Credit Portfolio Loss Modelling

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Foreword

This master thesis is the end result after several years of following the master programme in mathematics. When I started the master programma I was working at Rabobank. Doing a mathematics master was probably not really a first choice when working in a practical environment as a bank usually tends to be. However since I was always curious about how things like risk neutral pricing worked, I decided to start anyway. Although at times it was really tough working full time and doing a master I have never regretted starting it and learned a lot. I would like thank my thesis advisor Prof. Roberto Fernandez for taking over as a thesis advisor for the last part. I would also like to thank Dr. Karma Dajani for being my thesis advisor for the main part, but illness prevented her from continuing. I would also like to thank Dr. Cristian Spitoni for being the second reader. Furthermore I would like to thank Mâcé Mesters for accomodating me when I needed to follow classes and fruitful discussions. Finally I would like to thank my parents and my sister for their support over the years.

The subject of this thesis is credit porfolio loss modelling. This was a subject that I was working on as a Risk Researcher at the research department of Group Risk Management within Rabobank Nederland. I have used some of the results obtained then and tried to expand more on the theoretical parts. Credit portfolio loss modelling deals with modelling credit risk and finding methods to accurately determine quantiles of a loss distribution for the losses in the loan portfolio of a bank. We look only at the losses incurred by the bank resulting from clients not repaying their loans. The underlying model we use for losses on indivual loans is quite simple. However obtaining quantiles of portfolio losses appears not to be very straightforward. In this thesis we discuss some methods to obtain quantiles of a loss distribution and compare their performances on a number of portfolios.

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Chapter 1

Introduction

1.1 Credit risk

The main business for banks is to attract money from people and businesses and to lend out this money to other people or businesses. In order to make money the bank charges more interest on the loans it provides than on the money that it attracts such as savings. The main risk for those who have deposited their money with the bank is that the bank is unable to repay them their money. There can be a variety of reasons for the bank to be unable to repay the deposits. It can be the case that the bank has invested its money in long term loans that they cannot unwind at the moment the depositers want their money back. Another possibility is that the bank simply does not have the money due to misinvestments. It could be the case that a company that the bank has given a loan to does not repay the money at the moment it is due. In that case the bank could also not be able to repay its depositers. So a risk of a default on a loan given by the bank is also a risk for the depositers (or any other creditor) of the bank.
In general the bank charges more interest on its loans it gives than on the deposits it receives. So it can incur some losses on its loans, while still being able to repay the deposits and the interest on it. However if the losses are more extreme the bank will not be able to repay all of its deposits. The bank will than have to default on its obligation to repay the deposits. In case the depositers get wind of the bank incurring large losses and as a result not being able to repay the deposits will lead depositers to withdraw their money from the bank worsening the position of the bank. These events might lead to a fulfledged bank-run, toppling the bank fully. The failure of one bank may cause other banks to fail as well due to the entanglement of the banking and financial sector.
To shield the depositers from losses even when the bank faces some adverse conditions the banking regulators (e.g. the European Central Bank) require that banks have some buffers to absorb the losses up to some limit. The buffer is in the form of equity. This works as follows.
Example. Suppose the bank wants to give out 100 loans for 100 euro each (=notional amount). So it needs 100 $\times$ 100 euro = 10000 euro. Now suppose it takes 10000 euro worth of deposits. Now assume that the loans give a 10% interest rate and the bank pays 5% interest rate on the deposits. If all of the loans repay the notional and the interest the bank makes 5% (10%-5%) profit. The depositers get their promised 5% interest rate.
Case 1. Suppose now that 5 of the 100 loans repay the interest, but do not repay the notional of a 100 euro. So the bank gets the following 95 $\times$ 100 euro = 9500 euro of notional, 100 $\times$ 10 euro = 1000 euro of interest income. In total the bank gets 10500. The banks has to pay its depositers an amount of 10000+500=10500. In this case the bank can

just repay its depositers.
Case 2. Suppose now that 10 of the 100 loans repay the interest, but do not repay the notional of a 100 euro. So the bank gets the following 90 $\times$ 100 euro = 9000 euro of notional, 100 $\times$ 10 euro = 1000 euro of interest income. In total the bank gets 10000. The banks has to pay its depositers an amount of 10000+500=10500. In this case the bank cannot fully repay its depositers. It is short 500 euro and would default on its obligations due to 10 loans defaulting on their obligations to the bank.
Case 3. Now look at the case where not all the 10000 euro is funded by deposits, but also by equity. Equity does not have a fixed interest rate like deposits but only receives that income that is left after all of the depositers are paid what they are owed. Suppose we fund the 10000 euro by 9000 euro of deposits and 1000 euro of equity. Also let 10 loans only repay the interest, but not the notional. Now the bank receives 10000. The depositers are owed 9000$\times$1.10 = 9900. The bank has enough money to repay the depositers. However the equity holders only receive 100 euro (=10000-9900), while they invested a 1000 euro. So they incur a loss of 900 euro, this is a return of -90%, quite negative. The depositers however do not incur any loss in this case.
Case 4. In case all of the notional of the loans and the interest are repaid the bank receives 11000 euro. After repaying the depositers, 1100 euro is available to the equity holders. Given that they invested a 1000 euro they have in this case a return of 10%. This is much more than the 5% return that the depositers receive. The equity holders get a higher expected return as they are highly uncertain about their return. They could earn no money at all or even worse, losing a large part of their investment.
The above examples showed that in moderate adverse conditions (case 1) the bank can still repay its depositers. However in more extreme conditions (case 2) the bank defaults. When the bank uses equity to fund its loans (case 3) even in more extreme adverse conditions the bank can still repay its deposits. This means that depostors are more likelily to get their money back and this makes them less likely to withdraw their deposits from the bank.
The regulator of the European banking sector has deemed that banks have to have some level of equity such that its creditors (e.g. depositors, holders of bonds to the bank etc.) have a certain probability level of losing (part of) their money within a certain time frame. For regulatory purposes the banks need to hold equity (also called regulatory capital) at a level such that the probability that the value of the assets within 1 year is above that of the banks obligations to depositors (and bond holders) is not below 99.9%. In order to make this a bit more formal we make the following definitions. Let A_t be the assets at time t, E_t the equity (or capital) level at time t, B_t the debt level of the bank at time t. Note that with debt of the bank we mean the debt the bank has to other parties such as its depositors and e.g. bond holders. The assets of a bank are e.g. the loans, mortgages it issues. Then the regulator requires that

$$\mathbb{P}(A_{t+1} \geq B_{t+1}) \geq 99.9\% \tag{1.1}$$

Under the restriction that

$$A_t = E_t + B_t \tag{1.2}$$

$$B_{t+1} = (1+r)B_t \tag{1.3}$$

Where r is the interest paid by the bank on its debt. In our special banking case the value of B_{t+1} is known at time t. This could be for example the value of deposits at time t plus the interest rate on this amount. The interest rate is always known before hand so it is known by time t. The only random variable is A_{t+1}, the assets at time $t+1$. The assets of a bank consists of the loans and mortgages it issued, but also stocks and bonds in other companies it holds for trading and the things it owns such as buildings, computers etc. The majority of its assets are made up of loans it issued. So for our analysis we make the abstraction that the only assets a bank has are the loans it issued. These loans can be items like consumer loans, mortgages, student loans, lines of credit, revolver loans. In our analysis we will focus only on credit risk.

Here by credit risk we mean the risk that a client does not fully fullfill all the payments and timings thereof of the loan that are contractually due. So this refers to both the amount repaid and the time it is due.

Other risks such as changing interest rates we will not include.

1.2 Elements of credit risk

In this section we discuss what credit risk is and what elements play a role in it. The definition of credit risk is the uncertainty in the payments and the timings thereof of a loan as they are contractually due by the client taking the loan. So this could be for example not paying the full amount due at a certain moment or making payments later than required by the contract of the loan. When a client does not fulfill on its obligation of a loan, it is in default. The regulators definition of a default is when a client is 90 days overdue on its obligation or when it is unlikely to pay. In the regulators definition of default if a client pays 1 day too late it is not in default yet. However if it pays 90 days too late it is in default. Banks in Europe are forced to follow the regulators definition, so we will also take this as the definition of default. Admittantly, the 'unlikely to pay' phrase is a bit vague. We believe it to mean that if a client only has a 100 euro cash and has to pay a 1000 euro in 6 months and if it is virtually certain that it will not get any additional cash in between the client, it will certainly be in default in 6 months + 90 days. At this moment the client has not fallen behind on any payments, but the bank will almost certainly not receive the full 1000 euro payment. For our analysis the exact definition of default will not be that important.

1.3 Structuring credit risk

Now we discuss some elements in modelling credit risk. There are usually 3 elements that are used in modelling credit risk. **1:** The first elelment is the indicator of default, D. This variable indicates whether or not a default on a loan has occured within a certain period. This variable has value 0 if no default occured in a certain period and has value 1 if a default did indeed occur in that period. The probability of default, abbreviated as PD, is the probability that $D = 1$.
2: The second element is the loss given default (LGD). When a client defaults on a loan the bank has two options. The first is to renegotiate the terms of the contract. As the client is usually in financial trouble when it defaults this will be with less favorable conditions for the bank, so it takes a loss. Another option is to get a liquidator and sell the assets of the client in order to recoup the money the client owes the bank. In general this will also yield less than the clients owes the bank. The bank then incurs a loss on the loan. The loss the bank incurs after a default occurs is called, not surprisingly, the loss given default or LGD as it is usually abbreviated. The LGD is expressed as a percentage of another variable called the exposure at default, which we will discuss now.
3: The exposure at default (EAD) is the total amount that the bank is exposed to from that client at the moment of default. It is the amount that the bank has outstanding with the client, so not necessarily the amount the client failed to pay to the bank that triggered its default. So suppose a client has to make an annual interest payment of 5% on a 1 million euro mortgage loan. Suppose the mortgage is to be repaid in one single payment within 10 years. However suppose that this year the client failed to make the interest payment and is therefore in default. Now it failed to pay 50000 euro (=5%), but the outstanding

amount is 1 million euro (plus the interest owed). So the exposure at default was 1 million euro + 50000 euro = 1050000 euro. The interest payments that are due in the remaining 9 years we will not take into account anymore. So the loss on a loan can be expressed as $D \times LGD \times EAD$.

1.4 Modelling credit risk

In the previous section we saw the some important elements in the structure of credit risk, i.e. the default indicator, loss given default and exposure given default. So let us take the example of a loan in which the client has to pay the bank 1 euro after 1 year. So the $EAD = 1$ and is thus fixed. If the client defaults within that year the bank receives $(1 - LGD) \times 1$ so it loses $LGD \times 1$. Assume that LGD is also not random, but fixed. Then the loss L at the end of the year for the bank has the distribution

$$\mathbb{P}(L = 0) = \mathbb{P}(D = 0) = 1 - PD, \tag{1.4}$$
$$\mathbb{P}(L = LGD) = \mathbb{P}(D = 1) = PD. \tag{1.5}$$

So the distribution is like a Bernouilli one except that the possible outcomes are not $\{0, 1\}$ but rather $\{0, LGD\}$. The parameter PD is contained in the interval $[0, 1]$. The probability of default, PD, in this example is an externally determined parameter.

1.4.1 Merton model

A frequently used model of default is the so called Merton model. In its most basic form it goes as follows. Assume that we are now at the beginning of a period, say $t = 0$. The company whose default we want to model has some assets and a certain debt level at $t = 0$. Now suppose at $t = 1$ the company has to repay its full debt. It must use its assets to repay the debt. This means that if at $t = 1$ the asset level is at or above the debt level the company can repay all of its debt. If the asset level is below the debt level the company cannot repay all of its debt and is in default. To formalise this, let A_t be the asset level of the company at time t, B_t the debt level at time t, D_t the default indicator at t. Then for $t = 1$

$$A_1 < B_1 \Leftrightarrow D_1 = 1 \tag{1.6}$$
$$A_1 \geq B_1 \Leftrightarrow D_1 = 0. \tag{1.7}$$

We furthermore assume that B_1 is known at $t = 0$. The assets we model by means of a geometric Brownian motion, comparable to the Black-Scholes option pricing model. So the assets follow the stochastic differential equation

$$dA_t = \mu A_t dt + \sigma A_t dW_t. \tag{1.8}$$

Where W_t is a standard Brownian motion. So the asset value at time t is

$$A_t = A_0 \cdot e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}. \tag{1.9}$$

Here A_0 is assumed non-random. Then we can write

$$\begin{aligned}
\mathbb{P}(D_t = 1) =& \mathbb{P}(A_t < B_t) \\
=& \mathbb{P}(A_0 \cdot e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t} < B_t) \\
=& \mathbb{P}(\log(A_0) + (\mu - \frac{1}{2}\sigma^2)t + \sigma W_t < \log(B_t)) \\
=& \mathbb{P}\left(W_t < \frac{-\log(A_0) + \log(B_t) - (\mu - \frac{1}{2}\sigma^2)t}{\sigma}\right) \\
=& \Phi\left(\frac{-\log(A_0) + \log(B_t) - (\mu - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}\right).
\end{aligned} \tag{1.10}$$

Where Φ is the standard normal distribution function. In the above example we had $t = 1$, but we will keep t for now. So we can write

$$PD = \Phi\Big(\frac{-\log(A_0) + \log(B_t) - (\mu - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}\Big) \Leftrightarrow \tag{1.11}$$

$$\Phi^{-1}(PD) = \frac{-\log(A_0) + \log(B_t) - (\mu - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}} \tag{1.12}$$

Until now we have only looked at the default behaviour of one client or company. We could also look at the joint default behaviour of two or more clients. The Merton model framework also allows for this. Now let $A_{t,i}$ be the asset value at time t for client i where $i = 1, 2, ..., n$, $n \in \mathbb{N}$ and let $B_{t,i}$ be the debt value at time t for client i. Furthermore let the Brownian motions driving the asset values be dependent in the following way

$$dA_{t,i} = \mu_i A_{t,i} dt + \sigma_i A_{t,i}(\sqrt{\rho_i} dW_t + \sqrt{1 - \rho_i} d\varepsilon_{t,i}). \tag{1.13}$$

Where W_t is again a standard Brownian motion, $\varepsilon_{t,i}$, $i = 1, 2, ..., n$ is also a Brownian motion, independent of each other and W_t, and $\rho_i \in [0, 1]$. Here the parameter ρ_i is a client specific parameter determing how dependent the clients assets are on the common Brownian motion W_t. Since the Brownian motion W_t drives the asset values for all clients it is also called a 'common factor'. From the stochastic differential equation the process for the asset value follows as

$$A_{t,i} = A_{0,i} \cdot e^{(\mu_i - \frac{1}{2}\sigma_i^2)t + \sigma_i(\sqrt{\rho_i} W_t + \sqrt{1-\rho_i}\varepsilon_{t,i})}. \tag{1.14}$$

Now we can also look at the probability that two clients default together. That is for client i and client j, $i \neq j$. First define

$$\begin{aligned}
X_{t,i} &:= \sqrt{\rho_i} W_t + \sqrt{1 - \rho_i}\varepsilon_{t,i} \\
X_{t,j} &:= \sqrt{\rho_j} W_t + \sqrt{1 - \rho_j}\varepsilon_{t,j} \\
\Phi^{-1}(PD_i) &= \frac{\log(A_{0,i}) + (\mu_i - \frac{1}{2}\sigma_i^2)t - \log(B_{t,i})}{\sigma\sqrt{t}} \\
\Phi^{-1}(PD_j) &= \frac{\log(A_{0,j}) + (\mu_j - \frac{1}{2}\sigma_j^2)t - \log(B_{t,j})}{\sigma\sqrt{t}}. \qquad (1.15)
\end{aligned}$$

$$(1.16)$$

Then we have

$$
\begin{aligned}
&\mathbb{P}(A_{t,i} < B_{t,i}, A_{t,j} < B_{t,j}) = \\
&\mathbb{P}(A_{0,i} \cdot e^{(\mu_i - \frac{1}{2}\sigma_i^2)t + \sigma_i X_{t,i}} < B_{t,i}, A_{0,j} \cdot e^{(\mu_j - \frac{1}{2}\sigma_j^2)t + \sigma_j X_{t,j}} < B_{t,j}) = \\
&\mathbb{P}(X_{t,i} < \frac{-\log(A_{0,i}) - (\mu_i - \frac{1}{2}\sigma_i^2)t + \log(B_{t,i})}{\sigma}, X_{t,j} < \frac{-\log(A_{0,j}) - (\mu_j - \frac{1}{2}\sigma_j^2)t + \log(B_{t,j})}{\sigma}) = \\
&\mathbb{P}(X_{t,i}/\sqrt{t} < \Phi^{-1}(PD_i), X_{t,j}/\sqrt{t} < \Phi^{-1}(PD_j)).
\end{aligned} \tag{1.17}
$$

Now we look at $\mathbb{P}(X_{t,i}/\sqrt{t} < \Phi^{-1}(PD_i), X_{t,j}/\sqrt{t} < \Phi^{-1}(PD_j))$. It is easy to see that $(X_{t,i}/\sqrt{t}, X_{t,j}/\sqrt{t})$ is bivariate normal, since they are both linear functions of normally distributed random variables, with

$$
\begin{pmatrix} X_{t,i}/\sqrt{t} \\ X_{t,j}/\sqrt{t} \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \sqrt{\rho_i\rho_j} \\ \sqrt{\rho_i\rho_j} & 1 \end{pmatrix}\right). \tag{1.18}
$$

Another useful random variable is the conditional PD, where we condition on $W_t/\sqrt{t}$. Then

$$
\begin{aligned}
&\mathbb{P}(X_{t,i}/\sqrt{t} < \Phi^{-1}(PD_i)|W_t/\sqrt{t}) = \\
&\mathbb{P}((\sqrt{\rho_i}W_t + \sqrt{1-\rho_i}\varepsilon_{t,i})/\sqrt{t} < \Phi^{-1}(PD_i)|W_t/\sqrt{t}) = \\
&\mathbb{P}\left(\varepsilon_{t,i}/\sqrt{t} < \frac{\Phi^{-1}(PD_i) - \sqrt{\rho_i}W_t/\sqrt{t}}{\sqrt{1-\rho_i}} \,\middle|\, \frac{W_t}{\sqrt{t}}\right).
\end{aligned} \tag{1.19}
$$

Chapter 2

Credit portfolio loss modelling

In this chapter we set up a model for the credit losses of a portfolio of loans. By credit losses we mean the loss that is incurred due to clients not fulfilling their entire loan obligations. Consider a bank whose assets consist only of loans. We have seen in the previous chapter that for the bank to have a certain rating it needs to have a certain level of equity. The exact level of equity depends on the distribution of the assets of the bank, the loans in our case. To be more exact we would like to know certain quantiles of the distribution of the asset value. Given a certain reference point this is equivalent to finding the quantiles of the loss the bank incurs on the loans, via the relation *loss on loan = reference point - value of the loan*. The *reference point* in our set up is EAD. In case there is no default the *value of the loan* is just EAD and thus the loss is $EAD - EAD = 0$. In case there is a default then the *value of the loan* is $(1 - LGD) \cdot EAD$. The loss in case of a default is $EAD - (1 - LGD) \cdot EAD = LGD \cdot EAD$. Here the loss and value of the loans is at some future time and we assume that any interest payments due are included in the value of the EAD. In many cases this future time is 1 year after the current time. The loss on the loan portfolio is the sum of the losses on the individual loans. So if L_i is the loss on loan i and the portfolio consists of n loans so $i = 1, 2, ..., n$ then the total portfolio loss is $L^p = \sum_{i=1}^{n} L_i$. Our problem is finding the quantile of a sum of random variables L_i, which in general are dependent. This dependency rules out the use of standard methods such as the central limit theorem and the strong law of large numbers. However we will model the L_i such that they are conditionally independent and we will see the strong law of large number and the central limit theorem being used in certain cases.

2.1 A simple model for credit portfolio loss

We will start with a very simple model for our credit portfolio loss. Suppose our portfolio consist of n bullet loans. A bullet loan is a loan in which a client has to pay interest on the loan and a single repayment of the loan at the end of some period. We will even assume that the loans have to be repaid within 1 year and that the interest is 0 for now. We will also let the loss given default be fixed for every loan, but not necessarily the same. The same hold for the exposure at default. Suppose we start from $t = 0$ and the end of the period is $t = 1$, where time is measured in years. Now let D_i be the default indicator for loan i, where default is measured at $t = 1$. Let LGD_i and EAD_i be the loss given default and exposure at default for loan i. The loss for loan i at the end of the period is

$$L_i = \begin{cases} LGD_i \cdot EAD_i & \text{if } D_i = 1, \\ 0 & \text{if } D_i = 0. \end{cases} \tag{2.1}$$

The default indicator is modelled using the Merton model of the previous chapter. Since the end of the period is $t = 1$ this simplifies the formulas. Let Y be the common factor for all loans and ε_i the loan specific factor. Let Y be standard normally distributed and independent of all ε_i, $i = 1, 2, ..., n$. Also ε_i is standard normally distributed and ε_i is independent of ε_j for all $j = 1, 2, \ldots, n$ and $j \neq i$. Let p_i and ρ_i be a loan specific parameters with $p_i \in [0, 1]$ and $\rho_i \in [0, 1]$. Then we have that

$$D_i = \begin{cases} 1 & \text{if } \sqrt{\rho_i}Y + \sqrt{1-\rho_i}\varepsilon_i < \Phi^{-1}(p_i), \\ 0 & \text{if } \sqrt{\rho_i}Y + \sqrt{1-\rho_i}\varepsilon_i \geq \Phi^{-1}(p_i). \end{cases} \tag{2.2}$$

It is easy to see that $p_i = \mathbb{P}(D_i = 1)$. This is why we call p_i the unconditional probability of default. We will assume that the parameters p_i and ρ_i are given. Define the portfolio loss L^p as

$$L^p := \sum_{i=1}^{n} L_i. \tag{2.3}$$

Given our model above we can write the portfolio loss as a function of Y and ε_i, $i = 1, \ldots, n$. Let $I_{\{\cdot\}}$ be the indicator function. Using that $L_i := I_{\left\{\sqrt{\rho_i}Y+\sqrt{1-\rho_i}\varepsilon_i<\Phi^{-1}(p_i)\right\}} \cdot LGD_i \cdot EAD_i$ we get

$$L^p := \sum_{i=1}^{n} I_{\left\{\sqrt{\rho_i}Y+\sqrt{1-\rho_i}\varepsilon_i<\Phi^{-1}(p_i)\right\}} \cdot LGD_i \cdot EAD_i. \tag{2.4}$$

We see from the definition of the L_i that they are dependent on each other via Y. However if we condition on Y we see that the L_i in our model are independent of each other, since the only randomness is by the ε_i's and these are mutually independent.

2.2 Finding a quantile of sums of dependent random variables

In the previous section we modelled the random behaviour of our credit portfolio loss. It is still a rather simple model, but we will stick to it for now. In general finding the quantile of a sum of random variables can be quite hard, even when we assume them to be independent. In our case we have a sum of dependent discrete random variables making our case even harder. The q level quantile of L^p, $\alpha_q(L^p)$, is defined as

$$\alpha_q(L^p) := \inf\left\{ y | \int_{-\infty}^{y} dF_{L^p}(y) \geq q \right\}. \tag{2.5}$$

Where F_{L^p} is the distribution of L^p. We will discuss in this thesis a number of methods that approximate the quantile. The methods that we will investigate are the Saddlepoint method, De Hoog's algorithm, a wavelet based method, recursive algorithm and Monte Carlo simulation. The first 3 methods are based on inverting the Laplace transform of the distribution of the portfolio loss L^p via the Bromwich integral. The Monte Carlo simulation is a brute force simulation method simply generating a lot of losses to determine the distribution. The methods will be discussed in the next chapter.

Chapter 3

Methods based on Laplace transform inversion

In this chapter we will discuss some methods which are based the inversion of the Laplace transform of the distribution of the portfolio loss. First we define the Laplace transform for the loss on an individual loan. We use the model as presented in the previous chapter. So we have n loans, $L_i := D_i \cdot LGD_i \cdot EAD_i$ and $\mathbb{P}(D_i = 1) = p_i$, $\mathbb{P}(D_i = 0) = 1 - p_i$. Then the Laplace transform M_i of L_i is

$$M_i(t) := \mathbb{E}(e^{tL_i}). \tag{3.1}$$

Where $t \in \mathbb{C}$ such that the above expectation is well defined, which is the case if $\mathbb{E}(e^{\mathrm{Re}(t)L_i})$ exists. Then we have that

$$M_i(t) = (1 - p_i) + p_i e^{t \cdot LGD_i \cdot EAD_i}. \tag{3.2}$$

Another useful variable is the conditional probability of default, where we condition on Y. Recall that Y is standard normally distributed. So $\mathbb{P}(D_i = 1|Y) = c_i(Y)$, $\mathbb{P}(D_i = 0|Y) = 1 - c_i(Y)$ where

$$c_i(Y) := \Phi\left(\frac{\Phi^{-1}(p_i) - \sqrt{\rho_i}Y}{\sqrt{1-\rho_i}}\right). \tag{3.3}$$

This can be derived as follows.

$$\begin{aligned}
\mathbb{P}(D_i = 1|Y) &= \mathbb{P}(\sqrt{\rho_i}Y + \sqrt{1-\rho_i}\varepsilon_i < \Phi^{-1}(p_i)|Y) \\
&= \mathbb{P}\left(\varepsilon_i < \frac{\Phi^{-1}(p_i) - \sqrt{\rho_i}Y}{\sqrt{1-\rho_i}} \middle| Y\right) \\
&= \Phi\left(\frac{\Phi^{-1}(p_i) - \sqrt{\rho_i}Y}{\sqrt{1-\rho_i}}\right).
\end{aligned} \tag{3.4}$$

Then the conditional Laplace transform, conditional on Y, is defined as

$$M_i(t, Y) := \mathbb{E}(e^{tL_i}|Y). \tag{3.5}$$

Then using c_i we get that

$$M_i(t, Y) = (1 - c_i(Y)) + c_i(Y)e^{t \cdot LGD_i \cdot EAD_i}. \tag{3.6}$$

The Laplace transform for L^p is a bit more subtle. The Laplace transform has the property that for two independent random variables U, V with Laplace transforms M_U, M_V the Laplace transform for the sum $U+V$, M_{U+V}, satisfies $M_{U+V} = M_U M_V$. Now we do not have that the L_i's are independent, however conditional on Y they are independent. So we have that

$$\begin{aligned}
M_{L^p}(t) :&= \mathbb{E}(e^{tL^p}) \\
&= \mathbb{E}(e^{t\sum_{i=1}^n L_i}) \\
&= \mathbb{E}\left(\mathbb{E}(e^{t\sum_{i=1}^n L_i}|Y)\right) \\
&= \mathbb{E}\left(\mathbb{E}(\prod_{i=1}^n e^{tL_i}|Y)\right) \\
&= \mathbb{E}\left(\prod_{i=1}^n \mathbb{E}(e^{tL_i}|Y)\right) \\
&= \mathbb{E}\left(\prod_{i=1}^n M_i(t,Y)\right) \\
&= \int_{-\infty}^{\infty}\left(\prod_{i=1}^n M_i(t,u)\right) f_Y(u)du.
\end{aligned} \tag{3.7}$$

Where f_Y is the density of Y, which was a standard normal one. We used conditional independence in the fifth equality.

3.1 Saddlepoint approximation

This section describes the saddlepoint methodology. It is mainly based on [3]. We will first describe the method and then later fit our simple credit portfolio model to it. We start with the mean $\tilde{X}$ of n independent random variables $X_1, \ldots, X_n$. Then

$$\tilde{X} = \frac{1}{n}\sum_{i=1}^n X_i. \tag{3.8}$$

Where all the X_i's admit the same density g. Let the Laplace transform M for X_i be

$$M_{X_i}(T) = \int_{-\infty}^{\infty} e^{Tx} f(x)dx. \tag{3.9}$$

Define

$$K_{X_i}(T) := \log(M_{X_i}(T)). \tag{3.10}$$

For ease of notation we will leave out the subscript X_i for K and M. Let f_n be the density of $\tilde{X}$. The Bromwich integral gives the inversion formula for the density

$$f_n(\tilde{x}) = \frac{n}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty} e^{n(K(T)-T\tilde{x})}dT. \tag{3.11}$$

Where the integral runs over a vertical line in the complex plane which crosses the real line at τ. This relation holds for any $\tau \in \mathbb{R}$. Define the saddlepoint T_0 by

$$K'(T_0) = \tilde{x}. \tag{3.12}$$

Here $K'(T) := dK/dT$. Now we take the contour in the complex plane to integrate over to be $\gamma(t) = T_0 + it$ with $\gamma'(t) = i$ and let t go from $-\infty$ to ∞. Then we have that

$$\begin{aligned} f_n(\tilde{x}) &= \frac{n}{2\pi i}\int_{T_0-i\infty}^{T_0+i\infty} e^{n(K(T)-T\tilde{x})}dT \\ &= \frac{n}{2\pi i}\int_{-\infty}^{\infty} e^{n(K(\gamma(t))-\gamma(t)\tilde{x})}\gamma'(t)dt \\ &= \frac{n}{2\pi}\int_{-\infty}^{\infty} e^{n(K(\gamma(t))-\gamma(t)\tilde{x})}dt. \end{aligned} \tag{3.13}$$

We have that K and thus $K - T\tilde{x}$ is analytic on the complex plane, at least for the subset on which M is defined. Then the Taylor series expansion of $K - T\tilde{x}$ around T_0 exists and we obtain

$$\begin{aligned} K(T) - T\tilde{x} =& K(T_0) - T_0\tilde{x} + (K'(T_0) - \tilde{x})(T - T_0) + \frac{1}{2}K''(T_0)(T - T_0)^2 \\ &+ \frac{1}{6}K'''(T_0)(T - T_0)^3 + \frac{1}{24}K^{(4)}(T_0)(T - T_0)^4 + \cdots . \end{aligned} \tag{3.14}$$

Where $K^{(r)}$ is the rth derivative of K (also denoted with r primes for $r < 4$). Now fill in the contour $\gamma(t) = T_0 + it$ and use the fact that $K'(T_0) = \tilde{x}$ to get

$$\begin{aligned} K(T_0 + it) - (T_0 + it)\tilde{x} =& K(T_0) - T_0\tilde{x} + 0 + \frac{1}{2}K''(T_0)(it)^2 \\ &+ \frac{1}{6}K'''(T_0)(it)^3 + \frac{1}{24}K^{(4)}(T_0)(it)^4 + \cdots \\ =& K(T_0) - T_0\tilde{x} - \frac{1}{2}K''(T_0)t^2 \\ &+ \frac{1}{6}K'''(T_0)it^3 + \frac{1}{24}K^{(4)}(T_0)t^4 + \cdots . \end{aligned} \tag{3.15}$$

Then the integral becomes

$$\begin{aligned} &\frac{n}{2\pi}\int_{-\infty}^{\infty} e^{n(K(\gamma(t))-\gamma\tilde{x}}dt = \\ &\frac{n}{2\pi}e^{K(T_0)-T_0\tilde{x}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}nK''(T_0)t^2}e^{n(-\frac{1}{6}K'''(T_0)it^3+\frac{1}{24}K^{(4)}(T_0)t^4+\cdots)}dt. \end{aligned} \tag{3.16}$$

We expand the last exponent as

$$\begin{aligned} &e^{n(-\frac{1}{6}K'''(T_0)it^3+\frac{1}{24}K^{(4)}(T_0)t^4+\cdots)} = \\ &1+[n(-\frac{1}{6}K'''(T_0)it^3+\frac{1}{24}K^{(4)}(T_0)t^4+\cdots)]+\frac{1}{2}[-n(\frac{1}{6}K'''(T_0)it^3+\frac{1}{24}K^{(4)}(T_0)t^4+\cdots)]^2+\cdots . \end{aligned} \tag{3.17}$$

Where we used the expansion for the exponential function. Now in order to use 3.15 we must do some bookkeeping.We have the term $e^{-\frac{1}{2}nK''(T_0)t^2}$ which we would later on replace by $e^{-\frac{1}{2}v^2}$ so that we will apply the substitution $t = v/\sqrt{nK''(T_0)}$. We would like the integrand of 3.17 to be of order $O(n^{-2})$. So for $n(\frac{1}{6}K'''(T_0)it^3 + \frac{1}{24}K^{(4)}(T_0)t^4 + \cdots)$ we only need to write up to the 6th power. So we use $n(\frac{1}{6}K'''(T_0)it^3 + \frac{1}{24}K^{(4)}(T_0)t^4 + \frac{1}{120}K^{(5)}(T_0)t^5 + \frac{1}{720}K^{(6)}(T_0)t^6 + \cdots)$. For $[n(\frac{1}{6}K'''(T_0)it^3 + \frac{1}{24}K^{(4)}(T_0)t^4 + \cdots)]^2$ we need only up to third power. For $[n(\frac{1}{6}K'''(T_0)it^3 + \frac{1}{24}K^{(4)}(T_0)t^4 + \cdots)]^3$ we do not use any terms as well as for higher order terms. Then

$$e^{n(-\frac{1}{6}K'''(T_0)it^3+\frac{1}{24}K^{(4)}(T_0)t^4+\cdots)} \approx \tag{3.18}$$

$$1 - \frac{1}{6}nK'''(T_0)it^3 + \frac{1}{24}nK^{(4)}(T_0)t^4 - \frac{1}{2}\frac{1}{36}n^2K'''(T_0)t^6. \tag{3.19}$$

Then we get

$$\begin{aligned} f_n(\tilde{x}) &= \frac{n}{2\pi}e^{K(T_0)-T_0\tilde{x}} \\ &\times \int_{-\infty}^{\infty} e^{-\frac{1}{2}nK''(T_0)t^2} \\ &\times \left\{1 - \frac{1}{6}nK'''(T_0)it^3 + \frac{1}{24}nK^{(4)}(T_0)t^4 - \frac{1}{2}\frac{1}{36}n^2K'''(T_0)t^6 + \right. \\ &\left. \frac{1}{120}nK^{(5)}(T_0)it^5 - \frac{1}{720}nK^{(6)}(T_0)t^6 + \cdots\right\} dt. \end{aligned} \tag{3.20}$$

Now apply the substitution $t = v/\sqrt{nK''(T_0)}$ to obtain that

$$\begin{aligned} f_n(\tilde{x}) &= \frac{1}{2\pi}\frac{\sqrt{n}}{\sqrt{K''(T_0)}}e^{K(T_0)-T_0\tilde{x}} \\ &\times \int_{-\infty}^{\infty} e^{-\frac{1}{2}v^2} \\ &\times \left\{1 - \frac{1}{6}\frac{K'''(T_0)iv^3}{\sqrt{n}(K''(T_0))^{3/2}} + \frac{1}{n}\left(\frac{1}{24}\frac{K^{(4)}(T_0)v^4}{K''(T_0))^2} - \frac{1}{72}\frac{K'''(T_0)^2v^6}{K''(T_0))^3}\right)\right. \\ &\left. + \frac{1}{n^{3/2}}\frac{1}{120}\frac{K^{(5)}(T_0)iv^5}{K''(T_0))^{5/2}} - \frac{1}{n^2}\frac{1}{720}\frac{K^{(6)}(T_0)v^6}{K''(T_0))^3} + \cdots\right\} dv. \end{aligned} \tag{3.21}$$

The odd powers intergrate to 0 giving

$$\begin{aligned} f_n(\tilde{x}) &= \frac{\sqrt{n}}{\sqrt{2\pi K''(T_0)}}e^{K(T_0)-T_0\tilde{x}} \\ &\times \left\{1 + \frac{1}{n}\left(\frac{3}{24}\frac{K^{(4)}(T_0)}{K''(T_0))^2} - \frac{15}{72}\frac{K'''(T_0)^2}{K''(T_0))^3}\right) - \frac{1}{n^2}\frac{5}{720}\frac{K^{(6)}(T_0)}{K''(T_0))^3} + \cdots\right\}. \end{aligned} \tag{3.22}$$

For evaluating the even powers we used the fact that for even j and $j > 0$, it holds that for the moments of the standard normal density

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}v^2} v^j dv = \prod_{m=1}^{j/2} (2m-1). \tag{3.23}$$

If $j = 0$ the moment is 1 and for odd j the moment is 0. Using the definition $\lambda_r(T_0) := K^{(r)}/(K''(T_0))^{r/2}$ we can simplify 3.22 to

$$\begin{aligned} f_n(\tilde{x}) = & \frac{\sqrt{n}}{\sqrt{2\pi K''(T_0)}} e^{K(T_0)-T_0\tilde{x}} \\ & \times \left\{ 1 + \frac{1}{n} \left(\frac{1}{8}\lambda_4(T_0) - \frac{5}{24}\lambda_3^2(T_0) \right) + \mathcal{O}(n^{-2}) \right\}. \end{aligned} \tag{3.24}$$

3.2 Approximating a tail probability

For the tail probability $Q_n(\tilde{x}) := \mathbb{P}(\tilde{X} \geq \tilde{x})$ it holds that

$$Q_n(\tilde{x}) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{n(K(T)-T\tilde{x})} \frac{dT}{T} \quad , \quad c > 0. \tag{3.25}$$

This can be derived from 3.11 in the following way. Again let $c > 0$

$$\begin{aligned} Q_n(\tilde{x}) &= \int_{\tilde{x}}^{\infty} f_n(y) dy \\ &= \int_{\tilde{x}}^{\infty} \frac{n}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{n(K(T)-Ty)} dT dy \\ &= \int_{c-i\infty}^{c+i\infty} \int_{\tilde{x}}^{\infty} \frac{n}{2\pi i} e^{n(K(T)-Ty)} dy dT \\ &= \int_{c-i\infty}^{c+i\infty} \left[\frac{n}{2\pi i} \frac{1}{(-nT)} e^{n(K(T)-Ty)} \right]_{\tilde{x}}^{\infty} dT \\ &= \int_{c-i\infty}^{c+i\infty} \frac{1}{2\pi i} \frac{e^{n(K(T)-T\tilde{x})}}{T} dT. \end{aligned} \tag{3.26}$$

3.2.1 Approximation method 1

There two methods of getting an approximating formula for the above probability. The first method goes as follows. First get a Taylor expansion for $e^{n(K(T)-T\tilde{x})}$ similar to the one used in the previous section. Then we get (with $c > 0$ right of the singularity at 0)

$$
\begin{aligned}
Q_n(\tilde{x}) = & \frac{1}{2\pi i} e^{n(K(T_0)-T_0\tilde{x})} \\
& \times \int_{c-i\infty}^{c+i\infty} e^{-\frac{1}{2}nK''(T_0)(T-T_0)^2} \\
& \times \left\{ 1 + \frac{n}{6}K_0^{(3)}(T-T_0)^3 + \frac{n}{24}K_0^{(4)}(T-T_0)^4 + \frac{n}{120}K_0^{(5)}(T-T_0)^5 + \right. \\
& \left. \frac{n^2}{72}(K_0^{(3)})^2(T-T_0)^6 + \cdots \right\} \frac{dT}{T}.
\end{aligned} \tag{3.27}
$$

Where $K_0^{(i)} := K(T_0)^{(i)}$. The terms of the expansion can be found in the following way. Define

$$
I_r := \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{\frac{1}{2}z^2 - z_0 z}(z-z_0)^r \frac{dz}{z}. \tag{3.28}
$$

Where $\mathrm{Re}(z_0) > 0$. Then I_r satisfies the recurrence relations

$$
\begin{aligned}
I_{2m} &= -z_0 I_{2m-1} \\
I_{2m+1} &= (-1)^m a_m \phi(z_0) - z_0 I_{2m}.
\end{aligned}
$$

Where $I_0 = 1 - \Phi(z_0)$, $a_0 = 1$, $a_m = 1 \times 3 \times 5 \times \ldots \times (2m-1)$, ϕ, Φ are the standard normal density and distribution. The recurrence relations can be seen by

$$
\begin{aligned}
I_{r+1} :=& \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{\frac{1}{2}z^2 - z_0 z}(z-z_0)^{r+1} \frac{dz}{z} \\
=& \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{\frac{1}{2}z^2 - z_0 z}(z-z_0)^r (z-z_0) \frac{dz}{z} \\
=& - z_0 I_r + \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{\frac{1}{2}z^2 - z_0 z}(z-z_0)^r dz \\
=& - z_0 I_r + e^{\frac{1}{2}z_0^2} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{\frac{1}{2}(z-z_0)^2}(z-z_0)^r dz \\
=& - z_0 I_r + e^{\frac{1}{2}z_0^2} \frac{1}{2\pi i} \int_{z_0-i\infty}^{z_0+i\infty} e^{\frac{1}{2}(z-z_0)^2}(z-z_0)^r dz \\
=& - z_0 I_r + e^{\frac{1}{2}z_0^2} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\frac{1}{2}(it)^2}(it)^r dt \\
=& - z_0 I_r + e^{\frac{1}{2}z_0^2} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{1}{2}t^2} t^r i^r dt.
\end{aligned}
$$

We shifted the contour of integration from c to z_0. This is allowed as $e^{\frac{1}{2}(z-z_0)^2}(z-z_0)^r$ tends to zero for $z = c + it$ and t tends to $\pm\infty$. If r is even, say $r = 2m$, then

$$
\begin{aligned}
I_{r+1} =& - z_0 I_r + e^{\frac{1}{2}z_0^2} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{1}{2}t^2} t^{2m} (-1)^m dt \\
=& - z_0 I_r + \phi(z_0) a_m (-1)^m.
\end{aligned}
$$

Where

$$a_m = \int_{-\infty}^{\infty} \phi(t)t^{2m}dt = 1 \times 3 \times \ldots \times m. \tag{3.29}$$

When r is odd then by symmetry

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}t^2} t^r i^r dt = 0. \tag{3.30}$$

And $I_{r+1} = -z_0 I_r$. Combining all this gives the explicit formula

$$I_r = (-z_0)^r(1 - \Phi(z_0)) + (-1)^{r-1}\phi(z_0) \sum_{m=0}^{[\frac{1}{2}(r-1)]} (-1)^m a_m z_0^{r-2m-1}. \tag{3.31}$$

Writing out the formulas for $r = 1, 2, \ldots, 6$

$$\begin{aligned}
I_1 &= \phi(z_0) - z_0\left(1 - \Phi(z_0)\right) \\
I_2 &= -z_0\phi(z_0) + z_0^2\left(1 - \Phi(z_0)\right) \\
I_3 &= (z_0^2 - 1)\phi(z_0) - z_0^3\left(1 - \Phi(z_0)\right) \\
I_4 &= -(z_0^3 - z_0)\phi(z_0) + z_0^4\left(1 - \Phi(z_0)\right) \\
I_5 &= (z_0^4 - z_0^2 + 3)\phi(z_0) - z_0^5\left(1 - \Phi(z_0)\right) \\
I_6 &= (z_0^5 - z_0^3 + 3z_0)\phi(z_0) + z_0^6\left(1 - \Phi(z_0)\right).
\end{aligned} \tag{3.32}$$

For $r = 0$ we can see that I_0 is just the Bromwich integral as in 3.25 with $n = 1$ and $K(T) = \frac{1}{2}T^2$. The cumulant $K(T) = \frac{1}{2}T^2$ belongs to the standard normal distribution. Since the tail probability for a standard normal distribtion is $1 - \Phi$ we get that $I_0 = 1 - \Phi(z_0)$. In our case z_0 can be taken to be real, otherwise Φ has to be analytically continued over the complex plane. Now use the substition $T = z/\sqrt{nK''(T_0)}$ and $T_0 = z_0/\sqrt{nK''(T_0)}$. Then substituting this in 3.27 we get

$$\begin{aligned}
Q_n(\tilde{x}) = {} & e^{n(K(T_0) - T_0\tilde{x})} e^{\frac{1}{2}z_0^2} \\
& \times \left(I_0 + \frac{1}{n^{1/2}} \frac{1}{6} \frac{K_0^{(3)}}{(K_0'')^{3/2}} I_3 + \frac{1}{n} \frac{1}{24} \frac{K_0^{(4)}}{(K_0'')^2} I_4 + \right. \\
& \left. \frac{1}{n^{3/2}} \frac{1}{120} \frac{K_0^{(5)}}{(K_0'')^{5/2}} I_5 + \frac{1}{n} \frac{1}{72} \frac{(K_0^{(3)})^2}{(K_0'')^3} I_6 + \cdots \right).
\end{aligned} \tag{3.33}$$

Where

$$I_0 + \frac{1}{n^{1/2}}\frac{1}{6}\frac{K_0^{(3)}}{(K_0'')^{3/2}}I_3 + \frac{1}{n}\frac{1}{24}\frac{K_0^{(4)}}{(K_0'')^2}I_4 + \frac{1}{n^{3/2}}\frac{1}{120}\frac{K_0^{(5)}}{(K_0'')^{5/2}}I_5 + \frac{1}{n}\frac{1}{72}\frac{(K_0^{(3)})^2}{(K_0'')^3}I_6 + \cdots =$$
$$(1-\Phi(z_0)) + (1-\Phi(z_0))\frac{1}{n}\left\{\frac{1}{24}\frac{K_0^{(4)}}{(K_0'')^2}z_0^4 + \frac{1}{72}\frac{(K_0^{(3)})^2}{(K_0'')^3}z_0^6\right\}$$
$$-\phi(z_0)\frac{1}{n}\left\{\frac{1}{24}\frac{K_0^{(4)}}{(K_0'')^2}(z_0^3 - z_0) + \frac{1}{72}\frac{(K_0^{(3)})^2}{(K_0'')^3}(z_0^5 - z_0^3 + 3z_0)\right\}$$
$$+\phi(z_0)\frac{1}{n^{1/2}}\left\{\frac{1}{6}\frac{K_0^{(3)}}{(K_0'')^{3/2}}(z_0^2 - 1)\right\} - (1-\Phi(z_0))\frac{1}{n^{1/2}}\left\{\frac{1}{6}\frac{K_0^{(3)}}{(K_0'')^{3/2}}(z_0^3)\right\} + \mathcal{O}(n^{-3/2}). \tag{3.34}$$

The right hand side of the last equation equals

$$(1-\Phi(z_0))\left[1 - \frac{z_0^3}{6n^{1/2}}\frac{K_0^{(3)}}{(K_0'')^{3/2}} + \frac{1}{n}\left\{\frac{z_0^4}{24}\frac{K_0^{(4)}}{(K_0'')^2} + \frac{z_0^6}{72}\frac{(K_0^{(3)})^2}{(K_0'')^3}\right\}\right]$$
$$+\phi(z_0)\left[\frac{1}{n^{1/2}}\left\{\frac{1}{6}\frac{K_0^{(3)}}{(K_0'')^{3/2}}(z_0^2 - 1)\right\}\right.$$
$$\left.-\frac{1}{n}\left\{\frac{1}{24}\frac{K_0^{(4)}}{(K_0'')^2}(z_0^3 - z_0) + \frac{1}{72}\frac{(K_0^{(3)})^2}{(K_0'')^3}(z_0^5 - z_0^3 + 3z_0)\right\}\right] + \mathcal{O}(n^{-3/2}). \tag{3.35}$$

Then for $Q_n(\tilde{x})$ we have the approximation

$$Q_n(\tilde{x}) = e^{n(K(T_0) - T_0\tilde{x}) + \frac{1}{2}z_0^2}$$
$$\times\left((1-\Phi(z_0))\left[1 - \frac{z_0^3}{6n^{1/2}}\frac{K_0^{(3)}}{(K_0'')^{3/2}} + \frac{1}{n}\left\{\frac{z_0^4}{24}\frac{K_0^{(4)}}{(K_0'')^2} + \frac{z_0^6}{72}\frac{(K_0^{(3)})^2}{(K_0'')^3}\right\}\right]\right.$$
$$+\phi(z_0)\left[\frac{1}{n^{1/2}}\left\{\frac{1}{6}\frac{K_0^{(3)}}{(K_0'')^{3/2}}(z_0^2 - 1)\right\}\right.$$
$$\left.\left.-\frac{1}{n}\left\{\frac{1}{24}\frac{K_0^{(4)}}{(K_0'')^2}(z_0^3 - z_0) + \frac{1}{72}\frac{(K_0^{(3)})^2}{(K_0'')^3}(z_0^5 - z_0^3 + 3z_0)\right\}\right]\right) \times (1 + \mathcal{O}(n^{-3/2})). \tag{3.36}$$

This formula is valid if $\tilde{x} > E(X)$. In case that $\tilde{x} < E(X)$ and thus $T_0 < 0$ then we get that

$$Q_n(\tilde{x}) = 1 + \frac{1}{2\pi i}\int_{T_0 - i\infty}^{T_0 + i\infty} e^{n(K(T) - T\tilde{x})}\frac{dT}{T} \quad , \qquad T_0 > 0. \tag{3.37}$$

Where the 1 comes from pulling the contour from the positive part of the real line over the origin, but not crossing the pole at 0 leaving a cirle around $T = 0$ (also two line segments connecting the circle with the new contour, but these cancel). Then from complex analysis we know that for $\gamma(t) = e^{2\pi i t}, \quad t \in [0, 1)$ the integral

$$\frac{1}{2\pi i}\int_\gamma e^{n(K(T) - T\tilde{x})}\frac{dT}{T} = e^{n(K(0) - 0\tilde{x})} = 1. \tag{3.38}$$

For $\tilde{x} = E(X)$ we have

$$Q_n(\tilde{x}) = \frac{1}{2}1 + \frac{1}{2\pi i}\int_{T_0-i\infty}^{T_0+i\infty} e^{n(K(T)-T\tilde{x})}\frac{dt}{T}. \tag{3.39}$$

Also for I_r a similar thing holds. When $z_0 < 0$ and with the same definition for I_r namely

$$I_r := \frac{1}{2\pi i}\int_{z_0-i\infty}^{z_0+i\infty} e^{\frac{1}{2}z^2 - z_0 z}(z-z_0)^r\frac{dz}{z}.$$

The recursion in 3.29 still holds. The only thing that is different is I_0. To get I_0 we notice that for $c < 0$

$$\begin{aligned} I_0 :=& \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\frac{1}{2}z^2 - z_0 z}\frac{dz}{z} \\ =& -H(\gamma) + \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\frac{1}{2}z^2 - z_0 z}\frac{dz}{z}. \end{aligned} \tag{3.40}$$

Where $H(x) = 1$

$$H(x) = \begin{cases} 1 & x > 0 \\ \frac{1}{2} & x = 0 \\ 0 & x < 0 \end{cases} \tag{3.41}$$

When the contour is pulled over the singularity at $z = 0$ ($\gamma > 0$) we add -1 since we incure a loop around $z = 0$, $e^{-2\pi i t}$, $t \in [0, 2\pi]$, but it is clockwise so its value is $-e^0 = -1$. When $\gamma = 0$ we incur a semi-circle $e^{-2\pi i t}$, $t \in [\pi/2, 3\pi/2]$, clockwise around $z = 0$ and the integral over this semi-circle is $-1/2$. Then the general formula for I_0 becomes

$$I_0 = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\frac{1}{2}z^2 - z_0 z}\frac{dz}{z} = \begin{cases} -\Phi(z_0) & c < 0 \\ 1/2 - \Phi(z_0) & c = 0 \\ 1 - \Phi(z_0) & c > 0 \end{cases}$$

If we set $c = T_0$ and $z_0 = T_0$, so at the saddlepoint, we get as a general formula for I_r

$$I_r = (-z_0)^r(H(z_0) - \Phi(z_0)) + (-1)^{r-1}\phi(z_0)\sum_{m=0}^{[\frac{1}{2}(r-1)]}(-1)^m a_m z_0^{r-2m-1}. \tag{3.42}$$

This all together gives the general saddlepoint formula for any T_0

$$\begin{aligned} Q_n(\tilde{x}) =& H(-T_0) + e^{n(K(T_0)-T_0\tilde{x}) + \frac{1}{2}z_0^2} \\ &\times \Bigg((H(T_0) - \Phi(z_0)) \left[1 - \frac{z_0^3}{6n^{1/2}}\frac{K_0^{(3)}}{(K_0'')^{3/2}} + \frac{1}{n}\left\{ \frac{z_0^4}{24}\frac{K_0^{(4)}}{(K_0'')^2} + \frac{z_0^6}{72}\frac{(K_0^{(3)})^2}{(K_0'')^3} \right\} \right] \\ &+ \phi(z_0)\left[\frac{1}{n^{1/2}}\left\{ \frac{1}{6}\frac{K_0^{(3)}}{(K_0'')^{3/2}}(z_0^2 - 1) \right\} \right. \\ &\left. -\frac{1}{n}\left\{ \frac{1}{24}\frac{K_0^{(4)}}{(K_0'')^2}(z_0^3 - z_0) + \frac{1}{72}\frac{(K_0^{(3)})^2}{(K_0'')^3}(z_0^5 - z_0^3 + 3z_0) \right\} \right] \Bigg) \times (1 + \mathcal{O}(n^{-3/2})). \end{aligned} \tag{3.43}$$

3.2.2 Approximation method 2

The second method is also called the Lugannani-Rice formula. The method described is due to Bleistein (1966) or uses an idea by him. The basic idea is to find a transformation which describes the local behaviour of the function $K(T)-T\tilde{x}$ over a region containing both $T=\hat{T}$ and $T=0$ when $\hat{T}$ is small. Such a tranformation is given by

$$\frac{1}{2}(W-\hat{W})^2 = K(T) - T\tilde{x} - K(\hat{T}) + \hat{T}\tilde{x}. \tag{3.44}$$

Where $\hat{W}$ is chosen such that $K(\hat{T}) - \hat{W}\tilde{x} = -\frac{1}{2}\hat{W}^2$. So we get that

$$\frac{1}{2}W^2 - \hat{W}W = K(T) - TK'(\hat{T}). \tag{3.45}$$

The local behaviour of $K(T)-TK'(\hat{T})$ which is zero at $T=0$, since $K(0)=0$, and has zero derivative at $\hat{T}$ is reproduced by a quadratic in W with similar behaviour at $\hat{W}$ and $W=0$. It is easy to see that $\frac{1}{2}W^2 - \hat{W}W$ is zero for $W=0$ and the derivative $W-\hat{W}$ is zero at $\hat{W}$. The inversion formula 3.25 transforms into

$$Q_n(\tilde{x}) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{n(\frac{1}{2}W^2-\hat{W}W)}\left(\frac{1}{T}\frac{dT}{dW}\right)dW \quad , \quad c>0. \tag{3.46}$$

When T is small then $W \sim AT$, since T is small also W is small and neither the derivative of $K(T)-TK'(\hat{T})$ nor that of $\frac{1}{2}W^2-\hat{W}W$ is zero at $T=0$ and $W=0$ respectively. Then by using the implicit function theorem we get

$$A = -\frac{\left.\frac{d(K(T)-TK'(\hat{T})-(\frac{1}{2}W^2-\hat{W}W))}{dT}\right|_{T=0}}{\left.\frac{d(K(T)-TK'(\hat{T})-(\frac{1}{2}W^2-\hat{W}W))}{dW}\right|_{W=0}} = \frac{K'(\hat{T})-K'(0)}{\hat{W}} = \frac{\tilde{x}-E(X)}{\hat{W}}. \tag{3.47}$$

When $\tilde{x} \neq E(X)$ and $A = (K''(0))^{1/2}$ (Daniels has $A=(K''(0))^{-1/2}$) when $\tilde{x}=E(X)$. The last part can be seen by letting $\hat{T}\to 0$. Then

$$\lim_{\hat{T}\to 0} A = \lim_{\hat{T}\to 0}\frac{K'(\hat{T})-K'(0)}{\hat{W}} = \lim_{\hat{T}\to 0}\frac{K''(\hat{T})}{d\hat{W}/d\hat{T}}. \tag{3.48}$$

The limit of the derivative $d\hat{W}/d\hat{T}$ can be determined as follows.

$$\begin{aligned}\frac{d\hat{W}}{d\hat{T}} &= \frac{2\hat{T}K''(\hat{T})}{2\sqrt{2(\hat{T}K''(\hat{T})-K(\hat{T}))}}\\ &= \frac{\hat{T}K''(\hat{T})}{\sqrt{2(K'(0)\hat{T}+K''(0)\hat{T}^2+\mathcal{O}(\hat{T}^3)) - 2(K(0)+K'(0)\hat{T}+\frac{1}{2}K''(0)\hat{T}^2+\mathcal{O}(\hat{T}^3))}}\\ &= \frac{K''(\hat{T})}{\sqrt{K''(0)+\mathcal{O}(\hat{T})}}.\end{aligned} \tag{3.49}$$

Now let if we let $\hat{T}\to 0$ the derivative becomes

$$\lim_{\hat{T}\to 0}\frac{d\hat{W}}{d\hat{T}} = \lim_{\hat{T}\to 0}\frac{K''(\hat{T})}{\sqrt{K''(0)+\mathcal{O}(\hat{T})}} = \sqrt{K''(0)}. \tag{3.50}$$

Then $T^{-1}dT/dW \sim W^{-1}$ as $T^{-1}dT/dW = (A/W)A^{-1} = 1/W$ for small W. Since dT/dW is analytic in a neighbourhood of $W = 0$ due to $K(T)$ being analytic, so is $T^{-1}dT/dW - W^{-1}$ and 3.51 can be written as

$$Q_n(\tilde{x}) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{n(\frac{1}{2}W^2-\hat{W}W)}\frac{dW}{W} + e^{-n\frac{1}{2}\hat{W}^2}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{n\frac{1}{2}(W-\hat{W})^2}\left(\frac{1}{T}\frac{dT}{dW}-\frac{1}{W}\right)dW. \tag{3.51}$$

The singularity has been isolated into the first term, which has the value $1-\Phi(W\hat{n^{1/2}})$. In the second term $T^{-1}dTdW - W^{-1}$ is expanded about $\hat{W}$. We will define

$$G_1 := e^{-n\frac{1}{2}\hat{W}^2}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{n\frac{1}{2}(W-\hat{W})^2}\frac{1}{T}\frac{dT}{dW}dW$$
$$G_2 := e^{-n\frac{1}{2}\hat{W}^2}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{n\frac{1}{2}(W-\hat{W})^2}\frac{dW}{W}.$$

We start by analysing G_1 first. By reverting back to using T we get that

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{n(\frac{1}{2}W^2-\hat{W}W)}\frac{1}{T}\frac{dT}{dW}dW =$$
$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{n(K(T)-T\tilde{x})}\frac{dT}{T}.$$

Here we can use the expansion of $e^{n(K(T)-T\tilde{x})}$ and $1/T$. From the density we saw that $e^{n(K(T)-T\tilde{x})}$ could be expanded around $\hat{T}$ when $c = \hat{T}$. Define

$$g(t) := e^{n(K(\hat{T})-\hat{T}\tilde{x})}e^{-\frac{1}{2}nK''(\hat{T})t^2}. \tag{3.52}$$

Then we get

$$\begin{aligned} &g(t)e^{n\left(-\frac{1}{6}K'''(\hat{T})it^3+\frac{1}{24}K^{(4)}(\hat{T})t^4+\cdots\right)} = \\ &g(t)\left\{1+n\left[-\frac{1}{6}K'''(\hat{T})it^3+\frac{1}{24}K^{(4)}(\hat{T})t^4+\cdots\right]\right. \\ &\left.+\frac{1}{2}n^2\left[-\frac{1}{6}K'''(\hat{T})it^3+\frac{1}{24}K^{(4)}(\hat{T})t^4+\cdots\right]^2+\cdots\right\}. \end{aligned} \tag{3.53}$$

Now we also expand $1/T$ around $\hat{T}$

$$\frac{1}{\hat{T}+it} = \frac{1}{\hat{T}} - \frac{it}{\hat{T}^2} + \frac{(it)^2}{\hat{T}^3} - \frac{(it)^3}{\hat{T}^4} + \cdots = \frac{1}{\hat{T}} - \frac{it}{\hat{T}^2} - \frac{t^2 2}{\hat{T}^3} + \frac{it^3}{\hat{T}^4} + \cdots. \tag{3.54}$$

Then we get

$$g(t)\left\{1+n\left[-\frac{1}{6}K'''(\hat{T})it^3+\frac{1}{24}K^{(4)}(\hat{T})t^4+\cdots\right]\right.$$
$$\left.+\frac{1}{2}n^2\left[-\frac{1}{6}K'''(\hat{T})it^3+\frac{1}{24}K^{(4)}(\hat{T})t^4+\cdots\right]^2+\cdots\right\}$$
$$\times\left\{\frac{1}{\hat{T}}-\frac{it}{\hat{T}^2}-\frac{t^2 2}{\hat{T}^3}+\frac{it^3}{\hat{T}^4}+\cdots\right\}=$$
$$g(t)\left\{\frac{1}{\hat{T}}\left(1+\frac{n}{24}K^{(4)}(\hat{T})t^4-\frac{1}{2}\frac{n^2}{36}\left(K^{(3)}(\hat{T})\right)^2t^6\right)\right.$$
$$\left.-\frac{1}{\hat{T}^2}\left(\frac{n}{6}K^{(3)}(\hat{T})t^4\right)-\frac{t^2}{\hat{T}^3}+P(t)+\mathcal{O}(n^{-2})\right\}. \tag{3.55}$$

Where P is a polynomial in which every part has odd degree. Upon integration this will yield zero, i.e.

$$\int_{-\infty}^{\infty}g(t)P(t)dt=0. \tag{3.56}$$

Now use the substitution

$$t=\frac{v}{\sqrt{nK''(\hat{T})}}. \tag{3.57}$$

This gives

$$g(t)\left\{\frac{1}{\hat{T}}\left(1+\frac{n}{24}K^{(4)}(\hat{T})t^4-\frac{1}{2}\frac{n^2}{36}\left(K^{(3)}(\hat{T})\right)^2t^6\right)-\frac{1}{\hat{T}^2}\left(\frac{n}{6}K^{(3)}(\hat{T})t^4\right)-\frac{t^2}{\hat{T}^3}+P(t)+\mathcal{O}(n^{-2})\right\}=$$
$$\frac{e^{n(K(\hat{T})-\hat{T}\tilde{x})}e^{-\frac{1}{2}v^2}}{\sqrt{nK''(\hat{T})}}\left\{\frac{1}{\hat{T}}\left(1+\frac{1}{24n}\frac{K^{(4)}(\hat{T})}{K''(\hat{T})^2}v^4-\frac{1}{72n}\frac{\left(K^{(3)}(\hat{T})\right)^2}{K''(\hat{T})^3}v^6\right)\right.$$
$$\left.-\frac{1}{\hat{T}^2}\left(\frac{1}{6n}\frac{K^{(3)}(\hat{T})}{K''(\hat{T})^2}v^4\right)-\frac{v^2}{nK''(\hat{T})\hat{T}^3}+\mathcal{O}(n^{-2})\right\}.$$

Now we take the integral (ignoring the term $1/(2\pi)$ still)

$$
\begin{aligned}
&\frac{e^{n(K(\hat{T})-\hat{T}\tilde{x})}}{\hat{T}\sqrt{nK''(\hat{T})}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}v^2}\left\{1+\frac{1}{n}\left(\frac{1}{24}\frac{K^{(4)}(\hat{T})}{K''(\hat{T})^2}v^4-\frac{1}{72}\frac{\left(K^{(3)}(\hat{T})\right)^2}{K''(\hat{T})^3}v^6\right.\right.\\
&\left.\left.-\frac{1}{\hat{T}}\frac{1}{6}\frac{K^{(3)}(\hat{T})}{K''(\hat{T})^2}v^4-\frac{v^2}{K''(\hat{T})\hat{T}^2}\right)+\mathcal{O}(n^{-2})\right\}dv=\\
&\frac{e^{n(K(\hat{T})-\hat{T}\tilde{x})}}{\hat{T}\sqrt{nK''(\hat{T})}}\sqrt{2\pi}\left\{1+\frac{1}{n}\left(\frac{3}{24}\frac{K^{(4)}(\hat{T})}{K''(\hat{T})^2}-\frac{15}{72}\frac{\left(K^{(3)}(\hat{T})\right)^2}{K''(\hat{T})^3}\right.\right.\\
&\left.\left.-\frac{3}{6\hat{T}}\frac{K^{(3)}(\hat{T})}{K''(\hat{T})^2}-\frac{1}{K''(\hat{T})\hat{T}^2}\right)+\mathcal{O}(n^{-2})\right\}=\\
&\frac{e^{n(K(\hat{T})-\hat{T}\tilde{x})}\sqrt{2\pi}}{\hat{T}\sqrt{nK''(\hat{T})}}\left\{1+\frac{1}{n}\left(\frac{1}{8}\lambda_4-\frac{5}{24}\lambda_3^2-\frac{1}{2\hat{T}}\frac{\lambda_3}{K''(\hat{T})^{1/2}}-\frac{1}{K''(\hat{T})\hat{T}^2}\right)+\mathcal{O}(n^{-2})\right\}. \quad (3.58)
\end{aligned}
$$

Now we can define G_1

$$
G_1 =\frac{1}{2\pi}\frac{e^{n(K(\hat{T})-\hat{T}\tilde{x})}\sqrt{2\pi}}{\hat{T}\sqrt{nK''(\hat{T})}}\left\{1+\frac{1}{n}\left(\frac{1}{8}\lambda_4-\frac{5}{24}\lambda_3^2-\frac{1}{2\hat{T}}\frac{\lambda_3}{K''(\hat{T})^{1/2}}-\frac{1}{K''(\hat{T})\hat{T}^2}\right)+\mathcal{O}(n^{-2})\right\} \quad (3.59)
$$

$$
=\frac{e^{n(K(\hat{T})-\hat{T}\tilde{x})}}{\hat{T}\sqrt{2\pi nK''(\hat{T})}}\left\{1+\frac{1}{n}\left(\frac{1}{8}\lambda_4-\frac{5}{24}\lambda_3^2-\frac{1}{2\hat{T}}\frac{\lambda_3}{K''(\hat{T})^{1/2}}-\frac{1}{K''(\hat{T})\hat{T}^2}\right)+\mathcal{O}(n^{-2})\right\}. \quad (3.60)
$$

Now we expand the integrand of G_2 around $\hat{W}$

$$
\frac{1}{W}=\frac{1}{\hat{W}}-\frac{W-\hat{W}}{\hat{W}^2}+\frac{(W-\hat{W})^2}{\hat{W}^2}+\cdots. \quad (3.61)
$$

Then we get

$$
G_2=\frac{1}{2\pi i}e^{-\frac{1}{2}n\hat{W}^2}\int_{c-i\infty}^{c+i\infty} e^{-\frac{1}{2}n(W-\hat{W})^2}\left\{\frac{1}{\hat{W}}-\frac{W-\hat{W}}{\hat{W}^2}+\frac{(W-\hat{W})^2}{\hat{W}^2}+\cdots\right\}dW. \quad (3.62)
$$

Using the substitution $z=\sqrt(n)(W-\hat{W})$ we get

$$
\int_{c-i\infty}^{c+i\infty} e^{-\frac{1}{2}n(W-\hat{W}^2)^2}(W-\hat{W})^{k+1}\frac{dW}{W-\hat{W}}=\frac{1}{n^{(k+1)/2}}\int_{c+\hat{W}-i\infty}^{c+\hat{W}+i\infty} e^{\frac{1}{2}z^2}z^{k+1}\frac{dz}{z}. \quad (3.63)
$$

Define

$$
I_{k+1}:=\int_{c+\hat{W}-i\infty}^{c+\hat{W}+i\infty} e^{\frac{1}{2}z^2}z^{k+1}\frac{dz}{z}. \quad (3.64)
$$

Applying 3.28 where we let $\hat{z} \to 0$ and take $\mathrm{Re}(c + \hat{W}) > 0$. The $I_{k+1} = 0$ if k is odd. So the powers with $k+1$ is odd vanish. For k even, with $k = 2m$, we get that

$$I_{k+1} = (-1)^m a_m \phi(0). \tag{3.65}$$

Where $a_m = 1 \times 3 \times \ldots \times (2m-1)$. This gives

$$\begin{aligned}
&e^{-\frac{1}{2}n\hat{W}^2} \int_{c-i\infty}^{c+i\infty} e^{\frac{1}{2}n(W-\hat{W}^2)^2} \left\{ \frac{1}{\hat{W}} - \frac{W-\hat{W}}{\hat{W}^2} + \frac{(W-\hat{W})^2}{\hat{W}^2} + \cdots \right\} dW = \\
&e^{-\frac{1}{2}n\hat{W}^2} \left\{ \frac{a_0\phi(0)}{n^{1/2}\hat{W}} - \frac{a_1\phi(0)}{n^{3/2}\hat{W}^3} + \frac{a_2\phi(0)}{n^{5/2}\hat{W}^5} - \frac{a_3\phi(0)}{n^{7/2}\hat{W}^7} + \cdots \right\} = \\
&e^{-\frac{1}{2}n\hat{W}^2} \phi(0) \left\{ \frac{1}{n^{1/2}\hat{W}} - \frac{1}{n^{3/2}\hat{W}^3} + \frac{1 \times 3}{n^{5/2}\hat{W}^5} - \frac{1 \times 3 \times 5}{n^{7/2}\hat{W}^7} + \cdots \right\} = \\
&e^{-\frac{1}{2}n\hat{W}^2} \frac{1}{\sqrt{2\pi}} \left\{ \frac{1}{n^{1/2}\hat{W}} - \frac{1}{n^{3/2}\hat{W}^3} + \frac{1 \times 3}{n^{5/2}\hat{W}^5} - \frac{1 \times 3 \times 5}{n^{7/2}\hat{W}^7} + \cdots \right\} = \\
&\phi(\sqrt{n}\hat{W}) \left\{ \frac{1}{n^{1/2}\hat{W}} - \frac{1}{n^{3/2}\hat{W}^3} + \frac{1 \times 3}{n^{5/2}\hat{W}^5} - \frac{1 \times 3 \times 5}{n^{7/2}\hat{W}^7} + \cdots \right\}.
\end{aligned} \tag{3.66}$$

When we combine G_1 and G_2 we get

$$\begin{aligned}
G_1 - G_2 =& \frac{e^{n(K(\hat{T}) - \hat{T}\tilde{x})}}{\hat{T}\sqrt{2\pi n K''(\hat{T})}} \left\{ 1 + \frac{1}{n}\left(\frac{1}{8}\lambda_4 - \frac{5}{24}\lambda_3^2 - \frac{1}{2\hat{T}} \frac{\lambda_3}{K''(\hat{T})^{1/2}} - \frac{1}{K''(\hat{T})\hat{T}^2} \right) + \mathcal{O}(n^{-2}) \right\} \\
&- \phi(\sqrt{n}\hat{W}) \left\{ \frac{1}{n^{1/2}\hat{W}} - \frac{1}{n^{3/2}\hat{W}^3} + \mathcal{O}(n^{-5/2}) \right\} \\
=& \frac{\phi(\sqrt{n}\hat{W}}{\hat{T}\sqrt{K''(\hat{T})}} \frac{1}{\sqrt{n}} \left\{ 1 + \frac{1}{n}\left(\frac{1}{8}\lambda_4 - \frac{5}{24}\lambda_3^2 - \frac{1}{2\hat{T}} \frac{\lambda_3}{K''(\hat{T})^{1/2}} - \frac{1}{K''(\hat{T})\hat{T}^2} \right) + \mathcal{O}(n^{-2}) \right\} \\
&- \phi(\sqrt{n}\hat{W}) \left\{ \frac{1}{n^{1/2}\hat{W}} - \frac{1}{n^{3/2}\hat{W}^3} + \mathcal{O}(n^{-5/2}) \right\}.
\end{aligned} \tag{3.67}$$

Now define $\hat{U} := \hat{T}\sqrt{K''(\hat{T})}$ then we get

$$\begin{aligned}
G_1 - G_2 =& \phi(\sqrt{n}\hat{W}) \left\{ \frac{1}{\sqrt{n}} \left(\frac{1}{\hat{U}} - \frac{1}{\hat{W}} \right) + \frac{1}{n^{3/2}} \left(\frac{1}{\hat{U}} \left(\frac{1}{8}\lambda_4 - \frac{5}{24}\lambda_3^2 - \frac{1}{2\hat{T}} \frac{\lambda_3}{K''(\hat{T})^{1/2}} - \frac{1}{K''(\hat{T})\hat{T}^2} \right) \right.\right. \\
&\left.\left. + \frac{1}{\hat{W}^3} \right) + \mathcal{O}(n^{-5/2}) \right\} \\
=& \phi(\sqrt{n}\hat{W}) \left\{ \frac{1}{\sqrt{n}} \left(\frac{1}{\hat{U}} - \frac{1}{\hat{W}} \right) + \frac{1}{n^{3/2}} \left(\frac{1}{\hat{U}} \left(\frac{1}{8}\lambda_4 - \frac{5}{24}\lambda_3^2 \right) - \frac{\lambda_3}{2\hat{U}^2} - \frac{1}{\hat{U}^3} + \frac{1}{\hat{W}^3} \right) + \mathcal{O}(n^{-5/2}) \right\}.
\end{aligned} \tag{3.68}$$

Combining all these results we get

$$
\begin{aligned}
Q_n(\tilde{x}) =& 1-\Phi(n^{1/2}\hat(W)) \\
&+\phi(\sqrt{n}\hat{W})\left\{\frac{1}{\sqrt{n}}\left(\frac{1}{\hat{U}}-\frac{1}{\hat{W}}\right)+\frac{1}{n^{3/2}}\left(\frac{1}{\hat{U}}\left(\frac{1}{8}\lambda_4-\frac{5}{24}\lambda_3^2\right)-\frac{\lambda_3}{2\hat{U}^2}-\frac{1}{\hat{U}^3}+\frac{1}{\hat{W}^3}\right)\right. \\
&\left.+\mathcal{O}(n^{-5/2})\right\} \qquad (3.69) \\
=& 1-\Phi(n^{1/2}\hat(W))+\phi(\sqrt{n}\hat{W})\left\{\frac{b_0}{n^{1/2}}+\frac{b_1}{n^{3/2}}+\cdots+\frac{b_k}{n^{k+1/2}}+\mathcal{O}(n^{-k-1-1/2})\right\}. \qquad (3.70)
\end{aligned}
$$

Where

$$
b_0=\frac{1}{\hat{U}}-\frac{1}{\hat{W}} \quad , \quad b_1=\frac{1}{\hat{U}}\left(\frac{1}{8}\lambda_4-\frac{5}{24}\lambda_3^2\right)-\frac{\lambda_3}{2\hat{U}^2}-\frac{1}{\hat{U}^3}+\frac{1}{\hat{W}^3}. \qquad (3.71)
$$

When $\hat{U}$, and hence $\hat{W}$, is small the terms that make up $b_0, b_1, \ldots$ look like they become infinitely large. However by writing $\hat{W}$ as an expansion in powers of $\hat{U}$ we see that the b_0, b_1 are in fact bounded around $\hat{T}=0$. First write

$$
\hat{W}=\hat{U}-\frac{1}{6}\lambda_3\hat{U}^2+\frac{1}{24}(\lambda_4-\frac{1}{3}\lambda_3^2)\hat{U}^3-\frac{1}{24}\left(\frac{1}{5}\lambda_5-\frac{1}{6}\lambda_3\lambda_4+\frac{1}{18}\lambda_3^2\right)\hat{U}^4+\cdots. \qquad (3.72)
$$

We can write

$$
b_0=\frac{1}{\hat{U}}-\frac{1}{\hat{W}}=\frac{\hat{W}-\hat{U}}{\hat{W}\hat{U}}. \qquad (3.73)
$$

Then $\hat{W}-\hat{U}=-\frac{1}{6}\lambda_3\hat{U}^2+\mathcal{O}(\hat{U}^3)$ and $\hat{W}\hat{U}=\hat{U}^2+\mathcal{O}(\hat{U}^3)$. This means that around $\hat{U}=0$ b_0 is bounded. Doing the same for b_1 we get that

$$
b_1=\frac{\hat{U}^2\hat{W}^3\left(\frac{1}{8}\lambda_4-\frac{5}{24}\lambda_3^2\right)-\frac{\lambda_3}{2}\hat{U}\hat{W}^3-\hat{W}^3+\hat{U}^3}{\hat{W}^3\hat{U}^3}. \qquad (3.74)
$$

Where

$$
\begin{aligned}
\left(\frac{1}{8}\lambda_4-\frac{5}{24}\lambda_3^2\right)\hat{U}^2\hat{W}^3 &= \left(\frac{1}{8}\lambda_4-\frac{5}{24}\lambda_3^2\right)\hat{U}^5+\mathcal{O}(\hat{U}^6) \\
-\frac{1}{2}\lambda_3\hat{U}\hat{W}^3 &= -\frac{1}{2}\lambda_3(\hat{U}^4-\frac{3}{6}\lambda_3\hat{U}^4)+\mathcal{O}(\hat{U}^6) \\
-\hat{W}^3 &= -\left(\hat{U}^3-\frac{1}{2}\lambda_3\hat{U}^4+\frac{1}{12}\lambda_3^2\hat{U}^5+\frac{3}{24}(\lambda_4-\frac{1}{3}\lambda_3^2)\hat{U}^5\right)+\mathcal{O}(\hat{U}^6) \\
\hat{U}^3 &= \hat{U}^3.
\end{aligned}
$$

If we sum the left hand side and the right hand side we get that

$$
\hat{U}^2\hat{W}^3\left(\frac{1}{8}\lambda_4-\frac{5}{24}\lambda_3^2\right)-\frac{\lambda_3}{2}\hat{U}\hat{W}^3-\hat{W}^3+\hat{U}^3=\mathcal{O}(\hat{U}^6). \qquad (3.75)
$$

Together with $\hat{W}^3\hat{U}^3=\hat{U}^6+\mathcal{O}(\hat{U}^7)$ we get that b_1 is also bounded around $\hat{T}=0$.

3.2.3 Using the saddlepoint approximation

We now have two formulas for the approximation of the excess probability. The first one is

$$\begin{aligned}
Q_n(\tilde{x}) &= H(-T_0) + e^{n(K(T_0)-T_0\tilde{x})+\frac{1}{2}z_0^2} \\
&\times \left((H(T_0) - \Phi(z_0)) \left[1 - \frac{z_0^3}{6n^{1/2}} \frac{K_0^{(3)}}{(K_0'')^{3/2}} + \frac{1}{n} \left\{ \frac{z_0^4}{24} \frac{K_0^{(4)}}{(K_0'')^2} + \frac{z_0^6}{72} \frac{(K_0^{(3)})^2}{(K_0'')^3} \right\} \right] \right. \\
&+ \phi(z_0) \left[\frac{1}{n^{1/2}} \left\{ \frac{1}{6} \frac{K_0^{(3)}}{(K_0'')^{3/2}} (z_0^2 - 1) \right\} \right. \\
&\left. \left. - \frac{1}{n} \left\{ \frac{1}{24} \frac{K_0^{(4)}}{(K_0'')^2} (z_0^3 - z_0) + \frac{1}{72} \frac{(K_0^{(3)})^2}{(K_0'')^3} (z_0^5 - z_0^3 + 3z_0) \right\} \right] \right) \times (1 + \mathcal{O}(n^{-3/2})).
\end{aligned} \tag{3.76}$$

The second one is

$$\begin{aligned}
Q_n(\tilde{x}) =& 1 - \Phi(n^{1/2}\hat{(}W)) \\
&+ \phi(\sqrt{n}\hat{W}) \left\{ \frac{1}{\sqrt{n}} \left(\frac{1}{\hat{U}} - \frac{1}{\hat{W}} \right) + \frac{1}{n^{3/2}} \left(\frac{1}{\hat{U}} \left(\frac{1}{8}\lambda_4 - \frac{5}{24}\lambda_3^2 \right) - \frac{\lambda_3}{2\hat{U}^2} - \frac{1}{\hat{U}^3} + \frac{1}{\hat{W}^3} \right) \right. \\
&\left. + \mathcal{O}(n^{-5/2}) \right\} \qquad (3.77) \\
=& 1 - \Phi(n^{1/2}\hat{(}W)) + \phi(\sqrt{n}\hat{W}) \left\{ \frac{b_0}{n^{1/2}} + \frac{b_1}{n^{3/2}} + \cdots + \frac{b_k}{n^{k+1/2}} + \mathcal{O}(n^{-k-1-1/2}) \right\}. \qquad (3.78)
\end{aligned}$$

These approximate the probability $\mathbb{P}(\tilde{X} \geq \tilde{x})$. These approximations are suited to when $\tilde{x}$ is located in the tail region. The approximation refers to a sample mean when the sample is an i.i.d. sample. In our case of a credit portfolio we have neither independence nor an identical distribution for all off the loan losses. We can condition on the common factor. Then all the loan losses are independent. By taking the formulas and insert $n = 1$ and letting $M(T) = \prod_{i=1}^{n} M_i(T, Y)$, where $M_i(T, Y)$ is the conditional moment generating function for loan loss i and n is the original value (not 1). We can use the saddlepoint approximations with $K(T) := \log(M(T))$. In using the distribution given the common factor we have to use the conditional PD $c_i(Y)$ where

$$c_i(Y) := \Phi\left(\frac{\Phi^{-1}(p_i) + \sqrt{\rho_i}Y}{\sqrt{1-\rho_i}} \right). \tag{3.79}$$

Then the probability $\mathbb{P}(L^p > u)$ can be calculated by

$$\mathbb{P}(L^p > u) = \mathbb{E}\left(\mathbb{P}\left(L^p > u \middle| Y \right)\right) = \int_{-\infty}^{\infty} \mathbb{P}\left(L^p > u \middle| y \right) f(y) dy. \tag{3.80}$$

Where f is the density of the common factor Y. Here $\mathbb{P}\left(L^p > u \middle| y \right)$ can be approximated by $Q_{n=1}(u)$, where we use $M(T) = \prod_{i=1}^{n} M_i(T, Y)$. Then we get that

$$\mathbb{P}(L^p > u) \approx \int_{-\infty}^{\infty} Q_{n=1}(u) f(y) dy. \tag{3.81}$$

Where the dependence of $Q_{n=1}(u)$ on Y is implicit via the conditional Laplace transform.

3.2.4 Calculating cumulants

The saddlepoint approximation formulas depend on derivatives of cumulant K. Since the cumulant K is the sum of the cumulants K_i of the individual cumulants of the loan losses, i.e. $K(T) = \sum_{i=1}^{n} K_i(T)$. Where $K_i(T)$ is the cumulant of loan loss L_i conditional on the common factor Y. Then

$$K_i(T, Y) = \log\left(M_i(T, Y)\right). \tag{3.82}$$

We define for the ease of notation

$$\frac{\partial^r K_i(T, Y)}{\partial T^r} := K_i^{(r)}(T, Y) \tag{3.83}$$

$$\frac{\partial^r M_i(T, Y)}{\partial T^r} := M_i^{(r)}(T, Y) \tag{3.84}$$

Then we have

$$\begin{aligned}
K_i^{(1)}(T, Y) =& \frac{M_i^{(1)}(T, Y)}{M_i(T, Y)} \\
K_i^{(2)}(T, Y) =& \frac{M_i^{(2)}(T, Y)}{M_i(T, Y)} - \left(K_i^{(1)}(T, Y)\right)^2 \\
K_i^{(3)}(T, Y) =& \frac{M_i^{(3)}(T, Y)}{M_i(T, Y)} - 3K_i^{(1)}(T, Y)K_i^{(2)}(T, Y) - \left(K_i^{(1)}(T, Y)\right)^3 \\
K_i^{(4)}(T, Y) =& \frac{M_i^{(4)}(T, Y)}{M_i(T, Y)} - \Big(4K_i^{(1)}(T, Y)K_i^{(3)}(T, Y) + 3K_i^{(2)}(T, Y)K_i^{(2)}(T, Y) \\
& +6K_i^{(2)}(T, Y)\left(K_i^{(1)}(T, Y)\right)^2 + \left(K_i^{(1)}(T, Y)\right)^4\Big)
\end{aligned} \tag{3.85}$$

If we use our Merton model with only two possible states (default or non-default) we have that

$$M_i(T, Y) = 1 - c_i(Y) + c_i(Y)e^{LGD_i \cdot EAD_i}. \tag{3.86}$$

The partial derivatives of $M_i(T, Y)$ with respect to T are then for $r = 1, 2, \ldots$

$$M_i^{(r)}(T, Y) = (LGD_i \cdot EAD_i)^r c_i(Y)e^{TLGD_i \cdot EAD_i}, \tag{3.87}$$

3.3 Saddlepoint contour

Suppose we have a function of the form $e^{f(z)}$ that we want to integrate over some contour and use the real part of the integral. We also use an approximation of f that works well around a neighbourhood of some point w. For our approximating to give a good approximation of the integral we would like to have that the real part of f on the contour has a maximum at w and decreases rapidly away from w at either direction. Define $z = x + iy$ and $u(x, y) = \mathrm{Re}(f(z))$

and $v(x,y) = \text{Im}(f(z))$. If we are at point $z = x + iy$ the direction in which u decreases or increases the most is $[\partial u(x,y)/\partial x, \partial u(x,y)/\partial y]$. If we have a path $h(t) + ig(t), \quad t \in \mathbb{R}$ and at t we have that $dh(t)/dt\partial u(x,y)/\partial x$ and $dg(t)/dt = \partial u(x,y)/\partial y$ then with $z = h(t)+ig(t)$

$$
\begin{aligned}
\frac{dv(h(t),g(t)}{dt} &= \frac{\partial v(h(t),g(t)}{\partial x}\frac{dh(t)}{dt} + \frac{\partial v(h(t),g(t)}{\partial y}\frac{dg(t)}{dt} \\
&= \frac{\partial v(h(t),g(t)}{\partial x}\frac{\partial u(x,y)}{dx} + \frac{\partial v(h(t),g(t)}{\partial y}\frac{\partial u(x,y)}{dy} \\
&= 0.
\end{aligned}
\tag{3.88}
$$

Where the last equality is due to the Cauchy-Riemann equations. This means that the imaginary part of f must be zero on the contour where the real parts decreases or increases the fastest. Take the case $f(z) := n(K(z) - \tilde{z(x)})$. For our case we know that if z is on the real axis then f must be real as well. We also know for that $f'' > 0$ as $K''(z) > 0$ for any real z. Although the imaginary part of f is zero and therefor constant on the real axis it is not a good contour as we have a minimum here. If we have a contour $h(t) + ig(t)$ that crosses the real axis at w_0, but $f'(w_0) \neq 0$, then we do not have that v is constant along that contour. This is becaus the only direction in which v is constant is that on the real axis. In case that $f'(w) = 0$ then there could be other paths running through w for which v is constant along the path. If we move along the direction $[\partial u(x,y)/\partial x, \partial u(x,y)/\partial y]$ we want the have that u is decreasing away from a point w at which u should have a maximum. We also know that $\partial^2 u(x,y)/\partial x^2 > 0$ for $y = 0$. Then by an application of the Cauchy-Riemann equations we know that $\partial^2 u(x,y)/\partial x^2 + \partial^2 u(x,y)/\partial y^2 = 0$ giving $\partial^2 u(x,y)/\partial y^2 < 0$ on $y = 0$. We have the original contour $w_0 + it, \quad t \in (-\infty, \infty)$. We can deform this contour such that it the function f has the right properties of quick decrease after a maximum on this deformed contour. So we apply the following strategy for our new contour γ. For some $\varepsilon > 0$, where ε is small, we take that $\gamma_0(t) = w_0 + it$ for $t \in [-\varepsilon, \varepsilon]$. Here w_0 is any of the real valued solutions to $w_0 := \{z | K'(z) - \tilde{x} = 0\}$. Then from the points $w_0 + i\varepsilon$ and $w_0 - i\varepsilon$ we continue on a contour for which the imaginary part of f is constant, call this contours γ_1 and γ_2. Then we have that $\text{Im}(f(\gamma_1(t))) = \text{Im}(f(w_0 + i\varepsilon)), \quad t \in (\varepsilon, t_1]$ and $\text{Im}(f(\gamma_2(t))) = \text{Im}(f(w_0 - i\varepsilon)), \quad t \in [-t_2, -\varepsilon)$ for some $0 < t_1, t_2 < \infty$. After γ_1 we take a contour, $\gamma_3(t), \quad t \in (t_1, t_3]$, back to the original contour and after γ_2 we take a contour, $\gamma_4(t), \quad t \in (-t_4, -t_2]$ back to the original contour. For γ_1 and γ_2 we want to have that at the ends of the contours, i.e. at $\gamma_1(t_3)$ and $\gamma_2(t_4)$, that the the real part of f is small. We also let $\varepsilon \to 0$. Then $\text{Im}(f(w_0 + i\varepsilon)) \to \text{Im}(f(w_0))$. Also we have that along our new contour u is decreasing at least before entering contours γ_3 or γ_4. This can be seen as follows. Along the contour γ_0 we have that u is downward sloping except at $\gamma_0 = w_0$. In order for u to be upward sloping again on γ_1 or γ_2 we need to have crossed a stationary point where $f' = 0$. In our case there is only one stationary point w_0 and we have that u is decreasing on some contour γ where $u(\gamma(t)) < u(w_0)$ it cannot come to a point w_0 such that $u(\gamma(t)) = u(w_0)$. We also have to choose t_1, t_2 such that after these points u is negligible.

3.4 Wavelets

This section is due to [7]. They employ wavelets to numerically invert the Laplace transform. We will use the Merton default model described in the previous chapter. Also in this case the LGD and EAD are taken to be fixed and there is one common risk factor. Again let L_i be the loss on loan i, L^p the credit portfolio loss with $L^p = \sum_i L_i$. First some definitions. let F be the cumulative distribution function of L^p (we also denote it just as L). Now assume without loss of generality that $\sum_{i=1}^{n} EAD_i = 1$. Suppose

$$F = \begin{cases} \bar{F}(x), & \text{if } 0 \leq x \leq 1, \\ 1, & \text{if } x > 1, \end{cases} \tag{3.89}$$

For some $\bar{F}$ defined on $[0, 1]$.

3.4.1 The Haar basis wavelets system

Consider the space of measurable functions defined on $\mathbb{R}$ called $L^2(\mathbb{R})$ defined as $L^2(\mathbb{R}) := \{f : \int_{-\infty}^{\infty} |f(x)|^2 dx < \infty\}$. A general structure for wavelets in $L^2(\mathbb{R})$ is called a Multi-resolution Analysis (MRA). We start with a family of closed nested subspaces

$$\ldots \subset V_{-2} \subset V_{-1} \subset V_{-0} \subset V_1 \subset V_2. \tag{3.90}$$

in $L^2(\mathbb{R})$ where

$$\bigcap_{j \in \mathbb{Z}} V_j = 0, \qquad \overline{\bigcup_{j \in \mathbb{Z}} V_j} = L^2(\mathbb{R}). \tag{3.91}$$

This means that the intersection of all the subspaces is the zero function and the closure of their union is the space $L^2(\mathbb{R})$. Also

$$f(x) \in V_j \Leftrightarrow f(2x) \in V_{j+1}. \tag{3.92}$$

If these conditions are met, then there exists a function $\phi \in V_0$ such that $\{\phi_{j,k}\}_{k \in \mathbb{Z}}$ is an orthonormal basis of V_j, where

$$\phi_{j,k}(x) = 2^{j/2} \phi(2^j x - k). \tag{3.93}$$

This means that, the function ϕ, called the father function, will generate an orthonormal basis for each subspace V_j. Then we define W_j such that $V_{j+1} = V_j \oplus W_j$. This says that W_j is the space of functions that is in V_{j+1}, but not in V_j. Then $L^2(\mathbb{R}) = \sum_j \oplus W_j$. Then there exists a function $\psi \in W_0$ such that $\{\psi_{j,k}\}_{k \in \mathbb{Z}}$ is an orthonormal basis of W_j and $\{\psi_{j,k}\}_{k \in \mathbb{Z}}$ is a wavelet basis of $L^2(\mathbb{R})$, where

$$\begin{aligned} \psi_{j,k}(x) &= 2^{j/2} \psi(2^j x - k), \\ \int_{-\infty}^{\infty} \psi(x) dx &= 0. \end{aligned} \tag{3.94}$$

The function ψ is called the mother function and the $\{\psi_{j,k}\}_{k \in \mathbb{Z}}$ are the wavelet functions. For any function $f \in L^2(\mathbb{R})$ a projection map of $L^2(\mathbb{R})$ onto V_m

$$\mathcal{P}_m : L^2(\mathbb{R}) \to V_m. \tag{3.95}$$

is defined by

$$\mathcal{P}_m f(x) = \sum_{j=-\infty}^{m-1} \sum_{k=-\infty}^{k=+\infty} d_{j,k}\psi_{j,k}(x) = \sum_{k\in\mathbb{Z}} c_{m,k}\phi_{m,k}(x) \tag{3.96}$$

Where $d_{j,k} = \int_{-\infty}^{+\infty} f(x)\psi_{j,k}(x)dx$ are the wavelet coefficients and $c_{m,k} = \int_{-\infty}^{+\infty} f(x)\phi_{m,k}(x)dx$ are the scaling coefficients. Considering higher values of m, using more terms, the truncation becomes a better approximation of f. For the analysis we will use Haar wavelets. For these wavelets the space V_j are all functions that are constant on a interval of the form $\left[\frac{k}{2^j}, \frac{k+1}{2^j}\right)$ for all integers k. Then

$$\phi(x) \begin{cases} 1, & \text{if } 0 \leq x < 1, \\ 0, & \text{otherwise.} \end{cases} \tag{3.97}$$

and

$$\psi(x) \begin{cases} 1, & \text{if } 0 \leq x < \frac{1}{2}, \\ -1, & \text{if } \frac{1}{2} \leq x < 1, \\ 0, & \text{otherwise.} \end{cases} \tag{3.98}$$

3.4.2 Haar wavelets approximation

Since the CDF $\bar{F}$ defined above is in our case a discontinuous step function the Haar wavelets are a very suitable wavelet to approximate this function.

Laplace transform inversion

Since $\bar{F} \in L^2([0,1])$ and using the theory of MRA we can approximate $\bar{F}$ on $[0,1]$ by sum of scaling functions

$$\bar{F}(x) \approx \sum_{k=0}^{2^m-1} c_{m,k}\phi_{m,k}(x), \tag{3.99}$$

and

$$\bar{F}(x) = \lim_{m\to+\infty} \sum_{k=0}^{2^m-1} c_{m,k}\phi_{m,k}(x), \tag{3.100}$$

For the unconditional Laplace transform M_L we have that

$$M_L(s) := \mathbb{E}\left(e^{-sL}\right) = \int_0^{+\infty} e^{-sx} dF(x), \tag{3.101}$$

Where the last integral is with respect to the counting measure on the set $x_1, \ldots, x_N$, which is the set of all possible values of L. Also for $Re(s) > 0$ and $x \geq 0$ we have that

$$e^{-sx} = \int_x^\infty se^{-st}dt, \tag{3.102}$$

Then with μ the counting measure on $x_1, \ldots, x_N$ and $dF = f(x)\mu(dx)$

$$\begin{aligned}
\int_0^\infty e^{-sx}dF(x) &= \int_0^\infty \int_x^\infty se^{-st}dtdF(x) \\
&= \int_0^\infty \int_x^\infty se^{-st}f(x)dt\mu(dx) \\
&= \int_0^\infty se^{-st} \int_0^t f(x)\mu(dx)dt \\
&= \int_0^\infty se^{-st}F(t)dt \\
&= \int_0^1 se^{-st}\bar{F}(t)dt + \int_1^\infty se^{-st}dt \\
&= \int_0^1 se^{-st}\bar{F}(t)dt + e^{-s}.
\end{aligned} \tag{3.103}$$

Where we used the fact that for $0 \leq x \leq 1$ $F(x) = \bar{F}(x)$ and for $x > 1$ $F(x) = 1$. If we then approximate the last part by inserting the expansion of $\bar{F}$ into scaling functions we get that

$$\begin{aligned}
M_L(s) &\approx e^{-s} + \int_0^1 se^{-st} \sum_{k=0}^{2^m-1} c_{m,k}\phi_{m,k}(t)dt \\
&= e^{-s} + 2^{\frac{m}{2}} s \sum_{k=0}^{2^m-1} c_{m,k}\tilde{\phi}_{m,k}(s).
\end{aligned}$$

Where

$$\tilde{\phi}_{m,k}(s) = \frac{1}{s}e^{-s\frac{k}{2^m}}\left(1 - e^{-s\frac{1}{2^m}}\right). \tag{3.104}$$

is the Laplace transform of the basis function $\phi_{m,k}$. We observe that $\tilde{\phi}_{m,k}(s) = \tilde{\phi}_{m,0}(s)e^{-s\frac{k}{2^m}}$ and making the change of variable $z = e^{-s\frac{1}{2^m}}$ we get that

$$Q(z) := \sum_{k=0}^{2^m-1} c_{m,k}z^k \approx \frac{M_L(-2^m \log(z)) - z^{2^m}}{2^{\frac{m}{2}}(1-z)}. \tag{3.105}$$

Where we used that $\tilde{\phi}_{m,0}(s) = (1-z)/s$. For $|z| < 1$ Q is analytic on this area. Also the singularity at $z = 0$ is removable since if we let the real part of s tend to infinity $M_l(s)$ will always converge to the same value, regardless how the real part converges to infinity or what the value of the imaginary part of s is. Then given Q we can obtain the value for $c_{m,k}$ using the Cauchy integral formula. Then

$$c_{m,k} = \frac{1}{2\pi i}\int_\gamma \frac{Q(z)}{z^{k+1}}dz, \qquad k = 0, 1, \ldots, 2^m - 1. \tag{3.106}$$

Where γ is a circle around the origin with radius r, $0 < r < 1$. If we make the change of variable $z = re^{iu}$, $0 < r < 1$ then

$$\begin{aligned} c_{m,k} &= \frac{1}{2\pi r^k}\int_0^{2\pi} \frac{Q(re^{iu})}{e^{iku}}du \\ &= \frac{1}{2\pi r^k}\int_0^{2\pi} \left[Re(Q(re^{iu}))\cos(ku) + Im(Q(re^{iu}))sin(ku)\right] du \\ &= \frac{2}{\pi r^k}\int_0^{\pi} Re(Q(re^{iu}))\cos(ku)du \end{aligned} \tag{3.107}$$

Where for second equality we use the fact that $c_{m,k}$ must be real valued. For the third equality that $\int_0^{2\pi}\cos(ku)\cos(mu)du = \int_0^{2\pi}\sin(ku)\sin(mu)du$ and $\int_0^{\pi}\cos(ku)\cos(mu)du = \int_\pi^{2\pi}\cos(ku)\cos(mu)du$ for $0 \le m \le k$.

Value-at-Risk computation

It is easy to see that

$$0 \le c_{m,k} \le 2^{-\frac{m}{2}}, \qquad k = 0, 1, \ldots, 2^m - 1 \tag{3.108}$$

as

$$c_{m,k} = \int_{-\infty}^{+\infty} \bar{F}(x)\phi_{m,k}(x)dx \le \int_0^{2^{-m}} 2^{m/2}dx = 2^{-m/2} \tag{3.109}$$

We also have that

$$0 \le c_{m,0} \le c_{m,1} \le \ldots \le c_{m,2^m-1} \tag{3.110}$$

This is due to the fact that $\phi_{m,k}$ shifts to the right as k increases and $\bar{F}(x)$ increases as x increases up to $x = 1$. We can approximate $\bar{F}$ by

$$\bar{F} \approx \sum_{k=0}^{2^m-1} c_{m,k}\phi_{m,k} = \sum_{k=0}^{2^m-1} 2^{\frac{m}{2}} c_{m,k} I_{\left[\frac{k}{2^m}, \frac{k+1}{2^m}\right)}(x) \tag{3.111}$$

Where I_A is the indicator function on the interval A. Calculating the q quantile l_q is easy now as we need to choose the smallest k such that $c_{m,k} \ge \alpha$ and then $l_q \in \left[\frac{k}{2^m}, \frac{k+1}{2^m}\right)$. Call this value k^*. Our desired quantile (or Value-at-Risk) will be $k^*/2^m$. We can search for $k*$ using a bisection algorithm so that we do not have to caculate all $c_{m,k}$ for all $k = 0, 1, \ldots, 2^m - 1$. By using a bi-section algorithm only for those k for $x \in \left[\frac{k}{2^m}, \frac{k+1}{2^m}\right)$ for the x value that is used in the bi-section algorithm needs to be calculated. We can determine the accuracy of l_q up to 2^{-m}.

Alternative method for determining the scaling coefficients

The scaling coefficients have been determined in the previous section by means of Cauchy's integral formula. We have that we need to determine $c_{m,k}$ for

$$Q(z) := \sum_{k=0}^{2^m-1} c_{m,k} z^k \approx \frac{M_L(-2^m \log(z)) - z^{2^m}}{2^{\frac{m}{2}}(1-z)} \tag{3.112}$$

Define

$$H(z) = \frac{M_L(-2^m \log(z)) - z^{2^m}}{2^{\frac{m}{2}}(1-z)} \tag{3.113}$$

If we wish to approximate H by a polynomial then the coefficients of this polynomial can also be determined by solving a linear system. We would like to have that $Q(z) \approx H(z)$. This holds for every $z \in D$ where D is some subset of $\mathbb{C}$. If we could calculate for some 2^m different values of z then using linear algebra to can obtain the $c_{m,k}$. The values of z can be chosen real. Now take some set $z_0, z_1, \ldots, z_{N-1}$, where $N = 2^m$. Define a vector $N \times 1$ y by $y_j = H(z_j)$, a vector b by $b_j = c_{m,j}$ and a $N \times N$ matrix X by $X_{s,t} = z_s^t$

$$b = X^{-1} y \tag{3.114}$$

Here $H(z)$ has to be calculated for many z values just like in the numerical Cauchy integral formula. With the cost of calculating the invese of X we get all $c_{m,k}$ for $k = 0, 1, \ldots, 2^m - 1$. The matrix X (or $X^\top$) is a so called Vandermonde matrix. There exist several algorithms to obtain the matrix inverse tailored to the case of a Vandermonde matrix.

3.5 De Hoog's method for numerical Laplace inversion

This part is based on [8]. In their article they address both the single common factor case as well as the multivariate common factor case. We shall address only the single common factor case. Let us start with some notation. Let X be the common factor. Again we use the Merton model for the default process. Let M_L be the Laplace transform of the portfolio loss and $M_L(X)$ the conditional Laplace trasnform of the portfolio loss, with $M_L(s) = \mathbb{E}(M_L(s,X))$. Let F_L be the portfolio distribution function define on $[0,1]$ by normalising the EADs such that $\sum_{j=1}^n EAD_j = 1$ where n is the total number of loans in the portfolio. Then by the Bromwich inversion formula we have that for $\gamma > 0$

$$\begin{aligned} F_L(x) &= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{sx}}{s} M_L(s)ds \\ &= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{sx}}{s} \mathbb{E}(M_L(s,X))ds. \end{aligned}$$

The key idea of De Hoog's algorithm is to divide the above interval of integration into small subintervals and evaluate the integral using the trapzoidal rule and obtain an infinite polynomial whose coefficients can be calculated. Then using a Padé approximation for this polynomial only relatively small number of the coefficients of the polynomial have to be calculated to get a fast convergence. If we divide the contour of the above integral into small intervals with width h. So the contour $\gamma + it$ with t ranging from $-\infty$ to ∞ is divided into small subcontours $[\gamma+ihk, \gamma+ih(k+1))$ with $k \in \mathbb{Z}$. Then we get for the approximation F_L^h of F_L

$$\begin{aligned} F_L^h(x) &:= \frac{h}{2\pi}\sum_{k=-\infty}^{\infty} \exp\{(\gamma+ikh)x\}\frac{M_L(\gamma+ikh)}{\gamma+ikh} \\ &= \frac{h}{\pi}\exp(\gamma x)\left[\frac{M_L(\gamma)}{2\gamma} + \sum_{k=1}^{\infty} Re\left\{\exp(ikhx)\frac{M_L(\gamma+ikh)}{\gamma+ikh}\right\}\right] \\ &= \frac{h}{\pi}\exp(\gamma x) Re\left(\sum_{k=0}^{\infty} s_k z_x^k\right). \end{aligned} \tag{3.115}$$

Where

$$\begin{aligned} s_0 &:= \frac{M_L(\gamma)}{2\gamma}, \qquad s_k := \frac{M_L(\gamma+ikh)}{\gamma+ikh} \quad, k = 1,2,\dots \\ z_x &:= \exp(ihx). \end{aligned}$$

If we truncate the infinite sum to a finite one we get

$$F_L(x) \approx F_L^{h,N_t}(x) := \frac{h}{\pi}\exp(\gamma x) Re\left(\sum_{k=0}^{N_t} s_k z_x^k\right). \tag{3.116}$$

Where N_t is the truncation parameter. The convergence of F_L^{h,N_t} is very slow if F_L is discontinuous as F_L^{h,N_t} can be written as a triginometric series

$$F_L^{h,N_t}(x) := \frac{h}{\pi}\exp(\gamma x)\sum_{k=0}^{N_t}\{a_k\cos(khx) + b_k\sin(khx)\}. \tag{3.117}$$

Where $a_k - ib_k := s_k$. Then for any discontinuity there have to be a lot of cosine and sine functions summed to get a good approximation, so N_t has to be large. A faster convergence is obtained by using a Padé approximation for $\sum_{k=0}^{\infty} s_k z_x^k$. This is the subject of the next section.

3.5.1 Continued fraction expansion

A continued fraction is a quantity of the form

$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cdots}}}. \tag{3.118}$$

These are also denoted as

$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cdots \frac{a_n}{b_n}}}} = b_0 + \left[\frac{a_k}{b_k}\right]_{k=1}^{n}. \tag{3.119}$$

Where $\left[\frac{a_k}{b_k}\right]_{k=1}^{n}$ is assumed to be 0 if $n \leq 0$.

Definition 1. *For all $i \geq 0$ and $k \geq 0$,*

$$\left|H_k^{(i)}\right| \neq 0, \tag{3.120}$$

where $H_k^{(i)}$ is a Hankel matrix

$$H_k^{(i)} = \begin{bmatrix} s_i & s_{i+1} & \cdots & s_{i+k-1} \\ s_{i+1} & s_{i+2} & \cdots & s_{i+k} \\ \vdots & \vdots & \ddots & \vdots \\ s_{i+k-1} & s_{i+k} & \cdots & s_{i+2k-2} \end{bmatrix}, \tag{3.121}$$

and $\left|H_k^{(i)}\right|$ is its determinant with $\left|H_0^{(i)}\right| = 0$

Now we can give a continued fraction expansion given the following three lemmas. Proofs are given in the appendix.

Lemma 1. *Let $S(z)$ be a power series whose coefficients are given by s_i, namely*

$$S(z) = \sum_{k=0}^{\infty} s_k z^k, \tag{3.122}$$

and a sequence c_k by

$$c_0 = s_0, \quad c_{2k-1} = -\frac{|H_{k-1}^{(0)}||H_k^{(1)}|}{|H_k^{(0)}||H_{k-1}^{(1)}|}, \quad c_{2k} = -\frac{|H_{k+1}^{(0)}||H_{k-1}^{(1)}|}{|H_k^{(0)}||H_k^{(1)}|}, \quad k = 1, 2, \ldots \tag{3.123}$$

Then a continued fraction

$$C_n(z) := c_0 / \left(1 + \left[\frac{c_k z}{1}\right]_{k=1}^{n}\right), \tag{3.124}$$

is the $[\lfloor n/2 \rfloor \,/\, \lfloor (n+1)/2 \rfloor]$ Padé approximation to $S(z)$. Or equivalently, $C_n(z)$ is a rational function whose numerator and denominator are polynomials of degree $\lfloor n/2 \rfloor$ and $\lfloor (n+1)/2 \rfloor$ repectively and satisfies

$$S(z) - C_n(z) = \mathcal{O}(z^{n+1}) \tag{3.125}$$

Here $\lfloor \cdot \rfloor$ is the rounding down to the nearest integer and the definition of the Padé approximation will be given in the appendix. This lemma allows us to find the Padé approximation from the determinants of the Hankel matrices. However determining the determinants directly takes a lot time. The following lemma give an efficients method of obtaining the determinants.

Lemma 2. *QD algorithm Let $e_k^{(i)}$ and $d_k^{(i)}$ be the sequences defined by*

$$\begin{aligned} e_0^{(i)} &= 0, \quad q_1^{(i)} = s_{i+1}/s_i, \quad i = 0, 1, 2, \ldots \\ e_k^{(i)} &= e_{k-1}^{(i+1)} + q_k^{(i+1)} - q_k^{(i)}, \qquad i = 0, 1, 2, \ldots; k = 1, 2, \ldots \\ q_{k+1}^{(i)} &= q_k^{(i+1)} e_k^{(i+1)} / e_k^{(i)}, \qquad i = 0, 1, 2, \ldots; k = 1, 2, \ldots \end{aligned}$$

Then they satisfy

$$e_k^{(i)} = \frac{|H_{k+1}^{(i)}||H_{k-1}^{(i+1)}|}{|H_k^{(i)}||H_k^{(i+1)}|}, \quad q_k^{(i)} = \frac{|H_{k-1}^{(i)}||H_k^{(i+1)}|}{|H_k^{(i)}||H_{k-1}^{(i+1)}|}, \quad i = 0, 1, 2, \ldots; k = 1, 2, \ldots \tag{3.126}$$

The next lemma show how to quickly calculate the continued fraction

Lemma 3. *Let $A_n(z)$ and $B_n(z)$ be the sequences of polynomials defined by*

$$\begin{aligned} A_{-1}(z) = 0, \quad A_0(z) &= c_0, \quad B_{-1}(z) = 1, \quad B_0(z) = 1, \\ A_{n+2}(z) &= A_{n+1}(z) + c_{n+2} A_n(z), \qquad n = -1, 0, 1, 2, \ldots \\ B_{n+2}(z) &= B_{n+1}(z) + c_{n+2} B_n(z) \end{aligned} \tag{3.127}$$

Then $A_n(z)$ and $B_n(z)$ are polynomials of degree $\lfloor n/2 \rfloor$ and $\lfloor (n+1)/2 \rfloor$ with $B_n(0) = 1$ and satisfy

$$C_n(z) = \frac{A_n(z)}{B_n(z)} \tag{3.128}$$

Where $C_n(z)$ is defined by 3.124

Now we want to approximate $\sum_{k=0}^{\infty} s_k z_x^k$ by a Padé approximation for say degree N_t. Then we determine s_k for $k = 0, 1, \ldots, 2N_t$. Then with the help of lemma 2 we can determine the relevant determinants of the Hankel matrices to get $c_0, c_1, \ldots, c_{2N_t}$. With lemma 3 we get the polynomials A_{2N_t} and B_{2N_t} and thus $C_n(z)$. This is then the approximation to $\sum_{k=0}^{\infty} s_k z_x^k$. Using this approximation we can calculate an approximation to $F_L^h(x)$ by setting $z = e^{ihx}$ in $C_n(z)$. Call this approximation as before F_L^{h,N_t}. Where

$$F_L^{h,N_t}(x) := \frac{h}{\pi} e^{\gamma x} \mathrm{Re}\left(C_{2N_t}(e^{ihx})\right) \tag{3.129}$$

It is easy to calculate this approximation for different values of x. Since we have that $\sum_{j=1}^{n} EAD_j = 1$ it follows that $0 \leq x \leq 1$. Then a simple search algorithm would give us the smallest x-value such that $F_L^{h,N_t}(x) \geq q$ for some q, say $q = 99.99\%$. This x-value is then the quantile we are looking for.

3.6 Gordy-Vasicek method

In this section we will discuss the method developed by Gordy to obtain an analytic formula for the quantile of the credit portfolio loss. See [5] for a formal treatment. The method is applicable to portfolios which can be considered well-diversified. This means that all loans are small compared to the total portfolio. Also it works only if the losses are correlated through one single factor and independent conditional on this single common factor. The key idea of the method is that if we condition on the single common factor the independence allows us to use the strong law of large numbers. This means that the loss (conditional on the common factor) converges to its conditional mean. This mean only depends on the common factor and the portfolio loss becomes a function of the common factor. If this function satisfies some conditions, e.g. monotonicity, then the quantile of the portfolio loss is simply the function evaluated at the quantile of the common factor. The quantile of the common factor is usually easily determined as its distribution is usually modeled as a standard distribution such as the normal distribution. The monotonicity assumption is just an example, some other conditions also lead to the same conclusion. Let us set-up the model again in the setting of the Merton model. Assume we have a portfolio with n loans. Let LGD_i and EAD_i, $i = 1, 2, \ldots, n$ be non-random, but do not need to be same for all loan. Also the PDs PD_i, $i = 1, 2, \ldots, n$ may differ across loans and the same holds for the asset correlation ρ_i. The conditional PD on the common factor X is

$$PD_i(X) = \Phi\left(\frac{\Phi^{-1}(PD_i) + \sqrt{\rho_i}X}{\sqrt{1-\rho_i}}.\right) \tag{3.130}$$

Where Φ is the standard normal distribution and Φ^{-1} is the inverse function of Φ. Define

$$w_{i,n} := \frac{LGD_i \cdot EAD_i}{\sum_{j=1}^{n} LGD_j \cdot EAD_j}, \quad L_{i,n} = D_i \cdot w_{i,n}. \tag{3.131}$$

where D_i is the default indicator for loan i and L_i the relative loss on loan i. Then under the condition that

$$\lim_{n\to\infty} \sum_{i=1}^{n} w_{i,n}^2 = 0. \tag{3.132}$$

We have that conditional on X and using the strong law of large numbers

$$\sum_{j=1}^{n} L_i \to \sum_{j=1}^{n} w_{i,n} PD_i(X) \tag{3.133}$$

This function is monotonically increasing in X since Φ is monotonically increasing. Then we have that the portfolio loss L_n, $L_n := \sum_{j=1}^{n} L_i$ converges to a function H with $H(x) := \lim_{n\to\infty} \sum_{j=1}^{n} w_{i,n} PD_i(x)$. The quantile of L_n converges to the quantile of $H(X)$. The quantile of $H(X)$, due to monotonicity, is just $H(\alpha_q(X))$, where $\alpha_q(X)$ is the quantile of X of level q.

3.7 Monte Carlo Simulation

In this section we describe the method of Monte Carlo simulation to obtain estimates for quantiles. Suppose we have clients $1, 2, \ldots, n$ with specific default indicator, PD, LGD and EAD and asset correlation. Call these $D_i, PD_i, LGD_i, EAD_i, \rho_i$, $i = 1, \ldots, n$. Define the loss for client i, L_i, as $L_i := D_i \cdot LGD_i \cdot EAD_i$. Now define the portfolio loss L^P again as before

$$L^P = \sum_{i=1}^{n} L_i. \tag{3.134}$$

The model for generating defaults is the Merton model from the previous chapters. So a default occurs if a clients asset return it too low. Now we simulate the losses by simulating the asset return for every individual. Let r_i be the asset return for client i. This return is composed of a common part, call this Y, and a idiosynctric part ε_i. Here we have that $r_i = \sqrt{\rho_i} Y + \sqrt{1-\rho_i}\varepsilon_i$. We also have that the random variables X and $\varepsilon_i, \quad i = 1, \ldots, n$ are independent of each other and standard normally distributed. So the common factor is independent of the idiosyncratic variables and they are independent from each other. Default for client i occurs if $r_i < \Phi^{-1}(PD_i)$. Where Φ^{-1} is inverse standard normal distribution function. Then

$$D_i = \begin{cases} 1 & r_i < \Phi^{-1}(PD_i) \\ 0 & r_i \geq \Phi^{-1}(PD_i) \end{cases} \tag{3.135}$$

Now we draw S times a $(n+1) \times 1$ vector $(Y^{(s)}, \varepsilon_1^{(s)}, \ldots, \varepsilon_1^{(s)}))^\top$. Here the superscipt s denotes the variable in simulation run s with $s = 1, 2, \ldots, S$. Over the simulation run all variables are independent as well of course. So after simulating the vector $(Y^{(s)}, \varepsilon_1^{(s)}, \ldots, \varepsilon_1^{(s)}))^\top$ we can calculate all $D_i^{(s)}$ and hence the $L_i^{(s)}$ and $L^{(s),P}$. Then we have a set of simulated portfolio loss $L^{(s),P}, \quad s = 1, \ldots, S$. We use this set to create an empirical distribution function. From this empirical distribution function we can get an estimate for a quantile at level q. If q is close to 1 then in order toe get a stable and accurate estimate the number of simulations S needs to be high. The Monte Carlo simulation itself is easy to implement on a computer. However there are some drawbacks, such as a large number of simulations are needed and we may not know how close our estimate of the quantile is to the actual value. Because we are working with very small probabilities a small deviation in an estimated probability may cause huge deviations in the quantile.

3.8 Recursive method

In case the losses on loans are discrete values and independent then one can recursively build up the portfolio loss. Now in our Merton model the losses are independent conditional on the common factor Y. We then use the conditional $PD(Y)$ for the conditional distribution of the losses. Then we build up the conditional distribution of the portfolio loss L^P. The idea is an application of the convolution of two independent discrete random variables. Suppose we have two independent random variables X and Y that take values in $\mathbb{Z}$. Then for $k \in \mathbb{Z}$

$$\mathbb{P}(X + Y = k) = \sum_{j \in \mathbb{Z}} \mathbb{P}(X = k - j) \tag{3.136}$$

In our case the random variables take only two possible values. Define

$$c_i(Y) := \Phi\left(\frac{\Phi^{-1}(p_i) - \sqrt{\rho_i}Y}{\sqrt{1-\rho_i}}\right). \tag{3.137}$$

Where $p_i = PD_i$. Define $L_k^P := \sum_{j=1}^k L_i$, where $L_0 = 0$ a.s.. Then $L_{k+1}^P = L_k^P + L_{k+1}$. Suppose $L_i \in \{a_i, b_i\}$ with $a_i, b_i \in \mathbb{Z}$, $a_i \leq b_i$ and $\mathbb{P}(L_i = a_i|Y) = 1 - c_i(Y)$, $\mathbb{P}(L_i = b_i|Y) = c_i(Y)$ then for $m \in \mathbb{Z} \cap [\min\{0, \sum_{i=1}^n a_i\}, \sum_{i=1}^n b_i]$

$$\mathbb{P}(L_{k+1}^P = m|Y) = \mathbb{P}(L_k^P = m - a_{k+1}|Y)(1 - c_{k+1}(Y)) + \mathbb{P}(L_k^P = m - b_{k+1}|Y)c_{k+1}(Y). \tag{3.138}$$

So we iterate over k until $k + 1 = n$. Then we have the conditional distribtion of L_n^P for which $L_n^P = L^P$. We have that

$$\mathbb{P}(L^P = m) = \int_{-\infty}^{\infty} \mathbb{P}(L^P = m|y)f(y)dy. \tag{3.139}$$

Where f is the density of the common factor which is standard normal. We can do a numeric integration by calculating $\mathbb{P}(L_{k+1}^P = m|y)$ for $y \in \{y_0, y_2, \ldots, y_S\}$. Here we choose $y_1, y_2, \ldots, y_S$ such that they form a relevant partition of the real line. We could choose $y_0 = -5, y_S = 5$ and $y_s = -5 + 10s/S$. Then

$$\mathbb{P}(L^P = m) \approx \sum_{s=1}^{S-1} \frac{1}{2}\left[\mathbb{P}(L^P = m|y_s)f(y_s) + \mathbb{P}(L^P = m|y_{s+1})f(y_{s+1})\right](y_{s+1} - y_s) \tag{3.140}$$

In case the y_s are equally space the above formula simplifies to the standard trapezium rule. This should give an accurate unconditional distribution of L^P. The quantiles then follow easily as we have the full distribution over all m with $\sum_{i=1}^n a_i \leq m \leq \sum_{i=1}^n b_i$. For our case we have that $a_i = 0$ and $b_i = LGD_i$ for $i = 1, \ldots, n$.

Chapter 4

Adjustments

In this chapter we discuss some adjustments to some of the methods described in the previous chapter.

4.1 Smoothing

The first adjustment is that of smoothing. The losses of indivual loans are either 0 or $LGD \cdot EAD$. In case the portfolio consists of many small loans and one very large loan we would see that the distribution has a small hump at the tail where the large[1] loan defaults. Between the lower part of the distribution, where the large loan is not in default, and the tail the distribution function is almost flat. For flat parts the methods using Laplace inversion would be expected not to work well as they are a sum of periodic functions. One way of mitigating the flatness of the distribution is by making the loss on individual loans, particularly the large ones, more smooth. We do this by creating some fuzziness around the points 0 and $LGD \cdot EAD$. So adjust the distribution of the loss on loan i, L_i, as follows

$$\mathbb{P}(L_i \leq b) = (1 - PD_i)\Phi\left(\frac{b}{\sigma_i}\right) + PD_i\Phi\left(\frac{b - LGD_i \cdot EAD_i}{\sigma_i}\right). \tag{4.1}$$

Where Φ is the standard normal distribution function. Here σ_i is a loan specific smoothing parameter. The larger σ_i is the more smooth the loss will be. Also σ_i will likely depend on $LGD_i \cdot EAD_i$ to get a smoothing that can deal with large loans. Whether a loan is considered to be large depends on its weight, w_i, in the portfolio measured by

$$w_i = \frac{LGD_i \cdot EAD_i}{\sum_{j=1}^{n} LGD_j \cdot EAD_j} \tag{4.2}$$

The Laplace transform of the loss of loan i under this new probability distribution is

$$M_{L_i}(t) := \mathbb{E}(e^{L_i t}) = (1 - PD_i)e^{\frac{1}{2}\sigma_i^2 t^2} + PD_i e^{(LGD_i \cdot EAD_i)t + \frac{1}{2}\sigma_i^2 t^2} \tag{4.3}$$

The difficulty in this method is finding an optimal value for σ_i if there exists any. We would to be able to determine this optimal value based on parameters like PD, LGD, EAD, asset correlation, number of loans in the portfolio and quantile level. We would expect that σ_i has a positive relation with w_i, most likely a linear one.

[1] By 'large' we mean large relative to the total portfolio. In many banks a loan that is 2% of the portfolio would be considered large.

4.2 Contour of integration

Another way of obtaining a better approximation is to change the contour in the Bromwich integral. It our methods it is a vertical line in the complex plane. In [10] a deformation of the contour is discussed to obtain a better numerical approximation of the contour integral. Some conditions on the Laplace transform, F, do apply such that $|F(z)| \to 0$ as $|z| \to \infty$ for $Rez \leq \sigma_0$. Here σ_0 is such that all singularities of F lie to the left of σ_0. We apply the deformation of the contour to the De Hoog method. We would like the discretization of the contour to integral to remain such that it can be easily written as a (infinite) polynomial. I f we take a contour $\gamma(t) = x(t) + iy(t)$ then we have that using the trapezium rule

$$\begin{aligned}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{sx}}{s}M_L(s)ds &= \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{\gamma(t)x}}{\gamma(t)}M_L(\gamma(t))\gamma'(t)dt \\ &\approx \sum_{k=-\infty}^{k=\infty}\frac{e^{\gamma(t_k)x}}{\gamma(t_k)}M_L(\gamma(t_k))\gamma'(t_k)\frac{1}{2}(t_{k+1}-t_{k-1}).\end{aligned}$$

Here $\ldots < t_{-1} < t_0 < t_1 < t_2 < \ldots$ form a partition of $(-\infty, \infty)$. One way to proceed from here is to define

$$a_k := \frac{e^{\gamma(t_k)x}}{\gamma(t_k)}M_L(\gamma(t_k))\gamma'(t_k)\frac{1}{2}(t_{k+1}-t_{k-1}). \tag{4.4}$$

Furthermore define $S(z)$ as

$$S(z) := \sum_{k=-\infty}^{k=\infty} a_k z^k.$$

If we have that $\sum_{k=-\infty}^{k=\infty}|a_k| \leq \infty$ then $S(z)$ converges for $|z| \leq 1$. For $|z| > 1$ it might still converge depending on how fast the a_k's tend to zero. Then we can approximate S by means of Padé approximation and evaluate this approximation at $z = 1$. In this case we are not restricted in the contour $\gamma(t)$ other than that modulus of the integrand, $\frac{e^{zx}}{z}M_L(z)$ on the contour must converge to 0. In case that for some k that $a_k = 0$ we introduce two new polynomials S_1 and S_2 defined by

$$\begin{aligned}S_1(z) &:= \sum_{k=-\infty}^{k=\infty} b_k z^k, \\ S_1(z) &:= \sum_{k=-\infty}^{k=\infty} c_k z^k.\end{aligned}$$

Where

$$\begin{aligned}b_k &= \begin{cases} \frac{1}{2}a_k & a_k \neq 0 \\ 2^{-k} & a_k = 0 \end{cases} \\ c_k &= \begin{cases} \frac{1}{2}a_k & a_k \neq 0 \\ -2^{-k} & a_k = 0 \end{cases}.\end{aligned}$$

Then $S = S_1 + S_2$ and S_1 and S_2 can be approximated individually using the Padé approximation. The reason that in case that $a_k = 0$ for k's requires new polynomials is that in the algorithm for the Padé approximation we have the quotient a_{k+1}/a_k which would not be defined for $a_k = 0$.

Chapter 5

Simulation of results

In this chapter we investigate the performs of the methods by some numerical calculations. The methods we investigate are the recursive method, Moody-Vasicek method, Monte Carlo method, saddlepoint Lugannani-Rice (see 3.78),saddlepoint Lugannani-Rice smooth, saddlepoint (see 3.76), saddlepoint smooth, De Hoog algorithm, De Hoog algorithm smooth and wavelet method.

5.1 Parameters

For the Monte Carlo method we do 1 million simulations. For the recursive saddlepoint, wavelet, De Hoog algorithm we have 1001 points for the numeric integration. These are $y_s = -5+10s/1000$, $s = 0, 1, \ldots, 1000$. TThe range $[-5, 5]$ was chosen as outside these values the standard normal density is very small. For the De Hoog algorithm we also use $h = 1/(8 \cdot L_{max})$, $N_t = 1600$ and $\gamma = -\log(10^{-14})/L_{max})$. Here $L_{max} = \max\{EAD_i / \sum_{j=1}^{n} EAD_j, i = 1, 2, \ldots, n\}$. For the smoothing the parameter was $\sigma = 0.1$.

5.2 Test portfolios

We have tested six portfolio with various level of PDs, EADs and asset correlations. The LGD have been kept at 1 as these are fixed and can be incorporated in the EAD without loss of generality. The portfolios are build up in the following way.

Portfolio 1. Parameters:

$$\begin{aligned} n &= 1000 \\ PD_i &= 0.003, && i = 1, \ldots, n \\ EAD_i &= 1, && i = 1, \ldots, n \\ \rho_i &= 0.2, && i = 1, \ldots, n \end{aligned}$$

Portfolio 2. Parameters:

$$\begin{aligned} n &= 1000 \\ PD_i &= 0.01, && i = 1,\dots,n \\ EAD_i &= C/i, && i = 1,\dots,n \\ \rho_i &= 0.2, && i = 1,\dots,n \\ C &= \left(\sum_{i=1}^{n} \frac{1}{i}\right)^{-1} \end{aligned}$$

Portfolio 3. Parameters:

$$\begin{aligned} n &= 1000 \\ PD_i &= 0.001, && i = 1,\dots,n \\ EAD_i &= C/i, && i = 1,\dots,n \\ \rho_i &= 0.2, && i = 1,\dots,n \\ C &= \left(\sum_{i=1}^{n} \frac{1}{i}\right)^{-1} \end{aligned}$$

Portfolio 4. Parameters:

$$\begin{aligned} n &= 1001 \\ PD_i &= 0.001, && i = 1,\dots,n \\ EAD_i &= 1, && i = 1,\dots,n-1 \\ EAD_i &= 500, && i = n \\ \rho_i &= 0.2, && i = 1,\dots,n \end{aligned}$$

Portfolio 5. Parameters:

$$\begin{aligned} n &= 1001 \\ PD_i &= 0.01, && i = 1,\dots,n-1 \\ PD_i &= 0.0001, && i = n \\ EAD_i &= 1, && i = 1,\dots,n-1 \\ EAD_n &= 500, && i = n \\ \rho_i &= 0.2, && i = 1,\dots,n \end{aligned}$$

Portfolio 5. Parameters:

$$\begin{aligned} n &= 1001 \\ PD_i &= 0.01, && i = 1,\dots,n-1 \\ PD_i &= 0.0001, && i = n \\ EAD_i &= 1, && i = 1,\dots,n-1 \\ EAD_n &= 500, && i = n \\ \rho_i &= 0.05, && i = 1,\dots,n \end{aligned}$$

We compute quantiles at the level of 99.9% and 99.88%. The first level is the level as used in the Basel 2 regulation for regulatory capital and the second number is the internally used number for economic capital for Rabobank.

5.3 Results

The results of our calculations are summarized in the following tables. The smoothing for the wavelet method yielded such bad results that we did not included them. Also for the method of obtaining the wavelet scaling coefficients using matrix inversion yielded very bad results and are also not included. The contour deformation was not investigated in the numerics. For portfolio 2 and 3 the Recursive method could not be used as the exposures were not discrete and scaling led to a too large a grid to compute.

Portfolio 1	99.90%	99.99%
Recursive	62.0	111.0
Vasicek	60.4	109.6
Monte Carlo	62.0	117.0
Saddlepoint LR	61.8	111.6
Saddlepoint LR smooth	60.3	109.5
SaddlePoint	61.8	111.6
Saddlepoint smooth	60.8	110.7
De Hoog algorithm	62.1	111.9
De Hoog algorithm smooth	61.8	111.6
Wavelet	60.4	109.6

Figure 5.1: Overview of portfolio 1.

Portfolio 2	99.90%	99.99%
Recursive	NA	NA
Vasicek	0.136	0.219
Monte Carlo	0.204	0.294
Saddlepoint LR	0.205	0.296
Saddlepoint LR smooth	0.192	0.272
SaddlePoint	0.205	0.297
Saddlepoint smooth	0.191	0.274
De Hoog algorithm	0.063	0.065
De Hoog algorithm smooth	0.059	0.060
Wavelet	0.206	0.296

Figure 5.2: Overview of portfolio 2.

Portfolio 3	99.90%	99.99%
Recursive	NA	NA
Vasicek	0.027	0.054
Monte Carlo	0.101	0.150
Saddlepoint LR	0.133	0.153
Saddlepoint LR smooth	0.133	0.147
SaddlePoint	0.133	0.154
Saddlepoint smooth	0.133	0.146
De Hoog algorithm	0.073	0.126
De Hoog algorithm smooth	0.067	0.133
Wavelet	0.112	0.151

Figure 5.3: Overview of portfolio 3.

Portfolio 4	99.90%	99.99%
Recursive	506.0	713.0
Vasicek	294.8	580.7
Monte Carlo	506.0	707.0
Saddlepoint LR	495.9	702.9
Saddlepoint LR smooth	496.1	701.3
SaddlePoint	495.9	702.9
Saddlepoint smooth	496.2	702.0
De Hoog algorithm	485.7	726.3
De Hoog algorithm smooth	506.6	713.6
Wavelet	481.7	675.2

Figure 5.4: Overview of portfolio 4.

Portfolio 5	99.90%	99.99%
Recursive	1463.0	2294.0
Vasicek	1457.5	2297.6
Monte Carlo	1474.0	2379.0
Saddlepoint LR	1360.7	2204.5
Saddlepoint LR smooth	1354.8	2208.6
SaddlePoint	1360.7	2204.5
Saddlepoint smooth	1360.0	2203.7
De Hoog algorithm	1465.4	2087.1
De Hoog algorithm smooth	1500.2	2005.3
Wavelet	1376.5	2197.5

Figure 5.5: Overview of portfolio 5.

Portfolio 6	99.90%	99.99%
Recursive	477.0	666.0
Vasicek	467.4	626.4
Monte Carlo	481.0	673.0
Saddlepoint LR	376.9	566.6
Saddlepoint LR smooth	373.5	563.4
SaddlePoint	376.8	566.8
Saddlepoint smooth	374.6	565.0
De Hoog algorithm	496.8	682.5
De Hoog algorithm smooth	476.9	667.5
Wavelet	392.1	683.9

Figure 5.6: Overview of portfolio 6.

5.4 Conclusion

We see that for the uniform portfolio 1 that all methods perform reasonably if we take the Recursive method results as our benchmark. The Saddlepoint methods perform a bit better for the 99.99% level. For portfolio 2 with more diverse exposure levels the De Hoog algorithm and the Vasicek method are much less accurate, if we take the Monte Carlo simulation method as our benchmark. The other methods are in line with the Monte Carlo benchmark. The same can be said for portfolio 3. For portfolio 4 most methods perform reasonably well taking the Recursive method as a benchmark. The Wavelet method and the De Hoog algorithm are a bit off and the Vasicek method is again well off the mark. The same holds for portfolio 5. For portfolio 6 we see that the saddlepoint methods are far off the mark for both quantile levels. Also the Wavelet method does not do well. The De Hoog algorithm does quite well for this portfolio and somewhat surpisingly so does the Vasicek method for this portfolio with some concentration. We also see that smoothing does not improve the accuracy of a method in most cases. There is no method that consistently dominates the other methods, but the saddlepoint methods seem to be performing on average the best for a reasonably diversified portfolio. Given that the saddlepoint methods are also reasonably fast they seem like appropriate methods to use. In our analysis we have taken quite basic model

assumptions, one period default model, one common factor and non-random LGD. Things becomes a bit more difficult when we expand on some of model assumptions, especially when going to multifactor. The purpose of this thesis was to investigate the basics of certain approximation methods and investigate their relative performance. As of currently we know of no method that is superior to all other method for all possible types of credit portfolios.

Appendix A

Padé approximation

In a Padé approximation a function is approximated by a rational function. It usually gives better results than a Taylor approximation. Consider the function S that admits a power series expansion as

$$S(z) = \sum_{k=0}^{\infty} s_k z^k \tag{A.1}$$

and suppose that two polynomials

$$P_M(z) := \sum_{k=0}^{M} p_k z^k, \quad Q_N(z) := 1 + \sum_{k=1}^{N} p_k z^k, \tag{A.2}$$

are given. Then a rational function $P_M(z)/Q_N(z)$ is said to be the $[M/N]$ Padé approximation to $S(z)$ if it satisfies

$$S(z) - \frac{P_M(z)}{Q_N(z)} = \mathcal{O}(z^{M+N+1}) \tag{A.3}$$

Now write A.3 more explicitly in terms of s_k, p_k and q_k, for two cases $N = M$ and $N = M = 1$. Since A.3 is equivalent to

$$S(z)Q_N(z) - P_M(z) = \mathcal{O}(z^{M+N+1}), \tag{A.4}$$

since $Q_N(z)\mathcal{O}(z^{M+N+1}) = \mathcal{O}(z^{M+N+1})$. Then we have that

$$\begin{aligned} s_k + \sum_{l=1}^{k} s_{k-l} q_l - p_k &= 0, && k = 0, 1, \ldots, M, \\ s_k + \sum_{l=1}^{M} s_{k-l} q_l &= 0, && k = M+1, M+2, \ldots, 2M, \end{aligned}$$

for $M = N$ and

$$s_k + \sum_{l=1}^{k} s_{k-l} q_l - p_k = 0, \qquad k = 0, 1, \ldots, M,$$
$$s_k + \sum_{l=1}^{M+1} s_{k-l} q_l = 0, \qquad k = M+1, M+2, \ldots, 2M+1,$$

for $N = M + 1$. Solving these equations we get that

$$p_k = \frac{\begin{vmatrix} & H_M^{(1)} & & \begin{matrix} s_{M+1} \\ \vdots \\ s_{2M} \end{matrix} \\ s_{k-M} & \cdots & s_{k-1} & s_k \end{vmatrix}}{\left| H_M^{(1)} \right|} \quad k = 0, 1, \ldots, M,$$
$$q_k = \frac{\left(C_{M+1}^{(1)} \right)_{M+1, M-k+1}}{\left| H_M^{(1)} \right|} \quad k = 1, 2, \ldots, M. \tag{A.5}$$

Where $|\cdot|$ is the determinant, $(C_n^{(u)})_{i,j}$ denotes the (i, j)-cofactor [1] of $H_n^{(u)}$. This can be seen as follows. The q_k's satisfy the equations

$$\begin{bmatrix} s_1 & s_2 & \cdots & s_M \\ s_2 & s_3 & \cdots & s_{M+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_M & s_{M+1} & \cdots & s_{2M-1} \end{bmatrix} \begin{bmatrix} q_M \\ q_{M-1} \\ \vdots \\ q_1 \end{bmatrix} = - \begin{bmatrix} s_{M+1} \\ s_{M+2} \\ \vdots \\ s_{2M} \end{bmatrix}. \tag{A.6}$$

Notice that the vector of q_ks has as it's first element q_M. Now define

$$b_M := \begin{bmatrix} s_{M+1} \\ s_{M+2} \\ \vdots \\ s_{2M} \end{bmatrix} \tag{A.7}$$

Also define $H_M^{(1),j}$ as the matrix $H_M^{(1)}$ with the j-th column replaced by $-b_M$. Then by Cramer's rule we have that

$$q_k = \frac{\left| H_M^{(1),M-k+1} \right|}{\left| H_M^{(1)} \right|} \tag{A.8}$$

The determinant $\left| H_M^{(1),M-k+1} \right|$ is the $(M+1, M-k+1)$ cofactor of the matrix $H_{M+1}^{(1)}$. This can be seen as follows. The matrix $H_M^{(1),M-k+1}$ has $-b_M$ as it's $M-k+1$ column. Now we

[1] The (i, j)-cofactor of a matrix is the determinant of that matrix with the ith row and jth column removed multiplied by $(-1)^{i+j}$

want to move the column with b_M to the last position. We do this by multiple switching with the column to the right. Everytime we switch we have to multiply the determinant with -1. So if we start at position $M-k+1$ and we need to move to the last column with number M we need to do $k-1$ switches. Also we want to habe b_M as the last column. The we have to again multiply the determinant with -1. This gives $(-1)^{k-1+1}$. Note also that $(-1)^{k-1+1} = (-1)^{-k+2(M+1)}$. Then we have that

$$\left|H_M^{(1),M-k+1}\right| (-1)^{k-1+1} = \left|H_M^{(1)}\right| \Rightarrow \tag{A.9}$$

$$\left|H_M^{(1),M-k+1}\right| = (-1)^{k-1+1} \left|H_M^{(1)}\right| \Rightarrow \tag{A.10}$$

$$\left|H_M^{(1),M-k+1}\right| = (-1)^{-k+2(M+1)} \left|H_M^{(1)}\right| = \left(C_{M+1}^{(1)}\right)_{M+1,M-k+1}. \tag{A.11}$$

Now for p_k this can be seen as follows. Define

$$A_M := \begin{bmatrix} H_M^{(1)} & \begin{matrix} s_{M+1} \\ \vdots \\ s_{2M} \end{matrix} \end{bmatrix} \tag{A.12}$$

Also define A_M^j is A_M with column j removed. Then $|A_M^j| = |H_M^{(1),j}|$ for $j = 1, \ldots, M$ and $|A_M^M| = (-1)^{M-j}|H_M^{(1)}|$. Also $q_k = (-1)^{k-1}|A_M^{M-k+1}|/|H_M^{(1)}|$

$$p_k = \frac{\begin{vmatrix} & H_M^{(1)} & & \begin{matrix} s_{M+1} \\ \vdots \\ s_{2M} \end{matrix} \\ s_{k-M} & \cdots & s_{k-1} & s_k \end{vmatrix}}{|H_M^{(1)}|} = \tag{A.13}$$

$$\sum_{j=0}^{M} s_{k-j}(-1)^j \frac{|A_M^{M-j+1}|}{|H_M^{(1)}|} = s_k + \sum_{j=1}^{M} s_{k-j} q_j \tag{A.14}$$

For $N = M+1$ we have that

$$p_k = \frac{\begin{vmatrix} & H_{M+1}^{(0)} & & \begin{matrix} s_{M+1} \\ \vdots \\ s_{2M+1} \end{matrix} \\ s_{k-M-1} & \cdots & s_{k-1} & s_k \end{vmatrix}}{\left|H_M^{(1)}\right|} \quad k = 0, 1, \ldots, M,$$

$$q_k = \frac{\left(C_{M+2}^{(0)}\right)_{M+2,M-k+2}}{\left|H_{M+1}^{(0)}\right|} \quad k = 1, 2, \ldots, M+1. \tag{A.15}$$

From the above we can explicitly compute the coefficient of the first order term in A.4 as

$$s_{2M+1} + \sum_{k=1}^{M} s_{2M+1-k} q_k = \frac{|H_{M+1}^{(1)}|}{|H_M^{(1)}|}, \quad (N = M), s_{2M+2} + \sum_{k=1}^{M+1} s_{2M+2-k} q_k = \frac{|H_{M+2}^{(0)}|}{|H_{M+1}^{(0)}|}, \quad (N = M+1), \tag{A.16}$$

For the case $N = M$ this can be seen by the definition of the determinant $|H^{(1)}_{M+1}|$

$$
\begin{aligned}
|H^{(1)}_{M+1}| &= \sum_{k=0}^{M} s_{M-k+1} \left(C^{(1)}_{M+1}\right)_{M+1,M-k+1} \Rightarrow \\
\frac{|H^{(1)}_{M+1}|}{|H^{(1)}_{M}|} &= \sum_{k=0}^{M} s_{M-k+1} \frac{\left(C^{(1)}_{M+1}\right)_{M+1,M-k+1}}{|H^{(1)}_{M}|} = s_{2M+1} + \sum_{k=1}^{M} s_{M-k+1} q_k
\end{aligned}
$$

From this we obtain

$$
\begin{aligned}
S(z)Q_N(z) - P_M(z) &= \frac{|H^{(1)}_{M+1}|}{|H^{(1)}_{M}|} z^{2M+1} + \mathcal{O}(z^{2M+2}), \quad (N = M) \\
S(z)Q_N(z) - P_M(z) &= \frac{|H^{(0)}_{M+2}|}{|H^{(0)}_{M+1}|} z^{2M+2} + \mathcal{O}(z^{2M+3}), \quad (N = M+1).
\end{aligned}
$$

A.1 Proof of lemma 3

We will prove that

$$
C_n(z) = \frac{A_n(z)}{B_n(z)}, \tag{A.17}
$$

where C_n is as in lemma 1. For $n = 1$ it is easy to see that this is true as $C_1(z) = c_0/(1+c_1 z)$ and $A_1(z) = c_0$, $B_1(z) = 1 + c_1 z$. We proceed with a proof by induction. Assume the assertion holds for $n = k$, $k \geq 1$. Since $C_{k+1}(z)$ is obtained from $C_k(z)$ by replacing c_k with $c_k/(1 + c_{k+1})$ we get that

$$
C_{k+1} = \frac{A'_k(z)}{B'_k(z)} \tag{A.18}
$$

Where by using the recursion relation for A and B

$$
\begin{aligned}
A'_k(z) &:= A_{k-1}(z) + \frac{c_k z}{1 + c_{k+1}} A_{k-2}(z), \\
B'_k(z) &:= B_{k-1}(z) + \frac{c_k z}{1 + c_{k+1}} B_{k-2}(z),
\end{aligned}
$$

Then we obtain for $k + 1$

$$
\begin{aligned}
C_{k+1}(z) &= \frac{(1 + c_{k+1})A_{k-1}(z) + c_k z A_{k-2}(z)}{(1 + c_{k+1})B_{k-1}(z) + c_k z B_{k-2}(z)} \\
&= \frac{A_{k-1}(z) + c_k z A_{k-2}(z) + c_{k+1} A_{k-1}(z)}{B_{k-1}(z) + c_k z B_{k-2}(z) + c_{k+1} B_{k-1}(z)} \\
&= \frac{A_k(z) + c_{k+1} A_{k-1}(z)}{B_k(z) + c_{k+1} B_{k-1}(z)} \\
&= \frac{A_{k+1}(z)}{B_{k+1}(z)}
\end{aligned}
$$

This proves the lemma.

A.2 Proof of lemma 1

From lemma 3 $A_n(z)$ and $B_n(z)$ are polynomials of degree $\lfloor n/2 \lfloor$ and $\lfloor (n+1)/2 \lfloor$ respectively. and satisfy

$$C_n(z) = \frac{A_n(z)}{B_n(z)}, \quad n = 0, 1, 2, \ldots$$
$$B_n(z) = 1. \tag{A.19}$$

So we would expect that $C_n(z)$ would give the $[\lfloor n/2 \lfloor / \lfloor (n+1)/2 \lfloor]$ Padé approximation if the coefficients c_k are chosen correctly. This is true for the cases $n = 0$ and $n = 1$ if we choose

$$c_0 = s_0, \quad c_1 = -\frac{s_1}{s_0} = -\frac{|H_0^{(0)}||H_1^{(1)}|}{|H_1^{(0)}||H_0^{(1)}|}, \tag{A.20}$$

which follows from $|H_0^{(0)}| = 1$, $|H_0^{(1)}| = 1$ and the solution for q_1 in the Padé approximation in the previous section. Now we assume that for and integer $k \geq 0$, $C_{2k}(z)$ and $C_{2k+1}(z)$ are respectively the $[k/k]$ and $[k/k+1]$ Padé approximations to $S(z)$. Then we get from A.17 that

$$S(z)B_{2k}(z) - A_{2k}(z) = \frac{|H_{k+1}^{(1)}|}{|H_k^{(1)}|} z^{2k+1} + \mathcal{O}(z^{2k+2}),$$
$$S(z)B_{2k+1}(z) - A_{2k+1}(z) = \frac{|H_{k+2}^{(0)}|}{|H_{k+1}^{(0)}|} z^{2k+2} + \mathcal{O}(z^{2k+3}).$$

This gives using the recursion relation for A, B

$$\begin{aligned} S(z)B_{2k+2}(z) - A_{2k+2}(z) &= S(z)\left(B_{2k+1}(z) + c_{2k+2} z B_{2k}\right) - \left(A_{2k+1}(z) + c_{2k+2} z A_{2k}\right) \\ &= S(z)B_{2k+1}(z) - A_{2k+1}(z) + c_{2k+2} z \left(S(z)B_{2k} - A_{2k}\right) \\ &= \left(\frac{|H_{k+2}^{(0)}|}{|H_{k+1}^{(0)}|} + \frac{|H_{k+1}^{(1)}|}{|H_k^{(1)}|} c_{2k+2}\right) z^{2k+2} + \mathcal{O}(z^{2k+3}) \\ &= \mathcal{O}(z^{2k+3}). \end{aligned} \tag{A.21}$$

Since by the definition of c_{2k+2}

$$\frac{|H_{k+2}^{(0)}|}{|H_{k+1}^{(0)}|} + \frac{|H_{k+1}^{(1)}|}{|H_k^{(1)}|} c_{2k+2} = \frac{|H_{k+2}^{(0)}|}{|H_{k+1}^{(0)}|} - \frac{|H_{k+1}^{(1)}|}{|H_k^{(1)}|} \frac{|H_{k+1+1}^{(0)}||H_{k+1-1}^{(1)}|}{|H_{k+1}^{(0)}||H_{k+1}^{(1)}|} = 0 \tag{A.22}$$

Then $C_{2k+2}(z) = A_{2k+2}(z)/B_{2k+2}(z)$ is the $[k+1/k+1]$ Padé aprroximation. In a similar manner we can verify that $C_{2k+3}(z)$ is the $[k+1/k+2]$ Padé approximation to $S(z)$ if C_{2k+1} and C_{2k+2} are respectively the $[k/k+1]$ and $[k+1/k+1]$ Padé approximations to $S(z)$ for an integer $k \geq 0$.

A.3 Proof of lemma 2

This lemma can be proven by induction. For $k = 1$ we can show

$$\begin{aligned} q_1^{(i)} &= \frac{s_{i+1}}{s_i} = \frac{|H_0^{(i)}||H_1^{(i+1)}|}{|H_1^{(i)}||H_0^{(i+1)}|}, \\ e_1^{(i)} &= e_0^{(i+1)} + q_1^{(i+1)} - q_1^{(i)} = \frac{s_{i+2}}{s_{i+1}} - \frac{s_{i+1}}{s_i} = \frac{s_i s_{i+2} - s_{i+1}^2}{s_i s_{i+1}}. \end{aligned} \tag{A.23}$$

Now assume that 3.126 holds for all $k \leq l$ for some integer $l \geq 1$. The by the second equation of 3.126 for $k = l + 1$ we get that

$$\begin{aligned} q_{l+1}^{(i)} &= q_l^{(i+1)} e_l^{(i+1)} / e_l^{(i)} \\ &= \frac{|H_{l-1}^{(i+1)}||H_l^{(i+2)}|}{|H_l^{(i+1)}||H_{l-1}^{(i+2)}|} \cdot \frac{|H_{l+1}^{(i+1)}||H_{l-1}^{(i+2)}|}{|H_l^{(i+1)}||H_l^{(i+2)}|} \Big/ \frac{|H_{l+1}^{(i)}||H_{l-1}^{(i+1)}|}{|H_l^{(i)}||H_l^{(i)}|} \qquad = \frac{|H_l^{(i)}||H_{l+1}^{(i+1)}|}{|H_{l+1}^{(i)}||H_l^{(i+1)}|} \end{aligned}$$

The second equation in 3.126 is more difficult to prove for $k = l + 1$. Again note that

$$C_{2k+1}(z) = c_0 \Big/ \left(1 + \left[\frac{c_m z}{1}\right]_{m=1}^{2k+1}\right) \tag{A.24}$$

is the $[k/k+1]$ Padé approximation to $S_{i+1} = \sum_{k=0}^{\infty} s_{i+1+k} z^k$ if $k \leq l$ and

$$\begin{aligned} c_0 &= s_{i+1}, \quad c_1 = -q_1^{(i+1)}, \\ c_{2k} &= -e_k^{(i+1)}, c_{2k+1} = -q_{k+1}^{(i+1)}, \quad k = 1, 2, \ldots, l \end{aligned}$$

Then if we write the polynomial $B_{2k+1}(z)$, $k \leq l$, as

$$B_{2k+1}(z) = 1 + \sum_{m=1}^{k+1} q_m z^m, \tag{A.25}$$

its coefficients must satisfy

$$q_m = \frac{(C_{k+2}^{(i+1)})_{k+2,k-m+2}}{|H_{k+1}^{(i+1)}|}, \tag{A.26}$$

from the solution of the Padé approximation equations for q_m for the case $N = M + 1$. Now if we compare terms of z on the left and right hand side of the following equation we get by applying the recursion for B

$$\begin{aligned} B_{2l+1}(z) &= B_{2l}(z) + c_{2l+1} z B_{2l-1}(z) \\ &= B_{2l-1}(z) + c_{2l} z B_{2l-2}(z) + c_{2l+1} z B_{2l-1}(z). \end{aligned} \tag{A.27}$$

we get with $m = 1$

$$\frac{(C_{l+2}^{(i+1)})_{l+2,l-1+2}}{|H_{l+1}^{(i+1)}|} = \frac{(C_{l+1}^{(i+1)})_{l+1,l}}{|H_{l}^{(i+1)}|} + c_{2l} + c_{2l+1} \tag{A.28}$$

Then with $-c_{2l} = e_l^{(i+1)}$ and $-c_{2l+1} = q_{l+1}^{(i+1)}$ we get

$$e_l^{(i+1)} + q_{l+1}^{(i+1)} = \frac{(C_{l+1}^{(i+1)})_{l+1,l}}{|H_{l}^{(i+1)}|} - \frac{(C_{l+2}^{(i+1)})_{l+2,l-1+2}}{|H_{l+1}^{(i+1)}|} \tag{A.29}$$

Now if we redefine the c_k by

$$\begin{aligned} c_0 &= s_i, \quad c_1 = -q_1^{(i)}, \\ c_{2k} &= -e_k^{(i)}, c_{2k+1} = -q_{k+1}^{(i)}, \quad k = 1, 2, \ldots, l \\ c_{2l+2} &= -\frac{|H_{l+2}^{(i)}||H_{l}^{(i+1)}|}{|H_{l+1}^{(i)}||H_{l+1}^{(i+1)}|}. \end{aligned}$$

Then the continued fraction belonging to this set

$$C_{2k+2}(z) = c_0 / \left(1 + \left[\frac{c_m z}{1}\right]_{m=1}^{2k+2}\right), \tag{A.30}$$

with $k \leq l$ is the $[k+1/k+1]$ Padé approximation to $S_i(z) = \sum_{k=0}^{\infty} s_{i+k} z^k$ by lemma 1. In a similar manner the coefficients of

$$B_{2k+2}(z) = 1 + \sum_{m=1}^{k+1} q_m z^m, \tag{A.31}$$

satisfy

$$q_m = \frac{(C_{k+2}^{(i+1)})_{k+2,k-m+2}}{|H_{k+1}^{(i+1)}|}, \tag{A.32}$$

as the solution for the Padé approximation for the case $N = M$. If we again compare the coefficients for the term z in

$$\begin{aligned} B_{2l+2}(z) &= B_{2l+1}(z) + c_{2l+2} z B_{2l}(z) \\ &= B_{2l}(z) + c_{2l+1} z B_{2l-1}(z) + c_{2l+2} z B_{2l}(z). \end{aligned} \tag{A.33}$$

then we get that

$$\frac{(C_{l+2}^{(i+1)})_{l+2,l+1}}{|H_{l+1}^{(i+1)}|} = \frac{(C_{l+1}^{(i+1)})_{l+1,l}}{|H_{l}^{(i+1)}|} + c_{2l+1} + c_{2l+2}. \tag{A.34}$$

Then, using the special definition for c_{2l+2} and that $c_{2l+2} = -e_{l+1}^{(i)}$

$$q_{l+1}^{(i)} + \frac{|H_{l+2}^{(i)}||H_l^{(i+1)}|}{|H_{l+1}^{(i)}||H_{l+1}^{(i+1)}|} = \frac{(C_{l+1}^{(i+1)})_{l+1,l}}{|H_l^{(i+1)}|} - \frac{(C_{l+2}^{(i+1)})_{l+2,l+1}}{|H_{l+1}^{(i+1)}|} \tag{A.35}$$

Now subtract A.35 from A.29 to get

$$\begin{aligned} e_l^{(i)} &= e_l^{(i+1)} + q_{l+1}^{(i+1)} - q_{l+1}^{(i)} \\ &= \frac{|H_{l+2}^{(i)}||H_l^{(i+1)}|}{|H_{l+1}^{(i)}||H_{l+1}^{(i+1)}|} \end{aligned} \tag{A.36}$$

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