A note on consequence

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Abstract

This paper contains a detailed account of the notion of admissibility in the setting of consequence relations. It is proved that the two notions of admissibility used in the literature coincide, and it provides an extension to multi-conclusion consequence relations that is more general than the one usually encountered in the literature on admissibility. The notion of a rule scheme is introduced to cover rules with side conditions, and it is shown that what is generally understood under the extension of a consequence relation by a rule can be extended naturally to rule schemes. It is shown that such extensions correctly capture the intuitive idea of extending a logic by a rule.

Keywords: consequence relations, admissible rules, multi-conclusion logic.

1 Introduction

In this paper our aim is to provide a framework in which to reason about the admissibility of rules of inference. In most papers on admissibility the notion of consequence relation is taken as a foundation, and our aim in this paper is to provide arguments that support this choice. None of these arguments are very deep or completely new, but we feel it is worthwhile to present them in detail because in the literature they remain mostly implicit.

One of the incentives to spell out the details of what in most papers is discussed only briefly (and for good reasons) is the phenomenon, as pointed out in [7], that admissibility is defined in two ways in the literature, in what we call the *full* and the *strict* way. Given a theory or logic L and a rule R:

(full) R is admissible in L if L extended by R has the same theorems as L.

(strict) R is admissible in L if under all substitutions, whenever all premisses of R become theorems of L, then so does the conclusion.

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In informal explanations often the first definition is used, while the second one lends itself better for applications. Informally, it is quite easy to argue that the two definitions are the same, but if one wishes to make this precise, a lot of nasty details appear. In the full definition of admissibility, for example, one has to address the question what it means to extend a theory or logic by a rule. If the theory is given to us as a proof system in the same language as that in which the rule is formulated, then this might be quite straightforward, but if the theory is given in another way, say via a set of models or algebras, it is less clear what is meant. In this paper we describe what this means in detail, in a way that is applicable in many settings.

As is common in the literature on admissible rules, we choose consequence relations as a general framework. Since Tarski, consequence relations are traditionally used in the literature to capture reasoning in theories in a very general way, abstracting away from particular theories and particular syntax [13, 15]. We introduce the notion of a rule scheme (a rule with a set of substitutions) that captures, we think, what is generally meant by a rule of inference in a mathematical or logical context. Then we define what it means to extend a consequence relation by a set of rule schemes (a slight generalization of a similar notion that occurs at many places in the literature) and show in Proposition 3.3 that in this way, derivations in the extension indeed are derivations that consist of inferences in the original consequence relation and inferences based on the rules added in the extension. Thus supporting the claim that this is the correct way to define extensions.

Another aspect that has to be considered when trying to define admissibility, is the description of the substitutions that the strict definition of admissibility refers to. This description will be given in Section 2.5.

Finally, we address another issue in this paper, namely the analogue of the above definitions for multi-conclusion consequence relations. Such relations are useful in the setting of admissibility because they allow one to express the disjunction property, which is the property that for finite sets of formulas Δ, if Δ is derivable, then so is A for some $A \in \Delta$. If the consequence relation of a theory L has this property, then the analogues, for multi-conclusion rules R, of the above two definitions of admissibility are:

(dp-full) R is admissible in L if L extended by R has the same theorems as L.

(dp-strict) R is admissible in L if under all substitutions, whenever all premisses of R become theorems of L, then so does at least one formula in the conclusion.

However, if L does not have the disjunction property, these do not seem to be correct definitions of admissibility anymore. We think that the genuine multi-conclusion analogue of the full definition for admissibility is:

(full) R is admissible in L if L extended by R has the same multi-conclusion theorems as L.

Indeed, if the consequence relation has the disjunction property, this is equivalent to dp-full.

What the correct analogue of the strict definition is in the multi-conclusion case is not so clear. At first sight, one might guess something like this:

R is admissible in L if under all substitutions, whenever all premisses of R become theorems of L, then the conclusion becomes a multi-conlcusion theorem of L.

If a consequence relation has the disjunction property, this is equivalent to dp-strict. Otherwise this definition does not even seem to be equivalent to the full definition of admissibility for multi-conclusion consequence relations (Remark 4.5), which in Proposition 4.1 is shown to be equivalent to

(strict) $R = \Gamma/\Delta$ is admissible in L if under all substitutions σ and for all finite sets Σ, whenever $\sigma A, \Sigma$ is a multi-conlcusion theorem of L for all $A \in \Gamma$, then so is $\sigma\Delta, \Sigma$.

Thus this seems to be the strict analogue of the definition of admissibility for multi-conclusion consequence relations. Whether there are other reasonable strict definitions of admissibility is an issue we leave open for speculation and further research.

Metcalfe in [7] studies similar problems as the ones addressed in this paper, but then in an algebraic context.

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2 Consequence relations

When Tarski spoke in Paris at the International Congress of Scientific Philosophy in 1935 on logical consequence [13], he tried to characterize in all generality what it means for a sentence A to logically follow from a set of sentences Γ. He arrived at the definition that this is so if and only if every model of the sentences in Γ is a model of A. This led to the introduction and study of consequence relations, which are relations between sets of formulas and formulas, that satisfy reflexivity, transitivity and weakening. The Polish School has been particularly active in the area of consequence relations, which is no coincidence given Tarksi's Polish background, see [15] for an overview of its results.

Consequence relations play a central role in this paper, as we assume the theories or logics that we consider all to be given by a consequence relation. This is no great restriction as they cover almost all reasonable theories. As said in the introduction, in this paper we want to define in great detail what it means that a rule of inference is admissible in a certain theory. As this has more to do with consequence relations and the way in which they can be extended by rules, results on particular admissible rules for particular logics will be only mentioned in passing. For an overview of the area of admissible rules, the reader is referred to the literature, in particular to Rybakov's monograph [9]. For a brief overview of the main results on intermediate and modal logics in this area, see [5].

We start by defining what a consequence relation is. To maintain a certain level of generality we assume that there is a language $\mathcal{L}$, which is a set of symbols, and that there is a set of expressions $\mathcal{F}_{\mathcal{L}}$ in this language usually called *formulas*. In this way consequence relations can be about regular formulas as well as other expressions, such as sequents, as the following three examples show.

In the case of propositional logic, the language, $\mathcal{L}_\mathrm{p}$, consists of infinitely many propositional variables $p, q, r, \ldots$, the connectives $\neg, \to, \wedge, \vee$ and the constants $\top$ and $\bot$. The set of formulas $\mathcal{F}_{\mathcal{L}_\mathrm{p}}$ is defined as usual. The language, $\mathcal{L}_\mathrm{s}$, for sequents in propositional logic consists of $\mathcal{L}_\mathrm{p}$ extended with $\Rightarrow$. $\mathcal{F}_{\mathcal{L}_\mathrm{s}}$ consists of the sequents in $\mathcal{L}_\mathrm{p}$, that is, of all expressions $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite sets of formulas in $\mathcal{L}_\mathrm{p}$. Thus the formulas in $\mathcal{L}_\mathrm{s}$ are the sequents in $\mathcal{L}_\mathrm{p}$. The language, $\mathcal{L}_\mathrm{f}$, of predicate or first-order logic consists of predicates and functions, for every arity infinitely many, infinitely many variables, the connectives $\neg, \to, \wedge, \vee$, constants $\top$ and $\bot$ and quantifiers $\exists$, $\forall$.

2.1 Multi-conclusion consequence relations

Multi-conclusion consequence relations are relations $\vdash$ on sets of formulas. We write $\Gamma \vdash \Delta$ if the pair (Γ, Δ) belongs to the relation. We also write Γ/Δ for the pair (Γ, Δ), and $A, \Gamma \vdash \Delta, B$ for $\{A\} \cup \Gamma \vdash \{B\} \cup \Delta$. A *finitary multi-conclusion consequence relation* is a relation $\vdash$ on finite sets of formulas that satisfies for all finite sets of formulas $\Gamma, \Gamma', \Delta, \Delta'$ and formulas A:

reflexivity $A \vdash A$,

weakening if $\Gamma \vdash \Delta$, then $\Gamma', \Gamma \vdash \Delta, \Delta'$,

transitivity if $\Gamma \vdash \Delta, A$ and $\Gamma', A \vdash \Delta'$, then $\Gamma', \Gamma \vdash \Delta, \Delta'$.

In the older literature, such as [10], the first two properties are sometimes called *overlap* and *dilution*, respectively.

A *finitary single-conclusion consequence relation* is a relation between finite sets of formulas and formulas satisfying the single-conclusion variants of the three properties above. Thus for single-conclusion consequence relations the conclusion of a rule cannot be empty. Although most logics we discuss can be represented via a single-conclusion consequence relation, the multi-conclusion analogue allows us to express certain properties more naturally, such as the disjunction property discussed below. We often omit the word "finitary" in what follows and use (m.)c.r. as an abbreviation for "finitary (multi-conclusion) consequence relation". Given a m.c.r. $\vdash$, its single-conclusion fragment $\vdash_s$ is defined as

$$\Gamma \vdash_s A \equiv_{def} \Gamma \vdash A.$$

The minimal consequence relation $\vDash_\mathrm{m}$ is given by

$$\Gamma \vDash_\mathrm{m} \Delta \equiv_{def} \Gamma \cap \Delta \neq \emptyset.$$

A is a *theorem* if $\emptyset \vdash A$, which we write as $\vdash A$. The set of all theorems of a consequence relation is denoted by $\mathsf{Th}(\vdash)$ (called the *logical system* of the consequence relation in [15], page 46). Δ is a *multi-conclusion theorem* if $\vdash \Delta$, and the set of all multi-conclusion theorems is denoted by $\mathsf{Thm}(\vdash)$.

In [3] the multi-conclusion analogue of single-conclusion consequence relations are studied. The following observations can be found there. If a language does not contain conjunction, we use the abbreviation

$$\Gamma \vdash \bigwedge \Pi \equiv_{def} \forall A \in \Pi : \Gamma \vdash A.$$

Similarly, if the language does not contain disjunction, we use the abbrievation

$$\bigvee \Gamma \vdash \Pi \equiv_{def} \forall A \in \Gamma : A \vdash \Pi.$$

Any s.c.r. $\vdash$ has two natural multi-conclusion analogues:

$$\Gamma \vdash^{\mathsf{min}} \Delta \;\Leftrightarrow\; \exists A \in \Delta \; \Gamma \vdash A$$

$$\Gamma \vdash^{\mathsf{max}} \Delta \;\Leftrightarrow\; \forall \Pi \forall A (\Pi \vdash \bigwedge \Gamma \text{ and } \bigvee \Delta \vdash A \Rightarrow \Pi \vdash A).$$

It is not difficult to see that $\vdash^{\mathsf{min}}$ and $\vdash^{\mathsf{max}}$ are multi-conclusion consequence relations. The following lemma explains their name.

Lemma 2.2 [3] For all m.c.r. $\vdash'$ such that $\vdash'_s = \vdash$: $\vdash^{\mathsf{min}} \subseteq \vdash' \subseteq \vdash^{\mathsf{max}}$.

Proof $\vdash^{\mathsf{min}} \subseteq \vdash'$: if $\Gamma \vdash^{\mathsf{min}} \Delta$, then $\Gamma \vdash A$ for some $A \in \Delta$. Thus $\Gamma \vdash' A$.
$\vdash' \subseteq \vdash^{\mathsf{max}}$: if $\Gamma \vdash' \Delta$, then we have to show that $\Pi \vdash \bigwedge \Gamma$ and $\bigvee \Delta \vdash A$ implies $\Pi \vdash A$. Repeated application of cut to the assumption gives $\Pi \vdash' A$, and thus $\Pi \vdash A$. ⊣

2.3 Examples

The following examples of single-conclusion consequence relations are completely straightforward, but illustrate the general applicability of consequence relations nicely.
1. Given a logic L with theorems $\mathsf{Th}(\mathsf{L})$, then

$$\Gamma \vdash A \equiv_{def} A \in \Gamma \cup \mathsf{Th}(\mathsf{L})$$

defines a single-conclusion consequence relation which set of theorems is $\mathsf{Th}(\mathsf{L})$. There are other consequence relations which set of theorems is $\mathsf{Th}(\mathsf{L})$:

$$\Gamma \vdash A \equiv_{def} \begin{cases} A \in \mathsf{Th}(\mathsf{L}) & \text{if } \Gamma = \emptyset \\ \exists A' \in \Gamma : A' \to A \in \mathsf{Th}(\mathsf{L}) & \text{if } \Gamma \neq \emptyset. \end{cases}$$

This defines a consequence relation in case the logic satisfies that $A \to A$ is a theorem, and if $A' \to A$ and $A \to A''$ are theorems, then so is $A' \to A''$.
2. Many logics are given by a semantics such that

$$\Gamma \vdash A \equiv_{def} \text{in every model in which all formulas in } \Gamma \text{ hold, } A \text{ holds}$$

defines a consequence relation. Examples are the Kripke model semantics for modal and intermediate logics.
3. Suppose we consider sequents in propositional logic and the sequent calculus $\mathsf{G3i}$ from [14]. Then for sequents $S_0, S_1, \ldots, S_n$, the relation $\vdash$ defined as

$$\{S_1, \ldots, S_n\} \vdash S_0 \equiv_{def} S_0 \text{ is derivable from } \{S_1, \ldots, S_n\} \text{ in } \mathsf{G3i}$$

is a single-conclusion consequence relation. Thus the admissibility of Cut in $\mathsf{G3i}$ implies that $(\Gamma \Rightarrow A, \Delta), (\Gamma', A \Rightarrow \Delta') \vdash (\Gamma, \Gamma' \Rightarrow \Delta, \Delta')$ holds. Note that the transitivity, or cut, on the level of the consequence relation is another one than the cut rule on the level of sequents: ($\mathcal{S}$ and $\mathcal{S}'$ sets of sequents)

$$\frac{\mathcal{S} \vdash S \quad S, \mathcal{S}' \vdash S}{\mathcal{S}, \mathcal{S}' \vdash S} \ \mathsf{transitivity} \qquad \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma', A \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'} \ \mathsf{cut}$$

In terms of the meaning of sequents it might seem more natural to let the sequent arrow $\Rightarrow$ correspond to a consequence relation $\vdash$. In fact, one often encounters $\vdash$ denoting $\Rightarrow$ in sequent calculi. A sequent calculus then provides an axiomatization of a certain consequence relation. Here, however, consequence relations are used capture the various ways in which a logic can be given to us. Hence sequents are the "formulas", the elements, of the consequence relation.

2.4 The consequence relation of a logic

Suppose a logic L is given to us not as a consequence relation but in another way, for example by a class of models, an algebra, or a sequent calculus. What does it mean to say the a single-conclusion consequence relation captures the logic? That depends very much on the application one has in mind. But let us say that a consequence relation $\vdash$ *covers* a logic if $\mathsf{Th}(\vdash)$ equals the set of theorems of the logic, which we denote by $\mathsf{Th}(\mathsf{L})$.

Clearly, there are many consequence relations that cover a single logic L. The smallest such single-conclusion consequence relation $\vdash$ was already discussed above:

$$\Gamma \vdash A \Leftrightarrow A \in \Gamma \cup \mathsf{Th}(\mathsf{L}).$$

What the greatest consequence relation is that covers L we shall see in Section 4.

For propositional (modal) logics containing the theorems of IPC or predicate logics containing the theorems of IQC, we single out one particular single-conclusion consequence relation, denoted by $\vdash_{\mathsf{L}}$, that covers L as follows (where $\bigwedge \emptyset$ equals $\top$):

$$\Gamma \vdash_{\mathsf{L}} A \Leftrightarrow (\bigwedge \Gamma \to A) \in \mathsf{Th}(\mathsf{L}).$$

Note that this is a structural consequence relation if the logic itself is structural (closed under substitution). We also define one particular multi-conclusion consequence relation, also denoted by $\vdash_{\mathsf{L}}$, that covers L as follows:

$$\Gamma \vdash_{\mathsf{L}} \Delta \Leftrightarrow (\bigwedge \Gamma \to \bigvee \Delta) \in \mathsf{Th}(\mathsf{L}).$$

2.5 Substitutions

As we will be mostly interested in rules closed under certain substitutions, we need to explain which substitutions we consider where. If one would like to present the matter as formal as possible one would introduce two languages, the *meta-language* $\mathcal{L}_m$ and the *object-language* $\mathcal{L}_o$, where all elements of the meta-language belong to the object language except possibly the *meta-variables*. We use the word meta to distinguish the variables from the regular variables that may occur in the object language.

Substitutions are maps from formulas in the meta-language to formulas in the object-language that commute with all non-meta-variable symbols and are the identity on constants, if any are present. In this way every logic comes with a notion of meta-language and object-language and corresponding set of substitutions *Sub*. In the case of pure propositional or predicate logic the two languages are often mixed and considered as one. For example, substitutions in propositional logic are often considered to be maps on formulas commuting with the connectives. We will do so too in this paper where possible. So when talking about propositional or predicate logic, $\mathcal{L}_m = \mathcal{L}_o$ and the atoms, respectively the atomic formulas, have a double role in that they are considered as meta-variables (in the meta-language) as well as atoms (in the object-language).

In the case of predicate logic, there is a subtlety concerning the regular variables, as illustrated by the rule for the introduction of the universal quantifier

$$\frac{\begin{array}{c}\vdots\\ A(y)\end{array}}{\forall x A(x)}\ .$$

Here $A(x)$ is the meta-variable and we have to indicate what formulas are allowed to be substituted for it. Clearly, there have to be some requirements. For example, if $B(x)$ is substituted for $A(x)$, then $B(y)$ should be substituted for $A(y)$. Besides such obvious ones, it is for this exposition not important which particular requirements we impose, so we leave this open.

The rule above comes with a side condition as well, namely that y is not free in the assumptions. We can capture this condition by considering rule schemes instead of rules, as will be explained in Section 3.

A consequence relation is *structural* (*uniform* in [1]) if it satisfies

structurality if $\Gamma \vdash \Delta$, then $\sigma\Gamma \vdash \sigma\Delta$ for all $\sigma \in Sub$.

Typically, schematic systems are structural consequence relations, such as Gentzen calculi, Hilbert-style systems or natural deduction.

3 Rules

A *(multi-conclusion) rule* is an ordered pair of finite sets of formulas, written Γ/Δ or $\frac{\Gamma}{\Delta}$, where Γ and Δ are finite sets of formulas. It is called *single-conclusion* if $|\Delta| \leq 1$. For $R = \Gamma/\Delta$, σR is short for $\sigma\Gamma/\sigma\Delta$. and similarly for sets of rules. Rules Γ/Δ such that $\Gamma \vdash \Delta$ are the *rules of the consequence relation* $\vdash$ and $\mathsf{Ru}_\vdash$ is the set of all rules of $\vdash$.

Sometimes rules come with restrictions, such as the density rule

$$\frac{(A \to P) \vee (P \to B)}{(A \to B)} \quad (P \text{ atomic})$$

in Gödel logic or the universal quantifier rule $R\forall$ in the Gentzen calculus LK,

$$\frac{\Gamma \Rightarrow Ay, \Delta}{\Gamma \Rightarrow \forall x Ax, \Delta} \quad (y \text{ not free in } \Gamma\Delta).$$

To capture these rules in our setting we have to slightly extend our terminology to include the notion of a *rule scheme*, being a pair (R, S) consisting of a rule

R and a set of substitutions $S \subseteq Sub$. Like rules and sets of rules, rule schemes and sets of rule schemes will denoted by R and $\mathcal{R}$ respectively, trusting that it will always be clear from the context which variant is meant. A rule R is considered as a rule scheme $(R, \{\text{id}\})$, where id is the identity substitution. For $\sigma \in S$, a rule σR is called an *instance* of the rule scheme (R, S), as well as an *instance* of the rule R. We define the set of rules that are instances of a rule scheme in $\mathcal{R}$ as follows:

$$\mathsf{Ru}_{\mathcal{R}} \equiv_{def} \cup_{(R,S)\in\mathcal{R}}\{\sigma R \mid \sigma \in S\}.$$

Given a set $\mathcal{R}$ of rule schemes, the extension $\vdash^{\mathcal{R}}$ of a consequence relation $\vdash$ by these rule schemes is the smallest consequence relation extending $\vdash$ for which $\Gamma \vdash \Delta$ holds for all Γ/Δ in $\mathsf{Ru}_{\mathcal{R}}$. When all rule schemes in $\mathcal{R}'$ have the same set of substitutions S we sometimes write $\vdash^{\mathcal{R}}_S$ for $\vdash^{\mathcal{R}'}$, where $\mathcal{R}$ is the set of rules that occur in the schemes in $\mathcal{R}'$. In case of a single rule scheme (R, S) we write $\vdash^R_S$ for $\vdash^{\{(R,S)\}}$. In case of a set of rules schemes $\mathcal{R}'$ such that all sets of substitutions that occur in it are maximal, which means equal to *Sub*, we write $\vdash^{\mathcal{R}}$ for $\vdash^{\mathcal{R}'}$, where $\mathcal{R}$ is the set or rules that occur in the rules schemes in $\mathcal{R}'$.

Given a set of rule schemes $\mathcal{R}$, a sequence of formulas $A_1, \ldots, A_n$ is a *derivation* of Γ/Δ in $\mathcal{R}$ if $A_n \in \Delta$ and for all $A_i \notin \Gamma$ there are $i_1, \ldots, i_m < i$ such that $A_{i_1}, \ldots, A_{i_m}/A_i$ belongs to $\mathsf{Ru}_{\mathcal{R}}$. In this case we also say that *rule* Γ/Δ is *derivable* in $\mathcal{R}$ and in case Δ consists of one formula A, that A *follows* from Γ or that Γ *derives* A in $\mathcal{R}$. In [10] (p.25) the word *deducible* instead of derivable is used.

For an analogue of the notion of derivation for nonsaturated consequence relations we have to consider trees [10]. Here a *tree* T is a labelled tree in the usual sense with a root r that is below all other nodes. Every node k has a label $\text{lb}(k)$ which is a formula, except the root, that has a set of formulas as label. $\text{lb}(k\downarrow)$ is the set consisting of all labels that are formulas at k and the nodes below k. The *leafs* $\text{lf}(T)$ of T are the nodes with no successors. The *leafset* $\text{lb}(T)$ of the tree T is the set consisting of the formulas at the leafs of T.

Given a set of rule schemes $\mathcal{R}$, a tree T is a *multi-conclusion derivation* of Γ/Δ in $\mathcal{R}$ if T is finite, $\text{lb}(r) \subseteq \Gamma$, the leafset is a subset of Δ and for every node k with immediate successors $k_1, \ldots, k_n$ which labels do not belong to Γ, there is a set $\Gamma' \subseteq \text{lb}(k\downarrow)$ such that $\Gamma'/\{\text{lb}(k_i), \ldots, \text{lb}(k_n)\}$ belongs to $\mathsf{Ru}_{\mathcal{R}}$. If T consists of one node, $\text{lb}(r)$ is not empty.

3.1 The relation $\vdash^{\mathcal{R}}$

We show that in case the underlying consequence relation is *saturated*, meaning that

$$\Gamma \vdash \Delta \Rightarrow \exists A \in \Delta\, \Gamma \vdash A, \tag{1}$$

the consequence relation $\vdash^{\mathcal{R}}$ captures exactly the idea of adding the rule schemes $\mathcal{R}$ to the consequence relation: $\Gamma \vdash^{\mathcal{R}} \Delta$ if and only if Γ/Δ has a derivation in $\mathsf{Ru}_{\vdash} \cup \mathsf{Ru}_{\mathcal{R}}$. That is, $\Gamma \vdash^{\mathcal{R}} \Delta$ if and only if A can be derived from Γ using only inferences that are instances of the rule schemes in $\mathsf{Ru}_{\vdash}$ and $\mathcal{R}$.

If (1) only holds for empty Γ, but possibly not in general, $\vdash$ is said to have the *disjunction property*. Every single-conclusion consequence relation $\vdash$ is clearly saturated, and so is its multi-conclusion variant $\vdash^{\mathsf{min}}$.

Given a relation X on sets of formulas, the *weakening closure* and *transitive closure* of X are defined respectively as follows:

$$\begin{array}{lll}
\mathrm{wc}(X) & \equiv_{def} & \{(\Gamma \cup \Gamma', \Delta \cup \Delta') \mid (\Gamma, \Delta) \in X, \Gamma' \text{ and } \Delta' \text{ finite sets of formulas}\} \\
\mathrm{tc}_0(X) & \equiv_{def} & X \\
\mathrm{tc}_{i+1}(X) & \equiv_{def} & \mathrm{tc}_i(X) \cup \{(\Gamma \cup \Gamma', \Delta \cup \Delta') \mid \text{for some } A\text{: } (\Gamma, \Delta \cup \{A\}) \in \mathrm{tc}_i(X) \\
 & & \quad \text{and } (\Gamma' \cup \{A\}, \Delta') \in \mathrm{tc}_i(X)\} \\
\mathrm{tc}(X) & \equiv_{def} & \bigcup_i \mathrm{tc}_i(X).
\end{array}$$

Proposition 3.2 $\vdash^{\mathcal{R}} = \mathrm{tc}(\mathrm{wc}(\mathsf{Ru}_{\vdash} \cup \mathsf{Ru}_{\mathcal{R}}))$.

Proof Denote $\mathsf{Ru}_{\vdash} \cup \mathsf{Ru}_{\mathcal{R}}$ by X. We prove the lemma by showing that $\mathrm{tc}(\mathrm{wc}(X))$ is a consequence relation and that it is contained in $\vdash^{\mathcal{R}}$. The minimality condition on $\vdash^{\mathcal{R}}$ then implies that $\vdash^{\mathcal{R}}$ is actually equal to $\mathrm{tc}(\mathrm{wc}(X))$.
To see that $\mathrm{tc}(\mathrm{wc}(X))$ is contained in $\vdash^{\mathcal{R}}$ is easy. To show that it is a consequence relation, it suffices to show that it is closed under weakening and transitivity, that is, that $\mathrm{tc}(\mathrm{wc}(\mathrm{tc}(\mathrm{wc}(X)))) = \mathrm{tc}(\mathrm{wc}(X))$. Because of the definition of transitive closure, it suffices to show that $\mathrm{wc}(\mathrm{tc}(\mathrm{wc}(X))) = \mathrm{tc}(\mathrm{wc}(X))$. This follows if for all i, $\mathrm{wc}(\mathrm{tc}_i(\mathrm{wc}(X))) = \mathrm{tc}_i(\mathrm{wc}(X))$, the proof of which is straightforward. ⊣

Proposition 3.3 For any saturated consequence relation $\vdash$ and any set of single-conclusion rule schemes $\mathcal{R}$: $\Gamma \vdash^{\mathcal{R}} \Delta$ if and only if Γ/Δ has a derivation in $\mathsf{Ru}_{\vdash} \cup \mathsf{Ru}_{\mathcal{R}}$.

Proof For the direction from right to left it suffices to show that for any derivation $A_1, \ldots, A_m$ of Γ/Δ in $\mathsf{Ru}_{\vdash} \cup \mathsf{Ru}_{\mathcal{R}}$, for all i we have $\Gamma \vdash^{\mathcal{R}} A_i$, a proof that is left to the reader. For the other direction we use the equivalence from Lemma 3.2 stating that $\vdash^{\mathcal{R}}$ is equal to $\mathrm{tc}(\mathrm{wc}(\mathsf{Ru}_{\vdash} \cup \mathsf{Ru}_{\mathcal{R}}))$. Thus it suffices to show that any Γ/Δ in $\mathsf{Ru}_{\vdash} \cup \mathsf{Ru}_{\mathcal{R}}$ has a derivation in $\mathsf{Ru}_{\vdash} \cup \mathsf{Ru}_{\mathcal{R}}$, and if all rules in X have a derivation in $\mathsf{Ru}_{\vdash} \cup \mathsf{Ru}_{\mathcal{R}}$, then so do $\mathrm{wc}(X)$ and $\mathrm{tc}_i(X)$ for all i.
We only show that if all rules in $\mathrm{tc}_i(X)$ have a derivation in $\mathsf{Ru}_{\vdash} \cup \mathsf{Ru}_{\mathcal{R}}$, then so do all rules in $\mathrm{tc}_{i+1}(X)$, and leave the rest of the proof to the reader. Consider rules $\Gamma/\Delta, A$ and $\Gamma', A/\Delta'$ in $\mathrm{tc}_i(X)$ with respective derivations $A_1, \ldots, A_m$ and $B_1, \ldots, B_n$. We show that there exists a derivation of $\Gamma, \Gamma'/\Delta, \Delta'$ in $\mathsf{Ru}_{\vdash} \cup \mathsf{Ru}_{\mathcal{R}}$. If $A_m \neq A$, then $A_1, \ldots, A_m$ is a derivation of $\Gamma, \Gamma'/\Delta, \Delta'$. And if for no $i < n$, B_i equals A, then $B_1, \ldots, B_n$ is a derivation of $\Gamma, \Gamma'/\Delta, \Delta'$.
Therefore suppose that $A = A_m$ and that A occurs in $B_1, \ldots, B_{n-1}$. We show that $A_1, \ldots, A_m, B_1, \ldots, B_n$ is a derivation of $\Gamma, \Gamma'/\Delta, \Delta'$ in $\mathsf{Ru}_{\vdash} \cup \mathsf{Ru}_{\mathcal{R}}$. Consider a $C \notin \Gamma\Gamma'$ in the derivation. If C is A_i, then it follows immediately that there are $i_1, \ldots, i_k < i$ such that $A_{i_1}, \ldots, A_{i_k}/C$ is in $\mathsf{Ru}_{\vdash} \cup \mathsf{Ru}_{\mathcal{R}}$, as $A_1, \ldots, A_m$ is a derivation of $\Gamma/\Delta, A$. If $C = B_i$, then either $C \neq A$ or $C = A$. In the first case $C \notin \Gamma' \cup \{A\}$, and as $B_1, \ldots, B_n$ is a derivation of $\Gamma', A/\Delta'$, there are $i_1, \ldots, i_k < i$ such that $B_{i_1}, \ldots, B_{i_k}/C$ is in $\mathsf{Ru}_{\vdash} \cup \mathsf{Ru}_{\mathcal{R}}$. In the second case, A_m/C is in $\mathsf{Ru}_{\vdash}$ because consequence relations are reflexive. ⊣
Note that the above holds in particular for single-conclusion consequence relations and rule schemes. The following proposition is the analogue of Proposition 3.3 one for multi-conclusion consequence relations that are not saturated.

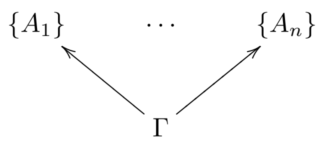
Proposition 3.4 For any consequence relation $\vdash$ and any set of rule schemes $\mathcal{R}$ that both do not contain rules with empty conclusion: $\Gamma \vdash^{\mathcal{R}} \Delta$ if and only if Γ/Δ has a multi-conclusion derivation in $\mathsf{Ru}_{\vdash} \cup \mathsf{Ru}_{\mathcal{R}}$.

Proof $\Leftarrow$ First observe that any multi-conclusion derivation T in $\mathsf{Ru}_{\vdash} \cup \mathsf{Ru}_{\mathcal{R}}$ of Γ/Δ is a multi-conclusion derivation in $\mathsf{Ru}_{\vdash} \cup \mathsf{Ru}_{\mathcal{R}}$ of $\Gamma/\mathrm{lb}(T)$ as well. As $\vdash^{\mathcal{R}}$ is closed under weakening, it therefore suffices to show that for any multi-conclusion derivation T of $\Gamma/\mathrm{lb}(T)$ in $\mathsf{Ru}_{\vdash} \cup \mathsf{Ru}_{\mathcal{R}}$ with root r, $\Gamma \vdash^{\mathcal{R}} \mathrm{lb}(T)$ if T has depth greater than one, and $\Gamma \vdash^{\mathcal{R}} A$ for all $A \in \mathrm{lb}(r)$ if T has depth one. We prove this with induction to the number of elements of T. If T consists of one node, r, we are in the second case, and indeed $\Gamma \vdash^{\mathcal{R}} A$ for all $A \in \mathrm{lb}(r)$, as $\vdash$ is reflexive and $\mathrm{lb}(r) \subseteq \Gamma$.

Suppose T has more than one node and let k be a node in T that has immediate successors $k_1, \ldots, k_m$, which are all leafs. Let $\Sigma = \{\mathrm{lb}(k_i) \mid i \leq n\}$ and let T^- be the result of removing the nodes $k_1, \ldots, k_m$ from T. Thus k is a leaf of T^-. If the label of some k_i belongs to Γ, $\Gamma \vdash^{\mathcal{R}} \mathrm{lf}(T)$ clearly holds by the reflexivity of $\vdash$. Therefore assume otherwise. By definition there exists $\Pi \subseteq \Gamma \cup \mathrm{lb}(k\!\downarrow)$ such that $\Pi/\Sigma \in \mathsf{Ru}_{\vdash} \cup \mathsf{Ru}_{\mathcal{R}}$. Let $\Sigma' = \mathrm{lf}(T^-)\backslash\{\mathrm{lb}(k)\}$, and let $\Pi\backslash\Gamma = \{B_1, \ldots, B_n\}$ and let l_i be the node in $k\downarrow$ with label B_i. T_i is the result of deleting all nodes above l_i from T^-. As T_i has less nodes than T, the induction hypothesis applies, proving that $\Gamma \vdash^{\mathcal{R}} \mathrm{lf}(T_i)$ for all i. By weakening, also $\Gamma \vdash^{\mathcal{R}} \{B_i\} \cup \Sigma'$. As $\Pi/\Sigma \in \mathsf{Ru}_{\vdash} \cup \mathsf{Ru}_{\mathcal{R}}$, closure under transitivity gives $\Gamma \vdash^{\mathcal{R}} \Sigma \cup \Sigma'$, that is, $\Gamma \vdash^{\mathcal{R}} \mathrm{lf}(T)$.

$\Rightarrow$ By Lemma 3.2 it suffices to show that any Γ/Δ in $\mathsf{Ru}_{\vdash} \cup \mathsf{Ru}_{\mathcal{R}}$ has a multi-conclusion derivation in $\mathsf{Ru}_{\vdash} \cup \mathsf{Ru}_{\mathcal{R}}$, and if all rules in X have a multi-conclusion derivation in $\mathsf{Ru}_{\vdash} \cup \mathsf{Ru}_{\mathcal{R}}$, the so do $\mathrm{wc}(X)$ and $\mathrm{tc}_i(X)$ for all i. We prove the first and the last case.

Suppose that Γ/Δ belongs to $\mathsf{Ru}_{\vdash} \cup \mathsf{Ru}_{\mathcal{R}}$, where $\Delta = \{A_1, \ldots, A_n\}$. Then the following tree is a multi-conclusion derivation of it.



To prove the last case, it suffices to show that if $\Gamma/\Delta, A$ and $\Gamma', A/\Delta'$ have respective multi-conclusion derivations T_1 and T_2 in $\mathsf{Ru}_{\vdash} \cup \mathsf{Ru}_{\mathcal{R}}$, then so does $\Gamma\Gamma'/\Delta\Delta'$. Let r_i be the root of T_i. If A does not occur in $\mathrm{lf}(T_1)$, then T_1 is a multi-conclusion derivation of $\Gamma, \Gamma'/\Delta, \Delta'$. And if A does not occur in $\mathrm{lb}(r_2)$, then T_2 is a multi-conclusion derivation of $\Gamma, \Gamma'/\Delta, \Delta'$. Therefore suppose that $A \in \mathrm{lf}(T_1) \cap \mathrm{lb}(r_2)$. Now let T be the tree obtained by glueing the root of T_2 to all the leafs of T_1 with label $\{A\}$, and let the label of this node remain $\{A\}$. All other labels remain as they were, except for the root, which receives label $\Gamma \cup \Gamma'$ in T. It is not difficult to see that T is a multi-conclusion derivation of $\Gamma, \Gamma'/\Delta, \Delta'$ in $\mathsf{Ru}_{\vdash} \cup \mathsf{Ru}_{\mathcal{R}}$. $\dashv$

3.5 Hilbert-type systems

Hilbert-style proof systems for propositional logics have the most straightforward representation in terms of consequence relations. In this section we only consider intermediate logics and single-conclusion consequence relations and rules. For intermediate logics, a Hilbert-style proof system is often given as a set of axioms and rules, but in our terminology these are rule schemes, where the set of substitutions for every rule is the maximal one, *Sub*. Proposition 3.3 has the following corollary, where derivability is defined at the beginning of Section 3, and $\vdash_{\mathrm{m}}$ stands for the minimal consequence relation: $\Gamma \vdash_{\mathrm{m}} \Delta$ if and only of $\Gamma \cap \Delta \neq \emptyset$.

Corollary 3.6 For any set of single-conclusion rule schemes $\mathcal{R}$: $\Gamma \vdash_{\mathrm{m}}^{\mathcal{R}} \Delta$ if and only if Γ/Δ has a derivation in $\mathsf{Ru}_{\mathcal{R}}$.

Thus for a Hilbert-type system $\mathcal{R}$, $\vdash_{\mathrm{m}}^{\mathcal{R}}$ not only covers L, but it does so in a faithful way: what is usually meant by "formula A follows from formulas Γ in the system" is exactly captured by derivability in $\vdash_{\mathrm{m}}^{\mathcal{R}}$.

Recall that for a logic L, the consequence relation $\vdash_{\mathsf{L}}$ is defined as $\Gamma \vdash_{\mathsf{L}} A$ if and only if $\bigwedge \Gamma \to A$. Suppose that L is an intermediate logic of which the Hilbert-type system $\mathcal{R}$ is an axiomatization. Then in general $\vdash_{\mathsf{L}}$ does not need to be equal to $\vdash_{\mathrm{m}}^{\mathcal{R}}$, as the former was only defined on the basis of the theorems of L and $\vdash_{\mathrm{m}}^{\mathcal{R}}$ takes the way formulas are inferred into account. For example, if $\mathcal{R}$ consists only of the theorems of L, thus only of axioms, then $\Gamma \vdash_{\mathrm{m}}^{\mathcal{R}} A$ will only hold if $A \in \Gamma$ or $A \in \mathsf{Th}(\mathsf{L})$, while $\Gamma \vdash_{\mathsf{L}} A$ will hold in many more cases. Thus showing that $\vdash_{\mathsf{L}} \not\subseteq \vdash_{\mathrm{m}}^{\mathcal{R}}$.

On the other hand, under some mild conditions, $\vdash_{\mathsf{L}} \subseteq \vdash_{\mathrm{m}}^{\mathcal{R}}$ does hold. Namely, in case $\mathcal{R}$ contains the rule schemes Modus Ponens $((B \to C, B/C), Sub)$ and $((B, C/B \wedge C), Sub)$. Because if so, we have $\Gamma \vdash_{\mathrm{m}}^{\mathcal{R}} \bigwedge \Gamma$, which in case $\bigwedge \Gamma \to A$ is a theorem of L, together with Modus Ponens and the fact that $\vdash_{\mathrm{m}}^{\mathcal{R}}$ is closed under Cut, gives $\Gamma \vdash_{\mathrm{m}}^{\mathcal{R}} A$.

For many intermediate logics there exist Hilbert-type systems $\mathcal{R}$ such that $\vdash_{\mathrm{m}}^{\mathcal{R}}$ is not contained in $\vdash_{\mathsf{L}}$. For example, for IPC the set of all admissible rules has this property, as we will see in Section 4.

4 Derivability and admissibility

In this section we introduce the notions of derivability and admissibility that were the reason for spelling out the details about consequence relations in the previous sections. Intuitively, derivable rules are the ones explicitly given by the consequence relation, while admissible rules can be used in proofs without changing the theorems that can be derived. Our definition of admissibility for multi-conclusion rules, according to the full view given in the introduction, is more general than the one found in the literature, which is usually the one based on the strict view. We will show that the two notions coincide for consequence relations with the disjunction property. In most case in the literature where the strict definition is used, the consequence relation has a disjunction property, in the intermediate case exactly the one used here, and in the modal case the *modal disjunction property* ($\Box\Delta$ being short for $\{\Box A \mid A \in \Delta\}$):

$$\vdash \Box\Delta \Rightarrow \exists A \in \Delta \vdash A.$$

We do not treat modal logics separately here, but just mention that in case the logic satisfies the modal disjunction property, an analogue of Corollary 4.2

below can be formulated.

Given a consequence relation $\vdash$, a rule $R = \Gamma/\Delta$ is *derivable* if $\Gamma \vdash \Delta$. Note that by Proposition 3.3 for saturated consequence relations this is equal to R being derivable in $\mathsf{Ru}_\vdash$, as defined in Section 3. We will use "derivable in $\mathsf{Ru}_\vdash$" and "derivable in $\vdash$" to express that $\Gamma \vdash \Delta$. The rule scheme (R, S) is *derivable* if for all $\sigma \in S$, σR is derivable. R is *admissible*, written $\Gamma \mathrel{|\!\sim} \Delta$, if $\mathsf{Thm}(\vdash) = \mathsf{Thm}(\vdash^R)$. The rule scheme (R, S) is *admissible*, written $\Gamma \mathrel{|\!\sim}_S \Delta$, if $\mathsf{Thm}(\vdash) = \mathsf{Thm}(\vdash^R_S)$. A set of rules (schemes) is admissible if all of its members are. Similarly for single-conclusion consequence relations. Observe that for consequence relations with the disjunction property, a rule scheme (R, S) is admissible if and only if $\mathsf{Th}(\vdash) = \mathsf{Th}(\vdash^R_S)$.

Note that for *Sub* being the set of all substitutions and Γ/Δ a rule, $\Gamma \mathrel{|\!\sim} \Delta$ is equivalent to $\Gamma \mathrel{|\!\sim}_{Sub} \Delta$, which is equivalent to the rule scheme $(\Gamma/\Delta, Sub)$ being admissible. The multi-conlcusion relation $\mathrel{|\!\sim}$ is defined as

$$\mathrel{|\!\sim} \ \equiv_{def} \{(\Gamma, \Delta) \mid \mathsf{Thm}(\vdash) = \mathsf{Thm}(\vdash^{\Gamma/\Delta})\}$$

and similarly for the single-conclusion relation.

A single-conclusion consequence relation $\vdash$ is *structurally complete* if all admissible single-conclusion rules (in the same language) are derivable, and *hereditarily structurally complete* if all extensions in the same language are structurally complete. A multi-conclusion consequence relation $\vdash$ is *universally complete* if all admissible multi-conclusion rules are derivable. Clearly, $\vdash$ is structurally complete if it coincides with $\mathrel{|\!\sim}$. And a single-conclusion consequence relation $\vdash$ is structurally complete if all proper extensions in the same language have new theorems.

Structural completeness depends very much on the particular consequence relation one uses for a logic. That is, a logic can have two consequence relations that both cover it, where the one is structurally complete, and the other is not. For example, as is well-known, $\vdash_{\mathsf{CPC}}$ is structurally complete and covers CPC (folklore, but for a proof see [5]). However, the minimal consequence relation covering CPC,

$$\Gamma \vdash A \Leftrightarrow A \in \Gamma \cup \mathsf{Th}(\mathsf{CPC}),$$

is certainly not structurally complete, as $\neg\neg A/A$ is admissible but nonderivable in it. More on classical logic in Section 4.7.

In contrast to structural completeness, admissibility solely depends on the (multi-conclusion) theorems of a consequence relation, as can be seen from the definition as well as Proposition 4.1 below. Using the developed terminology and precision, the full and strict view for single-conclusion consequence relations becomes for single-conclusion rule schemes (R, S):

(full) (R, S) is admissible in $\vdash$ if $\mathsf{Th}(\vdash) = \mathsf{Th}(\vdash^R_S)$.

(strict) (R, S) is admissible in $\vdash$ if under all substitutions in S, whenever all premisses of R become theorems of $\vdash$, then so does the conclusion.

As mentioned in the introduction, the full definition of admissibility is the one most often given when the notion is described informally. The strict definition, however, is the one most used in a technical setting. Corollary 4.2 proves they are the same in case the disjunction property holds.

Proposition 4.1 For every consequence relation $\vdash$:

$$\Gamma \mathrel{|\!\!\sim}_S \Delta \;\Leftrightarrow\; \forall\Sigma\forall\sigma \in S:\ (\forall A \in \Gamma \vdash \sigma A, \Sigma) \;\Rightarrow \vdash \sigma\Delta, \Sigma.$$

Proof Let $R = \Gamma/\Delta$.

$\Rightarrow$ Suppose $\Gamma \mathrel{|\!\!\sim}_S \Delta$, that is, $\mathsf{Thm}(\vdash) = \mathsf{Thm}(\vdash_S^R)$. If $\vdash \sigma A, \Sigma$ for all $A \in \Gamma$ and $\sigma \in S$, this means $\sigma A, \Sigma$ belongs to $\mathsf{Thm}(\vdash)$ for all $A \in \Gamma$. Hence $\sigma\Delta, \Sigma$ belongs to $\mathsf{Thm}(\vdash_S^R)$, and therefore $\sigma\Delta, \Sigma \in \mathsf{Thm}(\vdash)$, that is, $\vdash \sigma\Delta, \Sigma$.

$\Leftarrow$ Assuming the right side of the equivalence, we show that $\mathsf{Thm}(\vdash)$ equals $\mathsf{Thm}(\vdash_S^R)$. Therefore assume $\vdash_S^R \Delta$. By Proposition 3.4 there is a multi-conclusion derivation T of Δ in $\mathsf{Ru}_\vdash \cup \mathsf{Ru}_{\{(R,S)\}} = \mathsf{Ru}_\vdash \cup \{\sigma R \mid \sigma \in S\}$. Thus $\mathrm{lb}(T)$, the labels of the leafs of T, are contained in Δ and the label $\mathrm{lb}(r)$ at the root is empty. Let T_i be the tree obtained by omitting the nodes of depth larger than i in T. Then one can show with induction to i that $\vdash \mathrm{lb}(T_i)$ for $i > 1$, which will prove the proposition.

If $i = 2$ this is clear, using that $\mathrm{lb}(r)$ is empty. Suppose the depth of T is $i+1$ and let T^- be the result of omitting all nodes at depth $i+1$ in T. Given a node k in $\mathrm{lf}(T^-)$, let $\Delta(k)$ be the union of the labels of its immediate successors in T if it has any, otherwise $\Delta(k)$ is equal to $\mathrm{lb}(k)$. By definition, $\mathrm{lb}(T) = \bigcup\{\Delta(k) \mid k \in \mathrm{lf}(T^-)\}$. The induction hypothesis implies that $\vdash \mathrm{lb}(T^-)$. Using the definition of the trees it is easy to see that for all leafs $k \in \mathrm{lf}(T^-)$, $\mathrm{lb}(k) \vdash^{\mathcal{R}} \Delta(k)$. As $k \neq r$ for all leafs k, the sets $\mathrm{lb}(k)$ consist of at most one formula. This implies that $\vdash^{\mathcal{R}} \bigcup\{\Delta(k) \mid k \in \mathrm{lf}(T^-)\}$ holds, which is what we had to show. $\dashv$

This immediately gives the following.

Corollary 4.2 For every structural consequence relation $\vdash$ with the disjunction property:

$$\Gamma \mathrel{|\!\!\sim}_S \Delta \;\Leftrightarrow\; \forall\sigma \in S:\ (\forall A \in \Gamma \vdash \sigma A) \;\Rightarrow\; (\exists B \in \Delta \vdash \sigma B).$$

Therefore every single-conclusion structural consequence relation satisfies

$$\Gamma \mathrel{|\!\!\sim}_S A \;\Leftrightarrow\; \forall\sigma \in S:\ (\forall B \in \Gamma \vdash \sigma B) \;\Rightarrow\; (\vdash \sigma A).$$

The last proposition and corollary imply the following corollary, the proof of which is straightforward.

Corollary 4.3 Both in the single-conclusion and the multi-conclusion context, $\mathrel{|\!\!\sim}$ is a consequence relation. If the underlying consequence relation is structural, so is $\mathrel{|\!\!\sim}$.

Corollary 4.4 For consequence relations with the disjunction property, the greatest single-conclusion consequence relation in $\mathcal{L}$ with the same theorems as $\vdash$ is the single-conclusion $\mathrel{|\!\!\sim}$. For multi-conclusion consequence relations, the greatest multi-conclusion consequence relation in $\mathcal{L}$ with the same multi-conclusion theorems as $\vdash$ is the multi-conclusion $\mathrel{|\!\!\sim}$.

Proof We treat the general case and show that $\mathrel{|\!\!\sim}$ is the greatest consequence relation such that $\mathsf{Thm}(\vdash) = \mathsf{Thm}(\mathrel{|\!\!\sim})$. That $\mathsf{Thm}(\vdash)$ is contained in $\mathsf{Thm}(\mathrel{|\!\!\sim})$ is clear. For the other direction, suppose $\mathrel{|\!\!\sim} \Delta$. By Proposition 4.1, $\vdash \Delta$ follows, thus showing that $\mathsf{Thm}(\vdash) \supseteq \mathsf{Thm}(\mathrel{|\!\sim})$. If $\mathsf{Thm}(\vdash) = \mathsf{Thm}(\vdash')$ for some consequence relation $\vdash'$, then for all rule schemes (R,S) derivable in $\vdash'$, it follows that $\mathsf{Thm}(\vdash) = \mathsf{Thm}(\vdash^R_S)$, and thus that R is admissible. Thus proving that $\vdash'$ is contained in $\mathrel{|\!\sim}$. $\dashv$

Proposition 4.1 shows that via multi-conclusion consequence relations one can express the disjunction property: an intermediate logic has the disjunction property if and only if $\{p \vee q\}/\{p,q\}$ is admissible, and similarly for modal logic and the modal disjunction property $\{\Box p \vee \Box q\}/\{p,q\}$. This is one of the reasons to use multi-conclusion consequence relations rather than single-conclusion ones, as discussed in the first part of Section 2.

Remark 4.5 The full definition of admissibility for multi-conclusion rule schemes (R,S) as given informally in the introduction becomes in our terminology

(full) (R,S) is admissible in $\vdash$ if $\mathsf{Thm}(\vdash) = \mathsf{Thm}(\vdash^R_S)$.

Proposition 4.1 suggests that the full definition of admissibility is not equivalent to what might seem to be an analogue of the strict definition, as discussed in the introduction:

(R,S) is admissible in $\vdash$ if under all substitutions in S, whenever all premisses of R become multi-conclusion theorems of $\vdash$, then the conclusion becomes a multi-conclusion theorem of $\vdash$.

Here is an actual counter example. Let $\Gamma = \{p\}$ and $\Delta = \{q\}$ and let R be Γ/Δ and let S consist of the identity substitution. Let the multi-conclusion consequence relation $\vdash$ be the smallest one such that $\vdash \{p,q\}$ holds. Then it certainly holds that whenever the premiss of R becomes a theorem of $\vdash$, then so does the conclusion, as p is no theorem of $\vdash$. On the other hand, $\mathsf{Thm}(\vdash)$ is not equal to $\mathsf{Thm}(\vdash^R_S)$, as not even $\mathsf{Th}(\vdash)$ is equal to $\mathsf{Th}(\vdash^R_S)$, the former not containing q, while the latter does.

Based on Proposition 4.1 the strict analogue seems to be

(strict) $(\Gamma/\Delta, S)$ is admissible in $\vdash$ if for all $\sigma \in S$ and all finite Σ, whenever $\sigma A, \Sigma \in \mathsf{Thm}(\vdash)$ for all $A \in \Gamma$, then $\sigma\Delta, \Sigma \in \mathsf{Thm}(\vdash)$.

4.6 Bases

Given consequence relations $\vdash \subseteq \vdash'$, a set of rules $\mathcal{R}$ is a *basis* for $\vdash'$ over $\vdash$ if $\vdash^{\mathcal{R}} = \vdash'$. In particular, $\mathcal{R}$ is a basis of $\vdash^{\mathcal{R}}$. Thus $\mathcal{R}$ is a *basis* for the admissible rules of a given consequence relation $\vdash$ iff the rules in $\mathcal{R}$ are admissible in $\vdash$ and all admissible rules of $\vdash$ are derivable in $\vdash^{\mathcal{R}}$:

$$\mathrel{|\!\sim} \;=\; \vdash^{\mathcal{R}}.$$

This notion allows one to describe $\mathrel{|\!\sim}$ without having to include redundancies. For example, for intermediate or modal logics L, if the rule $R = A/B$ is admissible, then so is $A \wedge C/B \wedge C$, but we do not have to add the latter to the basis as it is derivable in $\vdash^R_{\mathsf{L}}$:

$$\dfrac{\dfrac{A \wedge C \vdash^R_{\mathsf{L}} A \quad A \vdash^R_{\mathsf{L}} B}{A \wedge C \vdash^R_{\mathsf{L}} B} \quad \dfrac{A \wedge C \vdash^R_{\mathsf{L}} C \quad B, C \vdash^R_{\mathsf{L}} B \wedge C}{B, A \wedge C \vdash^R_{\mathsf{L}} B \wedge C}}{A \wedge C \vdash^R_{\mathsf{L}} B \wedge C}$$

Here we use that $A \wedge C \vdash_{\mathsf{L}} A$ and $B, C \vdash_{\mathsf{L}} B \wedge C$ hold, which is the case for all logics L in which $A \wedge C \to A$ and $B \wedge C \to B \wedge C$ hold, by the definition of $\vdash_{\mathsf{L}}$.

There are in general no size restrictions on bases, but a basis is called *independent* if no rule is derivable from the others, and *finite* if it is finite.

Many intermediate and modal logics and fragments thereof are known to have nonderivable admissible rules, and for several an explicit basis for the admissible rules is known. We refer the reader to (the references in) [5, 9]. We finish the paper by discussing some well-known examples.

4.7 Examples

1. Probably the most famous admissible rule for intermediate logics is the *Harrop* or *Kreisel–Putnam* rule:

$$\frac{\neg A \to B \vee C}{(\neg A \to B) \vee (\neg A \to C)}\ \mathsf{HR}$$

Prucnal discovered the universal character of this rule, a result later strengthened by Minari and Wroński in [6]:

Theorem 4.8 [8] HR is admissible in any intermediate logic.

That HR is not always derivable follows from the underivability of the corresponding implication $(\neg A \to B \vee C) \to (\neg A \to B) \vee (\neg A \to C)$ in intuitionistic logic.

2. In Section 3 we encountered the density rule in predicate logic,

$$\frac{(A \to P) \vee (P \to B)}{(A \to B)}\quad (P \text{ atomic}),$$

in which it is required that P is an atomic formula. Here the A, B, P are all meta-variables. This rule, first introduced in [11, 12] in an axiomatization of first-order Gödel logic (or Intuitionistic Fuzzy Logic as many call it), has later been used in the setting of hypersequents, which are very well suited to express rules of this form. The density rule becomes

$$\frac{G \mid \Gamma \Rightarrow P \mid P, \Pi \Rightarrow \Delta}{G \mid \Gamma, \Pi \Rightarrow \Delta}\ \mathsf{Den}$$

Recall that when a rule is presented as a rule scheme, the corresponding set of substitutions is supposed to consist of all substitutions (in that setting). Let T be the set of rules of the hyper sequent system except density and let $\vdash$ denote the minimal consequence relation $\vdash_{\mathrm{m}}$. Let S_P be the set of substitutions that replace meta-variables by formulas, except for P, which is replaced by atomic formulas only. Then the rule scheme (Den, S_P) is admissible in $\vdash^T$ [11].

3. Recall that in Section 2.4 the multi-conclusion consequence relation $\vdash_{\mathsf{CPC}}$ is defined as

$$\Gamma \vdash_{\mathsf{CPC}} \Delta \equiv_{def} (\bigwedge \Gamma \to \bigvee \Delta) \in \mathsf{Th}(\mathsf{CPC}).$$

This consequence relation is structurally complete. For suppose $\Gamma \mid\!\sim_{\mathsf{CPC}} \Delta$. By Proposition 4.1 this implies that for all substitutions that map atoms to $\top$ or $\bot$, if $\bigwedge \Gamma$ becomes valid in CPC, then so does $\bigvee \Delta$. In other words, $(\bigwedge \Gamma \to \bigvee \Delta)$ is a theorem of CPC, and thus Γ/Δ is derivable in $\vdash_{\mathsf{CPC}}$.

One sometimes hears the remark that multi-conclusion CPC is not structurally complete. But arguments supporting this claim are always based on what in our opinion is the incorrect definition of admissibility for logics without the disjunction property, namely on dp-full or dp-strict. The above argument shows that reasonable consequence relations covering CPC are indeed structurally complete.

References

[1] A. Avron, Simple Consequence Relations, *Information and Computation* 92(1), 1991 pp. 105–139.

[2] A. Avron, Two Types of Multiple-Conculsion Systems, *Logic Journal of the IGPL* 6, 1998 pp. 695–717.

[3] K. Došen, On passing from singular to plural consequences, *Logic at Work: Essays Dedicated to the Memory of Helena Rasiowa*, E. Orlowska (ed.), Physica-Verlag, Heidelberg, 1999, pp.533-547.

[4] M. Baaz and R. Zach, Hypersequents and the proof theory of intuitionistic fuzzy logic, in *Proceedings of CSL 2000 - LNCS*, Springer-Verlag, 2000, pp. 187-201.

[5] R. Iemhoff, On rules, *Journal of Philosophical Logic*, to appear. http://www.phil.uu.nl/ iemhoff/papers.html

[6] P. Minari and A. Wronski, The property (HD) in intermediate logics: a partial solution of a problem of H. Ono, *Reports on Mathematical Logic* 22, 1988, pp. 21–25.

[7] G. Metcalfe, Admissible rules: from characterizations to applications, in *Proceedings of WoLLIC 2012*, *Lecture Notes in Computer Science* 7456, 2012, pp. 56–69.

[8] T. Prucnal, On two problems of Harvey Friedman, *Studia Logica* 38(3), 1979, p.247-262.

[9] V. Rybakov, *Admissibility of Logical Inference Rules*, Elsevier, 1997.

[10] D.J. Shoesmith and T.J. Smiley, *Multiple-Conclusion Logic*, Cambridge University Press, 1978.

[11] M. Takano, Another proof of the strong completeness of the intuitionistic fuzzy logic, *Tsukuba J. Math.* 11, 1987, pp. 101-105.

[12] G. Takeuti and T. Titani, Intuitionistic fuzzy logic and intuitionistic fuzzy set theory, *Journal of Symbolic Logic* 49(3), 1984, pp. 851-866.

[13] A. Tarski, Über den Begriff der logischen Folgerung, in *Actes du Congrès International de Philosophie Scientifique, fasc. 7 (Actualités Scientifiques et Industrielles 394)*, Paris: Hermann et Cie, 1936, pp. 1–11.

[14] A.S. Troelstra and H. Schwichtenberg, *Basic Proof Theory*, Cambridge University Press, 1996.

[15] R. Wójcicki, *Theory of Logical Calculi: Basic Theory of Consequence Operations*, Springer, 1988.