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Hermann Weyl's intuitionistic mathematics

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It is common knowledge that for a short while Hermann Weyl joined Brouwer in his pursuit of a revision of mathematics according to intuitionistic principles. There is, however, little information in the literature to shed light on Weyl's role, and in particular on Brouwer's reaction to Weyl's allegiance to the case of intuitionism. This short episode certainly raises a host of questions: how did Weyl come to give up his own program, spelled out in "*Das Kontinuum*" - how come Weyl was so well-informed about Brouwer's new intuitionism? - in what respect did Weyl's intuitionism differ from Brouwer's intuitionism? - what did Brouwer think of Weyl's views? - .... To some of these questions at least partial answers can be put forward on the basis of some of the available correspondence and notes.

Weyl entered the foundational controversy with a bang in 1920 with his sensational paper "On the new foundational crisis in mathematics"[1]. He had already made a name for himself in the foundation of mathematics in 1918 with his monograph "The Continuum"; this contained in addition to a technical logical-mathematical construction of the continuum, a fairly extensive discussion of the shortcomings of the traditional construction of the continuum on the basis of arbitrary - and hence also impredicative - Dedekind cuts. This book did not cause much of a stir in mathematics, that is to say, it was ritually quoted in the literature but, probably, little understood. It had to wait for a proper appreciation until the phenomenon of impredicativity was better understood[2] .

The paper "On the new foundational crisis in mathematics" had a totally different effect, it was the proverbial stone thrown into the quiet pond of mathematics. Weyl characterized it in retrospect with the somewhat apologetic words:

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1 The page numbers quoted below are from the original 1921 paper.

2 Cf. Feferman 1988.

> Only with some hesitation I acknowledge these lectures, which reflect in their style that was here and there really bombastic, the mood of exited times - the times immediately following the First World War.[3]

Indeed, Weyl's "New crisis" reads as a manifesto to the mathematical community, it uses an evocative language with a good many explicit references to the political and economic turmoil of the post war period. The opening sentences castigate the complacency of the mathematical community, which had not paid any serious attention to the potential dangers of the various paradoxes of Cantor, Russell, Richard, ...:

> The antinomies of set theory are usually considered as border conflicts that only concern the most remote provinces of the mathematical empire and that can in no manner endanger the inner solidity and security of the empire itself, of its proper central domains. Almost all statements that have been given on good authority concerning these disturbances of the peace (with the purpose to deny them or to smooth them out), do not, however, have the character of a conviction, clearly on itself based - born out of completely transparent evidence, but belong that sort of half to three-quarters honest self-deceptive attempts that one meets so often in political and philosophical thinking. Indeed: every earnest and honest reflection must lead to the insight that those disturbances in the border areas of mathematics must be judged as symptoms; in them it becomes apparent what the outwardly splendid and frictionless activity in the centre hides: the inner unreliability of the foundations on which the construction of the empire rests.

The readers of 1920 were all too familiar with the phenomenon, the German State with its skirmishes in the Baltic and its political instability, was a painful reminder of the self-deception of politics. Weyl used the political metaphor with great dexterity, he compared the classical use of existential statements with the use of paper money:

> The conception depicted above[4] only expresses the meaning which the general and

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3 Weyl 1955

4 i.e. the construction of a specific object versus its non-effective existence, DvD.

> existential propositions in fact have for us. In its light mathematics appears as a gigantic "paper economy". Real value, comparable to food products in the national economy, has only the direct, downright singular; everything general and all the existential statements participate only indirectly. And yet we mathematicians seldom think of cashing in this "paper money"! Not the existence theorem is the valuable thing, but the construction carried out in the proof. Mathematics is, as Brouwer sometimes says, more action than theory (*mehr ein Tun als eine Lehre*).

And the final clarion blast of Weyl , one that fired the imagination and fed the wrath of many a practising mathematician, rang through the next decade: "For this order can not in itself be maintained, as I have now convinced myself, **and Brouwer - that is the revolution!**"

Strong language, - the language of a man who finally has seen the true light!

Before analysing Weyl's views on intuitionistic mathematics, let us see how Weyl got involved with this particular branch of constructive mathematics.

Weyl was one of the outstanding mathematicians of his time, he was Brouwer's junior by 4 years, but he was already making a name for himself when Brouwer was still struggling for international recognition. In 1913 Weyl published one of the all-time successes of mathematics, his "The idea of Riemann surfaces", which established his name for good. Already quite early Brouwer and Weyl got into contact, in a letter of 16.5.1912 from Weyl to Klein a correspondence with Brouwer is mentioned[5]. With Brouwer attending conferences in Germany and regularly visiting Göttingen, they could hardly have missed each other. It seems most likely that those two men, who shared a vivid interest in the foundations of mathematics, discussed the various issues in private, probably already before the war. The first real indication in this respect is to be found in a letter from Brouwer to Fraenkel, in which he mentioned that during a stay in the summer vacation of 1919 in the Engadin in Switzerland, he had a number of

[5] In this letter Weyl sums up the various points of the Brouwer-Koebe controversy on the theory of automorphic functions.

personal discussions with Weyl. This piece of information solves much of the riddle of Weyl's early expertise of the new intuitionism of Brouwer that was conceived only in the war years . Indeed, reading only Brouwer's pre-war papers, one would have a very different picture of the "real" intuitionism. In the Swiss mountains Brouwer gave a one-man course to Weyl on the new insights that he himself had only acquired since 1914 and which were published in the communication of the Royal Academy at Amsterdam from 1918 onward. Weyl quickly grasped the main points of Brouwer's arguments and started to work out the consequences for himself. In May 1920 the paper on "the new crisis" was finished and a copy was sent to Brouwer:

> Zürich, 6.5.'20
>
> Dear Brouwer,
>
> Finally I have sent off the long promised [object] to you. It should not be taken as a scientific publication, but as a propaganda pamphlet, thence the size. I hope that you will find it suitable for this purpose, and moreover suited to shake the sleepers awake; for that purpose I want to publish it. I would be grateful for your views and comments. Did I include everything that you could only lend to me? If not, please let me know; the lecture on Formalism and Intuitionism[6] I already possess from an earlier date, at that time I did not pay attention to it or understand it ......

One can imagine how pleased Brouwer was, - here was one of the foremost members of the new generation of mathematicians, Hilbert's 'favourite son', who not only recognised the importance of Brouwer's concept of choice sequence for mathematics, but who was willing to give up his own program and to give active support to a wholesale renovation of mathematical practice! Of Brouwer's reaction only a rough draft, full of crossed-out parts, has survived. Nonetheless, one can reconstruct Brouwer's views from this scrap and from a few notes in pencil in the margin of Weyl's manuscript. The draft contains a number of comments on Weyl's manuscript,

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6 Brouwer 1912, 1914.

we will quote the various parts when discussing Weyl's paper. Here we translate the first few lines:

> The resolute scientific extending of your hand has given me an infinite pleasure. The lecture of your ms. was a continuous delight and your explanation, it seems to me, will also be clear and convincing for the public....
>
> That we judge differently on some side issues, will only stimulate the reader. Anyway, you are completely right in the formulation of these differences of opinion; in the restriction of the objects of mathematics you are as a matter of fact more radical than I am; one cannot discuss this however, these matters can only be decided by individual concentration.

As a strict schoolmaster, Brouwer could not forego the opportunity to spot a strategic flaw in the paper. After introducing the *functio continua*, Weyl, somewhat apologetic, remarked that "It should be stressed once more again that in the theorems of mathematics in certain cases such functions occur, general theorems about them are, however, never formulated. The general formulation of this notion is therefore only useful if one reflects on the meaning and proceedings of mathematics; for mathematics itself [*] , the content of its theorems they are not considered at all."

In the margin Brouwer indicated the insertion of "until now?"[7] at [*], and "except for the replacement of such a theorem (such as the Dirichlet principle according to Hilbert)?" at the end of the quotation.

In the above mentioned draft, Brouwer commented on the above lines:

> The whole purpose of your treatise seems me to be endangered by the end of the second paragraph of page 34[8]. After you have shaken awake the sleeper, he will say here to himself: "So the *authors* admits that the mathematical theorems proper are not influenced by his considerations? Then he should no longer disturb me!" and turns around and sleeps on. Thereby you do our cause injustice, for with the existence

7 *"bisherige?"*

8 p. 66

> theorem of the accumulation point of an infinite point set, many a classical existence theorem of a minimal function, and also the existential theorems of the geodetic line without the second differentiability condition, looses its justification!"

In this respect Weyl showed himself to be less radical than Brouwer, that is to say Brouwer did not fear the confrontation with the traditional practice of mathematics, where Weyl considered the fundamental considerations as somewhat less relevant for every day mathematics.

There are a few striking aspects to Weyl's version of intuitionism, one of them is the nature of the continuum and another one the role of logic. Weyl distinguished two sharply distinct views of the continuum, the atomistic one and the continuous one. In the first version the continuum is made up of individual real numbers which can be sharply distinguished. After defining specific relations on the natural numbers[9], and hence specific Dedekind cuts, "by the restricting of this notion, a bunch of individual points is, so to speak, picked from the flowing mush (*Brei*) of the continuum. The continuum is smashed to isolated elements, and the blending into each other of all of its parts is replaced by certain conceptual relations, based on the 'greater-smaller', between these isolated elements. Therefore I speak of an *atomic conception of the continuum* " (p. 46).

Indeed, the arithmetically definable continuum of "The Continuum" has this atomistic character. Almost immediately Weyl added, however, that "It has never been my opinion that the continuum given by intuition is a number system of Weyl; rather that analysis only needs such a system for its constructions and does not have to worry about the "continuum" poured in between." (p. 47).

After the discussion of "Weyl's continuum", the "New Crisis"-paper continued with an introduction of Brouwer's choice sequences as a basis for the continuum. Real numbers are given by infinite sequences of shrinking intervals, and thus the need arises to further specify the notion of sequence. Weyl allows two sorts of sequences: those

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9 "definite Relationen", p. 46.

given by a law - they stand for the individual points of the continuum, and the free choice sequences - which determine a continuum of "becoming" sequences (*werdende Folgen*[10]), i.e. sequences of intervals that are freely chosen and hence cannot in a predetermined way point at an element of the continuum. In the most liberal sense of the word, these sequences remain 'in statu nascendi' forever. In Weyl's words:

> "It is a first basic insight of Brouwer that the sequence which is emerging (*werdend*) through free choice acts is a possible object of mathematical concept formation. Where the *law* φ, which determines a sequence up to infinity, represents the single real number, the "choice sequence, restricted by no law in its freedom of development, represents the continuum" (p. 50)
>
> "The remark of Brouwer is simple but deep: here a "continuum" arises in which indeed the single real numbers fit, but which itself does by no means dissolve into a set of completed existing real numbers, rather it is a *medium of free becoming.*" (p. 50)

Weyl's evocative description of the "flowing" continuum is almost an artistic performance, he dared to put into writing what Brouwer would only describe in discussions and lectures. The image of a continuum in a continuous state of creation (becoming) was a decisive step forward in the quest for something truly "continuous", i.e. something that cannot be interrupted or broken into pieces, let alone be built up from the atomic particles called points!

There are basically two fundamental issues in Weyl's exposition of intuitionism:

(1) the properties of choice sequences,

(2) the consequences for logic.

As to the first issue, let us compare Brouwer's notion and Weyl's side by side:

Brouwer (1918): A (*Menge*) spread is a *law*, according to which each time another number complex of the sequence ξ (i.e. $\mathbb{N}$) is chosen, each of these choices either

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[10] The translation of the German "werdende Folge" is somewhat problematic. Various adjectives, such as *becoming, emerging, in statu nascendi* have been used.

generates a certain sign or nothing, or causes the blocking of the process and the definite destruction of its result, where for each n after each non-blocked sequence of n-1 choices at least on number complex can be indicated, which, when chosen as the n-th number-complex does not cause the blocking of the process. Each sequence of signs (which thus is in general not presentable as finished) is called an *element of the spread.*"

In the next line the word "choice sequence" is introduced for the above elements. Weyl (1920) did not give a definition of 'choice sequence' but rather introduced them "on the way". A few quotations are listed below:

(1) The difficulty is in the notion of sequence. If any viewpoint at all is at the basis of contemporary analysis, from which its propositions and proofs can be understood, then it is this: a sequence comes into existence by the successive arbitrary choices of its individual numbers; the result of these infinitely many choice acts is given in finished form, and with respect to the finished infinite sequence I can ask e.g. if 1 occurs among its numbers. But this stand-point is nonsensical and untenable; for the inexhaustibility is inherent to the essence of the infinite. When on the contrary a sequence is created step by step by free acts of choice, then it should be considered as "*becoming*" (growing), and only those properties can be stated about a becoming choice sequence, for which the decision "yes or no" (...) is already made when the sequence has reached a certain point." (p. 49).

(2) *a sequence in which each choice is totally free* (p. 52).

(3) a law which generates a natural number n from a becoming sequence , depending on the result of the choices, is necessarily such that the number n is determined as soon as a certain finite segment of the choice sequence is given in finished form, and it remains the same, no matter how far the choice sequence may develop further. (p. 51)

The main difference is that Brouwer does not elaborate the choice process and concentrates on the concept of spread, whereas Weyl does not consider spreads at all, but tries to get the choice process right.

At the time that Brouwer and Weyl were discussing the matter, the ideas concerning choice sequences were fairly implicit, and in view of Brouwer's use of (and examples of) spreads we may conclude that already at the level of choice sequences both parties diverged. Brouwer allowed choice sequences, given by a law (e.g. in the spread of the natural numbers (1918B, p.3), or finite spreads), whereas Weyl excluded lawlike sequence from the domain of choice sequences. At a first reading (after 1958) one might think that Weyl had lawless sequences in mind, absolutely freedom of choice, both (2) and (3) seem to point that way. However, he also allows choice sequences, built from other choice sequences, e.g. the sequence $m_1+ m_2+...+m_n$ obtained from the sequence m_n (p, 221). The reader will note that (3) looks very much like the principle of open data (for a function value), since it discusses the output of a function for a single choice sequence, and hence it points towards a strong degree of lawlessness.

We know now that this would yield the wrong class of choice sequences, but it would be highly unjust to demand the present day insights in a pioneering paper in 1920. It seems most plausible that the finer distinctions were not fathered by Weyl. It even is hard to guess in how far Brouwer knew exactly which pitfalls were to be found where on his road; nonetheless it is remarkable that he (almost) always formulated the right concepts and principles. The miracle therefore is not that Weyl's conception of choice sequence lacked coherence, but that Brouwer got it right!

In the case of (3) above, Brouwer had already in his 1918 paper formulated the correct principle:

(4) A law that assigns to each element g of C [the universal spread] an element h of A [the natural numbers], must have determined the element h completely after a certain initial segment α of the sequence of number complexes of g. (p. 13).

It should be added that Brouwer did not further justify this principle, but it shows that he realized that the universe of choice sequences was the home of a rich variety of sequences, ranging from lawlike to lawless, with all kinds of versions in between.

Brouwer's spreads may be thought of as trees with natural numbers at their nodes (a subset of the set of all finite sequences of natural numbers), where some particular nodes were forbidden (and hence also all nodes below them). The choice sequences could be visualized as infinite paths through the tree. In fact the choices made were in practice from a given domain, e.g. real numbers appeared as choice sequences of rational numbers, or of rational intervals. But the basic function of a spread was to regulate the possible choices, a matter not left till later. In Weyl's approach, all natural numbers were eligible, and only afterwards other sequences were generated by suitable mappings. From a topological point of view one could say that Weyl got the desired choice sequences by means of continuous mappings acting on Baire space. Hence there was no need for Weyl to restrict the possible choices in advance, the mappings would take care of the need for specific sequences. Hence also Weyl's preference for unrestricted choices.

The functions that did the work for Weyl were called by him.

(1) *functio discreta*: a law assigning numbers to numbers (a lawlike function from $\mathbb{N} \to \mathbb{N}$)

(2) *functio mixta*: a lawlike function from numbers to lawlike functions $(\mathbb{N} \to (\mathbb{N} \to \mathbb{N}))$, or a law assigning numbers to choice sequences. $((\mathbb{N} \to \mathbb{N}) \to \mathbb{N})$

(3) *functio continua*: a lawlike function from choice sequences to choice sequences $((\mathbb{N} \to \mathbb{N}) \to (\mathbb{N} \to \mathbb{N}))$.

The functio discreta is introduced by means of the unanalysed notion of a law. The functio mixta for the first case is given by a double sequence, and for the second case it is regulated by the following *basic insight (Wesenseinsicht)*:

"*According to this law there always is a moment for a becoming sequence, no matter how it develops, that it gives birth to a number*" (p. 64).

As a matter of fact, Weyl listed two generating principles for the functio mixta, but he explicitly warned the reader that he did not claim these principles to yield all cases of functio mixta.

The clauses for Weyl's functio mixta are in modern formulation:

(i) F, with $F(\alpha) = f(\bar{\alpha}k)$ is a functio mixta, where $f: N \to N$ is a a lawlike function and $k \varepsilon N$

(ii) if H and G are functio mixta then F with $F(\alpha) = H(G(\alpha), \alpha)$ is a functio mixta.

The class of functions given by the generating principles is evidently a proper subclass of the class K of Brouwer operations[11].

In the margin of the manuscript Brouwer noted: "this rule is indeed included in my notion of spread."

As in Brouwer's spread, Weyl allows partial functions, i.e. the function may be undefined for certain choice sequences, i.e. from the functio mixta law it must follow for each α that "at a certain argument (depending on α) either the generated number is available or the certainty exists that a deaf sequence[12] is considered, which will be barren in all eternity". Brouwer's comment: "from my viewpoint this is unneccessary". Indeed, on Brouwer's approach the underlying tree handles the question of being defined or not. No decisions concerning α's are involved.

The functio continua takes choice sequences to sequences, and it is defined by Weyl as a continuous operation:

"A functio continua is a law, according to which a becoming sequence of natural numbers created by free choice acts at every step which adds another term generates a certain number or nothing". (p. 65)

Since this definition does not guarantee that the resulting sequence will proceed indefinite, Weyl explicitly stipulated that eventually a new number will be generated. Here Brouwer commented, "superfluous from my point of view, it excludes unnecessarily discontinuous functions" (p. 65) This seems curious in view of Brouwer's later theorem that functions on the continuum (and also Baire space, i.e. the set of choice sequences) are continuous; the condition of Weyl, however, excluded partial functions, and they could easily be discontinuous.

Weyl obtained discontinuities by mimicking Brouwer's destruction clause. He had,

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[11] c.f. Troelstra-van Dalen 1988, Ch. 4, section 8.

[12] i.e. a sequence not in the domain of the functio mixta

however, to add a complicated clause to ensure closure under substitution (the problem of continuity and lazy computation). Brouwer again declared this clause superfluous for spreads. So far Weyl's view of choice sequences.

Weyl seemed to consider *first* choice sequences over all possible number-choices (all paths through the universal tree), and only *afterwards* to use continuous operations in order to get other families of choice sequence. In a way a plausible and even modern view, in the recent past these techniques have been used by Troelstra - van Dalen and Van der Hoeven to obtain suitable choice universes as "projections of lawless sequences". Needless to say that in 1920 one would not even dream of such matters.

Weyl's second personal innovation in intuitionistic mathematics did get a certain amount of attention. It was his reconsideration of the role of existential and general statements. In part II section 1, Weyl described his own "conversion" to Brouwer's rejection of the principle of the excluded third. He started by pointing out that for propositions involving choice sequences he had no difficulty accepting Brouwer's view, e.g. "the choice sequence α contains infinitely many 0's or finitely many 0's" is undecided beyond all hope, but for arithmetical statements the matter could be different. An existential statement could be decided as follows: check A(n) successively for 0, 1, 2, 3, ... "then this process will halt or not; *it is or is not the case*, without wavering and without a third possibility... Finally I found for myself the magic word. *An existential statement* - e.g. "there is an even number" - *is not a judgement in the proper sense at all, which states a state of affairs*; Existential - states of affairs are an empty invention of logicians." (p. 54).

Weyl termed such pseudo-statements "*judgement abstracts*", to be compared to "a paper which announces the presence of a treasure, without giving away its place." Similarly general statements are only *hints at judgements*. Hence they can not be negated, and "the possibility to formulate an 'axiom of the excluded third' with respect to them has disappeared." In this respect Weyl definitely is much more radical than Brouwer, indeed so radical that his viewpoint never was adopted in practice. The intuitionistic interpretation of "there exists" and "for all", according to the Brouwer-

Heyting-Kolmogorov was in the respect more convincing, and more manageable than Weyl's proposal.

Weyl's interpretation of existential statements was later echoed in Hilbert-Bernays, where they are explained as "incomplete statements", and in 1940 Kleene, in turn, took his cue from Hilbert - Bernays, when developping his "recursive realizability"-notion [Kleene 1973].

It is worth noting that Weyl's argument against the principle of the excluded third in the case of choice sequences, had a scholastic ring. It could possibly be constructed as the Aristotelian indeterminism of "the sea battle to-morrow". This could explain why he had rather un-intuitionistic qualms about the principle of the excluded middle for arithmetic statements, where the tense aspect is less forceful.

At the end of section 2 "The notion of a function", Weyl summed up the main foundational points:

> I derive from Brouwer 1. the basis, which is in all respects the essential, namely the idea of becoming sequences and the doubt of the principium tertii exclusi, 2. The idea of the functio continua. The functio mixta and the viewpoint, which I summarize in the following three theses, are mine:
>
> 1. The notion of sequence oscillates, depending on the logical context in which it occurs, between "law" and "choice", "being" and "becoming".
>
> 2. The general and existential statements are no judgements in the proper sense, they do not state states of affairs, but are judgement indications, resp. judgement abstracts.
>
> 3. Arithmetic and analysis only contain general statements about numbers and freely becoming sequences; no general function- and set theory of independent content.

There are two comments of Brouwer in the margin: - with respect to (1),

> for me "becoming sequence" is one nor the other; one considers the sequences from stand-point of a powerless spectator, who knows nothing about the question in how far the completion has been free"[13].

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[13] This is a surprising formulation for those who know Brouwer's solipsistic tendencies. Probably

- with respect to (2) and (3),

"I do, however, not agree with 2) and 3)."

The draft of the letter elaborates (1) a bit further:

Your assertion of p. 37[14], lines 3 - 6, which, as you know by the way, contradicts my view, should be explained a bit further. It seems to me that the reader, who has followed you so far, will have difficulty with this passage. Your functio discreta and functio mixta seem to me, just as your functio continua, to be contained in my concept of spread. My spread law can very well give right away for each choice sequence the certainty that once a sign has been generated, nothing will be generated any more.

Weyl's paper contains so much material that the prize-winner might easily escape the reader; after discussing the notion of a real function, Weyl concludes with "Above all there can be no other functions on the continuum than continuous functions" (p. 76). Did Weyl thus establish the continuity theorem years before Brouwer? The answer is 'no' for various reasons. In the first place Weyl defined real functions via mappings of the intervals of the choice sequence determining a real number to intervals in the image sequence. Hence such a function is continuous *by definition*. Weyl's argument for this definition is on the whole plausible: one should be able to find effectively approximations to the output from approximations to the input. Brouwer's continuity theorem, however, goes one step further, it establishes the continuity of a function from choice sequences to choice sequences or from reals to reals. So, where Weyl reduced the type of a function from **R** to **R** in advance to that of a function from $\mathbb{N}$ to $\mathbb{N}$ (initial segments to initial segments), Brouwer showed that this by necessity followed from the intuitionistic principles.

Furthermore, there is ample evidence that Weyl was, also in this issue, inspired by Brouwer. Brouwer had already quite early reached the conclusion that there could not

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Brouwer did not exert himself in being ultra dogmatic. In his courses he often introduced spreads as a two person game; it is not always easy to distinguish in how far Brouwer used didactic techniques and in how far his arguments were philosophical.

14 p. 71 of the paper.

be any full (i.e. total) discontinuous functions on **R**[15]: "Through this above mentioned theorem[16], which is an immediate consequence of the intuitionistic stand-point, which has since 1918 often been mentioned by me in lectures and in conversation, ... " ; and he was already looking out for a natural notion of a discontinuous function on (a subset of) **R**. This is substantiated by the draft mentioned above:

> With reference to your considerations on the notion of continuous function I would like to draw your attention to my notion of the *fully defined* (*volldefiniert*) function of the continuum. I mean by that a law which assigns to every point of a set equivalent to the continuum (*örtlich übereinstimmenden Punktspezies*) a further point of the continuum. Such a function can very well be discontinuous without being generated by a conjoining of continuous functions on separated continua; one can anyway operate with them in many ways (one can e.g. in certain cases integrate them without having information on their continuity or discontinuity).

After pointing out the absence of discontinuous functions on **R**, Weyl went on to say that:

> When the old analysis allowed the formation of discontinuous functions, it announced most clearly in this way, how far it is from grasping of the essence of the continuum. What one calls nowadays a discontinuous function, consists in fact (and also that is ultimately a return to older intuitions) of a number of functions on separated continua.

Brouwer added in the margin the remark:

> better, is a not everywhere defined function. The definition of the function does not have to be split in cases, as the pointwise discontinuous function shows.

Weyl then went on to illustrate the piecewise defined function $f(x)$ which is x for $x > 0$ and $-x$ for $x < 0$. This partial function, he remarked can be extended to the total function $|x|$, but the function g with $g(x) = 1$ for $x > 0$ and -1 for $x < 0$ cannot be extended to a total function.

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[15] cf. Brouwer 1927, p. 62

[16] Every full function is negatively continuous.

Brouwer noted, with respect to the latter part "very true! To be underlined, for this is what matters."

The conjecture of the continuity of all real functions was, according to Brouwer's letter to Fraenkel (28.1.1927), brought up in 1919: "In the summer of 1919 I have in the course of private discussions with Weyl in Engadin, as a result of which he was converted to my views, once in connection with the definition of continuous functions in § 1 of the first part of my "Founding set theory independent from the logical proposition of the excluded third"[17], ventilated and motivated the conjecture that these functions are the only ones consisting in the full continuum [......]. It can be only on the basis on this (and other, half-understood by Weyl) utterance of a conjecture by me, that since than the legend has been spread by Weyl, that it is self evident that in Brouwer's analysis none but continuous functions can exist".

The above comment of Brouwer is a bit out of tune, for reasons that have nothing to do with Weyl or his papers, - at the time of writing Brouwer had the (not wholly unjustified) impression that the credits in the foundations of mathematics were distributed in a rather off-handed way. But at least it makes it quite clear that Weyl had not quite grasped Brouwer's arguments about the continuous functions.

Not surprisingly, the available information on Brouwer's reaction raises as many questions as it provides answers. It makes clear that Brouwer did not share Weyl's radical rejection of existential and general statements, and it also formulates the objections against Weyl's notion of choice sequence. The question remains, however, why did Brouwer not give an exposition of his own views on these topics? Since we only have the draft of the letter, it cannot be excluded that the letter itself contained more extensive arguments. It seems more likely, however, that Brouwer did not wish to repeat his arguments of the Engadin discussion in 1919, and that he wanted to respect Weyl's version, as appears from the first paragraph of the draft:

> these matters can only be decided by individual concentration.

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[17] p. 12.

This probably explains why Brouwer did not provide more detailed comments on Weyl's wording of the continuity principle; for in Weyl's version of choice sequence, his "open data"-like formulation - "for a single choice sequence" rather than "for all choice sequences" - was plausible indeed.

It would also have been interesting to have Brouwer's view on the meaning of the quantifiers in writing; here also he respected Weyl's view and was content to just register his disagreement.

Apart from the "propaganda pamphlet" of 1920, Weyl figures only twice in the history of intuitionism. After this short episode Weyl practiced intuitionism only once more, in 1924 he gave a constructive proof of the fundamental theorem of algebra[18]. He remained a lifelong admirer of Brouwer and his intuitionism, but he distanciated himself from the actual practice and also from the foundational debate. When the Grundlagenstreit that ensued from the action in 1920, was well under way, he remained rather aloof. Only in 1928, when Hilbert attacked intuitionism during a seminar talk in Hamburg, Weyl came once more to the defence of Brouwer[19]. Following Hilbert's talk, Weyl took it upon him to straighten out a number of points, in particular the role of Brouwer's intuitionism in connection with Hilbert's program.[20] He pointed out that Brouwer had systematically pursued the investigation of the extent of the "contentual thought"[21].

> Brouwer required from mathematics, like everybody else, that its theorems (in Hilbert's way of expression), should be "real statements", meaningful truths. But he saw for the first time and to the full extent, how it had transgressed this limit of contentual thought everywhere by far. I think that we owe him gratitude for recognition of the limits of

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18 Weyl 1924.

19 Weyl 1928. Contributions to the discussion on the second lecture of Hilbert on the foundations of mathematics.

20 Weyl 1928.

21 *inhaltliches Denken.*

contentual thought"[22].

Weyl then went on to sketch Hilbert's program very briefly, concluding that "Also in the epistemological evaluation of the new situation thus created, nothing separates me any longer now from Hilbert." From the following passages, it appears that Weyl had Hilbert's conservative extension" program (or "real interpretation of ideal statements") in mind. He compared the situation with that in physics, where the (mathematical) theories were no longer in immediate contact with the phenomenal physical world, and quoted Hilbert's justification of the theory with an appeal to its success.

Although Weyl recognised the potential effects of Hilbert's program, which was still a real option at the time, he somewhat sadly reflected that "if a triumph of formalism over intuitionism were to be the case - as it seems to all appearances , then I see in that a decisive defeat of the philosophical approach of a phenomenology, which proves itself insufficient for the understanding of creative science, on the most primitive and most accessible area of knowledge, mathematics."

It is a bit hard to draw the right conclusion from this contribution, on the one hand Weyl felt the need to stand up for Brouwer, on the other hand he no longer felt compelled to stick to the intuitionist views on the role of formalism. One is tempted to view Weyl's statement as rather ahead of his time, i.e. his acceptation of the results of proof theory as a theoretical enterprise based on intuitionistic principles. This view certainly is compatible with the paper and also with later ones.

In his influential book "Philosophy of mathematics and Natural Science" (1949), which grew out of an early essay "The present state of knowledge in mathematics" (1925) and its subsequent extension "Philosophie der Mathematik und Naturwissenschaft" (1927), Weyl gave an penetrating survey of the foundational situation in mathematics, in which he discussed both intuitionism and formalism. The 1925 version still witnesses the enormous relief of having found a faithful theory of the continuum. The section on "Brouwer's intuitive mathematics" opens with the magistral

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[22]Weyl 1928, p. 86.

sentence

> "The ice field was broken into floes , and now the element of the flowing will soon completely be master over the rigid." L.E.J. Brouwer designs - and this is an accomplishment of the greatest epistemological importance - an exact mathematical theory of the continuum, which conceives it not as a rigid being, but as a medium of free becoming."[23]

The same paper contains also the first sign of his despair in view of the ultimate success:

> Mathematics attains with Brouwer the highest intuitive clarity; his doctrine is the idealism in mathematics thought through till the end. But with pain the mathematician sees the larger part of his towering theories fall apart.

Two years later Weyl expressed himself even more pessimistically:

> Mathematics attains with Brouwer the highest intuitive clarity. He succeeds in developing the beginnings of analysis in a natural way, retaining contact with intuition much closer than before. But one can not deny, that progressing to higher and more general theories, the fact that the simple principles of classical logic are not applicable finally results in a hardly bearable awkwardness. And with pain the mathematician sees that the larger part of his tower, which he thought to be joined from strong blocks, dissolves in smoke.[24]

We now, of course, know that this pessimism was somewhat exaggerated; things were not as bad as they seemed. In a sense Brouwer was himself to be blamed for the lack of popularity of his intuitionistic practice. Instead of polishing up the presentation of important parts of mathematics, say differential- and integral calculus, he preferred to concentrate on the basic phenomena of intuitionistic mathematics. This strategy was

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[23] "Die Eisdecke war in Schollen zerborsten, und jetzt ward das Element des Flieszenden bald vollendes Herr über das Feste." L.E.J. Brouwer entwirft - und das ist eine Leistung von der gröszten Erkenntnistheoretischen Tragweite - eine strenge mathematische Theorie des Kontinuums, die es nicht als starres Sein, sondern als Medium freien Werdens faszt" [Weyl 1928]

24 Weyl 1928.

not designed to win over the young practicing mathematicians, at best the singular devotee of foundational research. It was not before Bishop's book in 1967, that a convincing basic text became available.

Weyl never could quite part with his earlier intuitionistic past, he remained fair with respect to Brouwer's intuitionism in a period where intuitionism had become a minor curiosity in the larger context of mathematics.

In a larger review paper in 1946, he summed up his feelings:

> From this history one thing should be clear, we are less certain than ever about the ultimate foundations of (logic and) mathematics, like everybody and everything in the world to-day, we have our "crisis". We have had it for nearly fifty years. Outwardly it does not seem to hamper our daily work, and yet I for one confess that it has had a considerable practical influence on my mathematical life: it directed my interests to fields I considered relatively "safe", and it has been a constant drain on the enthusiasm and determination with which I pursued my research work. The experience is probably shared by other mathematicians who are not indifferent to what their scientific endeavours mean in the context of man's whole caring and knowing, suffering and creative existence in the world.

Weyl may have been right, maybe he was not alone in taking the "crisis in mathematics" seriously in a personal way. He certainly was the most prominent mathematician with a conscience sensitive to the cracks in the foundations of the building of mathematics.

The story had a sequel, - when Weyl was already fatally ill, Brouwer visited him in Zürich and the two had a long talk. At the parting of the two old friends, Weyl sadly remarked: "*Brouwer, es ist alles wieder schwankend geworden*" (Brouwer, it has all become wavering again).

Appendix

Draft Brouwer to Weyl - 1920 (undated)

Ihre rückhaltslose wissenschaftliche Handreichung hat mir eine unendliche Freude bereitet. Den Lektüre Ihres M.S. war mir ein fortwährender Genuss und Ihre Auseinandersetzung scheint mir - auch für das Publ. klar und überzeugend sein wird.......

Das wir beide über einige Nebensachen verschieden urteilen, wird auf die Leser nur anregend wirken. Allerdings haben Sie mit Ihrer Formulierung dieser Meinungsverschiedenheiten volkommen recht; in der Einschränkung des Objektes der Mathematik sind Sie tatsächlich radikaler als ich; es läszt sich aber darüber nicht diskutieren, diese Sachen sind nur durch individuelle Konzentration zu entscheiden.

Im Anschlusz zu Ihren Darlegungen über den Begriff der stetige Funktion möchte ich Sie auf meinem Begriff einer volldefinierten Funktion des Kontinuums aufmerksam machen.

Ich verstehe darunter einen Gesetz das jedem Punkte einer mit den Kontinuum örtlich übereinstimmenden Punktspezies einem weiteren Punkt des Kontinuums zuordnet. Eine solche Funktion kann sehr gut unstetig sein ohne irgendwie durch Zusammensetzung von stetigen Funktionen in getrennten Kontinuen erzeugt zu werden; man kann übrigens mit ihnen in mannigfacher Weise operieren. [man kann sie z.B. in gewissen Fallen integrieren ohne hinsichtlich ihre Stetigkeit oder Unstetigkeit Aufschlusz zu haben].

Abgesehen von unseren Differenzpunkten hatte ich noch folgende Bemerkungen:

Den Non-existential Sätzen (wozu z.B. die Mächtigkeitssätze auf S. 134 u. 43 des ersten Teiles meiner Abhandlung und auch Hilbertschen Endlichkeitssatz des vollen Invarianten Systems in seiner ersten Herleitung gehören) raümen Sie bei Ihrer Aufzählung der mathematischen Urteile gar keinen Platz ein. S. 3, Z. 8 (ebenso wie an der analogen Stelle auf S. 13 von "Das Kontinuum") ist mir den Sinn des Wortes "Sachkenntnissen" dunkel.

Durch den Schluss des zweiten Absatzes von S. 34 scheint mir den ganzen Zweck Ihrer Schrift gefährdet zu werden. Nachdem sie eben den schlafenden aufgerüttelt haben sagt er sich hier: "Also der Verf gibt zu, dass die wirklichen mathematischen

Sätze von seinen Darlegungen nicht berührt werden? Dann soll er mich weiter nicht stören!" und wendet sich ab und schläft weiter. Damit tun Sie aber unserer Sache Unrecht, denn mit dem Existenzsatz des Verdichtungspunkt einer unendlichen Punktmenge wird doch gleichzeitig manchem klassischen Existenztheorem einer Minimalfunktion wie auch dem Existenzialsatze des geodetischen Linie bei Fortlassung der zweiten Differenzierbarkeitsbedingungen den Boden entzogen!

Ihr S. 37 Z.3-6 aufgestellte, übrigens wie Sie wissen, meiner Auffassung widersprechende Behauptung sollte, wie ich glaube, etwas näher erklärt werden. Es scheint mir dass auch der Leser, der Ihnen bis dahin genau gefolgt ist, mit diesem Passus Mühe haben wird. Ihr Funktio diskreta und Funktio mixta scheinen mir eben so gut wie Ihre Funktio kontinua in meinen Mengenbegriff enthalten zu sein. Mein Mengengesetz kann ja sehr gut von vornherein für jede Wahlfolge sicherheit geben, dass nachdem einmal im Zeichen erzeugt ist, weiterhin immer wieder nichts erzeugt wird.

Auf Ihre Entscheidung zwischen Göttingen, Berlin und Zürich bin ich ungeheuer gespannt. Mögen Sie klar sehen, und die richtige Wahl treffen! Leicht wird Ihnen das nicht fallen!

Den gesandten Abzug Ihres Manuskripts darf ich wohl behalten? Sie brauchen mir nichts mehr zurückzusenden. Weil inzwischen einige meiner älteren Separaten neu abgedrückt worden sind, so möchte ich Sie um Mitteilung bitten, welche von meinen folgenden Publikationen Sie augenblicklich besitzen:

1. Intuitionisme en formalisme (Holländisch)
2. Intutionism and formalism (Englisch)
3. De onbetrouwbaarheid der logische principes (Holländisch)
4. Het wezen der meetkunde (Holländisch)

Ich kann Ihnen das eventuell fehlende jetz ergänzen.

Nochmal aufrichtigen Dank für die Freude und die Satisfaktion, die Ihre Schrift mir bereitet hat, herzliche Grüsse auch an Ihrer Frau und auf Wiedersehen!

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