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Embeddings of Heyting algebras revised version

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Why a revised version?

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It turns out that the fact that the Heyting algebra of Heyting's Arithmetic is RE, non-recursive, is an immediate corollary of the results of the earlier preprint. This corollary is now worked into the text.

A secondary improvement is that the proof of 3.7 is made more accessible.

Embeddings of Heyting algebras

Dick de Jongh &
Albert Visser

ABSTRACT: In this paper we study embeddings of Heyting algebras (*Ha*'s). It is pointed out that such embeddings are naturally connected with Derived Rules. We consider the *Ha*'s embeddable in the *Ha* of the Intuitionistic Propositional Calculus (IPC), i.e. the free *Ha* on $\aleph_0$ generators, those embeddable in the *Ha* of Heyting's arithmetic (HA) and those embeddable in the *Ha* of HA*, a 'natural' extension of HA. We prove the following theorems. The same *Ha*'s on finitely many generators are embeddable in the *Ha* of IPC and in the *Ha* of Boolean (or: Brouwerean) combinations of Σ-sentences of HA. The *Ha*'s on finitely many generators embeddable in the *Ha* of IPC are finitely axiomatizable. There is a non-recursive *Ha* on three generators, that can be embedded in the *Ha* of HA. Every recursively enumerable prime *Ha* is embeddable in the *Ha* of HA*.

1 Introduction

This paper sprung from an interest in the Heyting algebra's (*Ha*'s) of Constructive arithmetical theories. This interest was in its turn inspired by an interest in the Propositional Derived Rules of constructive arithmetical theories. We study and compare four specific *Ha*'s in some detail:

- The free *Ha* on $\aleph_0$ generators, in other words: the *Ha* $\mathfrak{H}_{\mathrm{IPC}}$ of the Intuitionistic Propositional Calculus (IPC).
- The *Ha* $\mathfrak{H}_{\mathrm{HA}}$ of Heyting's Arithmetic (HA).
- The *Ha* $\mathfrak{S}_{\mathrm{HA}}$ of $\mathrm{B}\Sigma_1$-sentences in HA (here $\mathrm{B}\Sigma_1$ is the set of Boolean (or perhaps more appropriately: Brouwerean) combinations of Σ_1-sentences).
- The *Ha* $\mathfrak{H}_{\mathrm{HA}^*}$ of HA*, an arithmetical theory studied in Visser[82].

We ask ourselves which RE *Ha*'s can be embedded in our target algebras. As we will see the answer to this question also determines what the Propositional Derived Rules for the various theories are. A reasonably complete answer has only been obtained for $\mathfrak{H}_{\mathrm{HA}}$*. All RE algebras of which one could reasonably expect it, i.e. those satisfying the property of primenes (corresponding to having the disjunction property), are embeddable in $\mathfrak{H}_{\mathrm{HA}}$*, and in consequence, only rules directly derivable in intuitionistic logic are rules under which HA* is closed. This property of HA* is a nice one —and in a surprising manner enables one to prove some properties of HA itself— but it does not seem to hold for more usual theories. Many of these algebras cannot be embedded in $\mathfrak{H}_{HA}$, nor in $\mathfrak{H}_{IPC}$, since both these theories validate additional rules not derivable in intuitionistic logic, the best known being:

$\neg A \to (B \vee C) \mathbin{/} (\neg A \to B) \vee (\neg A \to C)$. (Independence of Premiss Rule)

We will show that all *Ha*'s on finitely many generators embeddable in $\mathfrak{H}_{IPC}$ are finitely axiomatizable (i.e. are the *Ha*'s of finitely axiomatized IPC-theories). In contrast there is a non-recursive (and hence not finitely axiomatizable) *Ha* on three generators, that can be embedded in $\mathfrak{H}_{HA}$. It is an open question, whether there is a *finitely axiomatizable Ha* on finitely many generators, that can be embedded in $\mathfrak{H}_{HA}$, but not in $\mathfrak{H}_{IPC}$. It is also open whether HA and IPC have the same derived rules. On the other hand, we will show that the same *Ha*'s on finitely many generators are embeddable in $\mathfrak{S}_{HA}$ and in $\mathfrak{H}_{IPC}$. It follows that rules validated by HA, when one restricts oneself to substitutions of propositional combinations of Σ_1-sentences, and rules validated by IPC are the same. It is open whether $\mathfrak{S}_{HA}$ and $\mathfrak{H}_{IPC}$ are isomorphic. (We conjecture: *no*.)

We state some sample results with the places, where they can be found:

- Any RE prime *Ha* $\mathfrak{H}$ can be embedded in $\mathfrak{H}_{HA^*}$. (5.1)
- There are Σ_1-sentences A and B such that the subalgebra of $\mathfrak{H}_{HA^*}$ generated by A and B is RE, non-recursive. (7)
- There are Σ_1-sentences A and B and a sentence C, such that the subalgebra of $\mathfrak{H}_{HA}$ generated by A, B and C is RE, non-recursive. It follows that $\mathfrak{H}_{HA}$ is non-recursive. (7)
- Let $\mathfrak{H}$ be a *Ha* on finitely many generators, which is embeddable in $\mathfrak{H}_{IPC}$. Then $\mathfrak{H}$ is the *Ha* of a finitely axiomatizable IPC-theory. (2.3)
- Let $\mathfrak{H}$ be a *Ha* on finitely many generators. Then $\mathfrak{H}$ is embeddable in $\mathfrak{S}_{HA}$ iff $\mathfrak{H}$ is embeddable in $\mathfrak{H}_{IPC}$. (6.2)

The paper is organized as follows. In section 1 we define *Ha*'s in the presentation most useful to our purposes. In section 2 we introduce the notion of embedding and a connected notion of propositional formulas exactly provable for sentences of a theory. Propositional formulas with no iterations of implications on the left (NNIL formulas) turn out to play an important role. In section 3 and 4 necessary facts about HA and IPC, and HA* respectively, are given. In section 5 the above mentioned 'RE universality' of HA* is proved. In section 6 $\mathfrak{S}_{HA}$ is treated. Finally, in section 7 it is shown that, in consequence of the previous results, there is a *Ha* on two generators which is RE, but non-recursive, that can be embedded in $\mathfrak{H}_{HA^*}$, whereas such an algebra could never be embeddable in $\mathfrak{S}_{HA}$ or $\mathfrak{H}_{IPC}$. The theorem that there is a *Ha* on three generators which is RE, but non-recursive, that can be embedded in $\mathfrak{H}_{HA}$ is an immediate consequence of this last result. It follows that $\mathfrak{H}_{HA}$ is non-recursive.

1.1 Acknowledgements: Most of the results of this paper were obtained during an exceptionally inspiring visit of both authors to the Katedra Logiky of Prague University and the Institute of Computer and Information Science of the Czech Academy of Sciences. We thank the Prague logicians for their wonderful hospitality.

Some of the main methods employed in this paper were invented by Volodya Shavrukov (see Shavrukov[93]) and further developed and simplified by Domenico Zambella (see Zambella[92]). The work of Shavrukov and Zambella concerns embeddings of RE Diagonalizable algebras into Diagonalizable algebras of Classical arithmetical theories.

A major tool of the present paper is also Pitts's Uniform Interpolation Theorem (see Pitts[92]).

1.2 The classical case: Before going on, let's briefly look at the Boolean algebras of classical arithmetical theories. The Boolean algebras of all consistent RE arithmetical theories extending Q are isomorphic to the free Boolean algebra on $\aleph_0$ generators, i.e. to the Boolean algebra $\mathfrak{B}_{CPC}$ of the Classical Propositional Calculus (CPC). As far as we can trace this result is folklore. It follows from three observations. First: the Boolean algebras of all consistent RE arithmetical theories extending Q are countably infinite and (by Rosser's Theorem, see also 1.4) atomless. Second: $\mathfrak{B}_{CPC}$ is countably infinite and atomless. Third: all countably infinite atomless Boolean algebras are isomorphic.

It is not difficult to show that every countable Boolean algebra can be embedded into $\mathfrak{B}_{CPC}$.

1.3 Heyting algebras: A Heyting algebra (*Ha*) $\mathfrak{H}$ is a structure $\langle H,\wedge,\vee,\perp,\rightarrow\rangle$, where $\langle H,\wedge,\vee,\perp\rangle$ is a lattice with bottom $\perp$. We demand that $\mathfrak{H}$ is non-trivial, i.e. that H contains at least two elements. Let $x\leq y$ be defined by $x\vee y=y$. $\rightarrow$ is a binary operation satisfying: $x\wedge y\leq z \Leftrightarrow x\leq(y\rightarrow z)$. It is easily seen that if a partial order can be extended to a *Ha* such an extension is unique. *Ha*'s can be shown to be distributive lattices. Conversely every *finite* distributive lattice determines a Heyting algebra.

There are many good sources for *Ha*'s. We just mention van Troelstra & van Dalen[88b].

We will write:

- $\top := \perp\rightarrow\perp$,
- $\neg x := x\rightarrow\perp$,
- $x\leftrightarrow y := (x\rightarrow y)\wedge(y\rightarrow x)$,
- $\mathfrak{H}\models A(\mathbf{x}) :\Leftrightarrow A(\mathbf{x})=\top$,

  where A is a polynomial in $\wedge,\vee,\perp,\rightarrow$ and $\mathbf{x}$ is a sequence of elements of $\mathfrak{H}$.

Note that $\mathfrak{H}$ can be recovered from $\models$, since $A(\mathbf{x})=B(\mathbf{y}) \Leftrightarrow \mathfrak{H}\models A(\mathbf{x})\leftrightarrow B(\mathbf{y})$.

Define:

- $\mathfrak{f}:\mathfrak{H}\leqq\mathfrak{K} :\Leftrightarrow \mathfrak{f}$ is an embedding of $\mathfrak{H}$ into $\mathfrak{K}$
- $\mathfrak{H}\leqq\mathfrak{K} :\Leftrightarrow \mathfrak{f}:\mathfrak{H}\leqq\mathfrak{K}$ for some $\mathfrak{f}$
- $\mathfrak{H}\equiv\mathfrak{K} :\Leftrightarrow \mathfrak{H}\leqq\mathfrak{K}$ and $\mathfrak{K}\leqq\mathfrak{H}$

Clearly $\leqq$ is a preorder on *Ha*'s with induced equivalence relation $\equiv$.

1.3.1 Example: Equivalent *Ha*'s need not be isomorphic.

It is easily seen that any linear order with endpoints determines a *Ha*. E.g. we find: $x\rightarrow y := \top$ if $x\leq y$, $x\rightarrow y :=y$ if $y<x$. Moreover an embedding of linear orderings determines an embedding of *Ha*'s. Consider the algebras given by the real interval $[0,1]$ and by $[0,1/2]\cup\{1\}$. On the one hand these algebras are equivalent, on the other they are not isomorphic. ❍

Let T be any consistent theory in constructive propositional logic or in constructive predicate logic. We take $\mathfrak{H}_T$ to be the obvious *Ha* given by the T-provable equivalence classes. Sometimes we will consider only equivalence classes of a subset X of the language of T, which is closed under the propositional connectives. In this case we write: $\mathfrak{H}_T(X)$.

We can go from theory to algebra. Obviously it is sometimes natural to go back and recover theories from algebras. We introduce some notions relevant to this motion, which is executed by choosing a set of generators.

- A *numbered Ha* **H** is a pair $\langle \mathfrak{f},\mathfrak{H}\rangle$, where
  (i) $\mathfrak{f}$ is a function (not necessarily injective) from either n={0,...,n-1} or ω to $H_{\mathfrak{H}}$;
  (ii) $\mathfrak{H}$ is generated by the range of $\mathfrak{f}$.
- A numbered *Ha* is *finitely based* if dom($\mathfrak{f}$) is finite.
- $\|\mathfrak{H}\| := \{\langle \mathfrak{f},\mathfrak{K}\rangle \mid \langle \mathfrak{f},\mathfrak{K}\rangle$ is finitely based and $\mathfrak{K}\leqq\mathfrak{H}\}$
- $\mathfrak{L}_\nu$ is the language of IPC if $\nu=\omega$, and the language of IPC restricted to $p_0,\ldots,p_{n-1}$ if ν=n. We often write $\mathfrak{L}$ for $\mathfrak{L}_\omega$.
- For $A\in\mathfrak{L}_{\mathrm{dom}(\mathfrak{f})}$: $\mathbf{H}\models A :\Leftrightarrow \mathfrak{H}\models A[\mathfrak{f}]$, where $A[\mathfrak{f}]$ is the result of substituting $\mathfrak{f}i$ for p_i in A (for each relevant i). It is pleasant to use $\models$ also when A contains p_j for $j\notin\mathrm{dom}(\mathfrak{f})$. In this case we substitute $\top$ for p_j.
- $\mathrm{Th}(\mathbf{H}) := \{A\in\mathfrak{L}_{\mathrm{dom}(\mathfrak{f})} \mid \mathbf{H}\models A\}$.

1.3.2 **Fact:** Let $\mathbf{H}=\langle \mathfrak{f},\mathfrak{H}\rangle$ be a numbered *Ha*. Then $\mathfrak{H}$ is isomorphic to $\mathfrak{H}_{\mathrm{Th}(\mathbf{H})}$.

Proof: Trivial. ❑

Define:

- A numbered *Ha* **H** is RE (recursive) if Th(**H**) is RE (recursive).
- A *Ha* is RE (recursive) if it can be extended to an RE (recursive) numbered *Ha*.

Note that the *Ha* of an RE (recursive) theory is RE (recursive).

1.3.3 **Example:** Let $\mathbf{H}_{PA}:=\langle \mathfrak{f},\mathfrak{H}_{PA}\rangle$, where $\mathfrak{f}i$ is the equivalence class of an arithmetical sentence A if i is the Gödelnumber of A, and $\mathfrak{f}i$ is $\top$ if i is not the Gödelnumber of an arithmetical sentence. Let $\mathbf{H}_{CPC}:=\langle \mathfrak{g},\mathfrak{H}_{CPC}\rangle$, where $\mathfrak{g}i$ is the equivalence class of p_i. Then $\mathbf{H}_{PA}$ is RE, non-recursive and $\mathbf{H}_{IPC}$ is recursive. By 1.2, $\mathfrak{H}_{PA}$ and $\mathfrak{H}_{CPC}$ are isomorphic and hence $\mathfrak{H}_{PA}$ is recursive. ○

1.3.4 Fact: Suppose $\mathfrak{H}$ is RE (recursive) and suppose $\mathfrak{K} \leqq \mathfrak{H}$, where $\mathfrak{K}$ is a *Ha* on finitely many generators. Then $\mathfrak{K}$ is RE (recursive). In fact *every* finitely based numbered *Ha* $\langle \mathfrak{f}, \mathfrak{K} \rangle$ is recursive.

Proof: Obvious. ❑

Let T be any theory and let $\mathfrak{f}$ be a function from the propositional variables to the language of T. We write $A[\mathfrak{f}]$ for the result of substituting the $\mathfrak{f}(p_i)$ for p_i in A. Define:

- $A \models_T B :\Leftrightarrow \forall \mathfrak{f}\ T \vdash A[\mathfrak{f}] \Rightarrow T \vdash B[\mathfrak{f}]$.

We say that the inference form A to B is an *IPC-derived rule for T.* Since all derived rules we will consider in this paper are IPC-derived we will suppress the 'IPC'.

IPC-derived rules are studied in detail by V.V. Rybakov. A good reference is Rybakov[92], where it is shown that the IPC-derived rules for IPC are decidable.

1.3.5 Fact

i) $A \models_T B \Leftrightarrow \forall \mathbf{H} \in \|\mathfrak{H}_T\| \ (\mathbf{H} \models A \Rightarrow \mathbf{H} \models B)$

ii) $\|\mathfrak{H}_T\| \subseteq \|\mathfrak{H}_U\|$ and $A \models_U B \Rightarrow A \models_T B$.

iii) $\mathfrak{H} \leqq \mathfrak{K} \Rightarrow \|\mathfrak{H}\| \subseteq \|\mathfrak{K}\|$.

Proof: Obvious. ❑

1.4 The density of Heyting algebras of arithmetical theories

Evidently many properties of *Ha*'s are not captured by embeddability results (see example 1.3.1). Such properties are not the main subject of this paper, yet, they at least merit a brief comment here. Moreover many properties of the Boolean algebra of Classical arithmetical theories can be generalized to the constructive case. We briefly illustrate this for the property of density.

Let i-Q be the constructive version of Robinson's Arithmetic.

1.4.1 Fact: The *Ha* of a consistent RE extension of i-Q is dense, i.e. between every two points there is a third one.

Proof: Fix a consistent RE extension of i-Q, say T. Let $\Box$ stand for (the formalization of) provability in T. Consider any two sentences A and B such that $\Box(A \to B)$ and not $\Box(B \to A)$.

Interpolated Remark: The usual proof of this theorem for the classical case would be as follows. Take the Rosser sentence R of T+B+¬A. I.e. something like:

$$T \vdash R \leftrightarrow \Box((B \wedge \neg A) \to \neg R) \leq \Box((B \wedge \neg A) \to R),$$

holds. Here $\leq$ is the witness comparison relation, which is defined between formulas having an outer existential quantifier. There are two witness comparison relations, which are defined as follows:

- $(\exists x\, Dx \leq \exists y\, Ey) := \exists x\, (Dx \wedge \forall y{<}x\, \neg Ey)$,
- $(\exists x\, Dx < \exists y\, Ey) := \exists x\, (Dx \wedge \forall y{\leq}x\, \neg Ey)$.

The element between A and B will be: $C := (A \vee (B \wedge R))$. In constructive logic one cannot even conclude from the data that T+B+¬A is consistent. The correct constructive proof is just a slight variation on the classical argument. ❍ End of Remark

Define by the fixed point theorem a sentence R such that (verifiably in T):

$$R \leftrightarrow \Box((B \wedge R) \to A) \leq \Box(B \to (A \vee R)).$$

Let $S := \Box(B \to (A \vee R)) < \Box((B \wedge R) \to A)$ and $C := (A \vee (B \wedge R))$. Clearly $\Box(A \to C)$ and $\Box(C \to B)$.

We have:

$$\begin{aligned} \Box(C \to A) &\to \Box((B \wedge R) \to A) \\ &\to R \vee S. \end{aligned}$$

On the other hand:

$$\begin{aligned} \Box((B \wedge R) \to A) \wedge R &\to \Box((B \wedge R) \to A) \wedge \Box R \\ &\to \Box(B \to A). \end{aligned}$$

And:

$$\begin{aligned} \Box((B \wedge R) \to A) \wedge S &\to \Box((B \wedge R) \to A) \wedge \Box(B \to (A \vee R)) \\ &\to \Box(B \to A). \end{aligned}$$

Combining we find: $\Box(C \to A) \to \Box(B \to A)$. Ergo not $\Box(C \to A)$.

Also we have:

$$\begin{aligned}\Box(B\to C) &\to \Box(B\to(A\vee R))\\ &\to R\vee S.\end{aligned}$$

On the other hand:

$$\begin{aligned}\Box(B\to(A\vee R))\wedge S &\to \Box(B\to(A\vee R))\wedge\Box S\\ &\to \Box(B\to(A\vee R))\wedge\Box\neg R\\ &\to \Box(B\to A).\end{aligned}$$

And:

$$\begin{aligned}\Box(B\to(A\vee R))\wedge R &\to \Box(B\to(A\vee R))\wedge\Box((B\wedge R)\to A)\\ &\to \Box(B\to A).\end{aligned}$$

Combining, we find: $\Box(B\to C)\to\Box(B\to A)$. Ergo not $\Box(C\to A)$.❑

2 Embeddings of Heyting algebras in Free Heyting algebras

Every *Ha* on countably many generators is the homomorphic image of $\mathfrak{H}_{IPC}$. In other words: it is the *Ha* of some theory in IPC. On the other hand not every *Ha* on countably many generators can be embedded into $\mathfrak{H}_{IPC}$. First of all $\mathfrak{H}_{IPC}$ is prime, i.e.: $\mathfrak{H}_{IPC}\models(x\vee y)\Rightarrow(\mathfrak{H}_{IPC}\models x$ or $\mathfrak{H}_{IPC}\models y)$, or, in other words: IPC has the disjunction property. Clearly subalgebras inherit primeness. In this section we illustrate that many countable prime *Ha*'s are not embeddable in $\mathfrak{H}_{IPC}$. We provide some information about the *Ha*'s on finitely many generators that are embeddable in $\mathfrak{H}_{IPC}$. The problem of giving a neat characterization of the algebras embeddable in $\mathfrak{H}_{IPC}$ is still open.

Whenever '$\vdash$' is used without exhibiting a theory we intend provability in IPC.

2.1 Example: There are many non-trivial derived rules for IPC. For example:

- $\vdash(\neg\neg A\to A)\to(A\vee\neg A)\Rightarrow\ \vdash\neg\neg A\vee\neg A$ (De Jongh[82])
- $\vdash\neg A\to(B\vee C)\Rightarrow\ \vdash(\neg A\to B)\vee(\neg A\to C)$ (Independence of Premiss Rule)

This means that every embeddable algebra $\mathfrak{H}$ will satisfy:

- $\mathfrak{H}\models(\neg\neg A\to A)\to(A\vee\neg A)\Rightarrow\mathfrak{H}\models\neg\neg A\vee\neg A$
- $\mathfrak{H}\models\neg A\to(B\vee C)\Rightarrow\mathfrak{H}\models(\neg A\to B)\vee(\neg A\to C)$.

2.2 Example: We give an infinitary derived rule. Let $F_n(p)$ be an enumeration of the formulas presenting the non-top elements of the Rieger Nishimura Lattice. (For information about this lattice, see e.g.: Troelstra & van Dalen[88a], p49.) We have:

- (For all n $\vdash F_n(A) \to B$) $\Rightarrow \vdash B$.

It follows that in an embedded *Ha* for any x there can be no element between the $F_n(x)$ and the top.

Proof: Suppose for all n $\vdash F_n(A) \to B$. Let p be a propositional variable not in A and B. It follows that for all n: $\vdash F_n(p) \to ((p \leftrightarrow A) \to B)$. By Pitts[92] there is a uniform pre-interpolant of $((p \leftrightarrow A) \to B)$ w.r.t. to the variables in this formula unequal to p. This means that there is a formula C with just p free such that for any formula D containing no variables of A or B we have:

$$\vdash D \to ((p \leftrightarrow A) \to B) \Leftrightarrow \vdash D \to C.$$

(Following Pitts we could write the formula C as: $\forall \mathbf{q}((p \leftrightarrow A) \to B)$, where **q** represents the propositional variables in A,B.) It follows that for every n: $\vdash F_n(p) \to C$. Ergo (since C only contains p): $\vdash C$ and hence $\vdash ((p \leftrightarrow A) \to B)$. Substituting A for p we find: $\vdash B$. ❑

2.3 **Theorem:** Every *Ha* on finitely many generators that is embeddable in $\wp_{IPC}$ is the *Ha* of a finitely axiomatizable IPC theory.

Proof: Suppose the generators of the algebra go to $A_1,\ldots,A_n$. We have:

$$\vdash B(A_1,\ldots,A_n) \Leftrightarrow \vdash ((p_1 \leftrightarrow A_1) \wedge \ldots \wedge (p_n \leftrightarrow A_n)) \to B(p_1,\ldots,p_n).$$

We suppose that $\{p_1,\ldots,p_n\} \cap VAR(A_i) = \emptyset$ and $VAR(B) \subseteq \{p_1,\ldots,p_n\}$. Now let C be the Pittsean post-interpolant of $((p_1 \leftrightarrow A_1) \wedge \ldots \wedge (p_n \leftrightarrow A_n))$ w.r.t. the variables in the A_i. So, if these variables are **q**, we could write C as: $\exists \mathbf{q}((p_1 \leftrightarrow A_1) \wedge \ldots \wedge (p_n \leftrightarrow A_n))$. The only variables of C are the p_i and we have: $\vdash B(A_1,\ldots,A_n) \Leftrightarrow \vdash C \to B$. ❑

As we will see in 7.2 there are three elements of $\wp_{HA}$, such that the algebra generated by these elements is not recursive. A fortiori, this algebra is not finitely axiomatizable. Thus no analogue of 2.3 holds for $\wp_{HA}$.

Which formulas C are axioms of *Ha*'s on finitely many generators that are embeddable in $\wp_{IPC}$? We call such C *IPC-exactly provable*. In this paper we will abbreviate *IPC-exactly provable* by *exact*. So $C(p_1,\ldots,p_n)$ is exact if there are $A_1,\ldots,A_n$ such that for all $B(p_1,\ldots,p_n)$: $\vdash B(A_1,\ldots,A_n) \Leftrightarrow \vdash C \to B$. The notion of exactly provable formula was introduced in De Jongh[82].

Clearly by the above the exactly provable formulas are precisely those which are provably equivalent to Pitts' formulas of the form $\exists\mathbf{q}((p_1\leftrightarrow A_1)\wedge\ldots\wedge(p_n\leftrightarrow A_n))$, where $\mathbf{q}$ contains precisely the variables occurring in the A_i and where none of the p_j is in $\mathbf{q}$.

We say that A is *prime* if $\wp_{\mathrm{IPC+A}}$ is prime, i.e. IPC+A is consistent and IPC+A has the disjunction property:

- for all $B,C\in\mathfrak{L}\ \vdash A\to(B\vee C)\Rightarrow\ \vdash A\to B$ or $\vdash A\to C$.

In an alternative formulation, adhering to the convention that the empty disjunction is $\bot$, IPC is prime if for every finite set of formulas X:

$\vdash A\to\bigvee X\Rightarrow\exists B\in X\vdash A\to B.$

We give some properties of exact formulas and provide some special classes of such formulas. Our primary aim is to show that the prime NNIL-formulas are all exact. *NNIL-formulas* are formulas with No Nestings of Implications to the Left. Let's define NNIL more precisely. Let Sub(A) be the set of subformulas of A. We have:

- A is in NNIL iff for all $(B\to C)\in \mathrm{Sub}(A)$: B does not contain $\to$.

NNIL-formulas are studied in Visser[85], Renardel[86] and in BJRV[to appear]. The lemmas we give, can, however, also be used to establish exactness for more formulas than our target class. The result on NNIL will be used in the proof of 6.2.

2.4 **Observation:** If A is exact, then A is prime.

2.5 **Lemma:** Suppose that A is exact via $\mathfrak{f}$ and that $B[\mathfrak{f}]$ is exact via $\mathfrak{g}$, then $(A\wedge B)$ is exact via $\mathfrak{f}\circ\mathfrak{g}$.

Proof: We have:

$$\begin{aligned}\vdash (A\wedge B)\to C &\Leftrightarrow \vdash A\to(B\to C)\\ &\Leftrightarrow \vdash (B\to C)[\mathfrak{f}]\\ &\Leftrightarrow \vdash B[\mathfrak{f}]\to C[\mathfrak{f}]\\ &\Leftrightarrow \vdash C[\mathfrak{f}][\mathfrak{g}]\end{aligned}$$

❑

2.6 **Observation:** (i) p is exact via $[p:=\top]$. (ii). $(p\to A)$ is exact via $[p:=p\wedge A]$.

Proof: (i) is trivial. We prove (ii). Without loss of generality we may assume that p does not occur in A, since $\vdash(p\to A)\leftrightarrow(p\to A[p:=\top])$ and $\vdash(p\wedge A)\leftrightarrow(p\wedge A[p:=\top])$. We have:

$$\begin{aligned}\vdash (p\to A)\to C \;&\Leftrightarrow \vdash (p\leftrightarrow(p\wedge A)) \to C\\ &\Leftrightarrow \vdash C[p:=(p\wedge A)].\end{aligned}$$ ❑

We say that a formula is *confined* if it is a conjunction of formulas of the form $p\to B$. A formula is *strictly confined* if it is confined and if for any two distinct conjuncts the antecedent variables are different. (We consider $\top$ as the empty conjunction, so $\top$ is strictly confined).

2.7 **Corollary:** Any confined formula is exact.

Proof: Suppose A is confined. First rewrite A to a strictly confined formula A' by merging different conjuncts $p\to B$ and $p\to C$ to $p\to(B\wedge C)$. Suppose A' is of the form $(p\to D)\wedge E$. This formula is equivalent to $A'':=((p\to D[p:=\top])\wedge E)$. According to 2.5, 2.6 A'' is exact if $A^*:=E[p:=(p\wedge D[p:=\top])]$ is. Clearly A^* is again a strictly confined formula with less conjuncts than A'. Repeat the procedure till all conjuncts are eliminated and we end up with $\top$. $\top$ is exact by the identity substitution. ❑

Note that it follows that confined formulas are prime.

2.8 **Theorem:** Every prime NNIL-formula is exact.

2.8.1 **Lemma:** Suppose p does not occur in A. Then A is prime if $(p\wedge A)$ is.

Proof: Suppose $(p\wedge A)$ is prime. Let X be a finite set of formulas and suppose $\vdash A\to\bigvee X$. Without loss of generality we may assume that p does not occur in X. It follows that $\vdash(p\wedge A)\to\bigvee X$ and hence $\vdash(p\wedge A)\to B$ for some $B\in X$. By substituting $\top$ for p we find: $\vdash A\to B$. ❑

Proof of 2.8: Let A be a NNIL-formula. We will reduce A to a formula A_0. The formula A_0 satisfies one of the following properties: (i) A_0 is confined or (ii) A_0 is a prime NNIL-formula and has strictly less propositional variables than A. Moreover we have: if A_0 is exact, then A is exact. In the first case we are done, in the second case we repeat the procedure.

Step 1: We first remove $\top$ and $\perp$ from A by the obvious procedure. This only fails when we end up with either $\top$ or $\perp$. We cannot end up with $\perp$, since A was supposed to be prime and hence non-refutable. If we end up with $\top$, then A is exact by the identity substitution. If we do not end up with $\top$ go on to step 2.

Step 2: Write A in disjunctive normal form (treating the implications as atoms). Since A is prime, it is equivalent with one of its disjuncts, say A'. A' is a conjunction of atoms and implications. If the number of atoms is zero go on to step 3. Otherwise write A' in the form $p \wedge C$. Clearly $p \wedge C$ is equivalent to $p \wedge (C[p:=\top])$. Put $A_0 := C[p:=\top]$. Note that A_0 is again prime by 2.8.1 and that A is exact if A_0 is (by 2.5, 2.6).

Step 3: A' is a conjunction of implications. Reduce subformulas of the form $(B \wedge C) \to D$ to $(B \to (C \to D))$ and subformulas of the form $(B \vee C) \to D$ to $(B \to D) \wedge (C \to D)$. Repeat the procedure till no such subformulas are left. Let A_0 be the result. Since A' was in NNIL, clearly A_0 is confined. ❑

3 Some useful facts about IPC and HA

In this section we provide some technical preliminaries to the result of section 5.

We suppose the reader is familiar with Kripke models for IPC (see Troelstra & van Dalen[88a], or Smorynski[73]). To fix notations: a *Kripke model* is a structure $\mathbb{K} = \langle K, \leq, \models \rangle$, where K is a non-empty set of nodes, $\leq$ is a partial ordering, $\models$ is the atomic forcing relation: it is a relation between nodes and propositional atoms, satisfying: $k \leq k'$ and $k \models p \Rightarrow k' \models p$. The relation $\models$ can be extended to the full language of IPC in the standard way. We write $\mathbb{K} \models A$ for: $\forall k \in K\ k \models A$. A *rooted* Kripke model $\mathbb{K}$ is a structure $\langle K, k_0, \leq, \models \rangle$, where $\langle K, \leq, \models \rangle$ is a Kripke model and where $k_0 \in K$ is the bottom element w.r.t. $\leq$. For any $k \in K$ $\mathbb{K}[k]$ is the model $\langle K', k, \leq', \models' \rangle$, where $K' := \{k' \mid k \leq k'\}$ and where $\leq'$ and $\models'$ are the restrictions of $\leq$ respectively $\models$ to K'. (We will often simply write $\leq$ and $\models$ for $\leq'$ and $\models'$.)

3.1 The Henkin construction: A set X is *adequate* if it is finite, closed under subformulas and contains $\bot$. A set Γ is X-saturated if:

(i) $\Gamma \subseteq X$, (ii) $\Gamma \nvdash \bot$, (iii) $\Gamma \vdash A$, $A \in X \Rightarrow A \in \Gamma$,

(iv) $\Gamma \vdash (B \vee C)$, $(B \vee C) \in X \Rightarrow B \in \Gamma$ or $C \in \Gamma$.

The Henkin model $\mathbb{H}_X$ has as nodes the X-saturated sets and as accessibily relation $\subseteq$. The atomic forcing in the nodes is given by: $\Gamma \vDash p \Leftrightarrow p \in \Gamma$. We have by a standard argument: for $A \in X$: $\Gamma \vDash A \Leftrightarrow A \in \Gamma$.

3.2 Definitions

i) Let $\mathbf{K}$ be a set of Kripke models. $M(\mathbf{K})$ is the model with nodes $\langle k, \mathbb{K} \rangle$ for $k \in \mathbb{K} \in \mathbf{K}$ and ordering: $\langle k, \mathbb{K} \rangle \leq \langle m, \mathbb{M} \rangle :\Leftrightarrow \mathbb{K} = \mathbb{M}$ and $k \leq_{\mathbb{K}} m$. As atomic forcing we take:

- $\langle k, \mathbb{K} \rangle \vDash p :\Leftrightarrow k \vDash_{\mathbb{K}} p$.

(In practice we will forget the second components of the new nodes, pretending the domains to be disjoint already.)

ii) Let $\mathbb{K}$ be a Kripke model. $B(\mathbb{K})$ is the rooted model obtained by adding a new bottom $\mathfrak{b}$ to $\mathbb{K}$ and by taking: $\mathfrak{b} \vDash p :\Leftrightarrow \mathbb{K} \vDash p$. We write $\mathrm{Glue}(\mathbf{K}) := BM(\mathbf{K})$.

3.3 Push Down Lemma: Let X be adequate. Suppose Δ is X-saturated and $\mathbb{K} \vDash \Delta$. Then $\mathrm{Glue}(\mathbb{H}_X[\Delta], \mathbb{K}) \vDash \Delta$.

Proof: We show by induction on $A \in X$ that $\mathfrak{b} \vDash A \Leftrightarrow A \in \Delta$. The cases of atoms, conjunction and disjunction are trivial. If $(B \to C) \in X$ and $\mathfrak{b} \vDash (B \to C)$, then $\Delta \vDash (B \to C)$ and hence $(B \to C) \in \Delta$. Conversely suppose $(B \to C) \in \Delta$. If $\mathfrak{b} \nvDash B$, we are easily done. If $\mathfrak{b} \vDash B$, then $B \in \Delta$, hence $C \in \Delta$ and by the Induction Hypothesis: $\mathfrak{b} \vDash C$. ❑

We say that Δ is *prime* if it is consistent and:

for every $(C \vee D) \in \mathfrak{L}$: $\Delta \vdash (C \vee D) \Rightarrow \Delta \vdash C$ or $\Delta \vdash D$.

3.4 Theorem: Suppose X is adequate and Δ is X-saturated. then Δ is prime.

Proof: Δ is consistent by definition. Suppose $\Delta \vdash C \vee D$ and $\Delta \nvdash C$ and $\Delta \nvdash D$. Suppose $\mathbb{K} \vDash \Delta$, $\mathbb{K} \nvDash C$, $\mathbb{M} \vDash \Delta$ and $\mathbb{M} \nvDash D$. Consider $\mathrm{Glue}(\mathbb{H}_X(\Delta), \mathbb{K}, \mathbb{M})$. By 3.3 we have: $\mathfrak{b} \vDash \Delta$. On the other hand by persistence: $\mathfrak{b} \nvDash C$ and $\mathfrak{b} \nvDash D$. Contradiction. ❑

3.5 **A big model:** Construct a Henkin model by taking as nodes $\langle\Gamma,X\rangle$, where X is adequate and Γ is X-saturated. Take $\langle\Gamma,X\rangle \leq \langle\Delta,Y\rangle :\Leftrightarrow \Gamma\subseteq\Delta$ and $X\subseteq Y$. Also: $\langle\Gamma,X\rangle\models p :\Leftrightarrow p\in\Gamma$. Then for all $A\in\mathfrak{L}$: $\langle\Gamma,X\rangle\models A \Leftrightarrow \Gamma\vdash A$. The proof, which uses 3.4, is left to the industrious reader. ❍

3.6 **Formalization in HA:** We first formalize Kripke completeness for finite models in Peano Arithmetic (PA). Noting that the model existence theorem yields a multi-exponential bound E on the size of the Henkin model we formulate the result as follows: $PA \vdash \forall A((\forall\mathbb{K}\leq E(A)\ \mathbb{K}\models A) \rightarrow IPC\vdash A)$. Noting that the formula proved is Π_2, we see that by a theorem due to Kreisel: $HA\vdash\forall A((\forall\mathbb{K}\leq E(A)\ \mathbb{K}\models A) \rightarrow IPC\vdash A)$. Since the converse is readily verifiable in HA we find:

$$HA\vdash\forall A((\forall\mathbb{K}\leq E(A)\ \mathbb{K}\models A) \leftrightarrow IPC\vdash A).$$

So IPC-provability is decidable in HA.

In intuitionistic theories even subsets of the singleton set are not decidable. We, however, assume that the finite sets that we are using, e.g. in the construction of the Henkin model, are *coded as numbers* and hence provably finite and decidable. Under this convention whether a finite set is X-saturated or not becomes decidable, given the decidability of IPC-provability.

We leave it to the reader to verify 3.3 and 3.4 in HA (assuming **K** to be a finite set of finite models, etc.). Note that the reductio reasoning in 3.4 is harmless because of decidability. ❍

3.7 **Theorem:** Let X be a prime, RE set of IPC-formulas, closed under IPC-consequence. Without loss of generality we may assume that X is given by a recursive increasing sequence of finite approximations X_i. We assume that $X_0=\emptyset$. Say, this sequence is presented by the Δ_1-formula $\xi(i,x)$. Then we can represent X by a sequence $\langle Y_i,Z_i\rangle$, where:

i) $X=\bigcup Y_i$,

ii) $i<j \Rightarrow (Y_i\subseteq Y_j$ and $Z_i\subseteq Z_j)$

iii) Z_i is adequate

iv) Y_i is Z_i-saturated

Our sequence can be represented by a Δ_1-formula $\sigma(i,y,z)$ such that HA verifies the functionality of the sequence, plus (ii), (iii), (iv). Let Y be given as:

$Y := \{B \mid \exists i,y,z\, (\sigma(i,y,z) \wedge B \in y)\}$.

It follows by 3.4-3.6 that HA verifies that:

Y is prime, that $Y \subseteq X$ and that ((X is prime) $\to$ Y=X).

In fact we will only use in this paper that ((X is prime) $\to$ Y=X) is *classically true*, not that HA verifies this fact.

Fix an increasing sequence U_i of adequate sets such that for every A we can effectively find an i such that $Sub(A) \subseteq U_i$. Before proving 3.7, we provide a modest lemma. Define:

- $Sat(n,m,k) :\Leftrightarrow$ for all $(B \vee C) \in U_n$ $(X_m \vdash (B \vee C) \Rightarrow (X_k \vdash B$ or $X_k \vdash C))$.

Note that HA proves that Sat is decidable.

3.7.1 Lemma

i) HA $\vdash$ (X is prime) $\to \forall n,m \exists k\, Sat(n,m,k)$.

ii) HA $\vdash$ (X is prime) $\to \forall n,m \exists k\, (k \geq m \wedge Sat(n,k,k))$.

Proof of the Lemma: We reason informally in HA.

i) Since U_n is finite we can exhaustively enumerate the U_n-disjunctions $E \vee F$ proved by X_m. Since X is prime we can find for any such $E \vee F$ an i such that $X_i \vdash E$ or $X_i \vdash F$. By the collection pronciple (which is provable in HA) we can find an upper bound k of these i's.

ii) Define F as follows:

F(p) := the smallest $q \geq p$ such that Sat(n,p,q).

By (i) F is recursive. Let $N := |U_n|$, Consider the sequence $\langle m, F(m), \ldots, F^{N+2}(m) \rangle$. If this sequence were strictly increasing, there would be N+1 different disjunctions in U_n. This is impossible by the Pigeon Hole Principle for recursive injections and decidable finite sets, which is verifiable in HA. Since everything in sight is decidable, we may conclude that there is a k with $0 \leq k < N+2$ and Sat(n,k,k). ❑(Lemma)

Proof of 3.7: We reason informally in HA. Remember that our sets are really finite, decidable sets represented by numbers. We define weakly monotonic functions $f,g:\omega\rightarrow\omega$ and take $Z_i:=U_{fi}$ and $Y_i:=\{B\in Z_i | X_{gi}\vdash B\}$.

- $f0:=0$, $g0:=0$
- Consider U_{fn+1}. In case $\{B\in U_{fn+1} | X_{n+1}\vdash B\}$ is U_{fn+1}-saturated (i.o.w. if Sat(fn+1,n+1,n+1)), put f(n+1):=fn+1, g(n+1):=n+1. Otherwise f(n+1):=fn, g(n+1):=gn.

f and g are recursive, since, by 3.6, IPC-provability is (verifiably) decidable. By the formalization of 3.4, every Y_i is prime (in case i=0, this uses the fact that IPC is prime).

We prove ((X is prime) $\rightarrow$ Y=X). Suppose X is prime. We will show that both f and g tend to infinity (and hence Y=X). Consider any n. Let k be the smallest number such that $k\geq n+1$ and Sat(fn+1,k,k). Then evidently the first clause of the recursion step of the definitions of f and g will be activated at k. Hence for every n there is a k>n such that f and g increase at k. By a simple induction it follows that f and g tend to infinity.

❑

4 What is HA*?

In this section we describe the theory HA*. This theory was introduced in Visser[82]. HA* is to Beeson's fp-realizability (Beeson[75]) as Troelstra's HA+ECT_0 is to Kleene's r-realizability. This means that for a suitable variant of fp-realizability HA* is the set of sentences such that their fp-translations are provable in HA. The natural way to define HA* is by a fixed point construction as: HA plus the *Completeness Principle for HA**. (Here it is essential that the construction is verifiable in HA, see below.) The Completeness Principle can be viewed as an arithmetically interpreted modal principle. The Completeness Principle viewed modally is:

C $\quad\vdash A\rightarrow\Box A$

The Completeness Principle for a specific theory T is:

C[T] $\quad\vdash A\rightarrow\Box_T A$.

Here $\Box_T$ stands for the formalization of provability in T. In the statement of the principle the syntactical variable 'A' ranges over formulas. Free occurrences of variables inside the box are interpreted according to the following convention: $\Box_T A$ means $Prov_T(t(\mathbf{x}))$, where $t(\mathbf{x})$ is the term 'the Gödelnumber of the result of substituting the Gödelnumbers of the numerals of the **x**'s for the variables in A'.

We have:

$$HA^* = HA + C[HA^*].$$

We briefly review some of the results of Visser[82].

- Let $\mathfrak{A}$ be the smallest class closed under atoms and all connectives except implication, satisfying: $A\in\Sigma_1$ and $B\in\mathfrak{A} \Rightarrow (A\rightarrow B)\in\mathfrak{A}$. Note that modulo provable equivalence in HA all formulas of the classical arithmetical hierarchy in their standard form are in $\mathfrak{A}$. HA^* is conservative w.r.t. $\mathfrak{A}$ over HA.
- There are infinitely many incomparable T with $T=HA+C_T$. However if $T=HA+C_T$ *verifiably in HA*, then $T=HA^*$.
- Let KLS:=Kreisel-Lacombe-Shoenfield's Theorem on the continuity of the effective operations. We have: $HA^*\vdash$ KLS $\rightarrow \Box_{HA^*}\bot$. This immediately gives Beeson's result that HA$\nvdash$KLS (see Beeson[75]).

Consider the Löb conditions.

L1 $\quad\vdash A \Rightarrow\ \vdash\Box A$

L2 $\quad\vdash \Box(A\rightarrow B)\rightarrow(\Box A\rightarrow\Box B)$

L3 $\quad\vdash \Box A\rightarrow\Box\Box A$

L4 $\quad\vdash \Box(\Box A\rightarrow A)\rightarrow\Box A$

i-K is given by IPC+L1,L2. i-L is i-K+L3,L4. We write i-K{P} for the extension of i-K with some principle P. Note that i-L{C} is valid for provability interpretations in HA^*.

A principle closely connected to C is the Strong Löb Principle:

SL $\quad\vdash (\Box A\rightarrow A)\rightarrow A$

As a special case of SL we have: $\vdash\neg\neg\Box\bot$.

4.2 **Fact:** i-L{C} is interderivable with i-K{SL}.

Proof: L4 is immediate from SL. "i-K{SL}$\vdash$C":

$$\vdash A \rightarrow (\Box(A\wedge\Box A)\rightarrow(A\wedge\Box A))$$
$$\rightarrow A\wedge\Box A$$

$\rightarrow \Box A.$

"i-L{C}⊢SL":

$$\vdash (\Box A \rightarrow A) \rightarrow (\Box A \rightarrow A) \wedge \Box(\Box A \rightarrow A)$$
$$\rightarrow (\Box A \rightarrow A) \wedge \Box A$$
$$\rightarrow A. \qquad \square$$

i-L{C} is a kind of Kindergarten theory in which all the well-known syntactical results of Provability Logic have extremely simple versions. We add the proofs for completeness. 4.3-4.6 are not essential for the rest of the paper.

4.3 Substitution Lemma: In i-L{C} we have a very powerful substitution principle:

$S^{++} \quad \vdash (A \leftrightarrow B) \rightarrow (CA \leftrightarrow CB)$

Proof: By a simple induction on C. $\square$

We say that p occurs only modalized in A if all occurrences of p are in the scope of $\Box$.

4.4 Uniqueness of Fixed Points in i-L{C}: Suppose p occurs only modalized in Ap and q does not occur in Ap. We have in i-L{C}:

$$\vdash (p \leftrightarrow Ap) \wedge (q \leftrightarrow Aq) \rightarrow (\Box(p \leftrightarrow q) \rightarrow (Ap \leftrightarrow Aq)$$
$$\rightarrow (p \leftrightarrow q))$$
$$\rightarrow (p \leftrightarrow q). \qquad \square$$

4.5 Explicit Fixed Points in i-L{C}: Suppose p occurs only modalized in Ap. We show that Ap has fixed point $A\top$. We have:

$$\vdash A\top \rightarrow (A\top \leftrightarrow \top)$$
$$\rightarrow AA\top.$$
$$\vdash AA\top \rightarrow (\Box A\top \rightarrow \Box(A\top \leftrightarrow \top)$$
$$\rightarrow A\top)$$
$$\rightarrow A\top. \qquad \square$$

A formula of the modal language is closed if it contains no propositional variables. We define: $\Box^0\bot := \bot$, $\Box^{n+1}\bot := \Box\Box^n\bot$, $\Box^\omega\bot := \top$.

4.6 The Closed Fragment of i-L{C}: Every closed formula A is i-L{C}-provably equivalent to a formula of the form $\Box^{\alpha}\bot$ for $\alpha\leq\omega$.

Proof: The proof is by induction on A. We have:

$\vdash \top \leftrightarrow \Box^{\omega}\bot, \vdash \bot \leftrightarrow \Box^{0}\bot$

$\vdash \Box^{\alpha}\bot\wedge\Box^{\beta}\bot \leftrightarrow \Box^{\min(\alpha,\beta)}\bot$

$\vdash \Box^{\alpha}\bot\vee\Box^{\beta}\bot \leftrightarrow \Box^{\max(\alpha,\beta)}\bot$

$\vdash \Box^{\alpha}\bot\rightarrow\Box^{\beta}\bot \leftrightarrow \Box^{\alpha\rightarrow\beta}\bot$, where $(\alpha\rightarrow\beta){:=}\top$ if $\alpha\leq\beta$ and $(\alpha\rightarrow\beta){:=}\beta$ if $\beta<\alpha$.

Note that $\min(\alpha,\gamma)\leq\beta \Leftrightarrow \gamma\leq(\alpha\rightarrow\beta)$.

$\vdash \Box\Box^{\alpha}\bot \leftrightarrow \Box^{1+\alpha}\bot$ ❑

4.7 Open Problems

i) Is i-L{C} the provability logic of HA^*?

ii) Prove or refute: $HA\vdash\neg\neg KLS$.

5 A Shavrukov Style Embedding Theorem for HA^*

Shavrukov proved that every RE Diagonalizable algebra satisfying an appropriate Disjunction Property is embeddable in the Diagonalizable algebra of Peano Arithmetic. It is clear from section 3 that there is no analogous result for *Ha*'s and Heyting Arithmetic. In this section we show that an analogue can be obtained for the theory HA^*.

5.1 Theorem: Any RE prime *Ha* $\mathfrak{H}$ can be embedded in $\mathfrak{H}_{HA^*}$. Moreover the equivalence classes in the range of the embedding all contain a Σ_1-sentence.

Before proving the theorem we briefly look at an illustrative example to give the reader some feeling of how it is possible that an embedded algebra can completely consist of equivalence classes of Σ_1-sentences.

5.1.1 Example: Consider the algebra $\mathfrak{H}$, IPC-axiomatized by $\neg\neg p\rightarrow p$. To be precise: $\mathfrak{H}=\mathfrak{H}_{IPC+(\neg\neg p\rightarrow p)}(\mathfrak{L}_1)$. We have:

$\mathfrak{H}$ can be embedded into $\mathfrak{H}_{IPC}$ by e.g. $[p := \neg p]$;

$\mathfrak{H}$ can be embedded into $\mathfrak{H}_{HA}$ by e.g. $[p := \neg\Box_{HA}\bot]$.

On the other hand $\mathfrak{H}$ cannot be embedded into $\mathfrak{H}_{HA}$ by sending p to a Σ_1-sentence, since for any Σ_1-sentence B, we have:

$$\begin{aligned} HA \vdash \neg\neg B \to B &\Rightarrow HA \vdash B \vee \neg B \\ &\Rightarrow HA \vdash B \text{ or } HA \vdash \neg B. \end{aligned}$$

(The first implication is proved by applying the Friedman translation for $\neg B$ to $(\neg\neg B \to B)$. See e.g. Visser[85].)

We turn to HA^*. Let R be the ordinary Σ_1 Rosser sentence for HA^*. I.e.:

$R \leftrightarrow \Box_{HA^*}\neg R \leq \Box_{HA^*}R$.

Let $S := \Box_{HA^*}R < \Box_{HA^*}\neg R$. We have by the ordinary Rosser property:

$HA^* \nvdash R$ and $HA^* \nvdash S$.

On the other hand we have:

$HA^* \vdash \neg R \leftrightarrow S$ and and $HA^* \vdash \neg S \leftrightarrow R$.

We prove the first equivalence. "$\leftarrow$" Trivially $HA^* \vdash S \to \neg R$. "$\to$" Reason in HA^*. Suppose $\neg R$ and $\Box_{HA^*}S$. It follows from the second assumption that $\Box_{HA^*}\neg R$ and hence that $R \vee S$. Combining $R \vee S$ with our first assumption, we get: S. By SL we may drop the assumption $\Box_{HA^*}S$.

Using the above facts it is easy to see that the subalgebra of $\mathfrak{H}_{HA^*}$ generated by R is given by the non-equivalent Σ_1-sentences: $\bot$, R, S, $\Box_{HA^*}\bot$, $\top$. This algebra is clearly isomorphic to $\mathfrak{H}$. ❍

In 5.6.1 we show that it is definitely *not* the case in HA^*, that the Σ_1-sentences are closed under the Boolean operations (modulo provable equivalence).

Proof of 5.1: Let $\Box$ stand for $\Box_{HA^*}$ and Proof for Proof_{HA^*}.

Consider the following Kripke model $\mathbb{H}$, which is a variant of the Big Model of 3.5. Its nodes are of the form $\langle i,U,V\rangle$, where:

- $i \in \{0,1\}$
- V is an adequate set of formulas,
- $U \subseteq V$, U is V-saturated.

Define $\preccurlyeq$ and $\models$ as follows:

- $\langle i,U,V\rangle \preccurlyeq \langle j,W,T\rangle :\Leftrightarrow i \leq j$ and $U \subseteq W$ and $V \subseteq T$ and $(i=1 \Rightarrow V=T)$
- $\langle i,U,V\rangle \models p :\Leftrightarrow p \in U$.

Using 3.3 and 3.6 it is easy to see (in HA) that:

for any formula A: $\langle 0,U,V\rangle \models A \Leftrightarrow U\vdash A$,

for $A\in V$: $\langle 1,U,V\rangle \models A \Leftrightarrow U\vdash A$.

Note that it follows that the relation $k\models A$ is decidable.

Let $\langle Y_i,Z_i\rangle$ be an enumeration of a propositional theory presenting $\mathfrak{H}$, satisfying the properties promised in 3.7. We define a Solovay function $\mathfrak{h}$ from ω to the nodes of $\mathbb{H}$. $\mathfrak{s}x$, the *state* of $\mathfrak{h}$ at x, is defined as $(\mathfrak{h}x)_0$. $\mathfrak{s}0$ will be set at 0. Till a certain catastrophic Event happens, the state will remain 0 and $\mathfrak{h}$ will run upward through nodes $\langle 0,Y_i,Z_i\rangle$. As soon as (and *if*) the Event happens, the state will definitively move to 1 and our function runs upwards through nodes of the form $\langle 1,U,V\rangle$. Define by the Recursion Theorem $\mathfrak{h}$ as follows:

- $[A] :\leftrightarrow \exists x\ \mathfrak{h}x\models A$
- $\mathfrak{h}0 := \langle 0,Y_0,Z_0\rangle$
- $\mathfrak{h}(n+1) := k$ if $(*)$ Proof(n,[A]), $\mathfrak{h}n\not\models A$,
  k is a 1-node, $\mathfrak{h}n\preccurlyeq k$, k maximal such that $k\not\models A$
- $\mathfrak{h}(n+1) := \langle 0,Y_{n+1},Z_{n+1}\rangle$ if case $(*)$ does not obtain and $\mathfrak{s}n=0$
- $\mathfrak{h}(n+1) := \mathfrak{h}n$ if case $(*)$ does not obtain and $\mathfrak{s}n=1$.

Since $\models$ is (provably in HA) decidable, it follows that $\mathfrak{h}$ is a well defined recursive function.

Note that the catastrophic Event is the first time that $(*)$ obtains. Before the Event the function enumerates nodes representing better and better approximations of $\mathfrak{H}$. After the event it behaves like an ordinary Solovay function traveling upwards through a converse wellfounded (w.r.t. $<$) part of the model.

5.2 Lemma

$HA\vdash x\leq y \rightarrow \mathfrak{h}x\preccurlyeq\mathfrak{h}y$

$HA\vdash (x\leq y \wedge \mathfrak{h}x\models A) \rightarrow \mathfrak{h}y\models A$

Proof: Obvious. ❑

5.3 **Lemma:** HA$\vdash$ $\mathfrak{l}x\neq 0 \rightarrow \Box\exists y\ \mathfrak{h}x<\mathfrak{h}y$.

Proof: Reason in HA. Suppose $\mathfrak{l}x\neq 0$. $\mathfrak{h}$ must have arrived at $\mathfrak{h}x$ by case (*). So for some A and for some p<x: $\mathrm{Proof}_{HA^*}(p,[A])$, $\mathfrak{h}(p+1)=\mathfrak{h}x\nvDash A$. By Σ-completeness we have: $\Box\mathfrak{h}x\nvDash A$. Combining this with $\Box\exists y\ \mathfrak{h}y\vDash A$, we obtain, using 5.2, the desired result. ❑

5.4 **Lemma:** $\mathfrak{l}n=0$ for any n.

Proof: Suppose $\mathfrak{l}n\neq 0$. By 5.3: $\Box\exists y\ \mathfrak{h}n<\mathfrak{h}y$. Remember that HA^* is Π_2-conservative over HA. Thus HA^* will certainly satisfy Σ-reflection. It follows that for some m: $\mathfrak{h}n<\mathfrak{h}m$. Repeating the argument we can construct an infinite strictly ascending chain above $\mathfrak{h}n$. This contradicts $\mathfrak{l}n\neq 0$. ❑

5.5 **Lemma:** [.] commutes modulo HA^*-provability with the propositional connectives.

Proof: Reason in HA^*. Clearly $[\bot]\leftrightarrow\bot$ and $[\top]\leftrightarrow\top$.

Suppose $[A\wedge B]$, then for some x: $\mathfrak{h}x\vDash A\wedge B$. It follows that $\mathfrak{h}x\vDash A$ and $\mathfrak{h}x\vDash B$ and hence $[A]\wedge[B]$. Conversely suppose $[A]\wedge[B]$. Say $\mathfrak{h}y\vDash A$ and $\mathfrak{h}z\vDash B$. Let u:=max(y,z), then by 5.2: $\mathfrak{h}u\vDash A$ and $\mathfrak{h}u\vDash B$ and thus $\mathfrak{h}u\vDash A\wedge B$. We may conclude: $[A\wedge B]$.

Suppose $[A\vee B]$, then for some x: $\mathfrak{h}x\vDash A\vee B$. It follows that $\mathfrak{h}x\vDash A$ or $\mathfrak{h}x\vDash B$ and hence $[A]\vee[B]$. Conversely suppose $[A]\vee[B]$. Suppose e.g. $\mathfrak{h}y\vDash A$. It is immediate that also $\mathfrak{h}y\vDash A\vee B$ and so $[A\vee B]$. Similarly in case $\mathfrak{h}z\vDash B$.

Suppose $[A\rightarrow B]$ and $[A]$. Then for some x and y: $\mathfrak{h}x\vDash A\rightarrow B$ and $\mathfrak{h}y\vDash A$. Take u:=max(x,y). Clearly $\mathfrak{h}u\vDash A\rightarrow B$ and $\mathfrak{h}u\vDash A$. Ergo: $\mathfrak{h}u\vDash B$ and thus $[B]$. Conversely suppose $[A]\rightarrow[B]$. We show $[A\rightarrow B]$ using the SL. So we may also assume $\Box[A\rightarrow B]$. Suppose $\mathrm{Proof}(p,[A\rightarrow B])$. In case $\mathfrak{h}(p)\vDash(A\rightarrow B)$ we have $[A\rightarrow B]$. Suppose $\mathfrak{h}(p)\nvDash(A\rightarrow B)$. In this case $\mathfrak{h}(p+1)$ is a maximal $k\geqslant\mathfrak{h}p$ such that $k\nvDash(A\rightarrow B)$. It follows that $k\vDash A$ and $k\nvDash B$. From $\mathfrak{h}(p+1)=k\vDash A$, we have: $[A]$, and hence by assumption: $[B]$. But $[B]$ imediately implies: $[A\rightarrow B]$. So in both cases we find $[A\rightarrow B]$. By the SL we may drop the assumtion that $\Box[A \to B]$. ❑

We finish our proof of 5.1, by showing that: $A \in X \Leftrightarrow HA^* \vdash [A]$.

Suppose $A \in X$. Then for some n: $A \in Y_n$. By 5.4: $\mathfrak{s}n=0$ and hence $\mathfrak{h}n = \langle 0, Y_n, Z_n \rangle$. Ergo $\mathfrak{h}n \models A$ and so $HA^* \vdash \mathfrak{h}n \models A$ and thus $HA^* \vdash [A]$. Conversely suppose $HA^* \vdash [A]$. Say m codes a proof of [A]. Suppose $Y_m \nvdash A$. Since $\mathfrak{h}m = \langle 0, Y_m, Z_m \rangle$ it follows that $\mathfrak{h}m \nvDash A$. So clause (∗) would become active and the catastrophic Event would take place. But 5.4 tells us this cannot happen at a standard stage. ❑

5.6 Remarks on the proof: (i) The present proof combines the proof strategy from Zambella[92] with an idea from Visser[85] (on how to handle implication using the SL). In fact our proof follows Zambella's quite closely modulo some inessential stylistic differences (like our use of a kind of Henkin model).

ii) The proof cannot be extended in any obvious way to give a completeness theorem for the provability logic of HA^*, since nodes of our Henkin model where we have $\Box\bot$ also satisfy Excluded Third. But, of course, HA^* does not prove Excluded Third from $\Box\bot$.

iii) An attractive alternative formulation of the proof is to take, on the one hand, as nodes of the Henkin model the more traditional pairs $\langle U,V \rangle$, but to work, on the other hand, with two accessibility relations:

- $\langle U,V \rangle \preccurlyeq_0 \langle W,T \rangle$ $:\Leftrightarrow U \subseteq W$ and $V \subseteq T$
- $\langle U,V \rangle \preccurlyeq_1 \langle W,T \rangle$ $:\Leftrightarrow U \subseteq W$ and $V = T$

Corresponding to these different accessibility relations we have forcing relations $\models_0$ and $\models_1$. We define a suitably adapted Solovay function simultaneously with an auxiliary state function. Which accessibily relation and which forcing relation is relevant, will depend on the state. We leave it to the reader to work out more details.

iv) The [A]'s are Σ_1. So our embedding is into the Σ_1-formulas modulo HA^*-provable equivalence. The surprising property of the [A]'s is that they are closed under implication (modulo HA^*-provable equivalence). It is not true in general that the Σ-sentences of HA^* are closed under implication. This is immediate from the following well-known fact (which is a simple adaptation of Kripke's result on flexible sentences to the constructive case).

5.6.1 **Fact:** Let T be any consistent extension of HA. Then there is a Σ_1-sentence Ω, such that for no Σ_1-sentence S: $T\vdash\neg\Omega\leftrightarrow S$.

Proof: Let T be a consistent extension of HA. Take Ω such that:

$$HA \vdash \Omega \leftrightarrow True_\Sigma(\epsilon S.\Box_T(\neg\Omega\leftrightarrow S)).$$

Here $True_\Sigma$ is the usual truth-predicate for Σ_1-sentences and $\epsilon S.\Box_T(\neg\Omega\leftrightarrow S)$ is the first S such that $\Box_T(\neg\Omega\leftrightarrow S)$ that we find if we run through the T-proofs. Clearly $\Omega\in\Sigma_1$. Suppose for some $S'\in\Sigma_1$: $T\vdash\neg\Omega\leftrightarrow S'$. Let S be the first such S' that we encounter, when running through the T-proofs. We have: $HA\vdash S = \epsilon S'.\Box_T(\neg\Omega\leftrightarrow S')$ and hence: $HA\vdash\Omega\leftrightarrow True_\Sigma(S)$. We may conclude that $HA\vdash\Omega\leftrightarrow S$. On the other hand $T\vdash\neg\Omega\leftrightarrow S$. Ergo $T\vdash\bot$. Quod non. ❑

v) Note that the only place, where we used HA* in an essential way, is the application of SL to handle the case of implication in the proof of Lemma 5.5. These applications all have the form:

$$(\Box_{HA^*}S\rightarrow S)\rightarrow S \qquad\qquad \text{for } S\in\Sigma_1.$$

Let Tr_Σ be the Σ_1-truth predicate. Clearly all applications of SL, that we need, follow from the single sentence:

SL_0 $\quad \forall x\in\Sigma\,((\Box_{HA^*}Tr_\Sigma(x)\rightarrow Tr_\Sigma(x))\rightarrow Tr_\Sigma(x))$.

SL_0, in its turn, follows from SL, *since SL is a scheme in which we allow free variables.* A pleasant lazy notation for SL_0 is: $\forall S\ (\Box_{HA^*}S\rightarrow S)\rightarrow S$, where the variable 'S' ranges over Σ_1-sentences. Since HA proves Π_2-conservativity of HA* over HA, SL_0 is HA-provably equivalent to: $\forall S\ (\Box_{HA}S\rightarrow S)\rightarrow S$. The complexity of SL_0 is $\forall((\Sigma_1\rightarrow\Sigma_1)\rightarrow\Sigma_1)$, which is both a subclass of $\forall(\Pi_2\rightarrow\Sigma_1)$ and of $\forall B\Sigma_1$. By the preceding considerations we find:

5.6.2 **Fact:** Every RE *Ha* can be embedded in the *Ha* of $HA+SL_0$. ❍

Using the notation of 1.3, we have:

5.7 **Corollary:** $A\vDash_{HA^*}B \Leftrightarrow IPC\vdash(A\rightarrow B)$.

Proof: '$\Leftarrow$' is trivial. '$\Rightarrow$' Suppose IPC$\nvdash$(A$\rightarrow$B). Then there is a finite rooted Kripke model $\mathbb{K}$ such that $\mathbb{K}\vDash$A and $\mathbb{K}\nvDash$B. Let $\mathfrak{H}$ be the *Ha* of upwards closed sets of $\mathbb{K}$. Obviously $\mathfrak{H}$ is finite and hence RE. Embedding $\mathfrak{H}$ into $\mathfrak{H}_{HA^*}$ gives us an interpretation $\mathfrak{f}$ such that $HA^*\vdash A[\mathfrak{f}]$ and $HA^*\nvdash B[\mathfrak{f}]$. ❑

6 On Brouwerean combinations of Σ_1-sentences in HA

In this section we study $\mathfrak{S}_{HA} := \mathfrak{H}_{HA}(B\Sigma_1)$. We show that $\mathfrak{H}_{IPC}$ can be embedded in $\mathfrak{S}_{HA}$, and that the same *Ha*'s on finitely many generators are embeddable in $\mathfrak{S}_{HA}$ and in $\mathfrak{H}_{IPC}$.

6.1 Theorem: $\mathfrak{H}_{IPC}$ is embeddable in $\mathfrak{S}_{HA}$.

Proof: Let [.] give the embedding of 5.1 of $\mathfrak{H}_{IPC}$ into $\mathfrak{H}_{HA^*}$. Let $\mathfrak{f}$ be given by: $\mathfrak{f}p := [p]$. We have:

$$\begin{aligned} IPC\vdash A &\Rightarrow HA\vdash A[\mathfrak{f}] \\ &\Rightarrow HA^*\vdash A[\mathfrak{f}] \\ &\Rightarrow HA^*\vdash [A] \\ &\Rightarrow IPC\vdash A. \end{aligned}$$

❑

Note that 6.1 is the uniform version of De Jongh's Completeness Theorem for IPC w.r.t. interpretations in HA using the result of 5 using only Σ_1-sentences in the interpretation of the propositional variables.

6.2 Theorem: Let $\mathfrak{H}$ be a *Ha* on finitely many generators. Then $\mathfrak{H}$ is embeddable in $\mathfrak{S}_{HA}$ iff $\mathfrak{H}$ is embeddable in $\mathfrak{H}_{IPC}$. It follows immediately that the IPC-derived rules for IPC are equal to the IPC-derived rules for HA w.r.t. substitutions involving only $B\Sigma_1$-sentences.

To prove 6.2 we borrow three facts from Visser[85].

6.2.1 Fact: For each IPC-fromula A, there is a formula A^* in NNIL such that:

i) All propositional variables of A^* occur in A,

ii) For all B$\in$NNIL: IPC$\vdash$B$\rightarrow$A $\Leftrightarrow$ IPC$\vdash$B$\rightarrow A^*$.

Note that 6.2.1(ii) tells us in terms of $\mathfrak{H}_{IPC}$ that $\{B \in NNIL \mid B \leq A\}$ both has and contains a supremum A^*. Thus A^* is the *greatest lower NNIL-approximant* of A.

6.2.2 Fact: Let $\mathfrak{f}$ assign Σ_1-sentences to the propositional variables. Then for any propositional A: $HA \vdash A[\mathfrak{f}] \Rightarrow HA \vdash A^*[\mathfrak{f}]$.

6.2.3 Fact: The number of NNIL-formulas in $p_1,\ldots,p_m$ modulo IPC-provable equivalence is finite.

Proof of 6.2: Let $\mathfrak{H}$ be a *Ha* on finitely many generators.

Suppose $\mathfrak{H}$ is embeddable in $\mathfrak{H}_{IPC}$. By 6.1 $\mathfrak{H}_{IPC}$ is embeddable in $\mathfrak{S}_{HA}$ and hence $\mathfrak{H}$ is embeddable in $\mathfrak{S}_{HA}$.

Suppose $\mathfrak{H}$ is embeddable in $\mathfrak{S}_{HA}$. Let the generators of $\mathfrak{H}$ be $A_1,\ldots,A_n$. These generators are in their turn Boolean combinations of Σ_1-sentences, say, $S_1,\ldots,S_m$. So $A_i = B_i(S_1,\ldots,S_m)$ for some propositional B_i. Let $\mathfrak{K}$ be the subalgebra of $\mathfrak{S}_{HA}$ generated by $S_1,\ldots,S_m$. Since $\mathfrak{H}$ is embedded in $\mathfrak{K}$ by assigning B_i to p_i, it is sufficient to show that $\mathfrak{K}$ is embeddable in $\mathfrak{H}_{IPC}$. Let C^* be the greatest lower NNIL-approximant of C promised by 6.2.1. We find by 6.2.2:

$$HA \vdash C(S_1,\ldots,S_m) \Rightarrow HA \vdash C^*(S_1,\ldots,S_m).$$

So $\mathfrak{K} \models C \Rightarrow \mathfrak{K} \models C^*$. Since the set of NNIL-formulas in $p_1,\ldots,p_m$ is finite (modulo IPC-provable equivalence) by 6.2.3, there are only finitely many possible C^*. Let C^+ be the conjunction of the C^*. We find for D in $p_1,\ldots,p_m$, $\mathfrak{K} \models D \Leftrightarrow IPC \vdash C^+ \to D$. Clearly C^+ is a prime NNIL-formula. By 2.8 C^+ is exact. Ergo $\mathfrak{K}$ is embeddable in $\mathfrak{H}_{IPC}$. ❑

6.3 Open question: Is $\mathfrak{S}_{HA}$ isomorphic to $\mathfrak{H}_{IPC}$? We conjecture: *no*.

7 An RE, non-recursive *Ha* on two generators

We show that there is an RE, non-recursive *Ha* on two generators. It is sufficient to produce an infinite decidable set X of IPC-formulas in p,q such that:

- for every finite $X_0 \subseteq X$ and every $A \in X/X_0$: $X_0 \nvdash A$
- every finite $X_0 \subseteq X$ has the disjunction property

The desired algebra is obtained by taking an RE and non-recursive subset Y of X as axiomatization.

In de Jongh[80] infinite sequences are produced of finite rooted Kripke models $\mathbb{L}_i$, of formulas A_i (ψ_i^* in de Jongh[80], p107) and of formulas B_i (ψ_i in de Jongh[80], p107) such that:

- $\mathbb{L}_i \models A_j \Leftrightarrow i \neq j$
- $\mathbb{L}_i \models B_j \Leftrightarrow i = j$
- A_i is of the form $B_i \rightarrow C$ for some C
- It is decidable whether a formula is of the form A_i

(This result is originally due to Jankov, see Jankov[68])

We take X to be the set of A_i. Consider a finite $X_0 \subseteq X$ and $A \in X/X_0$. Suppose $A = A_i$. Then clearly $\mathbb{L}_i \models X_0$ and $\mathbb{L}_i \not\models A_i$. Hence $X_0 \nvdash A_i$.

To prove the Disjunction Property, consider any finite $X_0 \subseteq X$. Suppose $X_0 \vdash E \vee F$, but $X_0 \nvdash E$ and $X_0 \nvdash F$. Let $\mathbb{K} \models X_0$ and $\mathbb{K} \not\models E$ and $\mathbb{M} \models X_0$ and $\mathbb{M} \not\models F$. Let j be such that A_j is not in X_0. We have: $\mathbb{L}_j \models X_0$ and $\mathbb{L}_j \not\models B_i$ for A_i in X_0. Consider Glue($\mathbb{K}$,$\mathbb{M}$,$\mathbb{L}_j$). Clearly $\mathfrak{b} \not\models E$ and $\mathfrak{b} \not\models F$. Consider any $A_i \in X_0$. $\mathfrak{b} \not\models B_i$, since $\mathbb{L}_j \not\models B_i$. Since A_i is of the form $B_i \rightarrow C$ and $\mathbb{K}$, $\mathbb{M}$ and $\mathbb{L}_j$ all force A_i, it follows that $\mathfrak{b} \models A_i$. We may conclude that $\mathfrak{b} \models X_0$, but $\mathfrak{b} \not\models E$ and $\mathfrak{b} \not\models F$. A contradiction.

We draw some obvious conclusions from the existence of our *Ha*.

7.1 **Fact:** There are Σ_1-sentences A and B such that the subalgebra of $\mathfrak{H}_{HA^*}$ generated by A and B is RE, non-recursive.

Proof: Immediate. ❑

In contrast every finitely generated *Ha* embeddable in $\mathfrak{H}_{IPC}$ is decidable and similarly, by 6.2, for $\mathfrak{S}_{HA}$.

7.2 **Theorem:** There are Σ_1-sentences A and B and a $\forall((\Sigma_1 \rightarrow \Sigma_1) \rightarrow \Sigma_1)$-sentence C such that the subalgebra of $\mathfrak{H}_{HA}$ generated by A, B and C is RE, non-recursive. By 1.3.4 it follows, that $\mathfrak{H}_{HA}$ is non-recursive.

Proof: Let A and B be as in 7.1 and take $C:=SL_0$. Let the algebra generated by A and B in $\mathfrak{H}_{HA*}$ be $\mathfrak{F}$ and let the algebra generated by A, B and C in $\mathfrak{H}_{HA}$ be $\mathfrak{G}$. For all propositional D(p,q) we have: $\mathfrak{F}\models D(A,B) \Leftrightarrow \mathfrak{G}\models(C\to D(A,B))$. So if $\mathfrak{G}$ were recursive, then so would $\mathfrak{F}$. Quod non. ❑

7.3 Open Questions

i) Are there sentences A and B, such that the subalgebra of $\mathfrak{H}_{HA}$ generated by A and B is RE, non-recursive?

ii) Are there Σ_1-sentences A and B and a sentence C of complexity less than $\forall((\Sigma_1\to\Sigma_1)\to\Sigma_1)$ (e.g. Π_2), such that the subalgebra of $\mathfrak{H}_{HA}$ generated by A, B and C is RE, non-recursive?

iii) Is there a *finitely axiomatizable Ha* on finitely many generators, which is a subalgebra of $\mathfrak{H}_{HA}$ and not of $\mathfrak{H}_{IPC}$?

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