Department of Philosophy - Utrecht University

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Y. Venema

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Cylindric Modal Logic

Yde Venema
Department of Philosophy
Utrecht University
Heidelberglaan 8
3584 CS Utrecht
e-mail: yde@phil.ruu.nl

Abstract.

Treating the existential quantification $\exists v_i$ as a diamond $\Diamond_i$ and the identity $v_i = v_j$ as a constant δ_{ij}, we study restricted versions of first order logic as if they were modal formalisms. This approach is closely related to algebraic logic, as the Kripke frames of our system have the type of the atom structures of cylindric algebras; the full cylindric set algebras are the complex algebras of the intended multi-dimensional frames called cubes.

The main contribution of the paper is a characterization of these cube frames for the finite-dimensional case and as a consequence of the special form of this characterization, a completeness theorem for this class. These results lead to finite, though unorthodox derivation systems for several related formalisms, e.g. for the valid n-variable first order formulas, for type-free valid formulas and for the equational theory of the Representable Cylindric Algebras. The result for type-free valid formulas indicates a positive solution to problem 4.16 of Henkin, Monk & Tarski [16].

Keywords: algebraic logic, modal logic, logic with finitely many variables, completeness, derivation rules.

1980 Mathematical Subject Classification: 03B20, 03B45, 03C90, 03G15.

1 Introduction

In this paper we develop a modal formalism called cylindric modal logic we investigate its basic semantics and axiomatics. The motivation for introducing this formalism is twofold: first, it forms an interesting bridge over the gap between propositional formalisms and first-order logic. And second, the modal tools developed in studying cylindric modal logic will be applied to analyze some problems in algebraic logic.

To start with the first point, let us consider (multi-)modal logic; here *correspondence theory* (cf. van Benthem [7]) studies the relation between modal and classical formalisms as languages for the same class of Kripke structures. The usual direction in correspondence theory is to start with a variable-free operator-language, and then search for a fragment of first-order logic which is expressively equivalent to it. The aim that we set ourselves here is the converse: to devise and study a modal formalism which is equally expressive as first-order logic itself. In fact, in this paper we will show how the above-mentioned gap vanishes if we implement the following idea:

> we can restrict the syntax of first-order logic in such a way that it behaves like a propositional modal logic.

Obviously, the central idea in the 'modalization' of first-order logic is to *look at quantifiers as if they were modal operators*. Indeed, several authors have observed the resemblance between quantifiers and modal ($S5$-)operators; some references are listed in Kuhn [19].

Let us start with syntax: we will define a language that has two readings, both as a restricted version of first-order logic, and as a multi-modal logic. Suppose that we have a language of first-order logic with the constraints that there are α many variables (with α an arbitrary but fixed ordinal), and that the only admissible *atomic* formulas are of the form $v_i = v_j$ or $R_l(v_0 v_1 \dots v_i \dots)_{i<\alpha}$ — the motivation for adopting this particular restriction will be given below. For $\alpha < \omega$, we get a logic with finitely many variables. Such logics have been studied in the literature, for purely logical reasons (Henkin [15], Henkin, Monk & Tarski [16], Tarski-Givant [41], Sain [37], Monk [24]) or because of their relation with temporal logics in computer science (Gabbay [12], Immerman-Kozen [17], Venema [43]). For $\alpha \geq \omega$ the logic is sometimes called the finitary logic of infinitary relations, cf. Sain [34]. Note that as their order is fixed, the variables in atomic relational formulas do not provide any information. We may leave them out, writing R_l for $R_l(v_0 \dots v_i \dots)_{i<\alpha}$. This *restricted first-order logic* becomes *cylindric modal logic* if we replace the identity $v_i = v_j$ with the *modal constant* δ_{ij}, and the existential quantification $\exists v_i$ with the *diamond* $\Diamond_i$. In order not to confuse the reader with too much notation, we will use modal notation and terminology mainly, occasionally referring to the first-order interpretation for motivations or clarifications.

Definition 1.1 *Let α be an arbitrary but fixed ordinal with $2 \leq \alpha$. CML_α is the modal similarity type having constants δ_{ij} for $i, j < \alpha$ and diamonds $\Diamond_i$ for $i < \alpha$. For a set of propositional variables Q, the set of α-dimensional cylindric modal formulas in Q, or for short,* **α-formulas (in Q)**, *is built up as usual: the atomic formulas are the (modal or boolean) constants and the propositional variables, and a formula is either atomic or of the form $\neg\phi$, $\phi \vee \psi$ or $\Diamond_i\phi$, where ϕ, ψ are formulas. We use standard abbreviations like $\wedge$, $\rightarrow$ and $\Box_i$.*

Turning to semantics, we will give the basic declarative statement in first-order logic, viz.

$$\mathfrak{M} \models \phi[u]$$

the modal reading 'ϕ holds in $\mathfrak{M}$ at the *possible world* u'. Here the assignment u to variables of elements of the domain can be identified with the α-tuple $(u(v_0), \dots, u(v_i), \dots)_{i<\alpha}$, for a fixed ordinal α. Clearly then the intended semantics of our modal language has a multi-dimensional character: the universe of a model for CML_α is of the form ${}^\alpha U$ (all α-tuples over some base set U). Note that the interpretation functions of first-order logic, indicating for which tuples a predicate holds, will turn up as modal valuations, i.e. maps assigning a set of possible worlds to each propositional variable.

Now let us formulate the essential clauses of the truth definition for restricted first-order logic as follows:

$$\begin{array}{lcl} \mathfrak{M} \models v_i = v_j\, [x] & \iff & x \in D_{ij}, \\ \mathfrak{M} \models \exists v_i \phi\, [x] & \iff & \exists y (x \equiv_i y \,\&\, \mathfrak{M} \models \phi\, [y]). \end{array}$$

with D_{ij} being the set of α-tuples with identical i- and j-coordinates, and $x \equiv_i y$ holding between two α-tuples iff x and y differ at most in their i-th coordinate. The crucial observation, and in fact the basic observation underlying our whole enterprise, is that this truth definition is in fact of a *modal* nature: we may see D_{ij} and $\equiv_i$ as unary resp. binary *accessibility relations* on the α-dimensional universe.

Note however that any modal similarity type comes automatically with a (Kripke) semantics consisting of abstract relational frames, i.e. structures having an *arbitrary* $(n+1)$-ary accessibility relation for every (n-ary) modal operator. Thus we obtain two kinds of semantics for restricted first-order logic/ cylindric modal logic; of these, the truly α-dimensional frames or *cubes*[1] form a subclass of the relational α-frames:

Definition 1.2 *An* α**-frame** *is a structure* $\mathfrak{F} = (W, T_i, E_{ij})_{i,j<\alpha}$ *with every* $T_i \subseteq W \times W$ *and every* $E_{ij} \subseteq W$. *An* α**-model** *is a pair* $\mathfrak{M} = (\mathfrak{F}, V)$ *with* $\mathfrak{F}$ *an* α*-frame and* V *a* **valuation**, *i.e. a map assigning a subset of the universe of* $\mathfrak{F}$ *to each propositional variable in the language.*

The (ordinary, i.e. unrestricted) first-order language used to describe these structures, having monadic predicates E_{ij} *and dyadic predicates* T_i, $i, j < \alpha$, *is denoted by* F_α.

Truth *of a formula* ϕ *at a world* w *in the model* $\mathfrak{M}$ *is defined by the usual induction, e.g.*

$$\begin{array}{lcl} \mathfrak{M}, w \Vdash p & \iff & w \in V(p), \\ \mathfrak{M}, w \Vdash \delta_{ij} & \iff & w \in E_{ij}, \\ \mathfrak{M}, w \Vdash \Diamond_i \psi & \iff & \textit{there is a } v \textit{ with } wT_iv \textit{ and } \mathfrak{M}, v \Vdash \psi. \end{array}$$

If no confusion arises concerning the model involved, we may abbreviate $\mathfrak{M}, w \Vdash \phi$ *by* $w \Vdash \phi$.

Now let U *be some set; the* α*-frame* $\mathfrak{C}_\alpha(U) = ({}^\alpha U, \equiv_i, D_{ij})_{i,j<\alpha}$, *with*

$$\begin{array}{lcl} D_{ij} & = & \{x \in {}^\alpha U \mid x_i = x_j\}, \\ \equiv_i & = & \{(x, y) \in {}^\alpha U \times {}^\alpha U \mid \textit{for all } j \neq i,\ x_j = y_j\ \}, \end{array}$$

is called the α**-cube** *(or* **square**, *in case* $\alpha = 2$*)* **over** U. *The class of* α*-cubes is denoted by* C_α.

Validity *of a formula or set of formulas in a model/frame/class of frames is defined and denoted as usual, e.g.*

$$\mathsf{C}_n \models \phi \iff \textit{for all frames } \mathfrak{F} \textit{ in } \mathsf{C}_n, \textit{ all valuations } V \textit{ on } \mathfrak{F} \textit{ and all worlds } w \textit{ in } \mathfrak{F}\text{: } \mathfrak{F}, V, w \Vdash \phi.$$

Note that with this definition, a restricted first-order formula is valid (in the usual sense of model theory, i.e. valid in every appropriate structure) iff its cylindric modal version is cube-valid, i.e. valid in the class of cubes.

The modal perspective on first-order logic has two important sides: first, it gives us an alternative, rather more general, semantics for first-order logic: restricted first-order formulas can now be interpreted at arbitrary α-frames. And second, the modal approach allows us to analyze problems concerning restricted first-order logic by modal means, i.e. using results and techniques developed in the theory of modal logic.

The choice we made in adapting the syntax of first-order logic may seem to be rather arbitrary[2]; its motivation takes us to the second aim of the paper, viz. the application of insights and results obtained in the theory of modal logic to the field of *algebraic logic*. For an overview of the algebraic approach towards logic we refer to Németi [27].

The framework that we are working in is the *duality theory* between Relational Kripke Frames and Boolean Algebras with Operators, cf. Goldblatt [14]. Our guideline is that

[1] The name 'cubes' for the intended frames is taken from a paper [29] by Prijatelj who studies related structures modeling natural language phenomena.

[2] For an alternative option, where (for finite α) arbitrary atomic formulas $Rv_{i_0} \dots v_{i_{\alpha-1}}$ are allowed, we refer to Venema [45].

the algebras of polyadic relations can be found as the modal algebras of our system.

In particular, the design of Cylindric Modal Logic is such that the modal algebras of our system have the type of *Cylindric Algebras* (cf. Henkin, Monk & Tarski [16]); the cubes are the atom structures of the *full* cylindric set algebras. In order not to confuse the reader by introducing too many formalisms at once, we delay the discussion of the precise connection between Cylindric Modal Logic and Cylindric Algebras to subsection 4.2. Let it suffice here to mention that modulo some trivial syntactic translations, the cylindric modal theory of the α-cubes can be identified with the equational theory of the Representable Cylindric Algebras of dimension α.

Overview
Let us now move on to indicate the main themes and results of the paper. The next section is devoted to *characterization* results. First we show that the class of zigzagmorphic images of disjoint unions of squares is definable by a finite set of cylindric modal formulas. Then we give a characterization of the class of finite-dimensional cubes in the first-order frame language F_α. This definition is special in the sense that it allows a modal so-called $\pm$-characterization of the class of disjoint unions of cubes. $\pm$-Characterizations have a positive and a negative part, in the style of Venema [44]. In section 3 we treat axiomatizations of cube validity: first we give a finite axiomatization of the class of squares. Then we prove that the $\pm$-characterization of section 2 can be turned into an axiomatization of cube validity, for arbitrary dimensions. This axiomatization will be unorthodox in the sense that the *negative* part of the $\pm$-characterization will return as a so-called non-ξ *rule* in the axiomatization. In section 4, this modal completeness result will first be applied in other, related fields of logic. We obtain complete axiomatizations for the valid schemas of first-order logic and for typeless validity. Note that the latter result indicates a possible solution to Problem 4.16 of Henkin, Monk & Tarski [16]. Then we turn to algebraic logic, defining a finite derivation system which recursively enumerates the equational theory of the class of Representable Cylindric Algebras. We finish the section with a negative result concerning interpolation, stating that no sufficiently strong, finite orthodox axiom system of restricted first order logic has Craig's Interpolation Property. Finally, we give a short evaluation of the paper's approach, and mention some recent developments and questions for further research.

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2 Cylindric Modal Logic: characterization results

In this section, we are interested in the question of how to give a (syntactic) characterization of the cubes among the α-frames. In principle, one would aim for a positive modal characterization, i.e. a set Γ of modal α-formulas such that for any α-frame $\mathfrak{F}$, $\mathfrak{F} \models \Gamma$ iff $\mathfrak{F}$ is a cube. However, such a set Γ cannot be found; a simple reason for this is that C_α is not closed under taking disjoint unions or zigzag morphic images, while modally definable classes are. For readers unfamiliar with these notions, we give the definitions here.[3]

Definition 2.1 *Let $\mathfrak{F}$ and $\mathfrak{F}'$ be two α-frames, then a map $f : W \mapsto W'$ is a* **zigzag morphism**[4] *iff it satisfies the following properties, for all $i, j < \alpha$:*

[3] It may be helpful for algebraists to note that zigzag morphisms and disjoint unions are the frame equivalents of complete subalgebras resp. direct products.

[4] This notion is called a *bounded morphism* in Goldblatt [14].

(1) f *is a homomorphism, i.e.* $E_{ij}u$ *only if* $E'_{ij}fu$ *and* T_iuv *only if* T'_ifufv*, for all* $i,j<\alpha$*,*
(2) $E_{ij}u$ *if* $E'_{ij}fu$*,*
(3) If T'_ifuv' *then there is a* $v\in W$ *such that* T_iuv *and* $fv=v'$*.*

If such an f *is onto, we will call* $\mathfrak{F}'$ *a* zigzagmorphic image *of* $\mathfrak{F}$*.*

Let $\{\mathfrak{F}^k \mid k\in K\}$ *be a family of frames. If the universes* W_j *are mutually disjoint, the* **disjoint union** $\Sigma_{j\in J}\mathfrak{F}_j$ *is defined as the frame* $\mathfrak{F}=(W,T_i,E_{ij})_{i,j<\alpha}$ *with* W*,* T_i *and* E_{ij} *being the unions of the* W^k*,* T_i^k *and* E_{ij}^k *respectively; if some of the universes overlap we take the disjoint union of some canonically defined family of mutually disjoined isomorphic copies of the* $\mathfrak{F}_i$*'s.*

For a class K *of frames, let* $\mathbf{H_f}(\mathsf{K})$ *resp.* $\mathbf{P_f}(\mathsf{K})$ *denote the classes of zigzagmorphic images resp. disjoint unions of* K*-frames.*

To give a simple example of a zigzagmorphism, consider the map $f: Z\times Z\mapsto Z$ given by $f(x,y)=y-x$ (Z is the set of integers). It is straightforward to verify that f is a surjective zigzagmorphism from $\mathfrak{C}_2(Z)$ onto the 2-frame $\mathfrak{Z}'$ over Z defined by $T_0=T_1=Z\times Z$, $E_{00}=E_{11}=Z$ and $E_{01}=E_{10}=\{0\}$. By the preservation of modal validity under taking zigzagmorphic images it follows that the cylindric modal theory of the squares is valid in $\mathfrak{Z}'$, while clearly $\mathfrak{Z}'$ is not a square.

So, if we confine ourselves to positive characterizations, the highest we can aim for is a definition of the class of zigzagmorphic images of disjoint unions of cubes. For the two-dimensional case we can achieve this aim, as we shall now see. As the formulas characterizing $\mathbf{H_f}\mathbf{P_f}\mathsf{C}_2$ also play an essential role for higher dimensions, we give a definition for arbitrary α:

Definition 2.2 *Consider the following pairs of* α*-formulas and* F_α*-formulas:*

$(CM1_i)$	$p\to\Diamond_i p$	$(N1_i)$	$\forall x T_ixx$
$(CM2_i)$	$p\to\Box_i\Diamond_i p$	$(N2_i)$	$\forall xy(T_ixy\to T_iyx)$
$(CM3_i)$	$\Diamond_i\Diamond_i p\to\Diamond_i p$	$(N3_i)$	$\forall xyz((T_ixy\wedge T_iyz)\to T_ixz)$
$(CM4_{ij})$	$\Diamond_i\Diamond_j p\to\Diamond_j\Diamond_i p$	$(N4_{ij})$	$\forall xz(\exists y(T_ixy\wedge T_jyz)\to\exists u(T_jxu\wedge T_iuz))$
$(CM5_i)$	δ_{ii}	$(N5_i)$	$\forall x E_{ii}x$
$(CM6_{ij})$	$\Diamond_i(\delta_{ij}\wedge p)\to\Box_i(\delta_{ij}\to p))$	$(N6_{ij})$	$\forall xyz((T_ixy\wedge E_{ij}y\wedge T_ixz\wedge E_{ij}z)\to y=z)$
$(CM7_{ijk})$	$\delta_{ij}\leftrightarrow\Diamond_k(\delta_{ik}\wedge\delta_{kj})$	$(N7_{ijk})$	$\forall x(E_{ij}x\leftrightarrow\exists y(T_kxy\wedge E_{ik}y\wedge E_{kj}y))$
$(CM8_{ij})$	$(\delta_{ij}\wedge\Diamond_i(\neg p\wedge\Diamond_j p))$ $\to\Diamond_j(\neg\delta_{ij}\wedge\Diamond_i p)$	$(N8_{ij})$	$\forall xz(E_{ij}x\wedge(\exists y T_ixy\wedge T_jyz\wedge y\neq z)$ $\to\exists u(\neg E_{ij}u\wedge T_jxu\wedge T_iuz))$

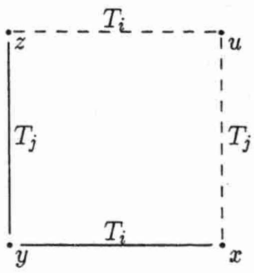
For finite α *we set* $CM1\equiv\bigwedge_i CM1_i$*, etc., taking* $CM4\equiv\bigwedge_{i,j}CM4_{ij}$*,* $CM6\equiv\bigwedge_{i\neq j}CM6_{ij}$*,* $CM7\equiv\bigwedge_{i,j,k}CM7_{ijk}$ *and* $CM8\equiv\bigwedge_{i\neq j}CM8_{ij}$*. If* $\alpha\geq\omega$*, we let* $CM1,\ldots,CM8$ *be the corresponding equation* schemata.

An α*-frame* $\mathfrak{F}$ *is called* **cylindric**[5] *if* $\mathfrak{F}\models CM1\ldots CM7$*,* **hypercylindric** *if* $CM1\ldots CM8$ *are valid in it. The class of (hyper)cylindric frames is denoted by* CF_α *(*HCF_α*).*

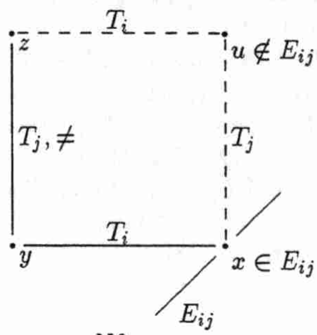
So, $N1_i$, $N2_i$ and $N3_i$ express that T_i is respectively reflexive, symmetric and transitive; together they state that T_i is an equivalence relation. $N6_{ij}$ then means that in every T_i-equivalence class there is *at most one* element on the diagonal E_{ij} ($i\neq j$). By $N5_j$ and $N7_{jji}$ one can show that every T_i-equivalence class contains *at least one* element on the diagonal E_{ij}. Taking these observations together, we find that every world in a cylindric frame has a unique T_i-successor on the E_{ij}-diagonal.

The meaning of $N4$ and $N8$ is best made clear by the following pictures:

[5] For a motivation of this terminology see subsection 4.2.



$N4_{ij}$



$N8_{ij}$

The reason why we did not confine ourselves to the modal formulas, but defined F_α-formulas as well, is that in the characterization theorems below, our working language will be F_α. Note that we are allowed to do so, because the formulas given in Definition 2.2 are pairwise equivalent:

Lemma 2.3 *Let $\mathfrak{F}$ be an α-frame. Then for $l = 1, \ldots, 8$ and $i, j, k < \alpha$:*

$$\mathfrak{F} \models CMl_{i(j(k))} \iff \mathfrak{F} \models Nl_{i(j(k))}.$$

Proof.
This lemma is a straightforward consequence of the correspondence part of the *Sahlqvist* theorem; cf. Sahlqvist [33], Venema [44] for more details. For readers unfamiliar with Sahlqvist's theorem, we will treat the equivalence for $l = 8$ as an example:
For $\Leftarrow$, assume that $\mathfrak{F} \models N8_{ij}$, and that $\mathfrak{F}, V, x \Vdash \delta_{ij} \wedge \Diamond_i(\neg p \wedge \Diamond_j p)$ for some valuation V. By the latter fact, there are y, z with $T_i xy$, $T_j yz$ and $y \Vdash \neg p$, $z \Vdash p$. Hence y must be different from z, so by $x \Vdash \delta_{ij}$ and our assumption, we find a u with $u \Vdash \neg\delta_{ij}$, $T_j xu$ and $T_i uz$. By $z \Vdash p$ we get $u \Vdash \Diamond_i p$, so we find $x \Vdash \Diamond_j(\neg\delta_{ij} \wedge \Diamond_i p)$, which is what we were after.
For $\Rightarrow$, suppose that $\mathfrak{F} \models CM8_{ij}$, and let $y \neq z$ be given such that $E_{ij}x$, $T_i xy$ and $T_j yz$. Now consider a valuation V on $\mathfrak{F}$, with $V(p) = \{z\}$. Unraveling the truth definition, we can show that $\mathfrak{F}, V, x \Vdash \delta_{ij} \wedge \Diamond_i(\neg p \wedge \Diamond_j p)$, so by $\mathfrak{F} \models CM8_{ij}$, we find $\mathfrak{F}, V, x \Vdash \Diamond_j(\neg\delta_{ij} \wedge \Diamond_i p)$. The truth definition gives us a u with $T_j xu$, $\neg E_{ij} u$ and $u \Vdash \Diamond_i p$. By the fact that z is the *only* world where p holds, this means $T_i uz$. But then we have proved that $\mathfrak{F} \models N8_{ij}$. □

Now we can state and prove our first characterization result, for $\alpha = 2$; it states that the hypercylindric 2-frames are precisely the disjoint unions of zigzag morphic images of squares:

Theorem 2.4

$$\mathsf{HCF}_2 = \mathbf{H_f P_f}\mathsf{C}_2.$$

Proof.
Clearly every square is hypercylindric, so $\mathbf{H_f P_f}\mathsf{C}_2 \subseteq \mathsf{HCF}_2$.
For the other direction, let $\mathfrak{F} = (W, T_0, T_1, E)$ be a hypercylindric 2-frame[6]; observe that the composition $T_0|T_1$ of T_0 and T_1 is an equivalence relation. Call a frame $\mathfrak{F}$ *connected* if this relation $T_0|T_1$ is total, *nice* if $\mathfrak{F}$ is connected and hypercylindric. It is an easy observation that every hypercylindric frame is a disjoint union of nice frames, so it suffices to show that

every nice frame is a zigzagmorphic image of a square.

So, let $\mathfrak{F} = (W, T_0, T_1, E)$ be a nice frame. Define λ as the maximum of $|W|$ and ω. We will define a chain $(f_\xi)_{\xi<\lambda}$ of potential zigzag morphisms (i.e. maps satisfying conditions (1) and (2)

[6] **It is straightforward to verify that in a hypercylindric frame we have $E_{ii} = W$ and $E_{ij} = E_{ji}$ (cf. (1) below); thus in our denotation of a hypercylindric 2-frame we suppress E_{00} and E_{11} and understand $E = E_{01} = E_{10}$.**

of Definition 2.1) such that the union f_λ of this chain is the desired zigzag. Every map f_ξ should be seen as an approximation[7] of f_λ.

Look at the set of *potential defects* $P = \lambda \times \lambda \times W \times \{0,1\}$. Call the quadruple $(\beta, \gamma, v, k) \in P$ a *defect* of a homomorphism $f : \xi \times \xi \mapsto W$ (where $\xi < \lambda$), if it defies one of the zigzag conditions in clause 3 of Definition 2.1, e.g. for $k = 1$: $(\beta, \gamma) \in \xi \times \xi$ and $T_1 f(\beta, \gamma) v$ while there is no $\gamma' \in \xi$ such that $f(\beta, \gamma') = v$; f is called *perfect* if it has no defects, i.e. if f is a zigzag morphism. Assume that P is well-ordered; then we may speak of the *first* defect of an imperfect potential zigzag morphism $f : \xi \times \xi \mapsto W$. By the following claim such a map has an extension lacking this defect:

Claim.

Let $f : \xi \times \xi \mapsto W$ be a potential zigzag morphism, (β, γ, v, k) a defect of f. Then there is a potential zigzagmorphism $g \supset f$, $g : (\xi + 1) \times (\xi + 1) \mapsto W$ such that (β, γ, v, k) is not a defect of g.

Proof of Claim. Without loss of generality we assume that $\beta = \gamma = 0$ and $k = 1$. We first set

$$\begin{aligned} g(\zeta, \eta) &= f(\zeta, \eta) \text{ for } \zeta, \eta < \xi, \\ g(0, \xi) &= v, \end{aligned}$$

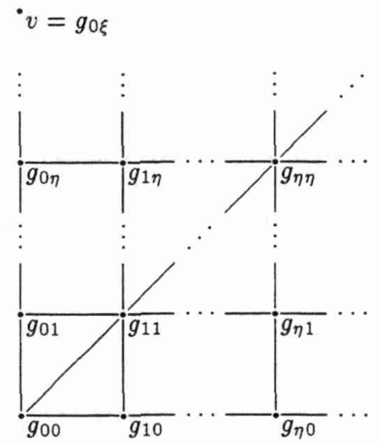
viz. the left picture below (where we denote $g(\zeta, \theta)$ by $g_{\zeta\theta}$).

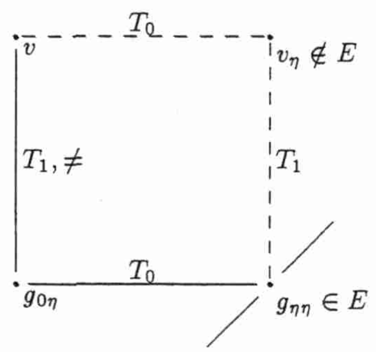
Next we are concerned with the $g(\eta, \xi)$, $0 < \eta < \xi$. By assumption we have $v \neq f(0, \eta)$, and as f is a potential zigzag morphism we get a situation as in the right picture. By $\mathfrak{F} \models N8_{01}$ and $Eg(\eta, \eta)$, $\mathfrak{F}$ has a $v_\eta \notin E$ with $T_0 v v_\eta$ and $T_1 v_\eta g(\eta, \eta)$. We define

$$g(\eta, \xi) = v_\eta,$$

and set $g(\xi, \xi)$ as the unique diagonal T_0-successor of any/all of the $g(\eta, \xi)$.

It is straightforward to verify that with this definition the part of g defined up till now satisfies both conditions (1) and (2).





For the definition of $g(\xi, \eta)$ $(\eta < \xi)$, we use the same trick as above to ensure $g(\xi, \eta) \notin E$: as $g(\xi, \xi)$ is in E and $g(\eta, \xi)$ is not, they cannot be identical. So $g(\xi, \eta)$ can be defined as any non-diagonal T_0-successor of $g(\eta, \eta)$ which is a T_1-successor of $g(\xi, \xi)$ (such a $g(\xi, \eta)$ exists by $N8_{10}$). *This proves the claim.*

Assuming that we have a canonical way to define a map f' which lacks the *first* defect of a potential zigzagmorphism f, we now define a chain of maps as follows:

$$\begin{aligned} f_0 &= \{((0,0), u)\} && \text{for some } u \text{ on the diagonal of } \mathfrak{F}, \\ f_{\xi+1} &= \begin{cases} f_\xi & \text{if } f_\xi \text{ is perfect,} \\ (f_\xi)' & \text{otherwise,} \end{cases} \\ f_\theta &= \bigcup_{\xi < \theta} f_\xi && \text{if } \theta \leq \lambda \text{ is a limit ordinal.} \end{aligned}$$

[7] The step-by-step method applied here has a long history in logic. It has been applied in model theory since the twenties, and it is well-known from both modal logic (cf. BURGESS [11] and algebraic logic (cf. HENKIN-MONK-TARSKI [16]).

It is now straightforward to verify that f_λ has the desired properties: first of all it is a potential zigzag morphism as all the maps in the chain are. Suppose that f_λ is not a zigzagmorphism, then there are quadruples in P witnessing this shortcoming. Let $\pi = (\beta, \gamma, v, k)$ be the first of these in the well-ordering of P, suppose its ordinal number is η. Take $\theta = max(\beta+1, \gamma+1)$, then π is a defect of f_θ. It need not be its first one, but there can be at most η problems before π that are more urgent. So π must be the first defect of $f_{\theta+\eta}$, whence it can not be a defect of $f_{\theta+\eta+1} = (f_{\theta+\eta})'$. But this gives a contradiction, since $f_{\theta+\eta+1} \subseteq f_\lambda$. So f_λ is a zigzag.
Finally, the proof that f_λ is surjective is straightforward by the connectedness of $\mathfrak{F}$. □

We now turn to the case where α is an arbitrary, finite ordinal n. If we confine ourselves to *positive* characterizations, then a nice result like Theorem 2.4 cannot be obtained: it is a fairly straightforward consequence of results by Monk [23] (resp. Andréka [1]) that there is no finite (resp. 'simple' infinite) set of modal formulas that characterize the class $\mathbf{H_f P_f}\mathsf{C}_n$ for $n > 2$. Therefore, we look for a different *kind* of characterization, viz. a $\pm$-characterization. In some sense, the result that we find in Theorem 2.11 is better than the one in Theorem 2.4, since it concerns the disjoint unions of cubes instead of the *zigzagmorphic images* of disjoint unions of cubes. The main part of the proof of Theorem 2.11 lies in a first-order characterization of the cubes to be given now. First we need some auxiliary definitions and notation concerning hypercylindric frames:

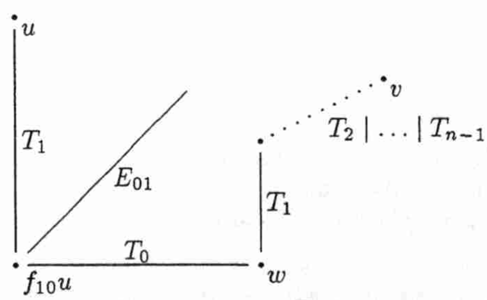
Definition 2.5 *For an arbitrary hypercylindric n-frame $\mathfrak{F} = (W, T_i, E_{ij})_{i,j<\alpha}$, define $f_{ij}(u)$, H_i^n, H^n and R^n as follows: $f_{ij}(u)$ is the unique v such that T_iuv and $E_{ij}v$. H^n (resp. H_i^n, $i < \alpha$) is the composition of all the T-relations, resp. all the T-relations minus T_i, i.e.*

$$\begin{aligned} H^n &= T_0 \mid T_1 \mid \ldots \mid T_{n-1}. \\ H_i^n &= T_0 \mid T_1 \mid \ldots \mid T_{i-1} \mid T_{i+1} \mid \ldots \mid T_{n-1}, \end{aligned}$$

For a world u in $\mathfrak{F}$, the set $H_i^n(u) = \{v \mid \mathfrak{F} \models H_i^n uv\}$ is called the i-hyperplane through u. R^n is given by

$$R^n = \{(u,v) \in W \times W \mid \mathfrak{F} \models \exists w \bigvee_{i \neq j} (T_i f_{ji} uw \wedge \neg E_{ij} w \wedge H_i^n wv)\}.$$

Note that in a hypercylindric frame all $H^n_{(i)}$-relations are equivalence relations, and that it does not matter in which way we compose the T_j to define them. Note too that the i-hyperplanes are the equivalence classes of H_i^n. An intuitive way to understand the definition of R^n is by the following figure depicting the case where $T_0 f_{10} uw \wedge \neg E_{01} w \wedge H_0^n wv$:



Here v lies in the hyperplane through w and 'orthogonal' to the 'line' T_0. The key observation is that if this picture is part of a cube, then the 0-th coordinate of u and v are *different.*

Now we reach the main theorem of this section. It states that the cubes are, modulo isomorphism, precisely the hypercylindric frames where R^n is the *inequality relation:*

Definition 2.6 *An n-frame is* **connected** *if the relation H^n is total, i.e. for all u, v in $\mathfrak{F}$ we have $\mathfrak{F} \models H^n uv$, and* R^n-**proper** *if $\mathfrak{F} \models \forall xy(xR^ny \leftrightarrow x \neq y)$.*

Theorem 2.7 *An n-frame is isomorphic to a cube iff it is hypercylindric and R^n-proper.*

Proof.

First some notational conventions: as we understand that n is fixed throughout the proof, we drop the superscript n when referring to the relations H^n, H_i^n or R^n; we will also find it convenient frequently to use infix notation for binary relations, i.e. uHv instead Huv.

$\boxed{\Rightarrow}$ It suffices to show that cubes are hypercylindric, and that $(*)$ R is the inequality relation in a cube. The first claim we leave to the reader; for $(*)$ we first discuss the meaning of f_{ij} and H_i in a cube. Let $u, v \in {}^nU$; then

$$\begin{array}{lcl} v = f_{ij}u & \iff & v_i = u_j \ \& \ v_k = u_k \text{ for all } k \neq i, \\ H_i uv & \iff & u_i = v_i, \end{array}$$

as an easy proof shows. Note that by the equivalence above, the term 'hyperplane' obtains its natural mathematical meaning.

The idea behind the proof of $(*)$ is that two n-tuples are different iff they differ in at least one coordinate: first assume that $u \neq v$. Without loss of generality we may assume that $u_0 \neq v_0$. Let w be the tuple $(v_0, u_0, u_2, \ldots, u_{n-1})$. Then we immediately have H_0wv and $\neg E_{10}w$. As $f_{10}u = (u_0, u_0, u_2, \ldots, u_{n-1})$, we also find $T_0 f_{10}uw$. But then we have obtained Ruv by definition of R.

For the other direction, assume that Ruv. Without loss of generality we may assume (take $i = 1$ and $j = 0$ in the definition of R) that there is a w with $T_0 f_{10}uw$, $\neg E_{10}w$ and H_0wv. By $T_0 f_{10}uw$ we find $w_1 = u_0$ (and $w_i = u_i$ for $1 < i < n$); H_0wv implies $w_0 = v_0$; so $\neg E_{10}w$ gives $v_0 = w_0 \neq w_1 = u_0$. But then $u \neq v$.

$\boxed{\Leftarrow}$ Before we prove the direction from right to left, we develop some basic theory concerning hypercylindric frames. To start with, there are some elementary facts that we need throughout the proof. These include: obvious properties like the fact that all T_i, H_i and H are equivalence relations; reformulations of the axioms, like $N7_{ijk} : E_{ij}x \to E_{kj}f_{ki}$; validities like $uT_iv \to uH_jv$ if $i \neq j$ or $uHv \to \exists w(uT_iwH_iv)$; etc. Such simple facts may be used without warning, or with a reference to *elementary hypercylindric theory.* We provide short proofs of the following claims:

$$\mathsf{HCF} \models E_{ij}x \leftrightarrow E_{ji}x. \tag{1}$$

For, assume $E_{ij}x$; by $N5_i$ and $N7_{iij}$, there is a y with T_jxy, $E_{ij}y$ and $E_{ji}y$. By $N6_{ij}$, $x = y$, so $E_{ji}x$.

$$\mathsf{HCF} \models E_{ij}u \to f_{ki}u = f_{kj}u. \tag{2}$$

For, assume $E_{ij}u$; by $N7_{ijk}$ and (1), there is a y with T_kuy, $E_{ki}y$ and $E_{kj}y$; it follows from elementary hypercylindric theory that $y = f_{ki}u$ and $y = f_{kj}u$.

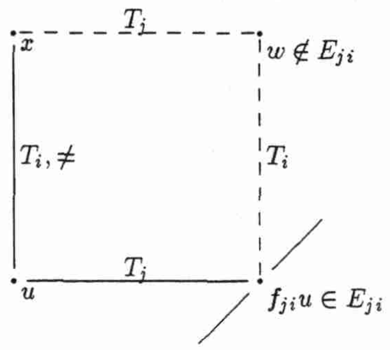
$$\mathsf{HCF} \models uT_iv \to f_{jk}uT_if_{jk}v. \tag{3}$$

First consider the special case where $v = f_{ik}u$. By $uT_ivT_jf_{jk}v$ and $N4$, there is a w with $uT_jwT_if_{jk}v$. By $E_{ik}v$, $N7_{ikj}$ and (1), $f_{jk}v$ is in E_{ji} and in E_{ik}, so by $N7_{jki}$ (from right to left), $E_{jk}w$. Thus by definition of f_{jk}, $w = f_{jk}u$.

Now for arbitrary u, v with uT_iv, the special case gives $f_{jk}uT_if_{jk}f_{ik}u$ and $f_{jk}vT_if_{jk}f_{ik}v$; however, uT_iv implies $f_{ik}u = f_{ik}v$ by elementary hypercylindric theory, so $f_{jk}uT_if_{jk}v$ follows immediately.

$$\mathsf{HCF} \models (R^n uv \vee u = v) \leftrightarrow Huv. \tag{4}$$

The implication from left to right is straightforward; for the other direction, assume that Huv and $u \neq v$ in a hypercylindric frame $\mathfrak{F}$. By definition of H, there are $x_0, x_1, \ldots, x_n$ such that $u = x_0T_0x_1T_1x_2 \ldots x_{n-1}T_{n-1}x_n = v$. Let i be the smallest index such that $x_i \neq x_{i+1}$; it follows that there is an x, viz. x_{i+1} such that uT_ixH_iv and $u \neq x$. Take a j with $j \neq i$. By $N8_{ji}$, there is a w with $\neg E_{ji}w$ and $f_{ji}uT_iwT_jx$, cf.



It follows that wH_ix, so by xH_iv we find that wH_iv. But then Ruv, by definition of R.

An immediate consequence of (4) is that

$$\text{hypercylindric } R^n\text{-proper frames are connected.} \tag{5}$$

The following property is further on referred to as the *disjoint hyperplanes*-property of R^n-proper frames. In a slightly different wording, it states that two different points on a 'line' ($u \neq v$ and T_iuv) belong to *disjoint* hyperplanes orthogonal to the 'line' T_i.

$$\text{If } \mathfrak{F} \text{ is hypercylindric and } R^n\text{-proper, then } \mathfrak{F} \models (uT_iv \wedge uH_iv) \rightarrow u = v. \tag{6}$$

Without loss of generality we may assume that $i = 0$. We will derive a contradiction from the assumption that in a R^n-proper frame there are u, v with uT_0v, uH_0v and $u \neq v$. From uT_0v and $u \neq v$, $N8_{10}$ yields a w with $f_{10}uT_0w$, vT_1w and $\neg E_{10}w$; uH_0v and vT_1w imply uH_0w. So we have found a w with $f_{10}uT_0w$, $\neg E_{10}w$ and uH_0w; in other words: Ruu. This gives the desired contradiction $u \neq u$.

Now turning to the representation theorem itself, let us start with giving the intuitive idea. Consider the cube $\mathfrak{C}$ over a set U; the key observation in the proof is that we may identify the i-th coordinate of a an n-tuple x in $\mathfrak{C}$ with the *i-hyperplane* through x. So the basic idea will be to represent a world x in a R^n-proper frame as the n-tuple $(H_0(x), \ldots, H_{n-1}(x))$. The problem is that if $i \neq j$, the sets of i- resp j-hyperplanes are different, while there can only be one base set U. The solution to this problem is to relate these disjoint sets of hyperplanes: in a cubic frame again, we can find the second coördinate of a world $u = (u_0, u_1, \ldots, u_{n-1})$ not only by considering the *one*-hyperplane $H_1(u)$, but also by looking at the *zero*-hyperplane of $(u_1, u_1, \ldots, u_{n-1}) = f_{01}(u)$. Therefore we may and will concentrate on 0-hyperplanes.

Let $\mathfrak{F} = (W, T_i, E_{ij})_{i,j<n}$ be a hypercylindric R^n-proper n-frame; set

$$U \text{ is the set of the 0-hyperplanes of } \mathfrak{F}.$$

Define the representation map $h : W \mapsto {}^nU$ by

$$h(u) = (H_0(u), H_0(f_{01}u), \ldots, H_0(f_{0,n-1}u)).$$

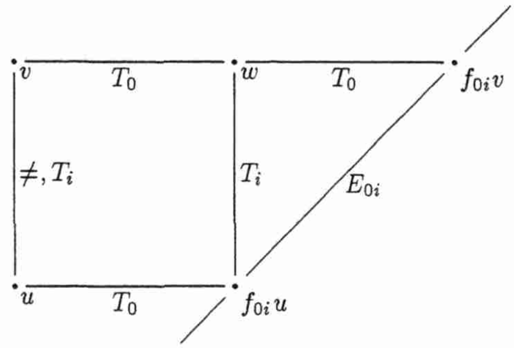
We will also write $h(u) = (h_0(u), \ldots, h_{n-1}(u))$, so $h_i(u)$ denotes $H_0(f_{0i}u)$.

Claim 1. h is injective.

Proof. Let u and u' be *different* worlds in $\mathfrak{F}$. As $\mathfrak{F}$ is connected by (5), there must be worlds $u_0, \ldots, u_n$ such that $u = u_0T_0u_1 \ldots u_{n-1}T_{n-1}u_n = u'$. As $u_0 \neq u_n$ there must be a *first* i with $u_i \neq u_{i+1}$. Distinguish cases:

(a) $i = 0$: we have $\neg H_0uu_1$ by (6), so $H_0(u) \neq H_0(u')$ and hence $h(u) \neq h(u')$.

(b) $i > 0$. Define $v = u_{i+1}$; then we have uT_iv, $u \neq v$. By $N8$ then there is a w with $vT_0wT_if_{0i}u$ and $w \notin E_{0i}$, viz.



We obtain $wT_0f_{0i}v$ and $w \neq f_{0i}v$. So by disjoint hyperplanes (6), $H_0(f_{0i}v) \neq H_0(w)$, while $H_0(w) = H_0(f_{0i}u)$. But $vT_{i+1} \mid \ldots \mid T_{n-1}u'$ implies, by several applications of (3), that $f_{0i}vT_{i+1} \mid \ldots \mid T_{n-1}f_{0i}u'$, so $H_0(f_{0i}v) = H_0(f_{0i}u')$.

Putting these facts together, we obtain $H_0(f_{0i}u) \neq H_0(f_{0i}u')$, so $h(u) \neq h(u')$. *This proves claim 1.*

Claim 2. h is a homomorphism.

Proof.
For the diagonals: if $u \in E_{ij}$, then $f_{0i}u = f_{0j}u$ by (2), so $h_i(u) = h_j(u)$ by definition of h.

For the cylinders: assume that uT_iu'; we have to show that $h_j(u) = h_j(u')$ for $j \neq i$. Distinguish cases:

If $i = 0$, then for $j \neq 0$ we have $f_{0j}u = f_{0j}u'$ by elementary hypercylindric theory, so $h_j(u) = h_j(u')$.

If $i \neq 0$, then uH_0u', so $h_0(u) = h_0(u')$. For $j \notin \{0, i\}$ we find $f_{0j}uT_if_{0j}u'$ by (3), so $H_0f_{0j}uf_{0j}u'$; this gives $h_j(u) = h_j(u')$ by definition. *This proves claim 2.*

Claim 3. h^{-1} is a homomorphism.

Proof.
For the diagonal, assume that $h_i(u) = h_j(u)$; we have to show that $f_{0i}u \in E_{ij}$. By assumption, $f_{0i}uH_0f_{0j}u$; by definition of the projection functions f_{0j} and f_{0i} we also have $f_{0i}uT_0f_{0j}u$. So by disjoint hyperplanes (6) we get $f_{0i}u = f_{0j}u$. Applying $N7_{ij0}$ from right to left, we find $E_{ij}f_{0i}u$.

For the cylinders, assume $h(u)T_ih(u')$, i.e. $h_j(u) = h_j(u')$ for $j \neq i$. Distinguish cases:

If $i = 0$, then by connectedness (5) there is a v with uT_0vH_0u'. We will show that $v = u'$, which immediately gives the desired uT_0u'.

By uT_0v and claim 2 we have $h_j(u) = h_j(v)$ for $j \neq 0$, so by the assumption, $h_j(v) = h_j(u')$ for $j \neq 0$. But we have $h_0(v) = h_0(u')$ as well, as vH_0u'. So $h(v) = h(u')$ and thus by injectivity of h it follows that $v = u'$.

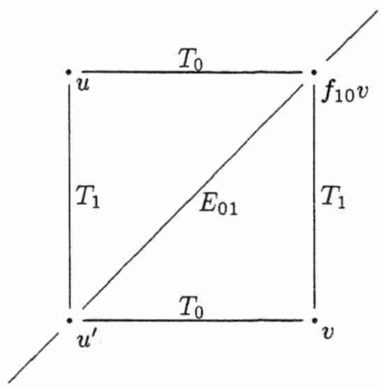
For the case where $i \neq 0$, we may assume without loss of generality that $i = 1$. By connectedness, there is a v with uT_1vH_1u'. Here too we will prove that $v = u'$, by showing that for all j we have $h_j(v) = h_j(u')$, and then use injectivity. For $j \neq 0, 1$ uT_1v implies $f_{0j}uT_1f_{0j}v$ by (2); this gives $H_0f_{0j}uf_{0j}v$, so by definition of h, $h_j(u) = h_j(v)$; by assumption $h_j(u) = h_j(u')$, so we find $h_j(v) = h_j(u')$. For $j = 0$ we have a similar story: uT_1v implies $h_0(v) = h_0(u')$. Finally, for $j = 1$, note that H_1vu' implies the existence of elements $x_2, x_3, \ldots x_{n-1}$ such that $vT_0x_2T_2x_3T_3 \ldots x_{n-1}T_{n-1}u'$. Then claim 2 implies that the images under h of $v_0, x_2, \ldots, x_{n-1}$ and u' have the same first coordinate: $h_1(v) = h_1(u')$. *This proves claim 3.*

Claim 4. h is surjective.

Proof.
We will prove that every $a \in {}^nU$ is the image under h of a world in $\mathfrak{F}$, by induction on the number k of coordinates differing from a_0.

For $k = 0$, let G be the 0-hyperplane in $\mathfrak{F}$ such that $a = (G, \ldots, G)$. We leave it to the reader to verify that there is an element u in G such that $E_{0i}u$ for all i. Clearly this u satisfies $u = f_{0i}u$ for all i, so $a = h(u)$.

For $k > 0$, assume (without loss of generality) that $a = (G, G_1, G_2, \ldots, G_k, G, \ldots, G)$. By the induction hypothesis, $a' = (G, G, G_2, \ldots, G_k, G, \ldots, G)$ is the h-image of some u' in $\mathfrak{F}$. By claim 3, $u' \in E_{01}$, and by connectedness, there is a $v \in G_1$ with $u' T_0 v$. $\mathfrak{F} \models N4$ implies the existence of a u with $u' T_1 u T_0 f_{10} v$, viz.



Now we verify that $h_i(u) = a_i$, for all i. For $i \neq 1$: by claim (2), uT_1u' implies $h_i(u) = h_i(u') = (a')_i = a_i$. For $i = 1$: $f_{01}u = f_{10}v$ holds by definition of u, so $h_1(u) = H_0(f_{01}u) = H(f_{10}v) = H_0(v) = G_1 = a_1$. So $h(u) = a$. *This proves claim 4.*

Thus we have proved that h is an isomorphism between $\mathfrak{F}$ and the n-cube over U. □

We would like to thank H. Andréka, I. Németi and I. Sain for bringing theorem 3.2.5 of Henkin, Monk & Tarski [16] to our attention. Via some (non-elementary) duality theory relating Boolean Algebras with Operators and Relational (Kripke) Frames, one can derive our characterization result 2.7 from this theorem, and vice versa.

The definition of R^n looks rather involved, so the reader may wonder why we did not choose a simpler formulation to define C_n. The reason for this is that the particular formulation allows us to give a nice kind of characterization of the class $\mathbf{P_f}\mathsf{C}_n$ of disjoint unions of cubes.

Definition 2.8 *Let* K *be a class of* α*-frames, and* ξ *an* α*-formula.* $\mathsf{K}_{-\xi}$ *is the class of frames in* K *such that for every world* w *in* $\mathfrak{F}$ *there is a valuation* V *satisfying* $\mathfrak{F}, V, w \Vdash \neg\xi$*. If* K *is of the form* Fr_Σ*, (i.e. for a frame* $\mathfrak{F}$ *we have* $\mathfrak{F} \in \mathsf{K}$ *iff every formula of* Σ *is valid in* $\mathfrak{F}$*), then we call the pair* (Σ, ξ) *a* **$\pm$-characterization** *of* $\mathsf{K}_{-\xi}$*, and we say that* $\mathsf{K}_{-\xi}$ *is* **$\pm$-definable**.

As an example, the reader could check that in ordinary modal logic, the class $\mathsf{K}_{-(p \to \Diamond p)}$ consists of precisely the *irreflexive* K-frames. The crucial step in the proof consists of showing that a world w in a frame $\mathfrak{F}$ is irreflexive only if the valuation V_w given by $V_w(p) = \{w\}$, satisfies $\mathfrak{F}, V_w, w \Vdash \neg(p \to \Diamond p)$.

$\pm$-Characterizations, and in particular axiomatizations of $\pm$-definable classes are studied in Venema [44], to which we refer for more details. Let us motivate the concept here by saying that under certain constraints, $\pm$-definable classes behave very nicely with respect to axiomatizability, allowing quite natural, though slightly unorthodox axiomatizations, as we will see in the next section. One of these constraints is that a certain modal operator called the *difference* operator, is definable over the class. The difference operator is a special diamond in the sense that it has a designated accessibility relation, viz. the inequality relation ($\neq$).

Definition 2.9 *Define the following abbreviated operator* D^n*:*

$$D_n\phi = \bigvee_{j \neq i} \Diamond_j(\delta_{ij} \wedge \Diamond_i(\neg\delta_{ij} \wedge \Diamond_0 \ldots \Diamond_{i-1}\Diamond_{i+1} \ldots \Diamond_{n-1}\phi)),$$

Note that D^n is defined to make the relation R^n act as its accessibility relation, i.e. for any α-model we have

$$\mathfrak{M}, u \Vdash D^n\phi \iff \text{there is a } v \text{ with } R^n uv \text{ and } \mathfrak{M}, v \Vdash \phi.$$

By Theorem 2.7 this implies that over the class of cubes D^n indeed behaves like the difference operator:

$$\mathfrak{M}, u \Vdash D^n\phi \iff \text{there is a } v \text{ with } u \neq v \text{ and } \mathfrak{M}, v \Vdash \phi.$$

We are now ready to state and prove the characterization result of the class $\mathbf{P_f}\mathsf{C}_n$. First define

Definition 2.10 *Let $\beta(p)$ be the formula $p \rightarrow D^n p$.*

Theorem 2.11 $\mathsf{HCF}_{n,-\beta} = \mathbf{P_f}\mathsf{C}_n$.

Proof.
Let $\mathfrak{F}$ be a hypercylindric frame. By Sahlqvist's theorem, we have for every u in $\mathfrak{F}$ that

$$R^n uu \iff \text{for all valuations } V\colon \mathfrak{F}, V, u \Vdash \beta. \tag{7}$$

So $\mathsf{HCF}_{n,-\beta}$ consists of all the hypercylindric frames where R^n is *irreflexive.*

Now to prove the theorem, let $\mathfrak{F}$ be a cube; by Theorem 2.7 it follows that in $\mathfrak{F}$, R^n is the inequality relation, so R^n is clearly irreflexive. But then R^n must also be irreflexive in a disjoint union of cubes, so $\mathbf{P_f}\mathsf{C}_n \subseteq \mathsf{HCF}_{n,-\beta}$.

For the other direction, we first apply some standard theory of modal logic yielding that every hypercylindric frame is isomorphic to the disjoint union of its connected subframes. Note that a connected subframe of a hypercylindric frame is hypercylindric itself. So, it suffices to prove that every connected frame in $\mathsf{HCF}_{n,-\beta}$ is isomorphic to a cube. Let $\mathfrak{F}$ be such a frame. By (7), R^n is irreflexive, so by (4) and the fact that H is total on $\mathfrak{F}$, R^n is indeed the inequality relation on $\mathfrak{F}$. Then by Theorem 2.7 $\mathfrak{F}$ is isomorphic to a cube. □

3 Cylindric Modal Logic: Axiomatics

In this section we will show how the characterization results of the previous section lead to complete axiomatizations for the modal theory of the cubes. As these axiomatizations are slightly unorthodox, at the end of the section we give a worked example how to use them to actually derive theorems.

We already mentioned that the *kind* of characterization that we gave for $\mathbf{P_f}\mathsf{C}_n$ in Theorem 2.7 lends itself for an unorthodox kind of axiomatization. Before giving the formal definitions of our derivation systems, let us discuss the basic idea behind the unorthodox derivation rule that we will use.

Let us consider the modal theory Θ of a given $\pm$-definable class $\mathsf{K} = \mathsf{Fr}_{\Sigma,-\xi}$. We will show that if a formula ψ is satisfiable in K, then so is $\psi \wedge \neg\xi$, provided that ψ and ξ do not share any propositional variables. For, suppose that there is a frame $\mathfrak{F}$ in K, a valuation V and a world w such that $\mathfrak{F}, V, w \Vdash \psi$. By our assumption that K was $\pm$-characterized by (Σ, ξ), there is a valuation V_1 such that $\mathfrak{F}, V_1, w \Vdash \neg\xi$. Now taking the valuation V' given by

$$V'(p) = \begin{cases} V(p) & \text{if } p \text{ occurs in } \psi, \\ V_1(p) & \text{otherwise,} \end{cases}$$

we obtain $\mathfrak{F}, V', w \Vdash \psi \wedge \neg\xi$, whence $\psi \wedge \neg\xi$ is satisfiable in K. The strategy is now to turn this semantic intuition into a syntactic devise, by a contraposition and formalization of the above remark; one might just as well have said that: if $\neg\xi(\vec{p}) \rightarrow \neg\psi$ is in Θ, and none of the $\vec{p}$ occur in ψ, then $\neg\psi \in \Theta$. In fact, what we have done here is to formulate a derivation rule, the *non-ξ rule*:

$$(N\xi R) \qquad \vdash \neg\xi(\vec{p}) \rightarrow \phi \Rightarrow \vdash \phi, \text{ if } \vec{p} \notin \phi.$$

The idea to let non-ξ-*rules* 'axiomatize' non-ξ properties, originates with Gabbay [12]. Whether adding a non-ξ rule to a complete axiomatization for a class K yields a complete axiomatization of the class $\mathsf{K}_{-\xi}$, is the main problem[8] concerning the non-ξ rules. Nevertheless, given certain constraints on the similarity type of the language and the syntactic form of the axioms, a positive general result was obtained in Venema [44]. This result can and will be applied here.

Definition 3.1 *Let A_α be the derivation system having as its axioms*

(CT) *all propositional tautologies*
$(DB_{\Box_i})$ $\Box_i(p \to q) \to (\Box_i p \to \Box_i q)$
(CM) $CM1, \ldots, CM8$.

and as its derivation rules, **Modus Ponens**, **Universal Generalization** *or* **Substitution**:

(MP) $\vdash \phi, \vdash \phi \to \psi \;\Rightarrow\; \psi$,
$(UG_{\Box_i})$ $\vdash \phi \;\Rightarrow \vdash \Box_i\phi$,
(SUB) $\vdash \phi \;\Rightarrow\; \sigma(\phi)$, *for any substitution σ of formulas for propositional variables in ϕ.*

For finite dimensions, A_n^+ is the derivation system A_n extended with the **Irreflexivity Rule for** D_n:

$$(IR_{D_n}) \qquad \vdash \neg\beta(p) \to \phi \;\Rightarrow \vdash \phi, \;\; if\, p \notin \phi.$$

A_ω^+ is defined as the system A_ω extended with the schema of rules $\{IR_{D_n} \mid n < \omega\}$. For $\alpha > \omega$ we add besides this set, the following schema:

$$\{\vdash \phi \;\Rightarrow \vdash \phi^\tau \mid \tau : \alpha \mapsto \alpha \text{ is a bijection}\},$$

where ϕ^τ is the formula obtained from ϕ by substituting $\Diamond_{\tau(i)}$ and $\delta_{\tau(i)\tau(j)}$ for every occurrence of $\Diamond_i$ resp δ_{ij}.

A **derivation** *in one of these systems is defined as a finite sequence $\phi_0, \ldots, \phi_n$ such that every ϕ_i is either an axiom or obtainable from $\phi_0, \ldots, \phi_{i-1}$ by a derivation rule. A* **theorem** *of is any formula that can appear as the last item of a derivation. Theoremhood of a formula ϕ in the system $A_\alpha^{(+)}$ is denoted by $\vdash_\alpha^{(+)} \phi$.*

In the sequel, we will frequently use the following result, stating that A_α is sound and complete with respect to hypercylindric frames:

Theorem 3.2 *Let α be an arbitrary ordinal and ϕ an α-formula. Then*

$$\vdash_\alpha \phi \iff \mathsf{HCF}_\alpha \models \phi.$$

Proof.
This follows by the *Sahlqvist* form of the axioms, cf. Sahlqvist [33], Sambin-Vaccaro [38] for more details. □

As an immediate consequence, we have a finite axiomatization of the cylindric modal theory of the *squares*:

Theorem 3.3 *COMPLETENESS FOR SQUARE VALIDITY*
Let ϕ a 2-formula. Then

$$\vdash_2 \phi \iff \mathsf{C}_2 \models \phi.$$

Proof.
Immediate by Theorem 2.4, Theorem 3.2 and the preservation result of modal validity under taking zigzagmorphic images and disjoint unions. □

For the general case, we need the unorthodox rules:

[8] Note that in modal logic, this question need not always have an affirmative answer: *characterizations* of frame classes lead not automatically to complete *axiomatizations*.

Theorem 3.4 *COMPLETENESS FOR CUBE VALIDITY*
Let α be an arbitrary ordinal and ϕ an α-formula. Then

$$\vdash^{+}_{\alpha} \phi \iff \mathsf{C}_{\alpha} \models \phi.$$

Proof.
We leave it to the reader to prove soundness. For completeness, we first treat the finite-dimensional case $\alpha = n$. Actually, this case is a straightforward corollary of Theorem 8.2 in [44] and the $\pm$-characterization result 2.7. There is a technical problem however: the mentioned theorem applies to similarity types where the difference operator is a *primitive* operator, while in our system D_n is a *defined* operator.

Therefore, our proof strategy is as follows: first (i) we extend the language CML_n with the difference operator D as a primitive symbol. We also extend A^{+}_{n} to a derivation system EA^{+}_{n}. This extended derivation system is in such a form that theorem 8.2 is directly applicable to it (ii). We thus obtain a completeness result for EA^{+}_{n} with respect to a certain class K of frames, which we will show (iii) to be identifiable with C_{α}. Finally (iv), we show that EA^{+}_{n} is conservative over A^{+}_{n}.

(i) Let us start with defining the language X, which is an extension of CML_n with a new unary operator D. We abbreviate $\underline{D}\phi = \neg D \neg \phi$. Note that in the Kripke semantics for this language, the frames are the α-frames augmented with a binary accessibility relation R_D for D. Only in the *intended* semantics, R_D is the inequality relation.

The derivation system EA^{+}_{n} is obtained by adding the following axioms and rules to A^{+}_{n}:

(DB_D)	$\underline{D}(p \to q) \to (\underline{D}p \to \underline{D}q)$
$(D1)$	$p \to \underline{D}Dp$
$(D2)$	$DDp \to (p \vee Dp)$
$(D3)$	$\Diamond_i p \to (p \vee Dp)$
(UG_D)	$\vdash \phi \Rightarrow \vdash \underline{D}\phi$
(IR_D)	$\vdash (p \wedge \neg Dp) \to \phi \Rightarrow \vdash \phi$, provided that p does not occur in ϕ.

(ii) Note that all axioms of this system are in Sahlqvist tense form (cf. [44]), that all diamonds are self-conjugate, and that D has indeed all the axioms and the rule IR_D needed to make it the *difference* operator. Therefore, Theorem 8.2 of VENEMA [44] yields that EA^{+}_{n} is sound and complete with respect to the class K_n of frames $\mathfrak{F}$ such that (1) the accessibility relation R_D of D is really the inequality relation, (2) all axioms of EA^{+}_{n} are valid in $\mathfrak{F}$ and (3) the 'accessibility' relation R_{D_n} of D_n is irreflexive.

(iii) In other words, K_n is the class of frames $\mathfrak{F} = (W, T_i, E_{ij}, R_D)_{i,j<\alpha}$ with $(W, T_i, E_{ij})_{i,j<\alpha}$ in $\mathsf{HCF}_{n,-\beta}$ and $\mathfrak{F} \models \forall xy(R_D xy \leftrightarrow x \neq y)$. But this means that for any CML_n-formula ϕ (i.e. an X-formula in which D does not occur), we have $\mathsf{K}_n \models \phi$ iff $\mathbf{P_f}\mathsf{C}_n \models \phi$. And as we have $\mathbf{P_f}\mathsf{C}_n \models \phi$ iff $\mathsf{C}_n \models \phi$, EA^{+}_{n} is complete with respect to the cube-theory of the cylindric modal fragment of X.

(iv) Finally, to show that EA^{+}_{n} is conservative over A^{+}_{n}, we first define an embedding translation $(\cdot)^{-}$:

$$\begin{array}{lcl} p^{-} & = & p, \\ (D\phi)^{-} & = & D_n\phi^{-}, \\ (\phi \wedge \psi)^{-} & = & \phi^{-} \wedge \psi^{-}, \\ (\Diamond_i \phi)^{-} & = & \Diamond_i \phi^{-}, \\ (\neg\phi)^{-} w & = & \neg\phi^{-}. \end{array}$$

The essential claim is that for all EA^{+}_{n}-formulas ϕ we have

$$EA^{+}_{n} \vdash \phi \iff A^{+}_{n} \vdash \phi^{-}. \tag{8}$$

The direction from right to left is trivial; the other direction is proved by induction on derivations in EA^{+}_{n}.

For the basic step, let ϕ be an *axiom* of EA^{+}_{n}. The only non-trivial cases are where ϕ is one of the D-axioms (including the distribution axiom for $\underline{D}$). By the completeness theorem 3.2 and the fact that A_n^+ is an extension of A_n, it suffices to show that $\mathsf{HCF}_n \models \phi^-$. We omit the rather trivial proof for the distribution axiom. For the axioms $D1 - D3$, we apply Sahlqvist's Theorem again: it suffices to show that the Sahlqvist *correspondents* (in F_α) of $D1^- - D3^-$ hold in HCF_n:

$$\begin{array}{ll} \mathsf{HCF}_n \models & \forall xy(R^n xy \rightarrow R^n yx) \\ \mathsf{HCF}_n \models & \forall xyz((R^n xy \wedge R^n yz) \rightarrow (x = z \vee R^n xz)) \\ \mathsf{HCF}_n \models & \forall xy(T_i xy \rightarrow (x = y \vee R^n xy)). \end{array} \tag{$*$}$$

Now the proofs of $(*)$ are all straightforward consequences of the fact that H^n is an equivalence relation, and of

$$\mathsf{HCF}_n \models (R^n uv \vee u = v) \leftrightarrow H^n uv. \tag{4}$$

which was established in the proof of Theorem 2.7.

For the induction step we need to consider one case only, viz. the necessitation rule for D. By successive applications of $UG_{\Box_i}$ and the $A_n^{(+)}$-derivable rules '$\vdash \phi \Rightarrow \vdash \delta_{ij} \rightarrow \phi$' and '$\vdash \phi \Rightarrow \vdash \neg\delta_{ij} \rightarrow \phi$', it is straightforward to derive $\vdash_n^+ \neg D_n \neg\phi$ from $\vdash_n^+ \phi$. This proves (8), and *finishes the proof for the finite-dimensional case.*

For infinite dimensions, we only treat the case where $\alpha = \omega$. Let ϕ be an ω-formula such that $\mathsf{C}_\omega \models \phi$. As there are only finitely many symbols occurring in ϕ, there is an $n < \omega$ such that ϕ is an n-formula. A relatively simple argument shows that for all ordinals β, γ:

$$\beta < \gamma \Rightarrow \text{ for all } \beta\text{-formulas } \psi\text{: } \mathsf{C}_\beta \models \psi \iff \mathsf{C}_\gamma \models \psi,$$

so we have $\mathsf{C}_n \models \phi$. By completeness of the finite-dimensional case we find $\vdash_n^+ \phi$; now $\vdash_\omega^+ \phi$ follows as A_ω^+ is an extension of A_n^+. □

Note that modulo the isomorphism on the formula algebra (i.e. replacing each propositional variable p_k by the predicate symbol R_k, each constant δ_{ij} by the identity $v_i = v_j$ and each diamond $\Diamond_i$ by the existential quantification $\exists v_i$), Theorem 3.4 is a completeness result for restricted first order logic as well.

In the remainder of this section we will show that the completeness proof for $\vdash_\alpha^+$ is more than just an abstract proof for the existence of a derivation system: one can actually work in it. In particular, we will give a cube-valid 3-formula ϕ for which we will prove that it is derivable in A_3^+, while it is not derivable in A_3.[9] The material in this part contains many essential contributions by H. Andréka, I. Sain and I. Németi. The first example of a derivation using (the algebraic version of) the D_n-irreflexivity rule was given by I. Sain [6].

First we define some further abbreviations:

$$\begin{array}{lcl} \boxplus\phi & = & \Box_0\Box_1\Box_2\phi \\ \bigcirc_{ij}\,\phi & = & \Diamond_i(\delta_{ij} \wedge \phi) \end{array}$$

Note that in a hypercylindric 3-frame, we may pretend that the relation H resp. the function f_{ij} are the accessiblity relation of $\boxplus$, resp. the accessiblity function of $\bigcirc_{ij}$:

$$\begin{array}{lcl} \mathfrak{M}, u \Vdash \boxplus\phi & \iff & \text{for every } v \text{ in } W\text{: if } Huv \text{ then } \mathfrak{M}, v \Vdash \phi, \\ \mathfrak{M}, u \Vdash \bigcirc_{ij}\,\phi & \iff & \mathfrak{M}, f_{ij}u \Vdash \phi. \end{array}$$

Note too that in a *connected* hypercylindric frame, H is total, whence $\boxplus$ is the *universal* modality, i.e. $\boxplus\phi$ holds at a world u iff ϕ holds everywhere in the model.

[9] The existence of such a formula follows from the fact that C_3 is not finitely axiomatizable by an orthodox system, i.e. a derivation system where MP, UG and SUB are the *only* derivation rules. This is the modal version of the famous non-finite axiomatizability result of the equational theory of the variety of Representable Cylindric Algebras, due to Monk.

Now consider the following CML_3-formulas:

$$\begin{array}{rcl}
\gamma' & = & (\Diamond_1\Diamond_2 r \wedge \mathsf{O}_{01}\Diamond_1\Diamond_2 r \wedge \mathsf{O}_{02}\Diamond_1\Diamond_2 r) \to (\delta_{01} \vee \delta_{02} \vee \delta_{12}) \\
\gamma & = & \boxplus\gamma' \\
\psi_i & = & \boxplus(r_i \to r) \wedge \boxplus(\Diamond_0 r \to \Diamond_0 r_i) \\
\psi' & = & \boxplus((r_0 \to \neg r_1) \wedge (r_1 \to \neg r_2) \wedge (r_2 \to \neg r_0)) \\
\psi & = & r_0 \wedge \psi_0 \wedge \psi_1 \wedge \psi_2 \wedge \psi' \\
\rho & = & \gamma \wedge \psi
\end{array}$$

Proposition 3.5 *Let ρ be as defined above. Then*

(i) $\mathsf{C}_3 \models \neg\rho$,

(ii) $A_3 \not\vdash \neg\rho$,

(iii) $A_3^+ \vdash \neg\rho$.

Proof.

(i) The basic intuition behind the definition of ρ is the following. Let R_i resp. R denote $V(r_i)$ resp. $V(r)$. In a 3-cubic model, γ expresses that the *domain* of R, (i.e. the set $\{s \in U \mid (s,t,u) \in R$ for some $t,u \in U\}$) has at most *two* elements; ψ states that R contains three disjoints parts R_0, R_1, R_2 in such a way that the domain of R contains at least *three* elements. So ρ is not satisfiable in a cube.

To give more details, let $\mathfrak{M} = ({}^3U, V)$ be a 3-cubic model, and suppose $\mathfrak{M}, (s_0,t,u) \Vdash \psi$. By $\mathfrak{M}, (s_0,t,u) \Vdash r_0$ and $\mathfrak{M} \models \psi_0$ we obtain $\mathfrak{M}, (s_0,t,u) \Vdash r$. Then by $\mathfrak{M} \models \psi_1 \wedge \psi_2$ there are s_1, s_2 such that for $i = 0,1,2$: $\mathfrak{M}, (s_i,t,u) \Vdash r_i \wedge r$; so s_0, s_1 and s_2 are mutually *distinct* by $\mathfrak{M} \models \psi'$. (This means that the domain of R contains at least three elements.)

So for the triple $\vec{s} = (s_0, s_1, s_2)$ we have $\vec{s} \Vdash \neg(\delta_{01} \vee \delta_{02} \vee \delta_{12})$.

We also have

$\vec{s} \Vdash \Diamond_1\Diamond_2 r$,

$\vec{s} \Vdash \mathsf{O}_{01}\Diamond_1\Diamond_2 r$ (as $(s_1,s_1,s_2) \Vdash \Diamond_1\Diamond_2 r$)

$\vec{s} \Vdash \mathsf{O}_{02}\Diamond_1\Diamond_2 r$ (as $(s_2,s_1,s_2) \Vdash \Diamond_1\Diamond_2 r$)

So $\vec{s} \Vdash \neg\gamma'$ whence $\mathfrak{M}, (s_0,t,u) \Vdash \neg\gamma$. But then ρ is not satisfiable in a cube.

(ii) Our second aim is to show that $\neg\rho$ is not derivable without the IR_{D_3}-rule. Note that by the completeness result for hypercylindric frames, it suffices to show that (†) $\mathsf{HCF}_3 \not\models \neg\rho$. We leave this part of the proof to the interested reader; it is not very difficult to prove (†) by algebraic means, using the well-known technique of *splitting* in cylindric algebras, cf. Henkin, Monk & Tarski [16].

(iii) We conclude the proof by showing that $\neg\rho$ is derivable if we have the new derivation rule at our disposal. We will not give the actual derivation, which would hardly give any insights; our task will be to prove

$$\mathsf{HCF}_3 \models (p \wedge \neg D_3 p) \to \neg\rho. \tag{9}$$

After establishing this, we reason as follows: by Theorem 3.2, $A_3 \vdash (p \wedge \neg D_3 p) \to \neg\rho$, so by one application of the D_3-irreflexivity rule: $A_3^+ \vdash \neg\rho$.

Below we frequently need the following fact on hypercylindric frames:

$$\mathsf{HCF} \models T_k xy \to (E_{ij}x \leftrightarrow E_{ij}y), \text{ if } k \notin \{i,j\}. \tag{10}$$

To prove (10), let $\mathfrak{F}$ be a hypercylindric frame, x and y in $\mathfrak{F}$, $k \notin \{i,j\}$. Assume that $T_k xy$ and $E_{ij}x$. By $N7_{ijk}$ (from left to right) there is a z (actually, $z = f_{ki}x$) such that $T_k xz$, $E_{ik}z$ and $E_{kj}z$. As T_k is an equivalence relation, we have $T_k yz$, so by using $N7_{ijk}$ again, but now in the other direction, we get $E_{ij}y$. This suffices to prove (10).

To prove (9), let $\mathfrak{F}$ be hypercylindric, V and a such that $\mathfrak{F}, V, a \Vdash \rho$. It suffices to prove that that $a \Vdash p \to D_3 p$ and to establish this, we will show that a is R^3-reflexive, i.e. $R^3 aa$. The reader is advised to follow the proof with a glance at the pictures below.

(I) As $a \Vdash \psi$, there are b, c in F with $aT_0 b$, $aT_0 c$, $a \Vdash r_0 \wedge \neg r_1 \wedge \neg r_2$, $b \Vdash \neg r_0 \wedge r_1 \wedge \neg r_2$ and $c \Vdash \neg r_0 \wedge \neg r_1 \wedge r_2$.

(II) Actually, all we need to remember is that a, b and c are distinct elements of the 'line' T_0 and that r holds at a, b and c.

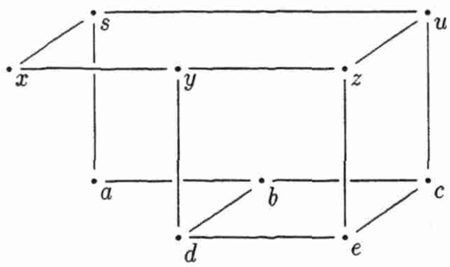
We will show that $\mathfrak{M} \models \gamma$ causes the *0-hyperplanes* through a and b to coincide.

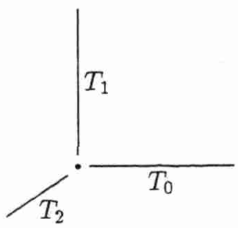
(III) Let $s = f_{10}a$, $d = f_{20}b$, i.e. $E_{01}s$, aT_1s and $E_{02}d$, bT_2d.

(IV) By $\mathfrak{F} \models N8_{10}$, aT_0c, $a \neq c$ and the definition of s, there is a u with $\neg E_{01}u$ and sT_0uT_1c. Likewise, there is an e satisfying and $\neg E_{02}e$ and dT_0eT_2c.

(V) By $\mathfrak{F} \models N4$, cT_2e and cT_1u there is a z with eT_1zT_2u. By (10), $\neg E_{01}u$ and uT_2z give $\neg E_{01}z$. Likewise we obtain $\neg E_{02}z$.

(VI) By $N4$ again, there are x, y with dT_1yT_0z and sT_2xT_0z, viz.





By (10), $E_{01}s$ and T_2xs imply $E_{10}x$. Likewise, $E_{02}d$ and T_1bd imply $E_{02}y$.

We will prove that the 0-hyperplanes through a and b intersect by showing $x = y$.

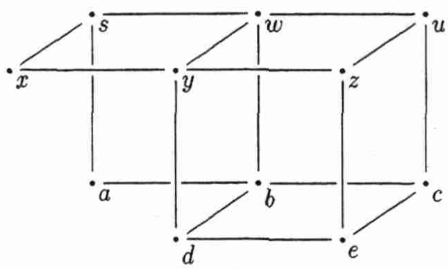
(VII) So we have that

$$\left.\begin{array}{lllll} a \Vdash r & \Rightarrow & s \Vdash \Diamond_1 r & \Rightarrow & x \Vdash \delta_{01} \wedge \Diamond_2\Diamond_1 r \\ b \Vdash r & \Rightarrow & d \Vdash \Diamond_2 r & \Rightarrow & y \Vdash \delta_{02} \wedge \Diamond_1\Diamond_2 r \\ c \Vdash r & \Rightarrow & e \Vdash \Diamond_2 r & \Rightarrow & z \Vdash \Diamond_1\Diamond_2 r \end{array}\right\} \Rightarrow z \Vdash \bigcirc_{01}\Diamond_1\Diamond_2 r \wedge \bigcirc_{02}\Diamond_1\Diamond_2 r \wedge \Diamond_1\Diamond_2 r$$

Now by the fact that $z \Vdash \gamma'$ (here we use the assumption $a \Vdash \gamma$) we obtain $z \Vdash \delta_{01} \vee \delta_{02} \vee \delta_{12}$. Then by (V) $z \Vdash \delta_{12}$.

(VIII) Now $E_{12}z$ and zT_0y give $E_{12}y$ by (10), so with $E_{02}y$ and $N7$ we get $E_{01}y$. We already had $E_{01}x$, so by xT_0y and $\mathfrak{F} \models N6$ we obtain $x = y$.

(IX) By $\mathfrak{F} \models N8$, $a \neq b$, T_0ab and the definition of s, there is a w with $\neg E_{01}w$ and sT_0wT_1b, viz.



(X) Now $wT_2y = xT_2sT_1a$ implies wH_0a, so we obtain (recall that $s = f_{10}a$)

$$f_{10}aT_0w, \ \neg E_{10}w \text{ and } wH_0a,$$

which gives R^3aa by definition of R^3. □

4 Applications to other logics and to algebraic logic.

In this section we discuss a number of formalisms which are closely related to cylindric modal logic, and for which the results obtained in the previous sections have nice applications. In the first subsection we treat type-free logic and schema validity of first order logic, and in the second, the connections with the algebraic theory of cylindric algebras; finally, we prove a result on interpolation and amalgamation properties.

4.1 Applications to other logics.

First we discuss *typeless* or *type-free* logic, which arises out of abstracting away from the ranks of relation symbols in ordinary first-order logic. Typeless logic is studied in e.g. Henkin-Monk-Tarski [16], Andréka-Gergely-Németi [2], Simon [39]. The syntax of typeless logic is identical to that of restricted first-order logic of dimension ω:

Definition 4.1 *The language L_{tf} of* **type-free** *or* **typeless** *logic is defined as follows, given a set of predicate symbols. An atomic formula is either an identity $v_i = v_j$ or a predicate symbol R_l, and a non-atomic typeless formulas is of the form $\neg\phi$, $\phi \vee \psi$ or $\exists v_i \phi$, where ϕ and ψ are typeless formulas.*

A **type** *for L_{tf} is a map $\rho : \omega \mapsto \omega$ assigning to each relation symbol R_l a finite* rank *$\rho(l)$. The ρ-instantiation ϕ^ρ of a type-free formula ϕ is the first-order formula obtained from ϕ by replacing all atomic (type-free) subformulas R_l by the ρ-typed $R_l(v_0 \ldots v_{\rho(l)-1})$.*

This idea of giving types to L^r_ω-formulas lies behind the *model theory* of typeless logic too:

Definition 4.2 *A* **model** *for L_{tf} is a pair $\mathfrak{M} = (U, V)$ such that there exists a type ρ with the property that $\mathfrak{M}$ is a structure for the restricted first-order logic of similarity type ρ (or equivalently, V is a function mapping every relation symbol R_l to a $\rho(l)$-ary relation on U).*

A typeless formula ϕ is **type-free valid in** *$\mathfrak{M}$, notation: $\mathfrak{M} \models_{tf} \phi$, if $\mathfrak{M}$ is of type ρ and ϕ^ρ is valid in $\mathfrak{M}$ in the usual classical sense; ϕ is* **type-free valid**, *notation: $\models_{tf} \phi$, if ϕ is valid in all models for L_{tf}.*

So a formula is type-free valid if it is valid in any model, no matter how we type the relation symbols of ϕ. A simple example: of the two typed instances $Pv_0v_1 \rightarrow \forall v_2 Pv_0v_1$ and $Pv_0v_1v_2 \rightarrow \forall v_2 Pv_0v_1v_2$ of the typeless formula $P \rightarrow \forall v_2 P$, the first is valid, but the second is not. Therefore $P \rightarrow \forall v_2 P$ is *not* a type-free valid formula.

Note that type-free validity is equivalent to ω-validity:

Lemma 4.3 *Let ϕ be an ω-formula. Then*

$$\models_{tf} \phi \iff \mathsf{C}_\omega \models \phi',$$

where ϕ' is the cylindric modal formula obtained by replacing all occurrences of $v_i = v_j$ and $\exists v_i$ by δ_{ij} resp. $\Diamond_i$.

Proof.
In a rather straightforward manner we can convert a typeless model $\mathfrak{M}$ into a model $\mathfrak{M}'$ over a ω-cube and vice versa, such that $\neg\phi$ is satisfiable in $\mathfrak{M}$ iff $\neg\phi'$ is satisfiable in $\mathfrak{M}'$. The theorem then follows immediately. □

Now the following completeness result for type-free validity is immediate:

Definition 4.4 *Let A^+_{tf} be the following derivation system for typeless formulas. Its axiom schemas are:*

(CT)	*all propositional tautologies*
(DB)	$\forall v_i(\phi \rightarrow \psi) \leftrightarrow (\forall v_i \phi \rightarrow \forall v_i \psi)$
$(CR1)$	$\phi \rightarrow \exists v_i \phi$
$(CR2)$	$\phi \rightarrow \forall v_i \exists v_i \phi$
$(CR3)$	$\exists v_i \exists v_i \phi \rightarrow \exists v_i \phi$
$(CR4)$	$\exists v_i \exists v_j \phi \rightarrow \exists v_j \exists v_i \phi$
$(CR5)$	$v_i = v_i$
$(CR6)$	$\exists v_i(v_i = v_j \wedge \phi) \rightarrow \forall v_i(v_i = v_j \rightarrow \phi))$
$(CR7)$	$v_i = v_j \leftrightarrow \exists v_k(v_i = v_k \wedge v_k = v_j)$
$(CR8)$	$(v_i = v_j \wedge \exists v_i(\neg\phi \wedge \exists v_j \phi)) \rightarrow \exists v_j(v_i \neq v_j \wedge \exists v_i \phi)$.

The derivation rules of A^+_{tf} are MP, SUB and UG (here: ϕ / $\forall v_i\phi$) and the schema of rules $\{IR^{tf}_{D_n} \mid n \leq \omega\}$, where $IR^{tf}_{D_n}$ is the appropriate version of IR_{D_n}, i.e.

$$(IR^{tf}_{D_n}) \qquad \vdash (P \wedge \neg D^{tf}_n(P)) \rightarrow \phi \Rightarrow \vdash \phi, \text{ if } P \notin \phi.$$

Here D^{tf}_n is the following abbreviation in L_{tf}:

$$D_n\phi = \bigvee_{j \neq i} \exists v_j(v_i = v_j \wedge \exists v_i(v_i \neq v_j \wedge \exists v_0 \ldots \exists v_{i-1}\exists v_{i+1} \ldots \exists v_{n-1}\phi)).$$

The theorem below indicates a possible solution to Problem 4.16 of Henkin-Monk-Tarski [16], as A^+_{tf} is a proof calculus for type-free valid formulas which involves only type-free valid formulas.

Theorem 4.5 *COMPLETENESS FOR TYPELESS LOGIC.*
Let ϕ be a typeless formula. Then

$$\vdash^+_{tf} \phi \iff \models_{tf} \phi.$$

Proof.
Immediate by Lemma 4.3 and Theorem 3.4. □

It is interesting to note the following: independently of our result, András Simon found a proof calculus for typeless validity (and thus, for the related notions), in which another kind of unorthodox derivation rule appears (cf. Simon [39]). Simon's methods seem to be complementary with ours in that he concentrates on infinite dimensional while we on finite dimensional cylindric algebras (cf. also the next subsection).

The second concept we (briefly) mention is that of *schema validity*, cf. Németi [26], Rybakov [32]. Formula schemas are used widely in logic, e.g. in *axiomatizations* of first-order logics: an example of such a formula schema is $\phi \rightarrow \exists v_i\phi$. Formally we set:

Definition 4.6 *Let Q_{fm} be a set of formula variables (i.e. variables ranging over formulas), and assume that we have a set $\{v_i \mid i \in \omega\}$ of individual variables.* **Formula schemas** *are defined by induction: (i) ϕ is a schema if $\phi \in Q_{fm}$, (ii) $v_i = v_j$ is a schema for $i, j < \omega$, (iii) $\exists v_i\sigma$, $\neg\sigma$, $\sigma \vee \xi$ are schemas if $i \in \omega$ and σ, ξ are schemas. An* **instance** *of a schema σ is any first-order formula we obtain by uniformly substituting first-order formulas for the formula variables in σ. A formula schema σ is* **valid** *if every instance of it is valid as a first-order formula.*

By Proposition 0.3 in Németi [26], schema validity is yet another variant of ω-validity or typeless validity: if we replace the formula variables with predicate symbols, a schema is valid iff the resulting type-free formula is typeless valid. Using this connection between typeless validity and schema validity, we see that the completeness theorem for typeless logic also yields a completeness theorem for the valid schemas of first-order logic. We leave the details to the interested reader.

4.2 Applications to algebraic logic.

Now we turn to the algebraization of the above logics (for the general idea of algebraizations we refer to Blok & Pigozzi [10] or Andréka, Németi & Sain [5]). We start with cylindric set algebras of dimension α. These are for restricted first-order logic what Boolean set algebras are for propositional logic; they are also the 'intended' modal algebras of *CML*.

Let us approach algebraic logic from the model-theoretic point of view: in this case one is interested in such operations on the power set algebra of models as are defined by the semantic truth definition of the connectives in the language. Consider for instance the case of restricted first-order logic; let for a model $\mathfrak{M}$, $\phi^{\mathfrak{M}}$ denote the set of n-tuples where ϕ holds. Then

$$\begin{array}{lcl} (\exists v_i\phi)^{\mathfrak{M}} & = & \{u \in^n U \mid \exists t\, [u \equiv_i t \;\&\; t \in \phi^{\mathfrak{M}}]\}, \\ (v_i = v_j)^{\mathfrak{M}} & = & D_{ij}, \\ (\neg\phi)^{\mathfrak{M}} & = & -\phi^{\mathfrak{M}} \; (= {}^{\alpha}U \setminus \phi^{\mathfrak{M}}), \\ \text{etc.} & & \end{array}$$

where $\equiv_i$ and D_{ij} are as defined in Definition 1.2. This inspires the following definition. (We denote the power set operation by $\mathcal{P}$.)

Definition 4.7 *Let U be some unspecified set, α an ordinal and $i < \alpha$. The* ***i-th cylindrification on*** *$\mathcal{P}({}^\alpha U)$ is the following operation C_i on $\mathcal{P}({}^\alpha U)$:*

$$C_i(X) = \{u \in {}^\alpha U \mid v \in X, \text{ for some } v \text{ with } u \equiv_i v\}.$$

The ***α-dimensional full cylindric set algebra on*** *U is the structure*

$$\mathfrak{Cs}_\alpha(U) = (\mathcal{P}({}^\alpha U), \cup, -, C_i, D_{ij})_{i,j<\alpha}.$$

As usual, the idea is now to abstract away from this concrete case of set-theoretically defined algebras.

Definition 4.8 *A* **cylindric type algebra of dimension** *α is a Boolean Algebra with Operators, i.e. an algebra of the form $\mathfrak{A} = (A, +, -, c_i, d_{ij})_{i,j<\alpha}$ with $(A, +, -)$ a Boolean Algebra, d_{ij} a constant and c_i a normal, additive unary operator, for all $i, j < \alpha$. Within this class we define the following classes of algebras: FCS_α is the class of full α-dimensional cylindric set algebras over some set U. The class RCA_α of* **representable cylindric algebras of dimension** *α is defined as* $\mathbf{ISP}(\mathsf{FCS}_\alpha)$*, i.e. isomorphic copies of subalgebras of products of full cylindric set algebras.*

The algebraic language used to describe these algebras is denoted by $\mathcal{L}_\alpha$, and by $Equ(\mathsf{K})$ we denote the set of $\mathcal{L}_\alpha$-equations that are valid in the class of algebras K.

It is not difficult to see that RCA_α consists of those algebras where elements represent 'real' α-ary relations. Tarski showed that RCA_α is closed under homomorphic images, i.e. it is the *variety* generated by FCS_α.

The connection with cylindric modal logic lies in the fact that cylindric type algebras of dimension α are the *complex algebras* of α-frames; complex algebras form one of the fundamental structural operations in the duality theory of relational frames and Boolean Algebras with Operators (cf. Goldblatt[14]). To explain what a complex algebra is, consider a relational frame $\mathfrak{F} = (W, R_i)_{i\in I}$. With each $n+1$-ary *relation* R_i we associate an n-ary *operation* f_{R_i} on the power set $\mathcal{P}(W)$:

$$f_{R_i}(X_1, \ldots, X_n) = \{x_0 \mid \exists x_1 \ldots x_n (R_i x_0 x_1 \ldots x_n \;\&\; \bigwedge_{1\leq i\leq n} x_i \in X_i)\}.$$

The complex algebra $\mathfrak{Cm}\mathfrak{F}$ of $\mathfrak{F}$ is given as

$$\mathfrak{Cm}\mathfrak{F} = (\mathcal{P}(W), \cup, -, f_{R_i})_{i\in I}.$$

For a class K of frames, we denote by $\mathbf{Cm}\mathsf{K}$ the class of complex algebras of frames in K. Now the essential observation is that the full cylindric set algebras are the complex algebras of the cubes:

Lemma 4.9

$$\mathbf{Cm}\mathsf{C}_\alpha = \mathsf{FCS}_\alpha.$$

Turning to axiomatics, the meaning of Lemma 4.9 is that finding complete derivation systems for cube-validity of modal formulas and for the 'true' cylindric equations (i.e. the equations valid in FCS_α) are really two halves of the same nut. First we have to give a syntactic translation from *CML*-formulas to $\mathcal{L}_\alpha$-terms:

Definition 4.10 *Let ϕ be an α-formula. The* corresponding $\mathcal{L}_\alpha$-term *ϕ^t of ϕ is defined by the following induction:*

$$\begin{array}{rcl}
(p_l)^t &=& x_l \\
(\delta_{ij})^t &=& d_{ij} \\
(\neg\phi)^t &=& -\phi^t \\
(\phi \vee \psi)^t &=& \phi^t + \psi^t \\
(\Diamond_i \phi)^t &=& c_i \phi^t.
\end{array}$$

The exact connection between validity of α-formulas and $\mathcal{L}_\alpha$-equations is given by the following lemma, of which the proof is trivial.

Lemma 4.11 *Let ϕ be an α-formula and K a class of α-frames; then*

$$\mathsf{K} \models \phi \iff \mathbf{Cm}\mathsf{K} \models \phi^t = 1.$$

The algebraic side of the above-mentioned nut has already been studied for some decennia. As a first approximation of the variety of representable cylindric algebras, Tarski proposed the finitely based variety of *Cylindric Algebras* (cf. the monograph Henkin, Monk & Tarski [16] for an extensive overview). We suggest to narrow down this class with one more axiom:

Definition 4.12 *Consider the following $\mathcal{L}_\alpha$-equations:*

$(C1_i)$	$c_i 0 = 0$
$(C2_i)$	$x \leq c_i x$
$(C3_i)$	$c_i(x \cdot c_i y) \leq c_i x \cdot c_i y$
$(C4_{ij}$	$c_i c_j x \leq c_j c_i x$
$(C5_i)$	$d_{ii} = 1$
$(C6_{ij})$	$c_i(d_{ij} \cdot x) \cdot c_i(d_{ij} \cdot -x)) = 0$
$(C7_{ijk})$	$d_{ij} = c_k(d_{ik} \cdot d_{kj})$
$(C8_{ij})$	$d_{ij} \cdot c_i(-x \cdot c_j x) \leq c_j(-d_{ij} \cdot c_i x)$

For finite α we set $C1 \equiv \bigwedge_i C1_i$, etc., taking $C4 \equiv \bigwedge_{i,j} C4_{ij}$, $C6 \equiv \bigwedge_{i \neq j} C6_{ij}$, $C7 \equiv \bigwedge_{i,j,k} C7_{ijk}$ and $C8 \equiv \bigwedge_{i \neq j} C8_{ij}$. If $\alpha \geq \omega$, we let $C1, \ldots, C8$ be the corresponding equation schemata.

An α-cylindric type algebra $\mathfrak{A}$ is a **cylindric algebra of dimension** *α (short: a CA_α), if $\mathfrak{A} \models C0, \ldots, C7$. The class of these algebras is denoted by CA_α. The class of* **hypercylindric algebras** *is denoted by HCA_α and consists of those cylindric algebras where $C8$ holds.*

It was realized very early in the development of cylindric algebraic theory that $C0, \ldots, C7$ do not suffice to axiomatize RCA_α, and one can also show that for $\alpha > 2$, adding $C8$ is not sufficient. Indeed, though $Equ(\mathsf{RCA}_\alpha)$ is known to be recursively enumerable, it was shown by Monk in [23] that for $\alpha > 2$, no finite schema of equations can generate $Equ(\mathsf{RCA}_\alpha)$, if one allows only the ordinary algebraic derivation rules; in the same article he gave a complete system with infinitely many axioms. Recently, Andréka [1] gave a very strong generalization of the negative result by Monk. Roughly speaking, she proved that if Σ is a set of universally quantified formulas axiomatizing the class RCA_α, $\alpha > 2$, then for all natural numbers n, and all ordinals $i < \alpha$, there are infinitely many axioms $\eta \in \Sigma$ such that η contains at least $\min(n, \alpha)$ operation symbols, more than n variables and a diagonal constant with index i. On the other hand, in [4] Andréka and Németi defined a finite schema of axioms and rules generating $Equ(\mathsf{RCA}_\alpha)$, but this system has an axiom which is not in equational form.

Let us see now how the results obtained in the previous sections can be applied in the theory of cylindric algebras. First of all, an easy result (which also explains some of our terminology concerning α-frames) is that the (hyper)cylindric algebras are the complex algebras of (hyper)cylindric frames:

Lemma 4.13 .
(i) $\mathsf{CA}_\alpha = \mathbf{Cm}(\mathsf{CF}_\alpha)$.
(ii) $\mathsf{HCA}_\alpha = \mathbf{Cm}(\mathsf{HCF}_\alpha)$.

Proof.
Immediate by the definitions and Lemma 4.11. □

As a corollary to this result, we can prove that RCA2 is finitely based:

Theorem 4.14

$$\mathsf{HCA}_2 = \mathsf{RCA}_2.$$

Proof.
It suffices to show that HCA_2 and RCA_2 validate the same equations of the form $t = 1$ (for t a $\mathcal{L}_\alpha$-term). Consider such an equation, and let ϕ be the CML_2-formula such that $\phi^t = t$. Then

$$\begin{array}{rcl} \mathsf{HCA}_2 \models t = 1 & \Longleftrightarrow & \mathsf{HCF}_2 \models \phi \\ & \Longleftrightarrow & \mathsf{C}_2 \models \phi \\ & \Longleftrightarrow & \mathsf{FCS}_2 \models t = 1 \\ & \Longleftrightarrow & \mathsf{RCA}_2 \models t = 1 \end{array}$$

where the first equivalence is by Lemma 4.13 and Lemma 4.11, the second by Theorem 2.4 and the preservation result of modal validity under taking zigzagmorphic images and disjoint unions; the third equivalence is again by Lemma 4.13 and Lemma 4.11 and the last one is by definition of RCA_2. □

It has been known for a long time that RCA_2 is finitely axiomatizable, cf. [16]. By a result of Henkin and Tarski, a cylindric algebra is representable iff the *Henkin equations*

$$c_j(x \cdot y \cdot c_i(x \cdot -y) \leq c_i(c_j x \cdot -d_{ij}) \qquad (C8'_{ij})$$

hold in it ($i \neq j$). It follows from Theorem 4.14 that over the class CA, the Henkin equations can be simplified to $C8_{ij}$:

$$d_{ij} \cdot c_i(-x \cdot c_j x) \leq c_j(-d_{ij} \cdot c_i x) \qquad (C8_{ij})$$

This result can also be proved directly, by using Sahlqvist correspondence theory; for details we refer the reader to de Rijke & Venema [31].

Finally, by 'algebraizing' the axiomatization $\vdash^+_\alpha$ of the previous section, we find a finite derivation system for the 'true' cylindric equations:

Definition 4.15 *First define the following $\mathcal{L}_n$-counterpart of the difference operator:*

$$d_n(x) = \bigvee_{j \neq i} c_j\big(d_{ij} \cdot c_i(-d_{ij} \cdot c_0 \ldots c_{i-1} c_{i+1} \ldots c_{n-1} x)\big).$$

Definition 4.16 *For an arbitrary ordinal α, let Σ_α be the smallest set of $\mathcal{L}_\alpha$-equations containing $C0, \ldots, C8$, which is closed under (1) ordinary algebraic deduction, (2) the following closure operations, for any n with $n \leq \min(\omega, \alpha)$:*

$$y \cdot -d_n(y) \leq t(x_0, \ldots, x_{n-1}) \ / \ t(x_0, \ldots, x_{n-1}) = 1$$
if y does not occur among the $\vec{x}$.

and (3) for $\alpha > \omega$, the rule

$$\eta^\tau \ / \ \eta,$$

where η^τ is the formula obtained from η by substituting $c_{\tau(i)}$ and $d_{\tau(i)\tau(j)}$ for every occurrence of c_i resp d_{ij}.

Theorem 4.17 *ENUMERATING 'TRUE' CYLINDRIC EQUATIONS*
For every ordinal α:

$$\Sigma_\alpha = \mathit{Equ}(\mathsf{RCA}_\alpha)$$

Proof.
Clearly Σ_α is the algebraic version of A^+_α (in the sense that $\vdash^+_\alpha \phi$ iff $\phi^t = 1 \in \Sigma_\alpha$), so the theorem is immediate by Theorem 3.4, Lemma 4.9 and the fact that RCA_α is the variety generated by FCS_α. □

4.3 Interpolation and Amalgamation

In Venema [44] we showed that the admissibility of a non-ξ rule over a logic Λ is related to the interpolation property of Λ. In a perhaps unexpected way, this connection can be used, together with the results of this paper, to prove results about interpolation of axiom systems for e.g. formalisms with finitely many variables. First some definitions:

Definition 4.18 *A derivation system Λ has the* **Craig Interpolation Property** *(CIP) if, whenever $\Lambda \vdash \phi \to \psi$ for some formulas ϕ and ψ, there is a formula χ in the common language of ϕ and ψ such that $\Lambda \vdash \phi \to \chi$ and $\Lambda \vdash \chi \to \psi$. This formula χ is called an* **interpolant** *of ϕ and ψ.*

For the formalisms treated in this paper, interpolation properties have been studied quite intensively, cf. Sain [37] for a recent overview. The following theorem states that for the predicate calculus with finitely many variables/finite-dimensional cylindric modal logic, no Hilbert style derivation system can have the Craig Interpolation Property if the system is an extension with finitely many axioms of the 'hypercylindric' logic A_n (cf. Definition 3.1). In fact, it states something slightly stronger:

Theorem 4.19 *Assume $n > 2$ and let Λ be a logic extending A_n by a set of axioms which (1) are valid in C_n and (2) use only finitely many propositional variables. Then Λ does not have CIP.*

Proof.
To derive a contradiction, suppose that Λ is a logic extending A_n in the way indicated above and that Λ does have the CIP.

Call a CML_n-formula *closed* if it does not contain any propositional variables (only constants). Let Λ' be the logic extending Λ with the following axioms: all closed formulas that are n-valid, and let Λ^+ be the derivation system extending Λ' with the D_n-irreflexivity rule. To start with, it is not hard to prove that

$$\Lambda' \text{ has the CIP too.} \tag{11}$$

and that

$$\Lambda^+ \text{ is sound and complete with respect to } \mathsf{C}_n. \tag{12}$$

Now we show that

$$\text{for \textit{closed} formulas, } \Lambda^+ \text{ is conservative over } \Lambda'. \tag{13}$$

For, let γ be closed such that $\Lambda^+ \vdash \gamma$. By soundness of Λ^+, γ is n-valid, so $\Lambda' \vdash \gamma$ by definition. Next we prove that the irreflexivity rule for D_n is *admissible* in Λ', i.e.

$$\text{If } \Lambda' \vdash (p \wedge \neg D_n p) \to \phi \text{ and } p \text{ does not occur in } \phi, \text{ then } \Lambda' \vdash \phi. \tag{14}$$

Suppose that $\Lambda' \vdash (p \wedge \neg D_n p) \to \phi$, with p not occurring in ϕ. By (11) there must be an interpolant γ. As the formulas $p \wedge \neg D_n p$ and ϕ have no proposition letters in common, γ must be closed. By $\Lambda' \vdash (p \wedge \neg D_n p) \to \gamma$ we get $\Lambda^+ \vdash \gamma$, so $\Lambda' \vdash \gamma$ by (13). But then $\Lambda' \vdash \phi$ by $\Lambda' \vdash \gamma \to \phi$.

However, (14) and (12) would imply that Λ' is a sound and complete axiom system for n-cube validity; but then its algebraic counterpart, which is a totally orthodox axiom system, would be a complete enumeration of the equations valid in RCA_n, while it has only finitely many axioms containing variables. This contradicts the strong non-finite axiomatizability result by Andréka [1], which is mentioned in subsection 4.2 □

It is also interesting to mention the algebraic version of the previous theorem. First we need a definition of the algebraic counterpart of the interpolation property:

Definition 4.20 *A class K of algebras is said to have the* **amalgamation property** *(AP), if for any $\mathfrak{A}$, $\mathfrak{B}_1$, $\mathfrak{B}_2$ in K with $\mathfrak{A} \subseteq \mathfrak{B}_i$, there are $\mathfrak{C}$ in K and embeddings $f_i : \mathfrak{B}_i \rightarrowtail \mathfrak{C}$ such that $f_1 \restriction \mathfrak{A} = f_2 \restriction \mathfrak{A}$.*

Theorem 4.21 *Let* K *be a finitely based variety with* $\mathsf{RCA}_n \subseteq \mathsf{K} \subseteq \mathsf{HCA}_n$*. Then* K *does not have* *AP*.

Proof.
Pigozzi showed in [28] that, for any logic Λ, Λ has *CIP* iff V_Λ has *AP*, where V_Λ is the class of algebras where the set of equations $\{\phi^t = 1 \mid \Lambda \vdash \phi\}$ is valid. Our result is therefore immediate by the previous theorem. □

5 Evaluation

In this section we briefly discuss the contribution of the paper with respect to the famous non-finitely axiomatizability results in algebraic logic. As we have mentioned before, it was generally assumed to follow from results by Monk and Andréka that the cylindric modal theory of the cubes cannot be axiomatized by a finite (MONK [23]) or indeed simple (ANDRÉKA [1]) set of axioms. The results in this paper show that the validity of this assumption depends on the *kind* of axiomatization one has in mind. If one allows unorthodox derivation rules like the irreflexivity rule for D^n, finite derivation systems are no longer out of reach: by adding rules that are non-standard from the traditional algebraic viewpoint, existing finite (and incomplete) axiomatizations can be turned into (finite) complete derivation systems. From this perspective, several remarks should be made, and various questions emerge:

1. To start with, we should mention that independently of our results, A. Simon found a proof calculus for typeless validity (and thus, for the related notions), in which another kind of unorthodox derivation rule appears (cf. SIMON [39]). Simon's method seems to be complementary with ours in that he concentrates on infinite dimensional while we on finite dimensional cylindric algebras.

2. The results concerning these unorthodox axiomatizations raise the philosophical question, what the criteria are for a natural, or acceptable axiomatization. The answer to this question will depend on the reasons one has to search for axiomatizations of a given class of structures. We feel that in the present context, the main function of an axiom system is to provide, in a compact and transparent way, information about a class of structures which is defined in a set-theoretical way (like the cubes or the cylindric set algebras). In this sense, we feel justified in saying that with the system A_n (axiomatizing the hypercylindric frames) one has enough *axioms*; the characterization results of section 2 express that, to jump from the hypercylindric frames to the (disjoint unions of) cubes, the only property needed is irreflexivity of the accessibility relation of the defined difference operator. So, if it turns out that this information cannot be provided by finitely many axioms, it should be done by a rule.

3. While the non-β rule has a clear interpretation in the modal context, its algebraic meaning seems to be less clear: note that the derivation system Σ_α (cf. 4.16) is a straightforward translation of the modal axiomatization. For some recent developments, in which non-ξ rules are linked up with so-called existential varieties (i.e. classes of algebras axiomatized by universal *and* universal-existential ($\forall\exists$) axioms, we refer the reader to MIKULÁS [22] or VENEMA [45].

4. It is interesting to note that, as usual in algebraic logic, the results about cylindric algebras can be translated to relation algebras as well, though not trivially so. In fact, the idea to use special derivation rules inspired by modal logic, to encompass non-finite axiomatizability results in algebraic logic, was applied to the theory of relation algebras first, witness VENEMA [42]. It is intriguing that the rules used in the relation algebraic and the cylindric algebraic context can be made to look very similar: the property bridging the gap between the finitely

based variety RA and the intended class FRA of full relation set algebras, is irreflexivity of the 'accessibility relation' of some defined difference operator. For details we refer to VENEMA [43].

5. The relations $\equiv_i$ and D_{ij} are only two examples of natural accessibility relations on cubes. For instance, in Venema [45] we treat the similarity type (i.e. set of modal operators of a given rank) *CMML* of cylindric mirror modal logic. *CMML* is an extension of *CML* with diamonds $\otimes_{ij}$ $(i, j > \alpha)$ having the following interpretation in cube models:

$$\mathfrak{C}, V, u \Vdash \otimes_{ij}\phi \text{ if } \mathfrak{C}, V, m_{ij}(u) \Vdash \phi,$$

where m_{ij} is the 'i, j-mirror' map defined by

$$v = m_{ij}(u) \iff v_i = u_j, v_j = u_i \;\&\; v_k = u_k \text{ for } k \neq i, j.$$

(The motivation behind this similarity type is that not only quantification, but also *substitution* of variables can be treated as a modal operator.)

Now consider the following problem: is there a modal similarity type of operators having a (first-order definable/permutation invariant/...) semantics over the cubes, such that cube validity becomes axiomatizable by an *orthodox* system having *finitely* many axioms (or axiom schemas in the infinite-dimensional case)?

This problem is (part of) the modal counterpart of the so-called *finitization quest* in algebraic logic, cf. Biró [9], Maddux [21], Németi [27], Sain [35], SIMON [40]. The outcome of these investigations seems to be that in a context of a *definable D-operator*, no natural extension of the similarity type allows a finite axiomatization, while there are some positive solutions for similarity types in which the D-operator cannot be defined. This leads us to formulate the conjecture (which would, if true, generalize results from relation algebras and cylindric algebras), that in sufficiently rich similarity types, definability of the D-operator is a *sufficient* reason for non-finite axiomatizability. To be a bit more precise: call a modal similarity type *hereditarily non-finitely axiomatizable* over a class K of relational frames, if for *no* reasonable[10] extension S' of S, the S'-logic of K allows a finite orthodox axiomatization. Our conjecture is then

> If S is a modal similarity type, and K a class of relational frames for S such that (i) the S-logic of K is not axiomatizable by a finite orthodox derivation system and (ii) over K, the difference operator is term-definable in S, then S is hereditary non-finite axiomatizable over K.

6. Note that definability of the D-operator is not a *necessary* condition for (hereditary) non-finite axiomatizability, as the following case shows. Let the diagonal-free reduct Df of *CML* be the similarity type *CML* without the diagonal constants δ_{ij}. This similarity type corresponds to the diagonal-free algebras of HENKIN, MONK & TARSKI [16], and to restricted first order logic without identity. Then over the class of cube frames, the difference operator is not definable, while the diagonal-free theory of the cubes is not finitely axiomatizable.

   In this perspective, it is interesting to note, that recently, Sz. Mikulás showed that Gabbay-style rules can be applied to Df as well (cf. MIKULÁS [22]). His results show that also without a definable D-operator, one can encompass negative results concerning finite axiomatizability by introducing unorthodox derivation rules.

[10]Note that with this definition, the precise meaning of the notion 'hereditary non-finitely axiomatizability' depends on the interpretation of the word 'reasonable'. In the literature, a communis opinio seems to arise that a 'reasonable' semantics of a modal or algebraic operator should be *permutation invariant*. We refer to NÉMETI for a more detailed discussion and a definition of the notion 'permutation invariant'.

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