite collection of R-module generators of $\mathbb{H}_\ell(V)_{A\mathfrak{g}}$. The remaining statements follow from 16. □

Remark 20. We expect the R-module $\mathbb{H}_\ell(V)_{A\mathfrak{g}}$ to be flat as well and this to be a consequence of a related property for the $UA\mathfrak{g}$-module $\mathbb{H}_\ell(V)$. Such a result, or rather an algebraic proof of it, might simplify the argument in [20] (see Section 6 for our version) which shows that the sheaf of covacua attached to a degenerating family of pointed curves is locally free.

Remark 21. It is clear from the definition that a system of $\mathfrak{g}$-equivariant isomorphisms $(\phi_i : V_i \cong V'_i)_{i\in I}$ of finite dimensional irreducible representations induces an isomorphism $\phi_* : \mathbb{H}_\ell(V)_{A\mathfrak{g}} \cong \mathbb{H}_\ell(V')_{A\mathfrak{g}}$. By Schur's lemma, each ϕ_i is unique up to scalar in k and hence the same is true for ϕ_*. We may rigidify the situation by fixing in each representation V_i and V'_i involved a highest weight orbit for the closed connected subgroup of linear transformations whose Lie algebra is the image of $\mathfrak{g}$: if we require that every ϕ_i respects these orbits, then ϕ_i is unique.

We can also say something if we are given a $\sigma \in \mathrm{Aut}(\mathfrak{g})$. This turns every representation V of $\mathfrak{g}$ into another one (denoted ${}^\sigma V$) that has the same underlying vector space V, by letting $X \in \mathfrak{g}$ act as $\sigma(X)$ on V. The extension $\hat\sigma$ of σ to $\widehat{L\mathfrak{g}}$ does the same with $\mathbb{H}_\ell(V)$. It follows that we have an identification of $\widehat{L\mathfrak{g}}$-modules:

$$Y_r f_r \circ \cdots \circ Y_1 f_1 \,{}^\sigma\!\otimes (\otimes_{i\in I} v_i) \in \mathbb{H}_\ell({}^\sigma V) \mapsto \sigma(Y_r) f_r \circ \cdots \circ \sigma(Y_1) f_1 \otimes (\otimes_{i\in I} v_i) \in {}^{\hat\sigma}\mathbb{H}_\ell(V).$$

Since σ preserves $A\mathfrak{g}$, this descends to an identification $\mathbb{H}_\ell(V^\sigma)_{A\mathfrak{g}} \cong \mathbb{H}_\ell(V)_{A\mathfrak{g}}$ of R-modules. It is clear from the definition above that this is also equivariant for the Segal-Sugawara representation and hence is an isomorphism of $\hat\theta^A_R$-modules.

Propagation principle. The following proposition is a bare version of what is known as the *propagation of vacua*; it essentially shows that trivial representations may be ignored (as long as some representations remain: if all are trivial, then we can get rid of all but one of them). If we do not care about the WZW-connection, then this is even true for nontrivial representations (a fact that can be found in Beauville [4]) so that we then essentially reduce the discussion to the case where I is a singleton.

Proposition 22. *Let $J \subsetneq I$ be such that A maps onto $\oplus_{j\in J} L_j/\mathcal{O}_j$. Denote by $B \subset A$ the kernel of the map $A \to \oplus_{j\in J} L_j/\mathfrak{m}_j \cong R^J$ (evidently an ideal) so that we have a surjective Lie homomorphism $B\mathfrak{g} \to (R\otimes_k \mathfrak{g})^J$ via which $B\mathfrak{g}$ acts on $R\otimes_k (\otimes_{j\in J} V_j)$. Then the map of $B\mathfrak{g}$-modules $\mathbb{H}_\ell(V|I-J)\otimes_k (\otimes_{j\in J} V_j) \to \mathbb{H}_\ell(V)$ induces an isomorphism on covariants:*

$$(\mathbb{H}_\ell(V|I-J) \otimes_k (\otimes_{j\in J} V_j))_{B\mathfrak{g}} \xrightarrow{\cong} \mathbb{H}_\ell(V)_{A\mathfrak{g}}.$$

If $\theta^{A,B}_R \subset \theta^A_R$ denotes the module of k-derivations $R \to R$ that lift to k-derivations $A \to A$ which preserve B (or equivalently, $\oplus_{j\in J}\mathfrak{m}_j$), and $\hat\theta^{A,B}_R \subset \hat\theta^A_R$ stands for the corresponding extension, then the above isomorphism of covariants is compatible with the action of $\hat\theta^{A,B}_R$ on both sides, provided that the representations V_j are trivial for $j \in J$.

Proof. For the first assertion it suffices to do the case when J is a singleton $\{o\}$. The hypotheses clearly imply that $\mathbb{H}_\ell(V|I-\{o\})\otimes V_o \to \mathbb{H}_\ell(V)_{A\mathfrak{g}}$ is onto. The kernel is easily shown to be $B\mathfrak{g}(\mathbb{H}_\ell(V|I-\{o\})\otimes V_o)$.

The second assertion follows in a straightforward manner from our definitions: if $\bar{D}\in\hat{\theta}_R^{A,B}$, then lift $\bar{D}$ to a k-derivation $D: A\to A$ which preserves B. This implies that D preserves each $\mathcal{O}_j$, $j\in J$. If we choose a parameter t_j for $\mathcal{O}_j$ so that $\mathcal{O}_j = R((t_j))$, then D takes in $\mathcal{O}_j$ the form $D^{(j)}_{\text{hor}} + D^{(j)}_{\text{vert}}$, with $D^{(j)}_{\text{hor}}$ the extension of $\bar{D}$ which kills t_j and $D^{(j)}_{\text{vert}} = c^{(j)}\partial/\partial t_j$ plus higher order terms with $c^{(j)}\in R$. The Sugawara action of $D^{(j)}_{\text{vert}}$ on the subspace $V_j\subset\mathbb{H}_\ell(V_j)$ is up to a factor in R given by $\sum_\kappa t_j^{-1}X_\kappa\circ X_\kappa$. But if V_j is the trivial representation, then this is evidently zero. The second assertion now follows. □

Remark 23. Our discussion of the genus zero case will show that the isomorphism of covariants generally fails to be compatible relative to the $\hat{\theta}_R^{A,B}$-action.

Remark 24. Proposition 22 is sometimes used in the opposite direction: if $\mathfrak{m}_o\subset A$ is a principal ideal with the property that for a generator $t\in\mathfrak{m}_o$, the $\mathfrak{m}_o$-adic completion of A gets identified with $R((t))$, then let $\tilde{I}$ be the disjoint union of I and $\{o\}$, $\tilde{V}$ the extension of V to $\tilde{I}$ which assigns to o the trivial representation and $\tilde{A} := A[t^{-1}]$. With $(\tilde{I},\{o\})$ taking the role of (I,J), we then find that $\mathbb{H}_\ell(V)_{A\mathfrak{g}}\cong\mathbb{H}_\ell(\tilde{V})_{\tilde{A}\mathfrak{g}}$.

5. Bundles of covacua

Spaces of covacua in families. We specialize the discussion of Section 4 to a more concrete geometric situation. This leads us to sheafify many of the notions we introduced earlier and in such cases we shall modify our notation (or its meaning) accordingly. Suppose given a proper and flat morphism between k-varieties $\pi:\mathcal{C}\to S$ whose base S is smooth and connected and whose fibers are reduced connected curves that have complete intersection singularities only (but we do not assume that $\mathcal{C}$ is smooth over k). Since the family is flat, the arithmetic genus of the fibers is locally constant, hence constant, say equal to g. We also suppose given disjoint sections x_i of π, indexed by the finite nonempty set I whose union $\cup_{i\in I}x_i(S)$ lies in the smooth part of $\mathcal{C}$ and meets every irreducible component of a fiber. The last condition ensures that if $j:\mathcal{C}^\circ := \mathcal{C}-\cup_{i\in I}x_i(S)\subset\mathcal{C}$ is the inclusion, then πj is an affine morphism.

We denote by $(\mathcal{O}_i,\mathfrak{m}_i)$ the formal completion of $\mathcal{O}_\mathcal{C}$ along $x_i(S)$, by $\mathcal{L}_i$ the subsheaf of fractions of $\mathcal{O}_i$ with denominator a local generator of $\mathfrak{m}_i$ and by $\mathcal{O}$, $\mathfrak{m}$ and $\mathcal{L}$ the corresponding direct sums. But we keep on using ω, θ, $\hat{\theta}$ etc. for their sheafified counterparts. So these are now all $\mathcal{O}_S$-modules and the residue pairing is also one of $\mathcal{O}_S$-modules: $r:\mathcal{L}\times\omega\to\mathcal{O}_S$. We write $\mathcal{A}$ for $\pi_*j_*j^*\mathcal{O}_\mathcal{C}$ (a sheaf of $\mathcal{O}_S$-algebras that is also equal to the direct image of $\mathcal{O}_{\mathcal{C}^\circ}$ on S) and often identify this with its image in $\mathcal{L}$. We denote by $\theta_{\mathcal{A}/S}$

the sheaf of $\mathcal{O}_S$-derivations $\mathcal{A} \to \mathcal{A}$ and by $\omega_{\mathcal{A}/S}$ for the sheaf $\pi_* j_* j^* \omega_{\mathcal{C}/S}$ (which is also the direct image on S of the relative dualizing sheaf of $\mathcal{C}^\circ/S$; if $\mathcal{C}^\circ$ is smooth, this is simply the sheaf of relative differentials). So $\omega_{\mathcal{A}/S}$ is torsion free and embeds therefore in ω.

Lemma 25. *The properties* A_1, A_2 *and* A_3 *hold for the sheaf* $\mathcal{A}$. *Precisely,*

- ($\mathcal{A}_1$) $\mathcal{A}$ *is as a sheaf of* $\mathcal{O}_S$*-algebras flat and of finite type,*
- ($\mathcal{A}_2$) $\mathcal{A} \cap \mathcal{O} = \mathcal{O}_S$ *and* $R^1\pi_*\mathcal{O}_\mathcal{C} = \mathcal{L}/(\mathcal{A} + \mathcal{O})$ *is locally free of rank* g,
- ($\mathcal{A}_3$) *we have* $\theta_{\mathcal{A}/S} = \mathrm{Hom}_\mathcal{A}(\omega_{\mathcal{A}/S}, \mathcal{A})$ *and* $\omega_{\mathcal{A}/S}$ *is the annihilator of* $\mathcal{A}$ *with respect to the residue pairing.*

Proof. Property $\mathcal{A}_1$ is clear. It is also clear that $\mathcal{O}_S = \pi_*\mathcal{O}_\mathcal{C} \to \mathcal{A} \cap \mathcal{O}$ is an isomorphism. The long exact sequence defined by the functor π_* applied to the short exact sequence

$$0 \to \mathcal{O}_\mathcal{C} \to j_* j^* \mathcal{O}_\mathcal{C} \to \mathcal{L}/\mathcal{O} \to 0$$

tells us that $R^1\pi_*\mathcal{O}_\mathcal{C} = \mathcal{L}/(\mathcal{A} + \mathcal{O})$; in particular, the latter is locally free of rank g. Hence $\mathcal{A}_2$ holds as well.

In order to verify $\mathcal{A}_3$, we note that $\pi_*\omega_{\mathcal{C}/S}$ is the $\mathcal{O}_S$-dual of $R^1\pi_*\mathcal{O}_S$, and hence is locally free of rank g. The first part of $\mathcal{A}_3$ follows from the corresponding local property $\theta_{\mathcal{C}/S} = \mathrm{Hom}_{\mathcal{O}_\mathcal{C}}(\omega_{\mathcal{C}/S}, \mathcal{O}_\mathcal{C})$ by applying $\pi_* j^*$ to either side. This local property is known to hold for families of curves with complete intersection singularities. (A proof under the assumption that $\mathcal{C}$ is smooth—which does not affect the generality, since π is locally the restriction of that case and both sides are compatible with base change—runs as follows: if $j' : \mathcal{C}' \subset \mathcal{C}$ denotes the locus where π is smooth, then its complement is of codimension ≥ 2 everywhere. Clearly, $\theta_{\mathcal{C}/S}$ is the $\mathcal{O}_\mathcal{C}$-dual of $\omega_{\mathcal{C}/S}$ on $\mathcal{C}'$ and since both are inert under $j'_* j'^*$, they are equal everywhere.)

The last assertion essentially restates the well-known fact that the polar part of a rational section of $\omega_{\mathcal{C}/S}$ must have zero residue sum, but can otherwise be arbitrary. More precisely, the image of $\omega_{\mathcal{A}/S}$ in $\omega/F^1\omega$ is the kernel of the residue map $\omega/F^1\omega \to \mathcal{O}_S$. The intersection $\omega_{\mathcal{A}/S} \cap F^1\omega$ is $\pi_*\omega_{\mathcal{C}/S}$ and is hence locally free of rank g. Since $(F^1\omega)^\perp = \mathcal{O}$, it follows that $(\omega_{\mathcal{A}/S})^\perp \cap \mathcal{O}$ and $\mathcal{L}/((\omega_{\mathcal{A}/S})^\perp + \mathcal{O})$ are locally free of rank 1 and g respectively. Since $\mathcal{A}$ has these properties also and is contained in $(\omega_{\mathcal{A}/S})^\perp$, we must have $\mathcal{A} = (\omega_{\mathcal{A}/S})^\perp$. □

For what follows one usually supposes that the fibers are stable I-pointed curves (meaning that every fiber of πj has only ordinary double points as singularities and has finite automorphism group) and is versal (so that the discriminant Δ_π of π is a reduced normal crossing divisor), but we shall not make these assumptions yet. Instead, we assume the considerable weaker property that the sections of the sheaf $\theta_S(\log \Delta_\pi)$ of vector fields on S tangent to Δ_π lift locally on S to vector fields on $\mathcal{C}$. (This is for instance the case if $\mathcal{C}$ is smooth and π is multi-transversal with respect to the (Thom) stratification of $\mathrm{Hom}(T\mathcal{C}, \pi^*TS)$ by rank [14].) Notice that we have a restriction homomorphism $\theta_S(\log \Delta_\pi) \otimes \mathcal{O}_{\Delta_\pi} \to \theta_{\Delta_\pi}$.

Let $\theta_{\mathcal{C},S} \subset \theta_{\mathcal{C}}$ denote the sheaf of derivations which preserve $\pi^*\mathcal{O}_S$. If we apply $\pi_* j_* j^*$ to the exact sequence $0 \to \theta_{\mathcal{C}/S} \to \theta_{\mathcal{C},S} \to \theta_{\mathcal{C},S}/\theta_{\mathcal{C}/S} \to 0$ and use our liftability assumption and the fact that πj is affine, we get the exact sequence

$$0 \to \theta_{\mathcal{A}} \to \theta_{\mathcal{A},S} \to \theta_S(\log \Delta_\pi) \to 0.$$

We defined $\hat{\theta}_{\mathcal{A},S}$ as the preimage of $\theta_{\mathcal{A},S}$ in $\hat{\theta}_{\mathcal{L},S}$ and $\hat{\theta}_S(\log \Delta_\pi)$ as the quotient $\hat{\theta}_{\mathcal{L},S}/\theta_{\mathcal{A}}$. These extend $\theta_{\mathcal{A},S}$ and θ_S by $c_0\mathcal{O}_S$. If we denote the *Hodge bundle*

$$\lambda := \lambda(\mathcal{C}/S) := \det(\pi_*\omega_{\mathcal{C}/S}),$$

then we see that $\hat{\theta}_S(\log \Delta_\pi)$ may be identified with the Lie sheaf $\mathcal{D}_1(\lambda)(\log \Delta_\pi)$ of first order differential operators $\lambda \to \lambda$ which preserve the subsheaf of sections vanishing on Δ_π.

Observe that $\mathcal{L}\mathfrak{g} = \mathfrak{g} \otimes_k \mathcal{L}$ is now a sheaf of Lie algebras over $\mathcal{O}_S$. The same applies to $\hat{\mathfrak{l}}$ and so we have a Virasoro extension $\hat{\theta}_S$ of θ_S by $c_0\mathcal{O}_S$. We have also defined $\mathcal{A}\mathfrak{g} = \mathfrak{g} \otimes_k \mathcal{A}$, which is a Lie subsheaf of $\mathcal{L}\mathfrak{g}$ as well as of $\widehat{\mathcal{L}\mathfrak{g}}$ and the Fock type $\widehat{\mathcal{L}\mathfrak{g}}$-module $\mathcal{F}_\ell(\mathfrak{g})$. The will also consider the sheaf of $\mathcal{A}\mathfrak{g}$-covariants in the latter,

$$\mathcal{F}_\ell(\mathfrak{g})_{\mathcal{C}/S} := \mathcal{F}_\ell(\mathfrak{g})_{\mathcal{A}\mathfrak{g}} = \mathcal{A}\mathfrak{g}\mathcal{F}_\ell(\mathfrak{g}) \backslash \mathcal{F}_\ell(\mathfrak{g}).$$

From Proposition 16 we get:

Corollary 26. *The representation of the Lie algebra $\hat{\theta}_{\mathcal{A},S}$ on $\mathcal{F}_\ell(\mathfrak{g})$ preserves $\mathcal{A}\mathfrak{g}\mathcal{F}_\ell(\mathfrak{g})$ and acts on $\mathcal{F}_\ell(\mathfrak{g})_{\mathcal{C}/S}$ via $\hat{\theta}(\log \Delta_\pi)$ with c_0 acting as multiplication by $(\ell + \check{h})^{-1}\ell \dim \mathfrak{g}$. This construction has a base change property along any smooth part S' of the discriminant in the sense that the residual action of $\hat{\theta}(\log \Delta_\pi)$ on $\mathcal{F}_\ell(\mathfrak{g})_{\mathcal{C}_{S'}/S'} \cong \mathcal{F}_\ell(\mathfrak{g})_{\mathcal{C}/S} \otimes \mathcal{O}_{S'}$ factors through $\hat{\theta}_{S'}$.*

The bundle of integrable representations $\mathcal{H}_\ell(V)$ over S is defined in the expected manner: it is obtained as a quotient of $\mathcal{F}_\ell(\mathfrak{g})$ in the way $\mathbb{H}_\ell(V)$ is obtained from $\mathbb{F}_\ell(\widehat{L\mathfrak{g}})$. We write $\mathcal{H}_\ell(V)_{\mathcal{C}/S}$ for $\mathcal{H}_\ell(V)_{\mathcal{A}\mathfrak{g}}$. The following theorem, which is mostly a summary of what we have done so far, is one of the main results of the theory.

Theorem 27 (WZW-connection)**.** *The $\mathcal{O}_S$-module $\mathcal{H}_\ell(V)_{\mathcal{C}/S}$ is of finite rank; it is also locally free over $S - \Delta_\pi$ and the Lie action of $\mathcal{D}_1(\lambda)(\log \Delta_\pi)$ on $\mathcal{H}_\ell(V)_{\mathcal{C}/S}$ defines a logarithmic λ-flat connection relative to Δ_π of weight $\frac{\ell}{2(\ell+\check{h})} \dim \mathfrak{g}$. The same base change property holds along the smooth part of the discriminant as in Corollary 26. Furthermore, any $\sigma \in \mathrm{Aut}(\mathfrak{g})$ determines an isomorphism of $\mathcal{D}_1(\lambda)(\log \Delta_\pi)$-modules $\mathcal{H}_\ell({}^\sigma V)_{\mathcal{C}/S} \cong \mathcal{H}_\ell(V)_{\mathcal{C}/S}$.*

Proof. The first assertion follows from 19. The action of $\hat{\theta}$ factors (locally) through $\mathcal{D}_1(\sqrt{\lambda})(\log \Delta_\pi)$ for some square root $\sqrt{\lambda}$ of λ and has then weight $(\ell + \check{h})^{-1}\ell \dim \mathfrak{g}$. This amounts to an action of $\mathcal{D}_1(\lambda)(\log \Delta_\pi)$ of half that weight. The last assertion follows from Corollary 13. The rest is clear except perhaps the local freeness of $\mathcal{H}_\ell(V)_{\mathcal{C}/S}$ on $S-\Delta_\pi$. But this follows from the local existence of a connection in the $\mathcal{O}_S$-module $\mathcal{H}_\ell(V)_{\mathcal{C}}$. □

So if $\Lambda^\times \to S$ denotes the $\mathbb{G}_m$-bundle that is associated to λ, then we have a flat connection on the pull-back of $\mathcal{H}_\ell(V)_{\mathcal{C}/S}$ to $\Lambda^\times|S-\Delta_\pi$ with fiber monodromy scalar multiplication by a root of unity of order $\frac{\ell}{2(\ell+\check{h})}\dim\mathfrak{g}$.

Propagation principle continued. In the preceding subsection we made the assumption throughout that a union of sections of $\mathcal{C}\to S$ is given to ensure that its complement is affine over S. However, the propagation principle permits us to abandon that assumption. In fact, this leads us to let $\mathbb{V}$ stand for any map which assigns to every S-valued point x of $\mathcal{C}$ an irreducible $\mathfrak{g}$-representation $\mathbb{V}_x$ of level $\le \ell$, subject to the condition that its *support*, Supp($\mathbb{V}$) (i.e., the union of the $x(S)$ for which $\mathbb{V}_x$ is generically not the trivial representation), is a trivial finite cover over S and contained in the locus where $\pi:\mathcal{C}\to S$ is smooth. We then might write $\mathcal{H}_\ell(\mathbb{V})$ for $\mathcal{H}_\ell(\mathbb{V}|_{\mathrm{Supp}(\mathbb{V})})$, but since $\mathcal{C}-\mathrm{Supp}(\mathbb{V})$ need not be affine over S, this does not yield the right notion of conformal block. We can find however, at least locally over S, additional pairwise disjoint sections of $\mathcal{C}\to S$ so that the complement $\mathcal{C}^\circ$ of their support and that of $\mathbb{V}$ is affine over S. Then we can form $\mathcal{H}_\ell(\mathbb{V}|\mathcal{C}-\mathcal{C}^\circ)$ and Proposition 22 shows that the resulting bundle of covacua $\mathcal{H}_\ell(\mathbb{V}|\mathcal{C}-\mathcal{C}^\circ)_{(\pi_*\mathcal{O}_{\mathcal{C}^\circ})\mathfrak{g}}$ with the projective connection is independent of the choices made. This suggests that we let $\mathcal{H}_\ell(\mathbb{V})$ resp. $\mathcal{H}_\ell(\mathbb{V})_{\mathcal{C}/S}$ stand for the sheaf associated to the presheaf

$$S\supset U\mapsto \varinjlim_{\tilde{S}}\mathcal{H}_\ell(\mathbb{V}|_{\tilde{S}})\ \text{resp.}\ \varinjlim_{\tilde{S}}\mathcal{H}_\ell(\mathbb{V}|_{\tilde{S}})_{\mathcal{C}_U/U},$$

where $\tilde{S}$ runs over the unions of pairwise disjoint sections as above. The latter, when twisted with the dual of $\det(\mathcal{C}/S)$, has, being a limit of presheaves with flat connections, a flat connection as well. It is clear that in this set-up there is also no need anymore to insist that the fibers of π be connected.

The genus zero case and the KZ-connection. We here assume C to be isomorphic to $\mathbb{P}^1$. Let $x_1,\dots,x_n\in C$ be distinct and contain Supp($\mathbb{V}$). Choose an affine coordinate z on C (which identifies C with $\mathbb{P}^1$) whose domain contains the x_i's and write z_i for $z(x_i)$. Notice that $t_\infty := z^{-1}$ may serve as a parameter for the local field at $z=\infty$. So if $\mathbb{H}_\ell(k)$ denotes the representation of $\widehat{\mathfrak{g}((z^{-1}))}$ attached to the trivial representation k of $\widehat{\mathfrak{g}((z^{-1}))}$, then by the propagation principle 22 we have $\mathbb{H}_\ell(\mathbb{V})_C=(V_1\otimes\cdots\otimes V_n\otimes\mathbb{H}_\ell(k))_{\mathfrak{g}[z]}$, where $\mathfrak{g}[z]$ acts on V_i for $i\le n$ via its evaluation at z_i. According to [9], the $\mathfrak{g}[z]$-homomorphism $U(\mathfrak{g}[z])\to\mathbb{H}_\ell(k)$ is surjective and its kernel is the left ideal generated by $(zX)^{1+\ell}$, where $X\in\mathfrak{g}$ generates a highest root line. This implies that $\mathbb{H}_\ell(\mathbb{V})_{\mathbb{P}^1}$ can be identified with a quotient of the space of $\mathfrak{g}$-covariants $(V_1\otimes\cdots\otimes V_n)_{\mathfrak{g}}$, namely its biggest quotient on which $(\sum_{i=1}^n z_iX^{(i)})^{1+\ell}$ acts trivially (where $X^{(i)}$ acts on V_i as X and on the other tensor factors V_j, $j \neq i$, as the identity). Now regard $z_1, \dots, z_n$ as variables. Our first observation is that a translation in $\mathbb{C}$ does not affect $\mathbb{H}_\ell(\mathbb{V})_C$: if $a \in \mathbb{C}$, then the actions of $\sum_{i=1}^n (z_i + a)X^{(i)}$ and $\sum_{i=1}^n z_i X^{(i)}$ on $V_1 \otimes \cdots \otimes V_n$ differ the action of $aX \in \mathfrak{g}$. So we always arrange that $z_1 + \cdots + z_n = 0$. Consider in $\mathbb{C}^n$ the hyperplane S_{n-1} defined by $z_1 + \cdots + z_n = 0$ and denote by S°_{n-1} the open subset of pairwise distinct n-tuples. Then the trivial family over S°_{n-1}, $\mathcal{C} := \mathbb{P}^1 \times S^\circ_{n-1}$, comes with $n+1$ 'tautological' sections (including the one at infinity) so that we also have defined $\mathcal{C}^\circ$. This determines a sheaf $\mathcal{H}_\ell(\mathbb{V})_{\mathcal{C}/S^\circ_{n-1}}$ over S°_{n-1}. According to the preceding, we have an exact sequence

$$(V_1 \otimes \cdots \otimes V_n)_\mathfrak{g} \otimes_k \mathcal{O}_{S^\circ_{n-1}} \to (V_1 \otimes \cdots \otimes V_n)_\mathfrak{g} \otimes_k \mathcal{O}_S \to \mathcal{H}_\ell(\mathbb{V})_{\mathcal{C}/S^\circ_{n-1}} \to 0,$$

where the first map is given by $(\sum_{i=1}^n z_i X^{(i)})^{1+\ell}$. We identify its WZW connection, or rather, a natural lift of that connection to $V_1 \otimes \cdots \otimes V_n \otimes_k \mathcal{O}_{S^\circ_{n-1}}$. In order to compute the covariant derivative with respect to the vector field $\partial_i := \frac{\partial}{\partial z_i}$ on S°_{n-1}, we follow our recipe and lift it to $C \times S^\circ_{n-1}$ in the obvious way (with zero component along C). We continue to denote that lift by ∂_i and determine its (Sugawara) action on $\mathcal{H}_\ell(\mathbb{V})$. We first observe that ∂_i is tangent to all the sections, except the ith. Near that section we decompose it as $(\frac{\partial}{\partial z} + \partial_i) - \frac{\partial}{\partial z}$, where the first term is tangent to the ith section and the second term is vertical. The action of the former is easily understood: its lift to $V_1 \otimes \cdots \otimes V_n \otimes_k \mathcal{O}_{S^\circ_{n-1}}$ acts as derivation with respect to z_i. The vertical term, $-\frac{\partial}{\partial z}$, acts via the Sugawara representation, that is, it acts on the ith slot as $-\frac{1}{\ell+\check{h}} \sum_\kappa X_\kappa (z - z_i)^{-1} \circ X_\kappa$ and as the identity on the others, in other words, acts as $-\frac{1}{\ell+\check{h}} \sum_\kappa X^{(i)}_\kappa (z - z_i)^{-1} \circ X^{(i)}_\kappa$. This action does not induce one in $V_1 \otimes \cdots \otimes V_n \otimes_k \mathcal{O}_{S^\circ_{n-1}}$. To make it so, we add to this the action by an element of $\mathfrak{g}[\mathcal{C}^\circ] U\widehat{\mathcal{L}\mathfrak{g}}$ (which of course will act trivially in $\mathcal{H}_\ell(\mathbb{V})_{\mathcal{C}/S^\circ_{n-1}}$), namely

$$\frac{1}{\ell+\check{h}} \sum_\kappa X_\kappa (z - z_i)^{-1} \circ X^{(i)}_\kappa = \frac{1}{\ell+\check{h}} \sum_{j,\kappa} \frac{1}{z - z_i} X^{(j)}_\kappa \circ X^{(i)}_\kappa.$$

Doing this for every i, then the modification acts in $V_1 \otimes \cdots \otimes V_n \otimes_k \mathcal{O}_{S^\circ_{n-1}}$ as

$$\frac{1}{\ell+\check{h}} \sum_{j \neq i} \frac{1}{z_j - z_i} X^{(j)}_\kappa X^{(i)}_\kappa.$$

Let us regard the Casimir element c as an element of $\mathfrak{g} \otimes_k \mathfrak{g}$, and denote by $c^{(i,j)}$ its action in $V_1 \otimes \cdots \otimes V_n$ on the ith and jth factor (since c is symmetric, we have $c^{(i,j)} = c^{(j,i)}$, so that we need not worry about the order here). We conclude that the WZW-connection is induced by the connection on $V_1\otimes\cdots\otimes V_n\otimes_k \mathcal{O}_{S^\circ_{n-1}}$ whose connection form is

$$\frac{1}{\ell+\check h}\sum_{i=1}^n\sum_{j\neq i}\frac{dz_i}{z_j-z_i}c^{(i,j)}=-\frac{1}{\ell+\check h}\sum_{1\leq i<j\leq n}\frac{d(z_i-z_j)}{z_i-z_j}c^{(i,j)}.$$

It commutes with the Lie action of $\mathfrak{g}$ on $V_1\otimes\cdots\otimes V_n$ and so the connection passes to one on $(V_1\otimes\cdots\otimes V_n)_{\mathfrak{g}}\otimes_k\mathcal{O}_{S^\circ_{n-1}}$. This lift of the WZW-connection is known as the *Knizhnik-Zamolodchikov connection.* It is not difficult to verify that it is flat (see for instance [11]), so that we have not just a projectively flat connection, but a genuine one.

Proposition 28. *The map* $(V_1\otimes\cdots\otimes V_n)_{\mathfrak{g}}\otimes_k\mathcal{O}_{S^\circ_{n-1}}\to\mathcal{H}_\ell(\mathbb{V})_{\mathcal{C}/S^\circ_{n-1}}$ *is an isomorphism for* $n=1,2$. *Hence for* $n=1$ *(resp.* $n=2$*),* $\mathcal{H}_\ell(\mathbb{V})_{\mathcal{C}/S^\circ_{n-1}}$ *is zero unless* V_0 *is the trivial representation (resp.* V_0 *and* V_1 *are each others dual), in which case it can be identified with* $\mathcal{O}_{S^\circ_{n-1}}$.

Proof. For $n=1$ this is clear. For $n=2$, the stalk of $\mathcal{H}_\ell(\mathbb{V})_{\mathcal{C}/S^\circ_1}$ at $(z,-z)$, $z\neq 0$, can be identified with the image in $(V_1\otimes V_2)_{\mathfrak{g}}$ of the kernel of $(zX^{(1)}-zX^{(2)})^{1+\ell}$ acting in $V_1\otimes V_2$. Since $X^{(1)}+X^{(2)}$ is zero in $(V_1\otimes V_2)_{\mathfrak{g}}$ and $(X^{(1)})^{1+\ell}$ is zero in V_1, this $(V_1\otimes V_2)_{\mathfrak{g}}$. □

Remark 29. A 3-pointed genus zero curve $(C\cong\mathbb{P}^1;x_1,x_2,x_3)$ has no moduli, and so we expect in this case an identification of $\mathbb{H}_\ell(\mathbb{V})_C$ also. Indeed, as is shown in [4], if V_1,V_2,V_3 are the associated irreducible $\mathfrak{g}$-representations of level $\leq\ell$, then $\mathbb{H}_\ell(\mathbb{V})_C$ is naturally identified with the biggest quotient of $V_1\otimes V_2\otimes V_3$ on which both $\mathfrak{g}$ and the endomorphisms $(z_1X^{(1)}+z_2X^{(2)}+z_3X^{(3)})^{1+\ell}$ act trivially for *all* values of (z_1,z_2,z_3). This last condition is of course equivalent to requiring that $X^p\otimes X^q\otimes X^r$ induces the zero map whenever $p+q+r>\ell$.

6. Factorization

In this section we consider the case when we are given a family $\pi_o:\mathcal{C}_o\to S_o$ of pointed curves of genus g with a smooth base germ $S_o=\mathrm{Spec}(R_o)$ (so R_o is a regular local ring) and for which we are given a section x_0 along which π_o has an ordinary double point. We assume that the fibers have no other singularities, in other words, that π_o is smooth outside x_0. After possibly making an étale base change of degree two we find a partial normalization $\nu:\tilde{\mathcal{C}}_o\to\mathcal{C}_o$ which separates the branches in the (strong) sense that ν is an isomorphism over the complement of $x_0(S_o)$ and x_0 has two disjoint lifts to $\mathcal{C}_o$ (which we shall denote by x_+ and x_-). In what follows we simply assume this to be already the case. There are two basic cases: the *nonseparating case*, where $\tilde{\mathcal{C}}_o/S_o$ is connected—in that case the fibers have genus $g-1$—and the *separating case*, where x_+ and x_- take values in different components $\tilde{\mathcal{C}}_\pm$ of $\tilde{C}_o$ such that the fiber genera $g_\pm$ of $\tilde{C}_\pm/S_o$ add up to g. Since the natural base of the WZW-connection is the $\mathbb{G}_m$-bundle defined by a determinant bundle (or a fractional power thereof), let us first recall what we get in the present case. The bundle of which we take the determinant is the direct image of the relative dualizing sheaf $\pi_{o*}\omega_{\mathcal{C}_o/S_o}$. This bundle contains the direct image of $\omega_{\tilde{\mathcal{C}}_o/S_o}$ and the two differ only at x_o: an element of $\omega_{\tilde{\mathcal{C}}_o/S_o,x_o}$ when pulled back under ν may have a simple pole at x_+ and x_- whose residues add up to zero. So we have a natural exact sequence

$$0 \to \nu_*\omega_{\tilde{\mathcal{C}}_o/S_o} \to \omega_{\mathcal{C}_o/S_o} \to \mathcal{O}_{S_o} \to 0,$$

where the last map is defined by taking the residue at x_+. If we take the direct image under π_o, we see that we have a natural injection $(\pi_o\nu)_*\omega_{\tilde{\mathcal{C}}_o/S_o} \to \pi_{o*}\omega_{\mathcal{C}_o/S_o}$. It is in fact an isomorphism in the separating case, whereas it has a cokernel naturally isomorphic to R_o in the nonseparating case. So after taking determinants we get in either case that $\lambda(\mathcal{C}_o/S_o) = \lambda(\tilde{\mathcal{C}}_o/S_o)$, where it is understood that in the separating case the right hand side equals $\lambda(\tilde{\mathcal{C}}_+/S_o) \otimes \lambda(\tilde{\mathcal{C}}_-/S_o)$.

We now also assume given a representation valued map $\mathbb{V}_o$ on the smooth part of $\mathcal{C}_o$ whose support is contained in a finite union of sections S_o so that we have defined $\mathcal{H}_\ell(\mathbb{V}_o)_{\mathcal{C}_o/S_o}$. A coarse version of the *factorization principle* expresses this R_o-module in terms of a space of covacua attached to the normalization $\tilde{\mathcal{C}}_o/S_o$. The more refined form describes it as a residue of a module of covacua on a smoothing of π_o and takes into account the flat connection.

Throughout this section $\Sigma_o \subset \mathcal{C}_o$ is a finite union of sections of $\mathcal{C}_o/S_o$ contained in the smooth part of $\mathcal{C}_o$, which contains the support of $\mathbb{V}_o$ and has the additional property that its complement $\mathcal{C}_o^\circ := \mathcal{C}_o - \Sigma_o$ is affine over S_o (this can always be arranged by adding some 'dummy' sections to the support of $\mathbb{V}_o$). We often identify Σ_o with its preimage in $\tilde{\mathcal{C}}_o$. Notice that $\tilde{\mathcal{C}}_o^\circ := \nu^{-1}\mathcal{C}_o^\circ = \tilde{\mathcal{C}}_o - \Sigma_o$ is also affine over S_o, being the normalization of an affine S_o-scheme. We write A_o resp. $\tilde{A}_o$ for their (coordinate) R_o-algebras.

Coarse version of the factorization property. Recall that P_ℓ denotes the set of isomorphism classes of irreducible representations of $\mathfrak{g}$ of level $\leq \ell$ and is invariant under dualization: if $\mu \in P_\ell$, then $\mu^* \in P_\ell$. Let V_μ be a $\mathfrak{g}$-representation in the equivalence class $\mu \in P_\ell$ and choose $\mathfrak{g}$-equivariant dualities

$$b_\mu : V_\mu \otimes V_{\mu^*} \to k,$$

where we assume that b_{μ^*} is the transpose of b_μ. Its transpose inverse $\check{b}_\mu \in V_\mu \otimes V_{\mu^*}$ then spans the line of $\mathfrak{g}$-invariants in $V_\mu \otimes V_{\mu^*}$.

Proposition 30. *Let $\tilde{\mathbb{V}}_{\mu,\mu^*}$ be the representation valued map on $\tilde{\mathcal{C}}_o$ which is constant equal to V_μ resp. V_{μ^*} on x_+ resp. x_- and is elsewhere equal to $\mathbb{V}_o$ (via the obvious identification defined by ν). Then the contractions $b_\mu : V_\mu \otimes V_{\mu^*} \to k$ define an isomorphism*

$$\oplus_{\mu\in P_\ell}\mathbb{H}_\ell(\tilde{\mathbb{V}}_{\mu,\mu^*})_{\tilde{\mathcal{C}}_o/S_o} \xrightarrow{\cong} \mathbb{H}_\ell(\mathbb{V}_o)_{\mathcal{C}_o/S_o}.$$

This is almost a formal consequence of:

Lemma 31. *Let* M *be a finite dimensional representation of* $\mathfrak{g}\times\mathfrak{g}$ *which is of level* $\le \ell$ *relative to both factors. If* M^δ *denotes that same space viewed as* $\mathfrak{g}$*-module with respect to the diagonal embedding* $\delta:\mathfrak{g}\to\mathfrak{g}\times\mathfrak{g}$*, then the contraction* $\oplus_{\mu\in P_\ell} M\otimes(V_\mu\boxtimes V_\mu^*)\to M$ *that on each summand is defined by* b_μ *(the symbol* $\boxtimes$ *stands for the exterior tensor product of representations) induces an isomorphism between covariants:*

$$\oplus_{\mu\in P_\ell}\big(M\otimes(V_\mu\boxtimes V_\mu^*)\big)_{\mathfrak{g}\times\mathfrak{g}}\xrightarrow{\cong} M^\delta_{\mathfrak{g}}.$$

Proof. Without loss of generality we may assume that M is irreducible, or more precisely, equal to $V_\lambda\boxtimes V_{\lambda'}$ for some $\lambda,\lambda'\in P_\ell$. Then $M^\delta=V_\lambda\otimes V_{\lambda'}$. By Schur's lemma, $M^\delta_{\mathfrak{g}}$ is one-dimensional if $\lambda'=\lambda^*$ and trivial otherwise. That same lemma applied to $\mathfrak{g}\times\mathfrak{g}$ shows that $(M\otimes(V_\mu\boxtimes V_\mu^*))_{\mathfrak{g}\times\mathfrak{g}}$ is zero unless $(\lambda,\lambda')=(\mu^*,\mu)$, in which case it is one-dimensional and maps isomorphically to M^δ. □

Proof of 30. Evaluation in x_0 resp. x_+,x_- define epimorphisms $A_o\to R_o$ resp. $\tilde A_o\to R_o\oplus R_o$ whose kernels may be identified by means of ν. We denote that common kernel by $\mathcal{I}$ and by B the algebra of regular functions on the smooth part of $\mathcal{C}_o^\circ$. This is also the algebra of regular functions on the complement of the two sections $x_\pm$ $\tilde{\mathcal{C}}_o^\circ$. If $\mathcal{I}\mathfrak{g}$ has the evident meaning, then the argument used to prove Proposition 19 shows that $M:=\mathbb{H}_\ell(\mathbb{V}_o|\Sigma_o)_{\mathcal{I}\mathfrak{g}}$ is an R_o-module of finite rank. It underlies a representation of $\mathfrak{g}\times\mathfrak{g}$ of level $\le\ell$ relative to both factors and is such that $M^\delta_{\mathfrak{g}}=\mathbb{H}_\ell(\mathbb{V}_o)_{A_o\mathfrak{g}}=\mathcal{H}_\ell(\mathbb{V}_o)_{\mathcal{C}_o/S_o}$. The assertion now follows from Lemma 31 and the argument used for the propagation principle which shows that $(M\otimes(V_\mu\boxtimes V_\mu^*))_{R_o\mathfrak{g}\times R_o\mathfrak{g}}=\mathbb{H}(\tilde{\mathbb{V}}_{\mu,\mu^*})_{B\mathfrak{g}}=\mathcal{H}_\ell(\tilde{\mathbb{V}}_{\mu,\mu^*})_{\tilde{\mathcal{C}}_o/S_o}$. □

A smoothing construction. In order to motivate the algebraic discussion that will follow, we choose generators $t_\pm$ of the ideals of the completed local R_o-algebras of $\tilde{\mathcal{C}}_o$ at $x_\pm$ and explain how they determine a *smoothing* of $\mathcal{C}_o/S_o$, that is, a way of making $\mathcal{C}_o$ the restriction over $S_o\times\{o\}$ of a flat morphism $\mathcal{C}\to S$, with $S:=S_o\times_k\Delta$ (the spectrum of $R:=R_o[[\tau]]$) which is smooth over $S-S_o$. The construction goes as follows: in the product $\tilde{\mathcal{C}}_o\times\Delta$, blow up $x_\pm\times\{o\}$ and let $\tilde{\mathcal{C}}$ be the formal neighborhood of the strict transform of $\tilde{\mathcal{C}}_o\times\{o\}$. So at the preimage of $x_\pm\times\{o\}$ we have on the strict transform of $\tilde{\mathcal{C}}\times\{o\}$ the formal S_o-chart $(t_\pm,\tau/t_\pm)$. Now let $\mathcal{C}$ be the quotient of $\tilde{\mathcal{C}}$ obtained by identifying these formal S_o-charts up to order: $(t_+,\tau/t_+)=(\tau/t_-,t_-)$, so that $(s_+,s_-):=(t_+,t_-)$ is now a formal S_o-chart of $\mathcal{C}$ on which we have $\tau=s_+s_-$ (in either domain τ represents the same regular function). We thus have defined a flat morphism $\mathcal{C}\to S_o\times\Delta=S$ (with τ as second component) with the stated properties.

Remark 32. If we were to work in the complex analytic category, then we could take for Δ the complex unit disk. The fiber of $\mathcal{C}/S$ over $(s,\tau)\in S_o\times\Delta$ is then obtained by removing from C_s the union of the two disks defined by

$|t_\pm| \le |\tau|$, followed by identification of the two closed annuli $|\tau| < |t_\pm| < 1$ by imposing the identity $t_+t_- = \tau$.

With a view toward a later application—namely, of extracting a topological quantum field theory from the WZW model—we note that there is even a limit if τ tends to zero if we keep its argument fixed. To see this, let us first observe that for $|\tau| < \frac{1}{2}$, the fiber is also obtained by removal of the union of the two open disks defined by $|t_\pm| < \sqrt{|\tau/2|}$, followed by the above identification of the two closed annuli $\sqrt{|\tau/2|} \le |t_\pm| \le \sqrt{|2\tau|}$. Now do a real oriented blow up $\hat{C}_s \to \tilde{C}_s$ of the points $x_\pm(s) \in \tilde{C}_s$. This means that the polar coordinates associated to $t_\pm$ are to be viewed as coordinates for the preimage of its domain on $\hat{C}_s$: $t_\pm = r_\pm\zeta_\pm$ with $|\zeta_\pm| = 1$ and $r_\pm \ge 0$ such that the exceptional set $\partial\hat{C}_s$ is defined by $r_\pm = 0$. Notice that $\partial\hat{C}_s$ is indeed the boundary of a surface; it has two components, each of which comes with a natural principal $U(1)$-action. If we write $\tau = \varepsilon\zeta$ accordingly with $|\zeta| = 1$ and $\varepsilon > 0$, then for $\sqrt{\varepsilon/2} \le r_\pm \le \sqrt{2\varepsilon}$, (r_+, ζ_+) must be identified with (r_-, ζ_-) precisely when $r_+r_- = \varepsilon$ and $\zeta_+\zeta_- = \zeta$. This has indeed a continuous extension over $\varepsilon = 0$, for then we just identify the two boundary circles corresponding to $r_\pm = 0$ by insisting that $\zeta_+\zeta_- = \zeta$. We thus obtain a family $\hat{\mathcal{C}} \to \hat{\Delta}$ over the real oriented blow up $\hat{\Delta} \to \Delta$ of Δ at its origin and whose fibers over $\partial\hat{\Delta}$ are as just described. The dependence of $\hat{\mathcal{C}}|\partial\hat{\Delta}$ is a priori on the coordinates $t_\pm$, but it is clear from the construction this dependence is in fact only via the (real) ray in $T_{x_+}\hat{C}_s \otimes T_{x_-}\hat{C}_s$ defined by $\frac{\partial}{\partial t_+}\big|_{x_+} \otimes_{\mathbb{C}} \frac{\partial}{\partial t_-}\big|_{x_-}$. The fibers of this family just differ by the way we identified the boundary circles and we thus see that the monodromy of the family is a positive Dehn twist defined by the welding circle. For later use we note that this construction takes place in the C^1-category: $\hat{\mathcal{C}}$ has a natural C^1-structure such that the projection to $\hat{\Delta}$ is C^1.

We should perhaps add that this has an algebro-geometric incarnation in terms of log structures and that $T_{x_+}\tilde{C}_s \otimes T_{x_-}\tilde{C}_s$ can be understood as the tangent space of the semi-universal deformation of the singular germ $(C_s, x(s))$ (equivalently, our data define a smooth point of the boundary divisor of some moduli stack $\overline{M}_{g,n}$ and $T_{x_+}\tilde{C}_s \otimes T_{x_-}\tilde{C}_s$ can be identified with its normal space).

We will denote by Σ the image of $\Sigma_o \times \Delta$ in both $\mathcal{C}$ and $\tilde{\mathcal{C}}$. In either case it is a union of sections over S. The representation valued map $\mathbb{V}_o$ on $\mathcal{C}_o$ is extended to $\mathcal{C}$ in the obvious way (so that its support is contained in Σ) and we denote this extension by $\mathbb{V}$. We let A stand for R-algebra of regular functions on $\mathcal{C}^o := \mathcal{C} - \Sigma$. Notice that $A_o = A/(\tau A)$ and that A embeds in $\tilde{A}_o[[\tau]]$.

The glueing tensor. Suppose that in the regular local algebra R we are given a subalgebra R_o and an element τ in the maximal ideal of R such that $R = R_o[[\tau]]$. Let L_+ and L_- be R-algebras, both isomorphic to $R((t))$. The 'ideal' in $L_\pm$ corresponding to $tR[[t]]$ is denoted by $\mathfrak{m}_\pm$. Let $L := L_+ \oplus L_-$ the direct sum as R-algebras. We assume given a closed R-subalgebra $\mathcal{O}_0 \subset L$ with the property that it can be topologically generated as a R_o-algebra by two generators s_+, s_- of the following type: there exist generators $t_\pm$ of $\mathfrak{m}_\pm$ such that $s_+ = (t_+, \tau/t_-)$ and $s_- = (\tau/t_+, t_-)$. So an element of $\mathcal{O}_0$ will then have the form

$$\sum_{m\geq 0, n\geq 0} a_{m,n} s_+^m s_-^n = \sum_{m\geq 0, n\geq 0} a_{m,n}(t_+^{m-n}\tau^n, t_-^{n-m}\tau^m) =$$
$$= \sum_{k\geq 0}\Big(\sum_{m\geq 0} a_{m,k} t_+^{m-k}, \sum_{n\geq 0} a_{k,n} t_-^{n-k}\Big)\tau^k =$$
$$= \sum_{n>m\geq 0} a_{n,m}\tau^n s_+^{m-n} + \sum_{m\geq 0} a_{m,m}\tau^m + \sum_{m>n\geq 0} a_{n,m}\tau^m s_-^{n-m},$$

with $a_{n,m} \in R_o$. Clearly, the coefficients $a_{m,n}$ can be arbitrary in R_o and the element in question is zero only when all $a_{m,n}$ are. So $\mathcal{O}_0$ is a copy of $R_o[[s_+, s_-]]$. The last identity shows that $\mathcal{O}_0$ is contained in the R-submodule generated by nonpositive powers of s_+ and s_-. A similar argument yields the following lemma and so the proof is left as an exercise.

Lemma 33. *Any continuous* R_o*-derivation of* $\mathcal{O}_0$ *which preserves* $\tau \in \mathcal{O}_0$ *extends uniquely to one of* L. *If we let* $D_k^\pm$ *stand for* $t_\pm^{k+1}\frac{\partial}{\partial t_\pm}$, *then it has there the form*

$$(D_0^+, 0) + \sum_{k\geq 0}\tau^k\Big(\sum_{m\geq 0} a_{m,k} D_{m-k}^+, \sum_{n\geq 0} a_{k,n} D_{n-k}^-\Big),$$

with $a_{m,n} \in R_o$.

We have defined $L\mathfrak{g}$ and its central extension $\widehat{L\mathfrak{g}}$. For $\mu \in P_\ell$, let $\mathbb{H}_\ell^\pm(V_\mu)$ denote the representation attached to V_μ of the central extension $\widehat{L_\pm\mathfrak{g}}$ of $L_\pm\mathfrak{g}$, so that the R-module $\mathbb{H}_\ell^+(V_\mu)\otimes_R \mathbb{H}_\ell^-(V_{\mu^*})$ is one of $\widehat{L\mathfrak{g}}$. These representations are defined over R_o (over k even) and so arise from a base change: $\mathbb{H}_\ell^\pm(V_\mu) = R\otimes_{R_o}\mathbb{H}_{o,\ell}^\pm(V_\mu)$ and likewise for their tensor product. The Casimir element c acts in V_μ as a scalar, a scalar we shall denote by c_μ. Observe that $c_{\mu^*} = c_\mu$. Its value is best expressed (and computed) in terms of a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and a system of positive roots relative to $\mathfrak{h}$: if we identify μ with its highest weight in $\mathfrak{h}^*$, then

$$c_\mu = c(\mu, \mu + 2\rho),$$

where ρ has the customary meaning as the half the sum of the positive roots. In particular, c_μ is a positive rational number (the denominator is in fact at most 3).

Lemma 34. *There exists a series* $\varepsilon^\mu = \sum_{d=0}^\infty \varepsilon_d^\mu \tau^d \in \mathbb{H}_\ell^+(V_\mu)\otimes_{R_o}\mathbb{H}_\ell^-(V_{\mu^*})[[\tau]]$ *(the* glueing tensor*) with constant term* $\varepsilon_0^\mu = \check{b}_\mu$ *that is annihilated by the image of* $\mathcal{O}_0\mathfrak{g}$ *in* $\widehat{L\mathfrak{g}}$. *Moreover, any continuous* R*-derivation* D *of* $\mathcal{O}_0$ *which preserves* τ *determines a* $\hat{D} \in \hat{\theta}$ *(relative to the Fock construction on the*

R-*algebra* L*) with the property that* ε^μ *is an eigenvector of* $\mathsf{T}_\mathfrak{g}(\hat{\mathsf{D}})$ *with eigenvalue* $-\frac{c_\mu}{2(\ell+\check{h})}$.

Proof. We first observe the generators $t_\pm$ of $\mathfrak{m}_\pm$ define a grading on all the relevant objects on which we have defined the associated filtration F (e.g., the degree zero summand of $\mathbb{H}_\ell(V_\mu)$ is $\mathsf{R}\otimes_k V_\mu$). It is known ([9], § 9.4) that the pairing $b_\mu : V_\mu \times V_{\mu^*} \to k$ extends (in fact, in a unique manner) to a perfect R-pairing

$$b_\mu : \mathbb{H}_\ell^+(V_\mu) \times \mathbb{H}_\ell^-(V_{\mu^*}) \to \mathsf{R}$$

with the property that $b_\mu(Xt_+^n u, u') + b_\mu(u, Xt_-^{-n}u') = 0$ for all $X \in \mathfrak{g}$ and $n \in \mathbb{Z}$. This formula implies that the restriction of b_μ to $\mathbb{H}_\ell^+(V_\mu)_{-d} \times \mathbb{H}_\ell^-(V_{\mu^*})_{-d'}$ is zero when $d \neq d'$ and is perfect when $d = d'$. So if $\varepsilon_d^\mu \in \mathbb{H}_\ell^+(V_\mu)_{-d} \otimes \mathbb{H}_\ell^-(V_{\mu^*})_{-d}$ denotes the latter's transpose inverse, then we have for all $n \in \mathbb{Z}$, $X \in \mathfrak{g}$ the following identity in $\mathbb{H}_\ell^+(V_\mu)_d \times \mathbb{H}_\ell^-(V_{\mu^*})_{-d-n}$:

$$(Xt_+^n \otimes 1)\varepsilon_{d+n}^\mu + (1 \otimes Xt_-^{-n})\varepsilon_d^\mu = 0.$$

This just says that $(Xt_+^n \otimes 1) + \tau^n(1 \otimes Xt_-^{-n})$ kills $\varepsilon^\mu := \sum_{d\geq 0} \varepsilon_d^\mu \tau^d$. Since $s_+^n = (t_+^n, \tau^n t_-^{-n})$, this amounts to saying that $Xs_+^n \in \mathcal{O}_0\mathfrak{g} \subset \widehat{L\mathfrak{g}}$ kills ε^μ. Likewise for Xs_-^n. Since any element of $\mathcal{O}_0$ lies in the R-submodule generated by the nonpositive powers of s_+ and s_-, it follows that ε^μ is killed by all of $\mathcal{O}_0\mathfrak{g}$.

The second statement is proved by a direct computation. If we use Lemma 33 to write D as an operator in L, then we find that it suffices to prove:

(i) $\tau^n \mathsf{T}_\mathfrak{g}(\hat{\mathsf{D}}_{m-n}^+) - \tau^m \mathsf{T}_\mathfrak{g}(\hat{\mathsf{D}}_{n-m}^-)$ kills ε^μ for all $m, n \geq 0$, and
(ii) $\mathsf{T}_\mathfrak{g}(\hat{\mathsf{D}}_0^+)(\varepsilon^\mu) = -\frac{c_\mu}{2(\ell+\check{h})}\varepsilon^\mu$.

As to (i), if we substitute

$$\mathsf{T}_\mathfrak{g}(\hat{\mathsf{D}}_{m-n}^+) = -\frac{1}{2(\ell+\check{h})}\sum_{j\in\mathbb{Z}}\sum_\kappa : X_\kappa t_+^{m-n-j} \circ X_\kappa t_+^j :$$

and do likewise for $\mathsf{T}_\mathfrak{g}(\hat{\mathsf{D}}_{n-m}^-)$, then this assertion follows easily.

For (ii) we first observe that $\mathsf{T}_\mathfrak{g}(\hat{\mathsf{D}}_0^+)$ preserves the grading of $\mathbb{H}_\ell^+(V_\mu)$ and acts on $\mathbb{H}_\ell^+(V_\mu)_0 = \mathsf{R}\otimes_k V_\mu$ as $-(2\ell+2\check{h})^{-1}\sum_\kappa X_\kappa \circ X_\kappa$. This is just multiplication by $-\frac{c_\mu}{2(\ell+\check{h})}$. For an element $u \in \mathbb{H}_\ell^+(V_\mu)_{-d}$ of the form $u = Y_r t_+^{-k_r} \circ \cdots \circ Y_1 t_+^{-k_1} \circ v$ with $v \in V_\mu$, and $Y_\rho \in \mathfrak{g}$ (so that $d = k_r + \cdots + k_1$), we have

$$\mathsf{T}_\mathfrak{g}(\hat{\mathsf{D}}_0^+)(u) = -du + Y_r t_+^{-k_r} \circ \cdots \circ Y_1 t_+^{-k_1} \circ \mathsf{T}_\mathfrak{g}(\hat{\mathsf{D}}_0^+)(v) = (-d - \frac{c_\mu}{2(\ell+\check{h})})u.$$

Since $\mathsf{D}_0^+(\tau^d) = d\tau^d$, it follows that $\varepsilon_d^\mu\tau^d$ is an eigenvector of $\mathsf{T}_\mathfrak{g}(\hat{\mathsf{D}}_0^+)$ with eigenvalue $-\frac{c_\mu}{2(\ell+\check{h})}$. □

Finer version of the factorization property. It is clear that our smoothing identifies the R-module $\mathbb{H}_\ell(\mathbb{V})$ with $\mathbb{H}_\ell(\mathbb{V}_o)[[\tau]]$. According to Proposition 19, $\mathcal{H}_\ell(\mathbb{V})_{\mathcal{C}/S} = \mathbb{H}_\ell(\mathbb{V})_{A\mathfrak{g}}$ is a finitely generated R-module. Since $A_o = A/\tau A$, the reduction of $\mathbb{H}_\ell(\mathbb{V})_{A\mathfrak{g}}$ modulo τ yields $\mathbb{H}_\ell(\mathbb{V}_o)_{A_o\mathfrak{g}} = \mathcal{H}_\ell(\mathbb{V}_o)_{\mathcal{C}_o/S_o}$. Proposition 30 identifies the latter with $\oplus_{\mu\in P_\ell}\mathcal{H}_\ell(\tilde{\mathbb{V}}_{\mu,\mu^*})_{\tilde{\mathcal{C}}_o/S_o}$. It is our goal to extend this identification to one of the space of covacua $\mathcal{H}_\ell(\mathbb{V})_{\mathcal{C}/S}$ with the pull-back of $\oplus_{\mu\in P_\ell}\mathcal{H}_\ell(\tilde{\mathbb{V}}_{\mu,\mu^*})_{\mathcal{C}_o/S_o}$ along the projection $\pi_{S_o} : S \to S_o$ and to identify the connection on that pull-back. This will imply among other things that $\mathcal{H}_\ell(\mathbb{V})_{\mathcal{C}/S}$ is a free R-module.

Theorem 35. *The* R*-homomorphism defined by tensoring with the glueing tensor,*

$$E = (E_\mu)_\mu : \mathbb{H}_\ell(\mathbb{V}) \to \oplus_{\mu\in P_l}\mathbb{H}_\ell(\tilde{\mathbb{V}}_{\mu,\mu^*})[[\tau]],$$
$$U = \sum_{k\geq 0} u_k\tau^k \mapsto \Big(U\hat{\otimes}_R\varepsilon^\mu = \sum_{k,d\geq 0} u_k\otimes\varepsilon^\mu_d\tau^{k+d}\Big)_\mu,$$

is also a map of $A\mathfrak{g}$*-representations if we let* $A\mathfrak{g}$ *act on the right hand side via the inclusion* $A\subset\tilde{A}_o[[\tau]]$*. The resulting* R*-homomorphism of covariants,*

$$E_{\mathcal{C}/S} : \mathbb{H}_\ell(\mathbb{V})_{A\mathfrak{g}} \to \oplus_{\mu\in P_l}\mathbb{H}_\ell(\tilde{\mathbb{V}}_{\mu,\mu^*})_{\tilde{A}_o\mathfrak{g}}[[\tau]],$$

is an isomorphism (so that $\mathbb{H}_\ell(\mathbb{V})_{A\mathfrak{g}}$ *is a free* R*-module). It is compatible with covariant differentiation with respect to* $\theta_S(\log S_o) = R[[\tau]]\otimes_{R_o}\theta_{R_o} + R[[\tau]]\tau\frac{d}{d\tau}$ *relative to the lift to* $\hat{\theta}_S(\log S_o)$ *of Lemma 34: it commutes with the action on* θ_{R_o}*, whereas* $\tau\frac{d}{d\tau}$ *respects each summand* $H_\ell(\tilde{\mathbb{V}}_{\mu,\mu^*})_{\tilde{A}\mathfrak{g}}[[\tau]]$ *and acts there as the first order differential operator* $\tau\frac{d}{d\tau} + \frac{c_\mu}{2(\ell+\check{h})}$.

Proof. The first statement is immediate from Lemma 34. So the map on covariants is defined and is R-linear. If we reduce $E_{\mathcal{C}/S}$ modulo τ, we get the map

$$\mathbb{H}_\ell(\mathbb{V}_o)_{A_o\mathfrak{g}} \to \oplus_{\mu\in P_l}\mathbb{H}_\ell(\tilde{\mathbb{V}}_{\mu,\mu^*})_{\tilde{A}_o\mathfrak{g}}, \quad u\mapsto\sum_{\mu\in P_\ell} u\otimes\varepsilon^\mu_0,$$

and observe that this is just the inverse of the isomorphism of Proposition 30. Since the range of $E_{\mathcal{C}/S}$ is a free R-module, this implies that $E_{\mathcal{C}/S}$ is an isomorphism.

The commutativity with the action of θ_{R_o} is clear. According to Corollary 13 covariant derivation with respect to $\tau\frac{d}{d\tau}$ in $\mathbb{H}_\ell(\mathbb{V})_{\mathcal{C}/S}$ is defined by means of a k-derivation D of A which lifts $\tau\frac{d}{d\tau}$: if we write $D = \tau\frac{d}{d\tau} + \sum_{n\geq 0}\tau^n D^{(n)}$, where $D^{(n)}$ is a vector field on the smooth part of $\mathcal{C}/S$, then the covariant derivative is induced by $T_\mathfrak{g}(\hat{D}) = \tau\frac{d}{d\tau} + \sum_{n\geq 0}\tau^n T_\mathfrak{g}(D^{(n)})$ acting on $\mathbb{H}_\ell(\mathbb{V}_o)[[\tau]]$. From the last clause of Lemma 34 we get that when $U\in\mathbb{H}_\ell(\mathbb{V}_o)[[\tau]]$,

$$\begin{aligned} &T_\mathfrak{g}(D)E_\mu(U) = T_\mathfrak{g}(D)(U\varepsilon^\mu) = \\ &\qquad = T_\mathfrak{g}(D)(U)\varepsilon^\mu - \frac{c_\mu}{2(\ell+\check{h})}U\varepsilon^\mu = E_\mu T_\mathfrak{g}(D)(U) - \frac{c_\mu}{2(\ell+\check{h})}E_\mu(U). \end{aligned}$$

Since $T_{\mathfrak{g}}(D)$ acts on $\mathbb{H}_\ell(\tilde{\mathbb{V}}_{\mu,\mu^*})_{\tilde{A}_o\mathfrak{g}}[[\tau]]$ as derivation by $\tau\frac{d}{d\tau}$, the last clause follows. □

Corollary 36. *The monodromy of the WZW connection acting on* $\mathcal{H}_\ell(\mathbb{V})_{\mathcal{C}/S}$ *has finite order and acts in the summand* $\mathcal{H}_\ell(\tilde{\mathbb{V}}_{\mu,\mu^*})_{\tilde{C}/S_o}[[\tau]]$ *as multiplication by the root of unity* $\exp(-\pi\sqrt{-1}\frac{c_\mu}{\ell+\check{h}})$.

Proof. The multivalued flat sections of $\mathcal{H}_\ell(\mathbb{V})_{\mathcal{C}/\Delta}$ decompose under $E_{\mathcal{C}/\Delta}$ as a direct sum labeled by P_ℓ. The summand corresponding to μ is the set of solutions of the differential equation $\tau\frac{d}{d\tau}U + \frac{c_\mu}{2(\ell+\check{h})}U = 0$. These are clearly of the form $u\tau^{-c_\mu/2(\ell+\check{h})}$ with $u \in \mathbb{H}_\ell(\tilde{\mathbb{V}}_{\mu,\mu^*})_{\tilde{A}_o\mathfrak{g}}$. If we let τ run over the unit circle, then we see that the monodromy is as asserted. Since $\frac{c_\mu}{\ell+\check{h}} \in \mathbb{Q}$, it has finite order. □

Remark 37. We use here the convention that the monodromy of the multivalued function z^α is $\exp(2\pi\alpha\sqrt{-1})$ (rather than $\exp(-2\pi\alpha\sqrt{-1})$). More pedantically: for us the monodromy is a *covariant* rather than a contravariant functor from the fundamental groupoid to a linear category.

7. The modular functor attached to the WZW model

We show here that the results of Section 6 lead to topological counterparts that take the form of (what is called) a modular functor in topological quantum field theory.

Defining the functor. For what follows, the most natural setting would probably be that of quasi-conformal surfaces, but we have chosen to work with the more familiar notion of C^1-surfaces. This forced us however to introduce the auxiliary notion of an infinitesimal collar below.

The main objects will be *compact oriented* surfaces endowed with a C^1-structure, possibly with boundary, but where we assume that each boundary component comes with a principal action of the unit circle $U(1)$ that is compatible with the orientation it receives from the surface. In the rest of this paper, we will simply refer to such an object as a *surface*.

An *infinitesimal collar* of a surface is a inward pointing (nowhere zero) vector field defined on the boundary only with the property that it is locally trivial in the sense that we can find local C^1-diffeomorphism (r,u) of a neighborhood of the boundary onto $[0,\varepsilon)\times U(1)$ which is compatible with the $U(1)$-action on the boundary and takes the vector field to $\partial/\partial r|_{\{0\}\times U(1)}$. The choice of such a vector field determines a basis for each tangent space (the second tangent vector field being the derivative of the $U(1)$-action) and so we may think of this as a first order extension of the given $U(1)$ action. Suppose given such an infinitesimally collared surface Σ and two of its boundary components B_+, B_-. Let us call a *glueing map* for this pair an *anti-isomorphism* $\phi : B_- \to B_+$, that is, a C^1-diffeomorphism with the property that $\phi(ub) = u^{-1}\phi(b)$ for all $b \in B_-$ and $u \in U(1)$. We call it thus, because if we use it to identify B_- with B_+, then we get a new (infinitesimally collared) surface Σ_ϕ without the need of making any further choices: the C^1-structure must be such that the normal vector fields become each others antipode. Similarly, the topological quotient $\check{\Sigma}$ of Σ obtained by contracting each of its boundary components also acquires a C^1-structure: a function on $\check{\Sigma}$ is differentiable precisely when its lift to Σ is C^1 and is such that its derivative evaluated on the infinitesimal collar of a boundary component is the representation of a linear map in polar coordinates.

Definition 38. We call a conformal structure on the interior of the infinitesimally collared surface Σ *admissible* if it is compatible with the given C^1-structure as well as with the infinitesimal collaring: for every boundary component either the conformal structure extends to the boundary or extends across its image in $\check{\Sigma}$ and we demand that in the first case the infinitesimal collaring be perpendicular to the boundary, and that in the second (cuspidal) case it maps to a $U(1)$-orbit in the tangent space.

This somewhat unconventional definition is in part motivated by the following observation. A conformal structure on a manifold is just a Riemann metric given up to multiplication by a continuous function. More precisely, it is a section of the bundle of positive quadratic forms modulo positive scalars on the tangent bundle. As the fibers of this bundle have a convex structure, so has its space of sections. This also holds in the present case with the given boundary conditions, in particular the space of admissible conformal structures is contractible. And this is still true if we restrict ourselves to the admissible conformal structures that are cuspidal at a prescribed union of boundary components. This makes it a tractable notion from the point of view of homotopy.

Definition 39. A $\mathfrak{g}$-*marking* of a surface Σ consists of giving a map V that assigns to every boundary component of Σ a finite dimensional irreducible represention of $\mathfrak{g}$. We then denote the resulting set of data by (Σ, V). We say that the $\mathfrak{g}$-marking is of level $\leq \ell$ if V takes values in representations of level $\leq \ell$.

Let (Σ, V) be $\mathfrak{g}$-marked surface. We first suppose Σ endowed with an infinitesimal collaring. Choose an admissible purely cuspidal conformal structure C with respect to this infinitesimal collaring. Then $\check{\Sigma}$ acquires a conformal structure and hence (since $\check{\Sigma}$ is oriented) the structure of a compact Riemann surface, or equivalently, a nonsingular complex projective curve. We hope the reader forgives us for denoting that curve by C as well. It comes with an injection $\pi_0(\partial\Sigma) \to C$. If V takes values in representations of level $\leq \ell$, then we have defined the space of covacua $\mathbb{H}_\ell(\mathbb{V})_C$; otherwise we set $\mathbb{H}_\ell(\mathbb{V})_C = 0$. For another choice of purely cuspidal admissible conformal structure C', we can find a path of such structures $(C_t)_{0\leq t\leq 1}$ connecting C

with C'. The projectively flat connection can be used to identify the corresponding projective spaces, and this identification is independent of the choice of path since they belong to the same homotopy class.

In order to lift this to the actual vector spaces, we need a 'rigging' of Σ as follows. Put $g := \dim H_1(\check{\Sigma};\mathbb{R})$ and denote by $\mathcal{L}(\Sigma) \subset \wedge^g H_1(\check{\Sigma};\mathbb{R})$ the set of $I \in \wedge^g H_1(\check{\Sigma};\mathbb{R})$ for which $L(I) := \ker(\wedge I : H_1(\check{\Sigma};\mathbb{R}) \to \wedge^{g+1} H_1(\check{\Sigma};\mathbb{R}))$ is a Lagrangian subspace (so that I is a generator of $\wedge^g L(I)$). Let us first assume that Σ is connected. It is known that if $g > 0$, then $\mathcal{L}(\Sigma)$ is connected, has infinite cyclic fundamental group with a canonical generator and is an orbit of the symplectic group $\mathrm{Sp}(H_1(\check{\Sigma};\mathbb{R}))$. For example, if $g = 1$, then $\mathcal{L}(\Sigma) = H_1(\check{\Sigma};\mathbb{R}) - \{0\} \cong \mathbb{R}^2 - \{0\}$. (If $g = 0$, then $\wedge^g H_1(\check{\Sigma};\mathbb{R}) = \mathbb{R}$ and so $\mathcal{L}(\Sigma)$ is canonically identified with $\mathbb{R} - \{0\}$.) An element of $\mathcal{L}(\Sigma)$ may arise if $\check{\Sigma}$ is given as the boundary of a compact oriented 3-manifold W: then the kernel of $H_1(\check{\Sigma};\mathbb{R}) \to H_1(W;\mathbb{R})$ is a Lagrangian sublattice and so an orientation of it yields an element of $\mathcal{L}(\Sigma)$.

Let $I \in \mathcal{L}(\Sigma)$. We note that every regular differential on C defines by integration a linear map $L(I) \to \mathbb{C}$ and the basic theory or Riemann surfaces tells us that we thus obtain a complex-linear isomorphism $H^0(C,\omega_C) \cong \mathrm{Hom}(L(I),\mathbb{C})$. Since this induces an isomorphism between $\det H^0(C,\omega_C)$ and $\mathrm{Hom}_{\mathbb{R}}(\det_{\mathbb{R}} L(I),\mathbb{C})$, the linear form that takes the value 1 in $I \in \det_{\mathbb{R}} L(I)$ yields a generator $I(C)$ of $\det H^0(C,\omega_C)$. For similar reasons, the arc $(C_t)_{0\le t\le 1}$ lifts to a section $t \in [0,1] \mapsto I(C_t) \in \det H^0(C_t,\omega_{C_t})$ of the determinant bundle and this in turn yields via Theorem 27 an identification of $\mathbb{H}_\ell(\mathbb{V})_C$ with $\mathbb{H}_\ell(\mathbb{V})_{C'}$. As this identification is canonical, we now have attached to the triple (Σ, V, I) and the infinitesimal collaring of Σ a well-defined finite dimensional complex vector space $H_\ell(\Sigma, V, I)$. Actually, the infinitesimal collaring is irrelevant, for the infinitesimal collarings make up an affine space over the vector space of vector fields on $\partial\Sigma$ and hence form a contractible set.

For $g = 0$, we shall always take for $I \in \mathcal{L}(\Sigma)$ the canonical element that corresponds to 1 under the identification $\mathcal{L}(\Sigma) \cong \mathbb{R} - \{0\}$ so that we then have a well-defined vector space $H_\ell(\Sigma, V)$. Proposition 28 tells us what we get in some simple cases:

Proposition 40. *For Σ a disk (resp. a cylinder), $H_\ell(\Sigma, V)$ is zero unless V is the trivial representation (resp. the two representations attached to the boundary are each other's contra-gradient), in which case it is canonically equal to $\mathbb{C}$.*

Now drop the assumption that Σ be connected and let $\Sigma_1, \dots, \Sigma_r$ enumerate its distinct connected components. If $I \in \mathcal{L}(\Sigma)$ corresponds to $I_1 \otimes_{\mathbb{R}} \cdots \otimes_{\mathbb{R}} I_r$ with $I_k \in \mathcal{L}(\Sigma_k)$, then the tensor product $H_\ell(\Sigma_1, V_1, I_1) \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} H_\ell(\Sigma_r, V_r, I_r)$ only depends on $I_1 \otimes_{\mathbb{R}} \cdots \otimes_{\mathbb{R}} I_r$ and so the preceding generalizes if we let $H_\ell(\Sigma, V, I)$ be this tensor product. We thus find:

Theorem 41. *Let (Σ, V) be a $\mathfrak{g}$-marked surface of level $\le \ell$. Then we have naturally defined on the Lagrangian manifold $\mathcal{L}(\Sigma)$ a local system $\mathbb{H}_\ell(\Sigma, V)$*

whose stalk at $I \in \mathcal{L}(\Sigma)$ *is* $H_\ell(\Sigma, V, I)$. *This construction is functorial with respect to automorphisms of* $\mathfrak{g}$ *so that for every* $\sigma \in \mathrm{Aut}(\mathfrak{g})$ *we have a natural isomorphism* $\mathbb{H}_\ell(\Sigma, {}^\sigma V) \cong \mathbb{H}_\ell(\Sigma, V)$.

Proof. The last assertion follows from the last clause of Theorem 27, the rest is clear from the preceding discussion. □

Remark 42. The natural involution of $\mathfrak{g}$ with respect to a choice of root data takes every finite dimensional $\mathfrak{g}$-representation into one equivalent to its contra-gradient. So for such an involution σ we obtain an isomorphism between $H_\ell(\Sigma, V^*, I)$ and $H_\ell(\Sigma, V, I)$, but beware that this involution is only unique up to inner automorphism. However, one expects that there exists a canonical perfect pairing (which therefore does not involve a choice of σ) $H_\ell(\overline{\Sigma}, V^*, I) \otimes H_\ell(\Sigma, V, I) \to \mathbb{C}$, where $\overline{\Sigma}$ stands for Σ with the opposite orientation.

Action of the centrally extended mapping class group. Let $\Gamma(\Sigma)$ denote the group of orientation preserving isotopies of Σ which leave each of its components and each boundary component invariant (but not necessarily point-wise). This is isomorphic to the usual mapping class group of the pair consisting of $\check{\Sigma}$ and the finite subset of $\check{\Sigma}$ that appears as the image of $\pi_0(\partial\Sigma)$. The above lemma shows that if (Σ, V) is a $\mathfrak{g}$-marked surface, then for every $I \in \mathcal{L}(\Sigma)$, the mapping class $\phi \in \Gamma(\Sigma)$ gives rise an isomorphism $\phi_\# :$ $H_\ell(\Sigma, V, I) \to H_\ell(\Sigma, V, \phi_* I)$. In other words, ϕ induces an automorphism of the local system $\mathbb{H}_\ell(\Sigma, V)$.

Assume Σ connected and $g > 0$ so that $\mathcal{L}(\Sigma)$ has infinite cyclic fundamental group. Fix a universal cover $\tilde{\mathcal{L}}(\Sigma) \to \mathcal{L}(\Sigma)$ and denote by $\tilde{H}_\ell(\Sigma, V)$ the space of sections of the pull-back of $\mathbb{H}(\Sigma, V)$ to this cover. The pairs $(\phi, \tilde{\phi}_\#)$ with ϕ a mapping class and $\tilde{\phi}_\# \in \mathrm{Aut}(\tilde{\mathcal{L}}(\Sigma))$ a lift of $\phi_\#$ define a central extension $\tilde{\Gamma}(\Sigma) \to \Gamma(\Sigma)$ of the mapping class group by $\mathbb{Z}$. We have arranged things in such a manner that this extension acts on $\tilde{H}_\ell(\Sigma, V)$ with the central element $2(\ell + \check{h}) \in \mathbb{Z}$ acting trivially. The central extension is clearly one that already lives on the automorphism group of $H_1(\check{\Sigma})$ (an integral symplectic group of genus g). The latter is known to produce the universal central extension of the symplectic group. It has an abstract description in terms of a 2-cocyle, known as the Maslov index.

The glueing property. Let B_+ and B_- be distinct boundary components of Σ and $\phi : B_+ \to B_-$ a glueing map so that we have an associated quotient surface Σ_ϕ. We show that there is a natural embedding of $\mathcal{L}(\Sigma_\phi)$ in $\mathcal{L}(\Sigma)$. If B_+ and B_- lie on distinct components, then we have a natural identification between the symplectic lattices $H_1(\check{\Sigma})$ and $H_1((\Sigma_\phi)\check{})$ and so we also have a natural identification of $\mathcal{L}(\Sigma)$ with $\mathcal{L}(\Sigma_\phi)$. If B_+ and B_- lie on the same component, then their common image B in Σ_ϕ has the property that the image $\langle B \rangle$ of $H_1(B) \to H_1((\Sigma_\phi)\check{})$ is a primitive rank one sublattice. If we denote by $\langle B \rangle^\perp$ the annihilator of $\langle B \rangle$ with respect to the intersection pairing, then $\langle B\rangle \subset \langle B\rangle^\perp$ and the symplectic lattice $\langle B\rangle^\perp/\langle B\rangle$ is naturally identified with $H_1(\check{\Sigma})$. Since the intersection pairing identifies $H_1((\Sigma_\phi)\check{})/\langle B\rangle^\perp$ with the dual of $\langle B\rangle$, we see that we have a natural embedding of $\mathcal{L}(\Sigma_\phi)$ in $\mathcal{L}(\Sigma)$.

If we now combine the discussion in Remark 32 with Theorem 35, we obtain

Theorem 43 (Glueing property)**.** *Suppose we have endowed* Σ_ϕ *with a* $\mathfrak{g}$*-marking, i.e., a map* $V : \pi_0(\partial\Sigma_\phi) = \pi_0(\partial\Sigma) - \{\{B_+\},\{B_-\}\} \to P_\ell$*. For* $\mu \in P_\ell$*, denote by* $V_{\mu,\mu^*} : \pi_0(\partial\Sigma) \to P_\ell$ *the extension of* V *to a* $\mathfrak{g}$*-marking of* Σ *which assigns to* B_+ *resp.* B_- *the value* λ *resp.* λ^**. Then the local system* $\mathbb{H}_\ell(\Sigma_\phi, V)$ *on* $\mathcal{L}(\Sigma_\phi)$ *can be naturally identified with the restriction of* $\oplus_{\lambda\in P_\ell}\mathbb{H}_\ell(\Sigma, V_{\mu,\mu^*})$ *with respect to the embedding of* $\mathcal{L}(\Sigma_\phi)$ *in* $\mathcal{L}(\Sigma)$ *defined above. Under this identification, the mapping class of* Σ_ϕ *obtained by the glueing maps* $\{\zeta\phi\}_{\zeta\in U(1)}$ *(a Dehn twist) acts on the summand* $H_\ell(\Sigma, V_{\mu,\mu^*})$ *as scalar multiplication by* $\exp\left(-\frac{\pi\sqrt{-1}c_\mu}{\ell+\check{h}}\right)$.

By repeated application of Theorem 43 in the opposite direction we can thus completely recover $\mathbb{H}_\ell(\Sigma, V)$ from a pair of pants decomposition of Σ: such a decomposition has a 3-holed sphere (a pair of pants) as its basic building block, a case that is taken care of by Remark 29. In particular we obtain a formula, at least in principle, for its dimension, known as the *Verlinde formula*. This process is nicely formalized by the notion of a fusion ring (see [4]). But if we wish to deal with the modular functor itself, then we are led to the representation theory of quantum groups. As we mentioned in the introduction, this has applications in knot theory via a threedimensional topological quantum field theory. For most of this we refer to the monograph of Turaev [21].

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