Department of Philosophy - Utrecht University

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M.D. Jumelet

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Department of Philosophy
Utrecht University
Heidelberglaan 8
3584 CS Utrecht
The Netherlands

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Strengthening Parikh's theorem for weak Systems of Arithmetic.

Marc Jumelet
Department of Philosophy,
Utrecht University
Heidelberglaan 8
3584 CS Utrecht
Marc.Jumelet@phil.ruu.nl

§1. Introduction. One of the basic results in the studies of weak systems of arithmetic is Parikh's theorem. In this paper we will be interested in an extension of this theorem and, most of all, in further applications of the model theory that is used to prove it. We will first restate Parikh's theorem for $I\Delta_0$: if $\theta(\bar{x}, \bar{y})$ is a Δ_0-formula and it is the case that $I\Delta_0 \vdash \forall \bar{x} \exists \bar{y}\, \theta(\bar{x}, \bar{y})$, then, for some term $t(\bar{x})$ in the language of $I\Delta_0$ we will have: $I\Delta_0 \vdash \forall \bar{x} \exists \bar{y} < t(\bar{x})\, \theta(\bar{x}, \bar{y})$. The original proof can be found in (2, Thms. 4.3, 4.4).

We can prove this theorem by means of a rather simple application of the compactness theorem. The main application of this theorem is in the proof of the fact that a weak theory of arithmetic like $I\Delta_0$ can not prove the totality of functions that have exponential growth. As it stands however, we can use the same kind of technique to prove similar results for Π_3-statements. We will prove these results for $I\Delta_0$, but they are easily seen to hold for other (weak) theories as well. We fix as our language the set {0, S, +, .} of non-logical symbols and stipulate that $I\Delta_0$ will be the theory obtained from Peano's arithmetic by restricting the induction scheme to Δ_0-formulae.

§2. Result. We will formulate a theorem for Π_3-statements that follow from $I\Delta_0$.

THEOREM 2.1. *Let* $A(x, y, z)$ *be a* Δ_0*-formula in the language we fixed above. Suppose* $I\Delta_0 \vdash \forall x \exists y \forall z A(x, y, z)$. *Then a sequence of terms* $t_1(v_1), t_2(v_1, v_2), \ldots, t_k(v_1, \ldots, v_k)$ *in the language* $\mathcal{L}_{I\Delta_0}$ *exists such that:*

$$I\Delta_0 \vdash \forall x_1 \exists y_1 < t_1(x_1) \forall x_2 \exists y_2 < t_2(x_1, x_2) \ldots \exists y_k < t_k(x_1, \ldots, x_k) \forall x_{k+1} [A(x_1, y_1, x_2) \vee \ldots \vee A(x_1, y_k, x_{k+1})].$$

REMARKS 2.2. It is easily seen that, by substituting in the above theorem for the formula $A(x, y, z)$ the formula $B(x, y) \wedge (z = z)$, we get Parikh's theorem back as it was formulated for Π_2-consequences of $I\Delta_0$. On the other hand, from

$$\forall x_1 \exists y_1 < t_1(x_1) \forall x_2 \exists y_2 < t_2(x_1, x_2) \ldots \exists y_k < t_k(x_1, \ldots, x_k) \forall x_{k+1} [A(x_1, y_1, x_2) \vee \ldots \vee A(x_1, y_k, x_{k+1})]$$

we can infer again $\forall x\exists y\forall z A(x, y, z)$, although the former statement may well be situated higher up in the arithmetical hierarchy. It has to be noted at this point that we lack the scheme of Σ-collection in $I\Delta_0$. Otherwise, in the presence of this scheme, the formula above would be equivalent to a Π_1-statement.

This result is a weak analogue of a similar result in the context of bounded arithmetic (cf. 1, Ch. V, Thm. 4.29).

PROOF. Assume that $I\Delta_0 \vdash \forall x\exists y\forall z A(x, y, z)$ for some Δ_0-predicate $A(x, y, z)$. Suppose, for a contradiction, that there is no sequence of terms $t_1(v_1), t_2(v_1, v_2), \ldots, t_k(v_1, \ldots, v_k)$ in the language $\mathcal{L}_{I\Delta_0}$, such that $I\Delta_0$ proves the expression:

$$\forall x_1\exists y_1<t_1(x_1)\forall x_2\exists y_2<t_2(x_1, x_2) \ldots\exists y_k<t_k(x_1, \ldots, x_k)\, \forall x_{k+1}$$
$$[A(x_1, y_1, x_2) \vee\ldots\vee A(x_1, y_k, x_{k+1})].$$

We add a new constant a_1 to the language $\mathcal{L}_{I\Delta_0}$ of $I\Delta_0$ and claim that the following theory is consistent: $I\Delta_0+\Gamma$, where Γ is the set of sentences of the form:

$$\neg\exists y_1<t_1(a_1)\forall x_2\exists y_2<t_2(a_1, x_2) \ldots\exists y_k<t_k(a_1, \ldots, x_k)\, \forall x_{k+1}$$
$$[A(a_1, y_1, x_2) \vee\ldots\vee A(a_1, y_k, x_{k+1})],$$

for any sequence $t_1(v_1), t_2(v_1, v_2), \ldots t_k(v_1, \ldots, v_k)$ in the language $\mathcal{L}_{I\Delta_0}$. We will first prove this claim: if $I\Delta_0+\Gamma$ had been inconsistent, there would have been a derivation in $I\Delta_0$ of a sentence of the form $\neg\gamma_1 \vee\ldots\vee \neg\gamma_m$ for some sequence $\gamma_1,\ldots, \gamma_m \in \Gamma$. Hence a set of terms $\{t^1_1(v_1),\ldots, t^1_{k_1}(v_1,\ldots, v_{k_1}),\ldots, t^m_1(v_1),\ldots, t^m_{k_m}(v_1,\ldots, v_{k_m})\}$ would have existed such that:

$$I\Delta_0 \vdash (\exists y_1< t^1_1(a_1)\forall x_2\exists y_2< t^1_2(a_1, x_2)\ldots\exists y_k<t^1_{k_1}(a_1, \ldots, x_{k_1})\forall x_{k_1+1}$$
$$[A(a_1, y_1, x_2) \vee\ldots\vee A(a_1, y_{k_1}, x_{k_1+1})])\vee\ldots\vee$$
$$(\exists y_1< t^m_1(a_1)\forall x_2\exists y_2< t^m_2(a_1, x_2)\ldots\exists y_{k_m}<t^m_{k_m}(a_1, \ldots, x_{k_m})\forall x_{k_m+1}$$
$$[A(a_1, y_1, x_2) \vee\ldots\vee A(a_1, y_{k_m}, x_{k_m+1})]).$$

Now let $u_1(v_1),\ldots, u_n(v_1,\ldots, v_n)$ for $n = \max\{k_1,\ldots, k_m\}$ be a set of terms such that for all $j \leq m$, $i \leq k_j$, $I\Delta_0$ proves $\forall v_1\ldots v_i\; u_i(v_1,\ldots, v_i)>t^j_i(v_1,\ldots, v_i)$. We can infer:

$$I\Delta_0 \vdash \exists y_1< u_1(a_1)\forall x_2\exists y_2< u_2(a_1, x_2)\ldots\exists y_n<u_n(a_1, \ldots, x_n)\forall x_{n+1}$$
$$[A(a_1, y_1, x_2) \vee\ldots\vee A(a_1, y_n, x_{n+1})],$$

and thus we can conclude:

$$I\Delta_0 \vdash \forall x_1\exists y_1< u_1(x_1)\forall x_2\exists y_2< u_2(x_1\, x_2)\ldots\exists y_n<u_n(x_1 \ldots, x_n)\forall x_{n+1}$$
$$[A(x_1, y_1, x_2) \vee\ldots\vee A(x_1, y_n, x_{n+1})]),$$

contrary to our original hypothesis that there could not be such a sequence of terms $u_1(v_1),\ldots, u_n(v_1,\ldots, v_n)$.

We can now conclude that there is a (non-standard) countable model $\mathfrak{A} \models I\Delta_0+\Gamma$. With this model as a start, we will infer that there is a model $\mathfrak{B} \models I\Delta_0$, such that the expression $\forall x\exists y\forall z A(x, y, z)$ is false in $\mathfrak{B}$. To this end we will define sets of elements of the domain of $\mathfrak{A}$ in order to constitute, in a step-by-step way, the domain of our new model. For this purpose, we need the following definition.

For every sequence of terms $\bar{u} := u_1(v_1), \ldots, u_n(v_1, \ldots, v_n)$ of progressive arity of length n, the set of *$\bar{u}$- sequences* in $\mathfrak{A}$ is defined as follows:

$\langle c_1, \ldots, c_n\rangle$ is a *$u_1(v_1), \ldots, u_n(v_1, \ldots, v_n)$ -sequence* in $\mathfrak{A}$ if and only if:

i. $\langle c_1, \ldots, c_n\rangle$ is a finite sequence of elements of dom($\mathfrak{A}$);

ii. a sequence $\langle a_1, \ldots, a_{n+1}\rangle$ of elements of dom($\mathfrak{A}$) exists such for all $j \in \mathbb{N}$, $1 \leq j \leq n$, $\mathfrak{A} \models c_j < u_j(a_1, \ldots, a_j)$ and $\mathfrak{A} \models \neg A(a_1, c_j, a_{j+1}) \wedge \neg \exists x < a_{j+1} \neg A(a_1, c_j, x)$.

As it is clear that, for any sequence $\langle c_1, \ldots, c_n\rangle$ that is a $\bar{u}$- sequence in $\mathfrak{A}$ for given terms $\bar{u}$, the sequence $\langle a_1, \ldots, a_{n+1}\rangle$ that satisfies the second condition of the definition above is unique, we will refer to this latter sequence with the term *witnessing sequence* for $\langle c_1, \ldots, c_n\rangle$. Here is what we should know about $\bar{u}$- sequences in $\mathfrak{A}$.

1. There is a sequence of terms $\bar{u}$ such that $\langle a_1\rangle$ is a $\bar{u}$- sequence in $\mathfrak{A}$.

2. If $\bar{u} = u_1(v_1), \ldots, u_n(v_1, \ldots, v_n)$ is a sequence of terms, $\langle c_1, \ldots, c_n\rangle$ is a $\bar{u}$- sequence in $\mathfrak{A}$, $\langle a_1, \ldots, a_{n+1}\rangle$ a witnessing sequence for $\langle c_1, \ldots, c_n\rangle$, $t(v_1, \ldots, v_{n+1})$ a term in $\mathcal{L}_{I\Delta_0}$, d an element in the domain of $\mathfrak{A}$ such that $\mathfrak{A} \models d \leq t(a_1, \ldots, a_{n+1})$, then a sequence of terms $\bar{u}' = u_1(v_1), \ldots, u_n(v_1, \ldots, v_n)$, $u_{n+1}(v_1, \ldots, v_{n+1})$ exists such that $\langle c_1, \ldots, c_n, d\rangle$ is a $\bar{u}'$-sequence.

1. This follows by taking $\bar{u} := u_1(v_1) = v_1 + 1$. Condition i is trivially fulfilled. As to condition ii, it suffices to remark that $\mathfrak{A}$ is a model for Γ and therefore it will be the case that $\mathfrak{A} \models \neg \exists y_1 < a_1 + 1 \forall x_2 A(a_1, y_1, x_2)$. We can conclude that the set $\{z \in \mathrm{dom}(\mathfrak{A}) \mid \mathfrak{A} \models \neg A(a_1, a_1, z)\}$ is Δ_0-defined (in $\mathfrak{A}$) and non-empty, so it will have a smallest element a_2 in dom($\mathfrak{A}$). The sequence $\langle a_1, a_2\rangle$ is a witnessing sequence for $\langle a_1\rangle$.

2. Let $\langle c_1, \ldots, c_n\rangle$ be a $u_1(v_1), \ldots, u_n(v_1, \ldots, v_n)$ - sequence in $\mathfrak{A}$, with witnessing sequence $\langle a_1, \ldots, a_{n+1}\rangle$. Let $\bar{u}'$ be the sequence $u_1(v_1), \ldots, u_n(v_1, \ldots, v_n), u_{n+1}(v_1, \ldots, v_{n+1})$, where $u_{n+1}(v_1, \ldots, v_{n+1})$ is exactly $t(v_1, \ldots, v_{n+1}) + 1$. In order to show that $\langle c_1, \ldots, c_n, d\rangle$ is a $\bar{u}'$-sequence in $\mathfrak{A}$, for d an element such that $\mathfrak{A} \models d \leq t(a_1, \ldots, a_{n+1})$, it is clearly sufficient to demonstrate that there is an element b in the domain of $\mathfrak{A}$ such that $\mathfrak{A} \models \neg A(a_1, d, b)$. Again, as $\mathfrak{A}$ is a model for Γ, we have:

$$\mathfrak{A} \models \forall y_1 < u_1(a_1) \exists x_2 \forall y_2 < u_2(a_1, x_2) \ldots \forall y_n < u_n(a_1, \ldots, x_n) \exists x_{n+1} \forall y_{n+1} < u_{n+1}(a_1, \ldots, x_{n+1}) \exists x_{n+2} [\neg A(a_1, y_1, x_2) \wedge \ldots \wedge \neg A(a_1, y_n, x_{n+1}) \wedge \neg A(a_1, y_{n+1}, x_{n+2})].$$

Because $\langle c_1, \ldots, c_n\rangle$ is a $\bar{u}$- sequence, we must have:

$\mathfrak{A} \models \exists x_2 \geq a_2 \forall y_2 < u_2(a_1, x_2) \ldots \forall y_n < u_n(a_1, \ldots, x_n) \exists x_{n+1}$
$\forall y_{n+1} < u_{n+1}(a_1, \ldots, x_{n+1}) \exists x_{n+2}$
$[\neg A(a_1, c_1, x_2) \wedge \ldots \wedge \neg A(a_1, y_n, x_{n+1}) \wedge \neg A(a_1, y_{n+1}, x_{n+2})]$.

Assuming that all terms in $\mathcal{L}_{I\Delta_0}$ are non-decreasing in any variable, we can conclude:

$\mathfrak{A} \models \forall y_2 < u_2(a_1, a_2) \ldots \forall y_n < u_n(a_1, a_2, \ldots, x_n) \exists x_{n+1}$
$\forall y_{n+1} < u_{n+1}(a_1, a_2, \ldots, x_{n+1}) \exists x_{n+2}$
$[\neg A(a_1, c_1, a_2) \wedge \ldots \wedge \neg A(a_1, y_n, x_{n+1}) \wedge \neg A(a_1, y_{n+1}, x_{n+2})]$.

By repeating this exactly n times, we obtain:

$\mathfrak{A} \models \forall y_{n+1} < u_{n+1}(a_1, a_2, \ldots, a_{n+1}) \exists x_{n+2} \neg A(a_1, y_{n+1}, x_{n+2})$.

On the other hand, by monotonicity, we have:

$\mathfrak{A} \models d \leq t(a_1, \ldots, a_{n+1})$ which is the same as $\mathfrak{A} \models d < u_{n+1}(a_1, a_2, \ldots, a_{n+1})$.

Evidently there is a smallest element b in the domain of $\mathfrak{A}$ such that $\neg A(a_1, d, b)$ is true in $\mathfrak{A}$. This completes 2.

We select elements in the domain of $\mathfrak{A}$ by means of the next definition.

$A := \{c \in \mathrm{dom}(\mathfrak{A}) : \exists\, \bar{u}$- sequence $<c_1, \ldots, c_n>$ in $\mathfrak{A}$ such that $\mathfrak{A} \models c \leq c_n\}$.

We will show that A is closed under multiplication. Since A is clearly downwards closed, it is sufficient to show that A is closed under squares. Let c be an element of dom($\mathfrak{A}$), such that $\mathfrak{A} \models c \leq c_n$ for some $\bar{u}$-sequence $<c_1, \ldots, c_n>$ in $\mathfrak{A}$. For the witnessing sequence $<a_1, \ldots, a_{n+1}>$ for $<c_1, \ldots, c_n>$, it is surely the case that $\mathfrak{A} \models c_n < u_n(a_1, \ldots, a_n)$ and hence $\mathfrak{A} \models c_n^2 < u_n(a_1, \ldots, a_n)^2$, so that we can invoke 2. above to produce a sequence of terms $\bar{u}'$ such that $<c_1, \ldots, c_n, c_n^2>$ is a $\bar{u}'$-sequence. We can now safely define our model $\mathfrak{B}$ as the substructure of $\mathfrak{A}$ with base set A. It is clear that $\mathfrak{B}$ will satisfy the same Δ_0- formulae as $\mathfrak{A}$ did satisfy and therefore it will again be a model of $I\Delta_0$. We now claim: $\mathfrak{B} \not\models \forall x \exists y \forall z A(x, y, z)$.

Consider the element a_1 in dom($\mathfrak{A}$). By 1. above we also have that a_1 is an element of dom($\mathfrak{B}$). We claim: $\mathfrak{B} \models \neg \exists y \forall z A(a_1, y, z)$. Suppose, for a contradiction that for some c_0 in dom($\mathfrak{B}$), $\mathfrak{B} \models \forall z A(a_1, c_0, z)$. Then there would be a sequence of terms $\bar{u} := u_1(v_1), \ldots, u_n(v_1, \ldots, v_n)$ and a $\bar{u}$- sequence $<c_1, \ldots, c_n>$ in $\mathfrak{A}$ such that for the same element c_0 in dom($\mathfrak{A}$) we have: $\mathfrak{A} \models c_0 \leq c_n$. By definition this implies: $\mathfrak{A} \models c_0 \leq u_n(a_1, \ldots, a_n)$. By 2. above, we can conclude that there is a sequence of terms $\bar{u}'$ such that $<c_1, \ldots, c_n, c_0>$ is a $\bar{u}'$-sequence. As a consequence, if $<a_1, \ldots, a_{n+1}, a>$ is a witnessing sequence for $<c_1, \ldots, c_n, c_0>$, then we trivially have $\mathfrak{A} \models a \leq a$, so for a certain sequence of terms $\bar{u}''$ $<c_1, \ldots, c_n, c_0, a>$ is a $\bar{u}''$-sequence. This means that a is in the domain of $\mathfrak{B}$, so we will have $\mathfrak{B} \models \neg A(a_1, c_0, a)$. This completes the proof.

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