Hyperbolic Space for Tourists

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Synopsis

We discuss how a creature accustomed to Euclidean space would fare in a world of hyperbolic or spherical geometry, and conversely. Various optical illusions and counterintuitive experiences arise, which can be explicated mathematically using plane models of these geometries.

What would we experience if we entered a hyperbolic universe? Hermann von Helmholtz, in his essay "The origin and meaning of geometrical axioms," offers the following answer:

> We can ... infer how the objects in a pseudospherical [i.e., hyperbolic] world, were it possible to enter one, would appear to an observer whose eye-measure and experiences of space had been gained like ours in Euclid's space. Such an observer would continue to look upon rays of light or the lines of vision as straight lines, such as are met with in flat space and as they really are in the spherical representation of pseudospherical geometry [i.e., the three-dimensional version of the projective disk model; see Figure 1]. The visual image of the objects in pseudospherical space would thus make the same impression upon him as if he were at the center of Beltrami's sphere. He would think he saw the most remote objects round about him at a finite distance, let us suppose a hundred feet off. But as he approached these distant objects, they would dilate before him ... while behind him they would contract. He would know that his eye judged wrongly. If he saw two straight lines that in his estimate ran parallel to his world's end, he would find on following them that the farther he advanced the more they diverged. [1, pages 316–317]

We intend to elaborate on these brief remarks made by Helmholtz. Helmholtz's basic assumption is that a creature entering a different world will distort the world he sees by embedding it in his own geometry in a geodesic-preserving manner. In doing so, the creature is acting normally; he interprets the world using the same geometry that he has always used, and he translates visual impressions into this geometry using the assumption that light travels in straight lines, as he has always done. So for example if a Euclidean creature entered a hyperbolic world he would take his visual impressions and force them into his Euclidean mind, i.e., he would represent hyperbolic geodesics in his mind as straight in the sense of Euclidean geometry, with other geometrical properties being distorted accordingly. Indeed, there is a famous model of hyperbolic geometry that represents hyperbolic geodesic as Euclidean lines: the projective disk model illustrated in Figure 1. Although straightness is preserved the model distorts sizes; all the tiles are of equal hyperbolic size.
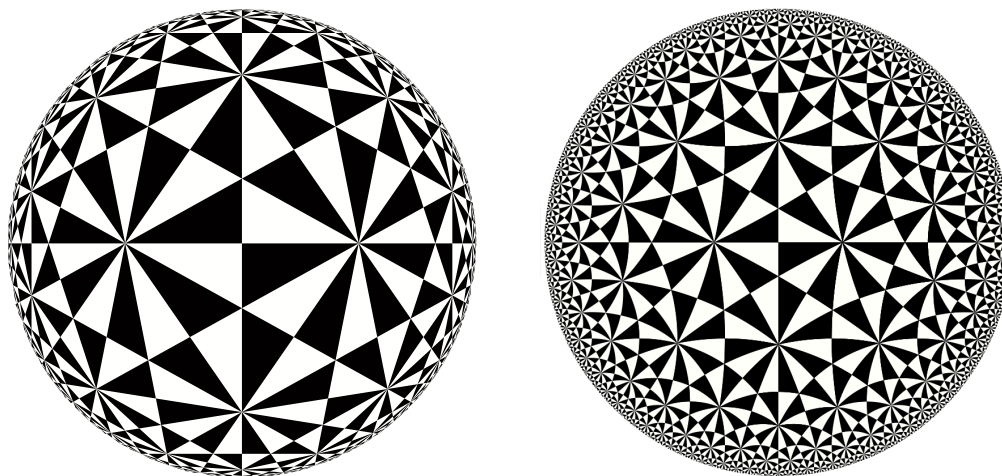

Figure 1: The projective disk model (left) and the conformal disk model (right) of hyperbolic geometry. (Figures generated in *Mathematica* using code in [2], chapter 22.)

With this figure in mind it is easy to confirm Helmholtz's account. We should imagine ourselves standing in the center of the disk, and as we walk we should picture ourselves remaining in the center while the ground moves underneath us like a treadmill. It will look to us as if a given tile in the distance is a fraction of the size of the tiles closest to us, but as we walk up to it this distant tile will come to occupy the central part of the figure so

it will appear as large as the ones we used to stand on, while those in turn would appear to shrink in size behind our backs. And if we set our eyes on some particular object that we intended to walk to, then we would find the walk to be much longer than expected, since we would judge, for example, the distance of five tiles ahead to be much less than one tile-length from our position. Thus, in general, we would misjudge objects as being closer and smaller than they actually are.

Unfortunately Helmholtz did not elaborate on the assumptions underlying his account, thus leaving a number of questions unanswered. In particular, why should we use the projective disk model rather than the conformal disk model? And how exactly is the hyperbolic size-distortion phenomenon related to our everyday experience in ordinary Euclidean space that objects in the distance appear smaller?

We shall attempt to answer these questions. First we should make it clear that we are not assuming that distances and sizes are judged in some mathematically well-defined manner (based on, for instance, angular measures of the light rays reaching the eye). Instead we are assuming that creatures judge distances and sizes in whatever manner they have always done so. This includes certain mathematical principles regarding incidence angles of light reaching the eye and other such measures, but it also involves various facts of experience and assumptions about the world.

Consider for example a tiled floor, as in Figure 2. When we see a perspective view of a tiled floor, such as the one on the left, we infer that all the tiles are of the same size. Mathematical principles play a part here; they enable us, among other effects, to tell true perspective from false perspective such as the one on the right.
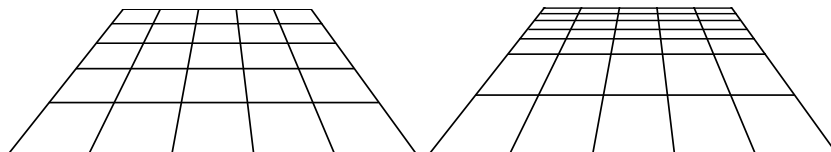

Figure 2: Accurate (left) and inaccurate (right) perspective views of a tiled floor.

There is, however, no strictly mathematical justification for the inference that the tiles are of equal size. After all, other geometrical arrangements

could give the exact same visual impression under certain simplifying assumptions such as our only having one eye, there being no "fog" or focal differences or other factors by which to judge if an object is near or far, etc. Indeed, the picture above and an actual tiled floor could give identical visual impressions under these assumptions. Our inference that we are looking at a tiled floor with equally-sized square tiles is entangled with mathematical principles, as well as experience and implicit assumptions about the types of objects one typically encounters in everyday life. We are assuming that all of this is carried over as we enter hyperbolic space.

Thus if we see something that looks like an accurate perspective view of a tiled floor, we will believe that we are looking at tiles of equal sizes. What type of geometrical configuration in a hyperbolic world would give us such a visual impression? In the projective disk model it is easy to draw a figure that would give this visual impression: simply draw a regular grid of Euclidean squares in the model. This grid would produce the desired visual impression, since light travels in straight lines (Euclidean lines in this model), so as we connect the points on the grid to the observer's eye point, we get a bundle of lines indistinguishable from the Euclidean case.

However, the conformal disk model also preserves geodesics through the midpoint. So could we not just as well have substituted the conformal disk for the projective disk in this argument? Strictly speaking, we could have, so far as pure visual data is concerned. But our ability to judge straightness goes beyond mere retinal geometry. Our perceptions of straightness are also mediated by a number of cultural clues, such as the trajectories of cars with all their wheels aligned, pedestrians in a hurry, objects in inertial motion, and the edges of buildings and their shadows. With the aid of such hints, the conformal disk situation is unlikely to fool us; we would probably dismiss the apparent impression of the tiled floor as a deliberate visual illusion of the type one finds in amusement park funhouses. In the projective disk case, on the other hand, no such excuses would be available to us, so we would buy the illusion of the tiled floor and the erroneous estimations of distance that go with it.

These arguments support Helmholtz's principle that a creature entering a different world will distort the world he sees by embedding it in his geometry in a geodesic-preserving manner. Now that this principle is established, we can amuse ourselves by extending it to other geometries.

Let us first consider spherical geometry. Once again Helmholtz offers us a few brief remarks:

> There would be an illusion of the opposite description, if, with eyes practised to measure in Euclid's space, we entered a spherical space of three dimensions. We should suppose the more distant objects to be more remote and larger than they are, and should find on approaching them that we reached them more quickly than we expected from their appearance. ... The strangest sight, however, in the spherical world would be the back of our own head. [1, pages 317–318]

Indeed we would see the back of our own heads, since geodesics in spherical geometry are great circles, and so there are geodesic paths from the back of our heads to our eyes. But how about the metric distortions? Is there a model similar to the hyperbolic case that can help us see why these distortions are of the type reported by Helmholtz? According to Helmholtz's principle, if we entered a spherical universe we would map it into our Euclidean minds in a geodesic-preserving manner. And in fact there is such a map: the gnomonic projection, that is, the projection from the center of the sphere onto a tangent plane (see Figure 3 for an example). Gnomonic projection is clearly geodesic-preserving since a geodesic on the sphere is the intersection of the sphere and a plane through its center, so its projection is the intersection of this plane with the tangent plane projected onto, a line.

Using Figure 3, we can confirm Helmholtz's account. Assuming that we stand at the center of our picture, we believe Sweden is as big as Africa, but if we were to walk to Sweden we would see that we were very much mistaken. And we believe things close to the poles to be almost infinitely far away, when in fact the distance is only a quarter of the circumference of the sphere, so we would be surprised to find that we could travel there quite easily. In sum, we misjudge objects as being further away and larger than they actually are.

A problematic aspect here is that only half the sphere is mapped onto the plane under gnomonic projection. Still, of course, light from the other half of the sphere would reach our eyes. How, then, would we interpret the light coming from that half? Where would it fit into our geometrical conception of the world? Or, for that matter, since we see the back of our own heads "in front" of us, how could we possibly conceive of a "distance" to it? This phenomenon is not as absurd as it first seems. In fact, it is analogous to our
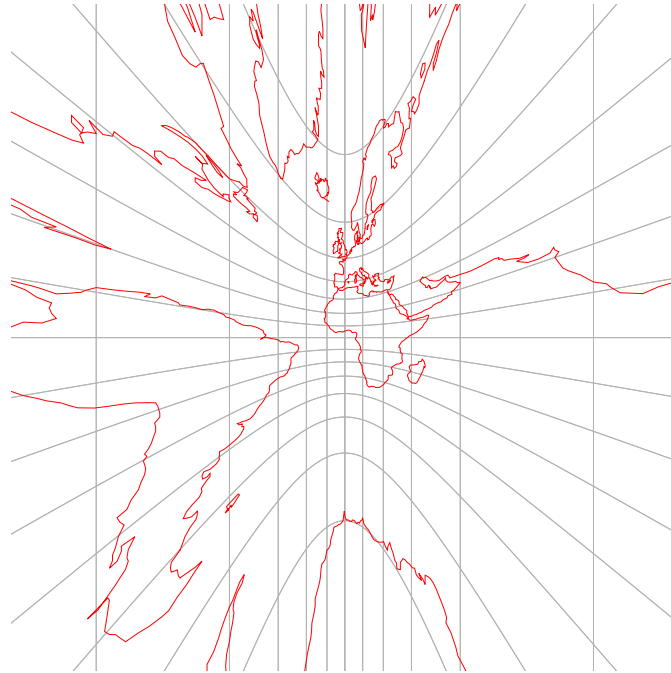
Figure 3: Gnomonic projection of the earth onto a plane tangent to it just south of Nigeria. (From [3].)

own everyday experience of looking at the sky. When we look at the sky, we see the stars and the sun and the moon, but our visual impressions do not come with any connotations of distance, as they do for local objects. There is a "metrical horizon," as it were, beyond which we cannot judge distance. Thus to a Euclidean creature who entered the spherical world, the far half of the sphere would be analogous to our night sky; they would see it but their metrical notions would not apply.

Now let us consider the reverse cases of creatures from other worlds entering our own. Again, the rule is that each individual creature would map our world into his own geometry in a geodesic-preserving manner.

For the case of a spherical creature this can be made quite vivid with the aid of a spherical mirror. If a spherical mirror (of the right size) is held against the illustration of gnomonic projection in Figure 3, touching it at the point of tangency just below Nigeria, the image in the mirror will be an almost perfect

globe as we are accustomed to seeing it. (You can try this for yourself using enlarged versions of the gnomonic projection figure. If you enlarge the figure to about the size of a newspaper spread then you can use a garden ornament mirror about the size of a bowling ball; with a smaller enlargement you can use an ordinary Christmas ball as the mirror.) This proves that looking in the mirror is approximately the inverse of gnomonic projection (since it "undoes" the original projection). Thus this map is almost geodesic-preserving too, so it can be used to figure out how the spherical creature would judge distances and sizes in our world. Clearly, then, the spherical creature would think that everything was rather close by (no more that a quarter of the circumference away) and very small (a mountain looks tiny when observed in a spherical mirror from afar—smaller than a nearby penny, for instance).

A hyperbolic creature would have the opposite misconceptions. Again the projective disk is a bridge between our two worlds since it is geodesic-preserving. But now it goes the other way; the hyperbolic creature would impose the metric of the projective disk on our world. Thus he would believe that objects that were only, say, a couple of inches away (or whatever the radius of his disk was) were almost infinitely far away (one would have to traverse so many tiles to get there) and very big (since it would cover so many tiles). How surprised he would be to find that he could walk there in just a step or two and that the objects were not as majestically large as he had imagined. Here again the "metric horizon" phenomenon is very noticeable. The hyperbolic creature can only judge distances within the radius of his disk. Everything else he would judge to be immeasurably far away, like the night sky is for us. His confusion in this regard would thus be comparable to ours if we suddenly realized that we could reach out and touch the moon by standing on our toes.

It remains to consider the cases where a hyperbolic creature enters a spherical world and vice versa. The required geodesic-preserving maps for these cases can be obtained by composition of the maps above, using Euclidean geometry as an intermediate stage. Thus to see how the spherical creature would experience a hyperbolic world, we simply need to look at the projective disk model in a spherical mirror touching the center of the disk. Hyperbolic straight = Euclidean straight in the projective disk model $\approx$ spherically straight under the mirror projection, according to our discussion thus far, so this gives an approximately geodesic-preserving spherical model of hyperbolic geometry. Thus the spherical creature would think that

everything was very close (within a tenth of the circumference perhaps, depending on the radii of the disk and the sphere) and tiny as compared to actual size (the tiles in the distance are deceptively small). Conversely, the hyperbolic creature's experience of spherical space is obtained by imposing the projective disk metric on the gnomonic projection of the sphere. In this way, spherically straight = straight on the gnomonic projection = straight in projective disk, so geodesics are preserved. Therefore the hyperbolic creature's illusions would be composed of the two illusions associated with these maps above; he would believe objects to be very far away (almost all of them even beyond the "metric horizon") and very big (since they would cover so many tiles).

References

[1] Hermann Helmholtz, "The origin and meaning of geometrical axioms," *Mind*, Volume 1 Number 3 (1876), pages 301–321.

[2] William T. Shaw, *Complex Analysis with Mathematica*, Cambridge University Press, Cambridge, 2006.

[3] Eric W. Weisstein, Gnomonic projection—From MathWorld; available at: http://mathworld.wolfram.com/GnomonicProjection.html, accessed on June 26, 2013.