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Dynamic Squares

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This paper examines various propositional logics in which the *dynamic implication* connective $\Rightarrow$ (discussed in Groenendijk and Stokhof's (1992) Dynamic Predicate Logic and Kamp's (1981) Discourse Representation Theory) plays the central role. Our approach is modal: the basic idea is to view $\Rightarrow$ as a binary modal operator in a two dimensional modal logic. Two dimensional modal logics are systems in which formulas are evaluated at *pairs* of points. Such logics provide natural setting for studying $\Rightarrow$, for we can think of the first point in the pair as an input state, and the second point in the pair as an output state.

Although the basic idea is very simple, as soon as one begins to explore it one faces a number of choices. First, what type of two dimensional system should be employed? For example, should we look at $\Rightarrow$ in the setting of 'full square' (or 'classical') two dimensional modal logics such as those studied by Segerberg (1980) and Venema (1991), or is a more general setting that uses 'relativised squares' (see van Benthem (1991a) and Marx *et al.* (1992)) appropriate? In addition, are there connections with Boolean Modal Logics such as those studied by Gargov *et al.* (1987) and de Rijke (1992a)? These distinctions are all explained in the course of the paper; for now it suffices to note that the choices involved are interesting ones. Somewhat loosely, opting for a 'square' two dimensional treatment of $\Rightarrow$ reflects most accurately Groenendijk and Stokhof's original system; viewing $\Rightarrow$ in a 'relativised' setting suggests modifications of the original system of both technical and philosophical interest; whereas making the simple notational change that leads to Boolean Modal Logic yields a system that is notationally natural and suggestive of further extensions.

A second choice that confronts us concerns the Boolean connectives (that is, $\wedge$, $\vee$, $\neg$, $\rightarrow$ and $\leftrightarrow$). Should they be present or not? They are not present in Groenendijk and Stokhof's system, and indeed if one adopts a 'strongly dynamic' perspective on natural language interpretation (such as that underlying the work of Vermeulen (1991, 1992) and Visser (1992a, 1992b)) this is a natural path to explore. On the other hand, in a number of publications van Benthem has emphasized the need to consider the interplay of static and dynamic notions (see van Benthem (1991a), for example). Viewed from this perspective, to discard the Boolean connectives is to discard static logic, which results in a lop-sided system which doesn't reflect the interesting interplay.[1]

This paper is more concerned with exploring the options than making the choices. By limiting our discussion to one relatively simple operator (namely $\Rightarrow$) we will be able to discuss the various forms of two dimensionality, the differences between systems with and without the Boolean operators, and, perhaps most importantly, to make clear why we think the two dimensional perspective is a natural one. We proceed as follows. In the first section we

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[1] The reader may be puzzled by this identification of 'static' with 'Boolean' — and indeed it is not obvious that this identification should be made. Nonetheless, it is an identification that is often made in the dynamics literature, and we refer the reader to van Benthem (1991a) for further discussion of the underlying ideas.

discuss dynamic implication, and show that it can be viewed as a two dimensional operator. In the second section we define a modal language L (the language of propositional calculus enriched with a single binary modality) and interpret it as a square two dimensional logic. We relate it to previously studied two dimensional systems and show how it can be embedded in the cylindric modal logic of Venema (1991). In the third section we reinterpret L, arguing that from a dynamic perspective it is is natural to consider the logic of 'relativised' squares. In the fourth section section we make a suggestive notational change: we start writing $\phi \Rightarrow \psi$ as $\langle\phi\rangle\psi$, thus treating the antecedent of dynamic implications as a 'program', and link our work with the ideas underlying Boolean Modal Logic. In the fifth section we axiomatise both the square and the relativised two dimensional logics of L. In the sixth section we investigate what happens when we drop the Boolean operators from L, and in the seventh we briefly discuss the relationship between our two dimensional Dynamic Implication Logics and other (propositional) dynamic logics of transitions. We conclude the paper with some general remarks and suggestions for further work.

Before going any further, we should mention that the idea of using two dimensional logics to analyse dynamic ideas is not entirely new, though as far as we are aware, the logic of dynamic implication has not previously been investigated. First, in a number of recent publications, van Benthem has proposed various modal analyses of dynamic phenomenon. Some of his suggestions are obviously two dimensional (see, for example, the comments in van Benthem (1991b, page 247) concerning Groenendijk and Stokhof's system), and some of his proposals for the development of 'arrow logic' concern precisely what we here have called 'relativised' two dimensional logic. Second, de Rijke (1992a, 1992b) has recently investigated a 'dynamic modal logic' in great detail. His system is one dimensional, but as he points out, because of its great expressive power it can simulate two dimensional patterns of evaluation. As he comments, this is important precisely because of the two dimensional flavour of many of the ideas underlying dynamic formalisms. All in all, the idea that two dimensional modal logic provides a natural setting for the study of dynamics seems to be an idea whose time has come. This paper provides some preliminary maps of the territory.

1 Dynamic implication

Dynamic implication is a logical connective present in a number of frameworks proposed over the last ten years to analyse such natural language phenomena as anaphora and presupposition. What these theories have in common is that in various ways, and to varying degrees, they take seriously the view that sentence meanings are active entities. On this view, sentence meanings, like computer programs, can be viewed as entities which transform state spaces. The meaning of a sentence is thus a set of *transitions*, or, to put it another way, a set of *pairs of states*. The first component of these pairs are thought of as *input states*, and the second component as *output states*, the states that result when the sentence is processed. This dynamic perspective on the interpretation of natural language has resulted in a number of attempts to devise both adequate state spaces and suitable formal languages for describing them. Influential work in this tradition includes the Discourse Representation Theory (DRT) of Kamp (1981), the File Change Semantics (FCS) of Heim (1982), Zeevat's algebraic reformulation of DRT (1991), and Groenendijk and Stokhof's (1992) Dynamic Predicate Logic (DPL).[2]

[2] The above account of dynamic interpretation is probably *not* uncontroversial. It is certainly an accurate account of how semanticists working with DPL view their formalism, but it is not clear that all practitioners of DRT would find it congenial. For a start, the idea of viewing DRT as a dynamic formalism seems to be relatively recent; previously DRT was considered to be interesting because it provided a *representational*

Somewhat informally, the idea of dynamic implication underlying these frameworks is as follows. First of all, for $\phi \Rightarrow \psi$ (ϕ *dynamically implies* ψ) to be accepted at an input/output pair (i, j), it must be the case that for any k, if ϕ is accepted at the input/output pair (i, k) then there exists a l such that ψ is accepted at (k, l). That is, for every output state k resulting from a successful execution of ϕ with respect to input i, we are guaranteed the existence of an output state l when ψ is executed with k as an input. In short, every successful execution of ϕ must lead to a successful execution of ψ. The second point is more subtle, but gives $\Rightarrow$ its flavour: dynamic implication, to use the terminology of Groenendijk and Stokhof, is a *test*. Although called 'dynamic implication', the task $\Rightarrow$ performs is simply to check whether the transition conditions just described hold or not. Executing a dynamic implication doesn't result in a state change, or, to put it another way, dynamic implications can only be accepted at those input/output pairs (i, j) where $i = j$.

It is helpful to think of $\Rightarrow$ geometrically. Suppose we are working in a two dimensional state space consisting of input/output pairs. Then $\phi \Rightarrow \psi$ exhibits the following pattern:

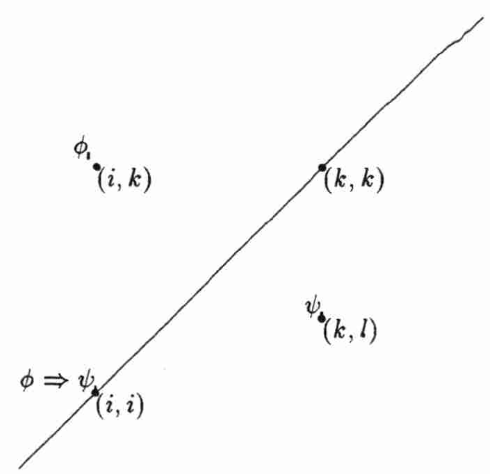

Figure 1.

That is, for $\phi \Rightarrow \psi$ to be accepted we must be standing on a point on the diagonal (here (i, i)), and whenever we move vertically and find a ϕ (here at (i, k)), then by moving moving horizontally to the diagonal and resuming vertical search we are guaranteed to find ψ (here at (k, l). Perhaps the most important part of this diagram is the horizontal move from (i, k) to (k, k). Its purpose is to turn k from an output state from the first search into an input state for the second. The other point worth noting is that searches (that is, executions of dynamic implications), always begin on the diagonal (for example, in the above diagram at (i, i)). This is the geometric content of the idea that dynamic implications are tests.

This evaluation pattern can be seen most clearly seen in Groenendijk and Stokhof's DPL. Syntactically, the language of DPL is that of first order logic. Groenendijk and Stokhof write

account of meaning (as opposed to a standard model theoretic one); and the idea that representation, not dynamics, is the important idea in DRT is still influential. Here we'll be content to point out that recent work by Visser (1992a, 1992b) and Vermeulen (1991, 1992) attempts to locate the heart of dynamics in the actual process of building up representations; thus it may well be that there is no essential difference between the dynamic and representational perspectives.

$\Rightarrow$ as $\rightarrow$, and regard their treatment as a dynamic reinterpretation of material implication. Semantically they work with two dimensional spaces consisting of all pairs of assignments of values to variables over a given first order model. (More precisely, given a first model $\mathbf{M}$, the state space over $\mathbf{M}$ consists of all pairs (f,g) of assignments of values to variables on $\mathbf{M}$.) The above evaluation pattern is then used to evaluate wffs of the form $\phi \Rightarrow \psi$ at pairs of assignments (f,g), and the idea that $\Rightarrow$ should be a test is explicitly stated and motivated. Much of the work that follows can be seen as a fairly direct abstraction from ideas clearly visible in DPL. However, it's worth noting that the same evaluation pattern also occurs in earlier frameworks such as Kamp's DRT and Heim's FCS. Consider DRT. Syntactically DRT allows 'boxes' (that is, DRSs) to be connected by $\Rightarrow$. Semantically, DRT (like DPL) also takes as its state space all pairs of valuations over a first order model; however DRT construes the above pattern in terms of *embeddings of* DRSs. We have that $\phi \Rightarrow \psi$ holds iff every successful embedding of ϕ in a first order model (the first vertical move) can be extended to a successful embedding of ψ (the second vertical move). For precise details consult Kamp (1981). On the other hand, although the concept of a test is implicitly present in DRT terms (the point is simply that conjoining two DRSs with $\Rightarrow$ neither introduces nor destroys discourse referents) the concept doesn't seem to have been isolated in the DRT tradition.

This paper is essentially a logical exploration of the above diagram. Semantically, we are going to abstract from both from the details of the various state spaces proposed in the literature, and from the actual mechanisms (such as embeddings) in terms of which $\Rightarrow$ has been defined, and instead work with the simple idea of 'square universes'. Syntactically we are going to abstract from DRSs and first order formulas and work with propositional languages. It should be clear that if we make these abstractions we are we are in the realm of some sort of (propositional) two dimensional modal logic. The state spaces will be two dimensional Kripke models, and $\Rightarrow$ will be nothing but a binary modal operator — albeit a modal operator with the peculiarity that it only works when used on the diagonal.

We think these are interesting abstractions to make for the following reasons. Firstly, although there is considerable disagreement as to the nature of 'dynamic interpretation' and tools best suited for capturing the concept, the idea of dynamic implication has proved a stable one. It thus seems sensible to explore its logic in a general setting. Secondly, we believe the setting of two dimensional modal logic is a technically natural one for exploring the ideas underlying dynamics, a point we will return to in the conclusion.

2 The square semantics

Having glimpsed $\Rightarrow$ in the setting of DPL and DRT, we now make the abstractions promised in the last section and view $\Rightarrow$ as an operator that scans propositional information distributed over a two dimensional state space. To be more precise, we are now going to treat $\Rightarrow$ as a binary modal operator in 'square' (or 'classical') two dimensional modal logic. Actually, we are not going to treat $\Rightarrow$ directly. Instead we take as primitive an operator $\rhd$ with the following pattern of evaluation:

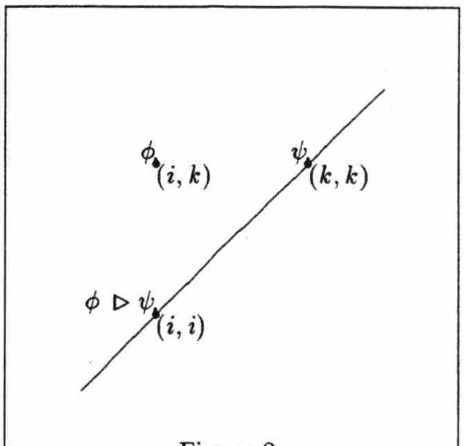

Figure 2.

Note that $\rhd$ is an operator that carries out the first two steps of $\Rightarrow$'s three step search. As we shall see, the third step (the final vertical scan that $\Rightarrow$ makes) is recoverable by means by means of an operator $\Diamond^t$ defined in terms of $\rhd$, thus $\Rightarrow$ is itself definable in our language. We'll also see that $\rhd$ is definable in terms of $\Rightarrow$, thus our decomposition of $\Rightarrow$ doesn't smuggle in any extra expressive power. There will be one other (rather trivial) difference between $\rhd$ and $\Rightarrow$. As the satisfaction definition below shows, the evaluation pattern for $\rhd$ is construed entirely existentially, and not semi-universally as was the case with the DPL definition of $\Rightarrow$. In modal terminology, we are taking a dyadic 'diamond' form as primitive. In the presence of full Boolean expressivity (which we are here assuming) nothing hangs on this choice. Later in the paper, when we do not allow ourselves the luxury of full Boolean expressivity, we shall have to deal directly with $\Rightarrow$.

Now for the details. We define the language L as follows. It has a denumerably infinite set ATOM of atomic symbols, which we typically write as p, q, r and so on; it has a truth functionally adequate supply of Boolean connectives (in what follows we treat $\wedge$ and $\neg$ as primitive, and the symbols $\vee$, $\rightarrow$, $\leftrightarrow$, $\top$ and $\bot$ as defined); it has a binary modal operator $\rhd$ and it has punctuation symbols) and (. The sets of wffs of L, WFF, is the smallest set containing ATOM such that if ϕ and $\psi \in$ WFF then so are $\neg\phi$, $(\phi \wedge \psi)$ and $(\phi \rhd \psi)$. We omit brackets whenever this won't result in ambiguity. We define the degree of a formula ϕ (that is, $deg(\phi)$) to be the number of logical connectives it contains. L, and various fragments and notational variants thereof, is the language we'll mostly be concerned with in this paper.

To interpret L we need a suitable notion of two dimensional state space. The work of the present section will be based on the concept of a (full) *square*:

Definition 2.1 (Squares) *If $\mathfrak{S}$ is a set of the form $S \times S$ for some non-empty set S, then $\mathfrak{S}$ is called a* **square**. □

Squares are the skeletons of state spaces; in traditional modal terminology, they are our frames. A square, together with an information distribution, is a *model*:

Definition 2.2 (Square models) *A* **square model** *$\mathfrak{M}$ is a pair $(\mathfrak{S}, V)$ where $\mathfrak{S}$ is a square and V is a valuation for L on $\mathfrak{S}$. (That is, V : ATOM $\longrightarrow Pow(\mathbf{S})$.)* □

For any model $\mathfrak{M}$ $(= (\mathfrak{S}, V))$ we inductively define the notion $\mathfrak{M}, (i,j) \models \phi$ (that is, the model $\mathfrak{M}$ satisfies ϕ at the input/output pair (i,j)) as follows:

$$
\begin{array}{lll}
\mathfrak{M}, (i,j) \models q & \text{iff} & (i,j) \in V(q), \ \text{ for all atoms } q \\
\mathfrak{M}, (i,j) \models \neg\phi & \text{iff} & \mathfrak{M}, (i,j) \not\models \phi \\
\mathfrak{M}, (i,j) \models \phi \wedge \psi & \text{iff} & \mathfrak{M}, (i,j) \models \phi \ \textit{ and } \ \mathfrak{M}, (i,j) \models \psi \\
\mathfrak{M}, (i,j) \models \phi \rhd \psi & \text{iff} & i = j \ \textit{ and } \ \exists k(\mathfrak{M}, (i,k) \models \phi \ \textit{ and } \ \mathfrak{M}, (k,k) \models \psi)
\end{array}
$$

If a wff ϕ of L is satisfiable at some pair in some model, then we say that ϕ is satisfiable. We defer defining our notion of validity until we have examined the language a little more closely. Consider the satisfaction clause for $\rhd$. Note that like $\Rightarrow$ it is a test (that is, it is a modal operator that only works on the diagonal) and that its evaluation pattern contains the characteristic horizontal move to the diagonal which turns an output state into an input state. The first thing to check is that these two ingredients (with the aid of our Booleans) enable us to recover $\Rightarrow$.

First note that $\top \rhd \top$ is true at all and only the diagonal points. So, we can abbreviate it to δ and regard it as a name for the diagonal. Next note that $\psi \rhd \top$ is true at precisely those diagonal points from which it is possible to find ψ by moving vertically. We abbreviate this to $\Phi^t \psi$. (The superscript t indicates that this vertically scanning existential modality is a test; it only works on the diagonal.) Now we define:

$$\phi \Rightarrow \psi =_{def} \delta \wedge \neg(\phi \rhd \neg\Phi^t\psi).$$

It is straightforward to check that this definition has the desired force. Conversely, given $\Rightarrow$, we can define δ by means of $\top \Rightarrow \top$, and thus can subsequently define:

$$\phi \rhd \psi =_{def} \delta \wedge \neg(\phi \Rightarrow \neg(\delta \wedge \psi)).$$

Once again, it is easy to check that this definition is correct.

We shall work with the following notion of validity:

Definition 2.3 (s-validity) *A wff ϕ is valid in a square model $\mathfrak{M}$ (notation: $\mathfrak{M} \models^s \phi$) iff for all pairs (i,j) in $\mathfrak{M}$ we have $\mathfrak{M}, (i,j) \models \phi$. If $\mathfrak{S}$ is a square and for all valuations V on $\mathfrak{S}$ we have $(\mathfrak{S}, V) \models^s \phi$, then ϕ is valid on $\mathfrak{S}$ (notation: $\mathfrak{S} \models^s \phi$). If ϕ is valid on all squares $\mathfrak{S}$ then ϕ is* **s-valid** *(notation: $\models^s \phi$).* □

This is a modally natural notion of validity, but it certainly not the only one one possible, especially if one wants to model dynamics. In particular, Groenendijk and Stokhof work with the following notion, which we'll here call gs-validity: a wff ϕ is gs-valid in a model $\mathfrak{M}$ iff for all inputs i there is an output k such that $\mathfrak{M}, (i,k) \models \phi$; and it is called gs-valid iff it is gs-valid in all models. Another natural notion given the prominence of the diagonal in the satisfaction definition is *diagonal validity*: a wff is d-valid in a model iff it is satisfied at all diagonal points in the model, and is diagonal valid simpliciter iff it is diagonal valid in all models. Now, because we have have full Boolean expressivity at our disposal it is easy to capture these other notions in terms of the usual modal notion. In particular we have that for all wffs ϕ, ϕ is gs-valid iff $\models^s \delta \rightarrow \Phi^t \phi$, and ϕ is d-valid iff $\models^s \delta \rightarrow \phi$. However, when we later drop the Booleans from L, we'll have to be more careful.

How does L relate to standard two dimensional languages? The most striking difference between $\rhd$ and other two dimensional languages is the fact that $\rhd$ is a test. With the exception of a handful of languages that employ a primitive constant symbol δ as a 'name' for the diagonal (for example the cylindric modal logic of Venema (1991) described below), such operators don't seem to have been treated. On the other hand, co-ordinate shuffling similar to that which takes place in $\rhd$'s horizontal shift have been considered. For example, Kuhn (1989) and Venema (1992) axiomatise various two dimensional languages containing the domino operator $\diamondplus$. This operator is interpreted on square models and has the following satisfaction clause:

$$\mathfrak{M},(i,j) \models \diamondplus\phi \text{ iff } \exists k(\mathfrak{M},(j,k) \models \phi)$$

Geometrically, evaluating $\diamondplus\phi$ at some pair causes a horizontal shift to the diagonal (thus shuffling j from output position to input position) followed by a vertical search for ψ. In short, $\diamondplus$ carries out the last two steps in the diagram for $\rhd$. On the other hand, the domino operator is not a test, thus we cannot use it unaided to mimic $\rhd$.

However there is a well understood two dimensional language which can mimic $\rhd$. As we shall now see, L can be regarded as a fragment of Venema's (1991) two dimensional cylindric modal logic. Syntactically two dimensional cylindric modal logic differs from L only in its choice of modal operators. Like L it has a set ATOM of atomic symbols, Boolean connectives and punctuation symbols, but instead of $\rhd$ it has a constant symbol (or 0-ary modal operator) δ and two unary modalities $\diamondvert$ and $\diamondminus$. The wffs are the smallest set containing all the atoms and δ that is closed under the Boolean operators and applications of $\diamondvert$ and $\diamondminus$. Semantically, the language is interpreted on square models. The satisfaction clauses for atoms and Boolean connectives are as for L, so it only remains to give the clauses for δ, $\diamondvert$ and $\diamondminus$. These are as follows:

$$\begin{array}{lll} \mathfrak{M},(i,j) \models \delta & \text{iff} & i=j \\ \mathfrak{M},(i,j) \models \diamondvert\phi & \text{iff} & \exists k(\mathfrak{M},(i,k) \models \phi)) \\ \mathfrak{M},(i,j) \models \diamondminus\phi & \text{iff} & \exists k(\mathfrak{M},(k,j) \models \phi)) \end{array}$$

Thus $\diamondvert$ and $\diamondminus$ allow us to scan vertically and horizontally respectively. Note that neither $\diamondvert$ nor $\diamondminus$ is a test; they work everywhere. However with the aid of the δ we can get the grip on the diagonal that is needed needed to define $\rhd$. In fact we embed L into L^C by means of the following translation C:

$$\begin{array}{lll} C(p) & = & p \\ C(\neg\phi) & = & \neg C(\phi) \\ C(\phi \wedge \psi) & = & C(\phi) \wedge C(\psi) \\ C(\phi \rhd \psi) & = & \delta \wedge (\diamondvert(C(\phi) \wedge \diamondminus(\delta \wedge C(\psi)))) \end{array}$$

Proposition 2.4 *Let $\mathfrak{M}$ be any square model and ϕ be any wff of L. Then for all pairs (i,j) in $\mathfrak{M}$ we have $\mathfrak{M},(i,j) \models \phi$ iff $\mathfrak{M},(i,j) \models C(\phi)$.*

Proof.
Induction on $deg(\phi)$. □

This leads to:

Theorem 2.5 *The satisfiability problem for L is decidable.*

Proof.
It is known that the satisfiability problem for two dimensional cylindric modal logic is decidable. (This follows from Corollary 3.2.66 of Henkin, Monk and Tarski (1985), the decidability of the satisfiability problem for equational theory of representable two dimensional cylindric algebras.) But the previous proposition shows that the problem of determining whether a wff ϕ of L is satisfiable reduces to the problem of of satisfying $C(\phi)$. As this is decidable we are through. □

The basic idea underlying our use of two dimensional modal logic should now be clear: concrete valuation spaces can be viewed as two dimensional Kripke models. Two general comments are called for. Firstly, this abstraction has already proved itself useful from the mathematical point of view. For example, the modal perspective on cylindric algebras (the algebraic structures underlying first order logic) has offered interesting handles on a number of problems in algebraic logic; see Venema (1991) for further discussion. Secondly, as we shall now explain, the abstraction links various general themes underlying modern dynamic semantics with pioneering ideas in formal semantics known as the California Theory of Reference.

Two dimensional modal logic can be traced back to brief remarks made by Richard Montague in "Universal Grammar". Montague treated meaning as a function of two arguments: a possible world and a context of use. The context of use component was to cope with the semantics of indexical expressions and free variables. The idea of two dimensional evaluation was explored in detail by Kamp (1971) in his study of the logic of the temporal indexical 'now'; and later work by Kaplan (see for example Kaplan (1977, 1979)) extended the treatment to a variety of other phenomena and explored its philosophical ramifications.

Now, in the literature just cited there is no mention of 'dynamic evaluation': however there is detailed discussion of context and its impact on semantics, and in addition, valuations are treated as part of context. Moreover, the main technical goal of the Californian tradition was to show that the behaviour of natural language indexicals could be mimicked in various modal extensions of classical logic by making use of the extra structure available in two (or multi) dimensional models. A similar pattern of ideas underlies much contemporary work in dynamics. For example, in Groenendijk and Stokhof (1988) the defence of DPL is couched in language highly reminiscent of the Californian tradition: they are at great pains, for example, to distinguish the contribution to content made by possible worlds and valuations. Further, the technical aim of DPL is to obtain an reinterpretation of first order logic that reflects natural language anaphoric phenomena by exploiting the extra structure available by working in the square of first order valuation spaces. It would be an exaggeration to claim that the ideas of dynamic interpretation are already to be found in the early writings of the Californian tradition;[3] nonetheless, at both the technical and conceptual level, there are parallels between the two traditions that deserve further study. The abstractions made in this paper indicate some of this common core.

3 The relativised semantics

In the previous section we gave a simple modal account of dynamic implication, and examined some of its basic properties. But, in spite of the simplicity of our account, we are now going to argue that some of its underlying assumptions may have been too strong; it may be more appropriate to work with the *relativised semantics.*

The step from square two dimensional logic to relativised two dimensional logic is technically simple. No syntactic changes are involved, and neither is the satisfaction definition altered: we simply enlarge the class of structures that count as models. Instead of insisting that models are pairs $(\mathfrak{S}, V)$ consisting of a square and a valuation, we also allow pairs of the form $(\mathfrak{R}, V)$, where $\mathfrak{R}$ is a non-empty subset of a square that is closed under projections to the diagonal, and V is a valuation. More precisely:

[3]There is at least one early paper which is explicitly 'dynamic', and uses two dimensional modal logic to modal this dynamism, namely Stalnaker's study of assertion (see Stalnaker (1978)). However this paper does not consider anaphoric phenomenon, and is better viewed as a precursor of 'update semantics' (see Veltman (1991)), rather than dynamism in the style of DRT, FCS, and DPL.

Definition 3.1 (Relativised squares and models) *If $\mathfrak{R}$ is a non-empty set such that $\mathfrak{R} \subseteq \mathfrak{S}$ for some square $\mathfrak{S}$, and for all $(i,j) \in \mathfrak{R}$, both (i,i) and (j,j) are in $\mathfrak{R}$, then $\mathfrak{R}$ is called a* **relativised square**. *If $\mathfrak{R}$ is a relativised square and V is a valuation on $\mathfrak{R}$ (that is, $V : \mathrm{ATOM} \longrightarrow Pow(\mathfrak{R})$) then $\mathfrak{M} = (\mathfrak{R}, V)$ is called a* **relativised model**. □

As has already been mentioned, the satisfaction definition is unchanged; we use the clauses given in the previous section, but now $\mathfrak{M}$ can be any relativised model, not just a square model. What *does* change is the definition of validity:

Definition 3.2 (r-validity) *A wff ϕ is valid in a relativised model $\mathfrak{M}$ (notation: $\mathfrak{M} \models^r \phi$) iff for all pairs (i,j) in $\mathfrak{M}$ we have $\mathfrak{M}, (i,j) \models \phi$. If $\mathfrak{R}$ is a relativised square and for all valuations V on $\mathfrak{R}$ we have $(\mathfrak{R}, V) \models^r \phi$, then ϕ is valid on $\mathfrak{R}$ (notation: $\mathfrak{R} \models^r \phi$). If ϕ is valid on all relativised squares $\mathfrak{R}$ then ϕ is* **r-valid** *(notation: $\models^r \phi$).* □

Proposition 3.3 *If $\models^r \phi$ then $\models^s \phi$, but not conversely.*

Proof.
That r-validity implies s-validity is trivial, for all square models are relativised models. To see that the converse is not true it suffices to exhibit a wff which is s-valid but not r-valid. It is straightforward to check that $(\top \rhd (\top \rhd p)) \rightarrow (\top \rhd p)$ is s-valid. However it is not r-valid, as the following model shows:

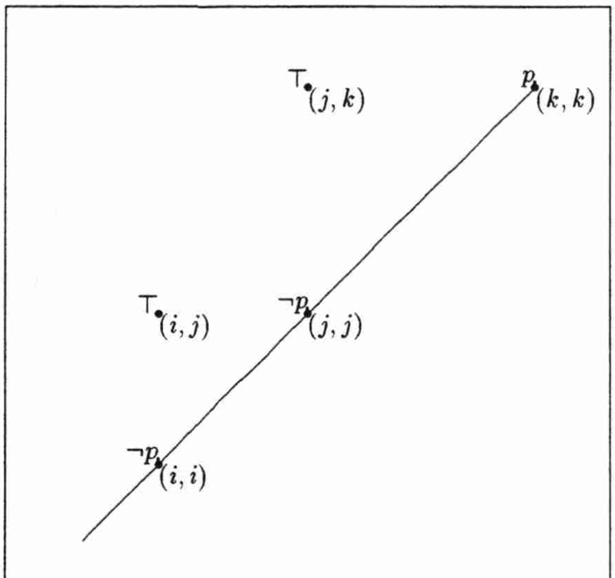

Figure 3.

The crucial thing to note about this picture is that the the pair (i,k) is missing. This means we cannot move from (i,i) to (k,k) in one step. As the reader can easily check, this 'hole' ensures that $(\top \rhd (\top \rhd p)) \rightarrow (\top \rhd p)$ is false at (i,i). □

Actually, the difference between the relativised and square semantics can be seen in another way: the relativised semantics admits (or suffers) non-trivial truth and validity preserving notions of substructure, while the square semantics does not.

Definition 3.4 *Let $\mathfrak{R}$ ($\subseteq S \times S$) be a relativised square and $\mathfrak{P}$ a non-empty subset of $\mathfrak{R}$ such that:*

1. $(\forall i,j,y \in S)[((i,j) \in \mathfrak{P}$ *and* $(i,y) \in \mathfrak{R})$ *implies* $(i,y) \in \mathfrak{P}]$

2. $(\forall i,j,x \in S)[((i,j) \in \mathfrak{P}$ *and* $(x,j) \in \mathfrak{R})$ *implies* $(x,j) \in \mathfrak{P}]$.

Then $\mathfrak{P}$ *is a generated relativised subsquare of* $\mathfrak{R}$. *If* $\mathfrak{P} \subset \mathfrak{R}$ *then* $\mathfrak{P}$ *is a proper relativised subsquare of* $\mathfrak{R}$. □

It is immediate from this definition that any generated relativised subsquare $\mathfrak{P}$ of $\mathfrak{R}$ is a relativised square; all that has to be checked is that $\mathfrak{P}$ is closed under projections to the diagonal, and this is clear. As a simple example consider the relativised square $\mathfrak{R} = \{(0,0),(0,1),(1,1),(2,2)\}$. $\mathfrak{R}$ has two proper generated relativised subsquares, namely $\{(0,0),(0,1),(1,1)\}$ and $\{(2,2)\}$. Note that no square $\mathfrak{S}$ has proper relativised subsquares. The notion of a generated relativised square gives rise to a notion of generated relativised submodel in the obvious way, and thus to the following lemma:

Proposition 3.5 (Generated submodel lemma) *Let* $\mathfrak{P}$ *be a generated relativised subsquare of* $\mathfrak{R}$, *and let* V^p *and* V^r *be valuations on* $\mathfrak{P}$ *and* $\mathfrak{R}$ *respectively such that for all atoms* q, $V^p(q) = V^r(q) \cap \mathfrak{P}$. *Then for all* $(i,j) \in \mathfrak{P}$, $(\mathfrak{P}, V^p),(i,j) \models \phi$ *iff* $(\mathfrak{R}, V^r),(i,j) \models \phi$.

Proof.
Induction on $deg(\phi)$. □

Thus satisfaction is transferred to generated submodels. This fact leads, quite routinely, to a generated submodel result: for any wff ϕ, if $\mathfrak{P}$ is a generated relativised subframe of $\mathfrak{R}$ then $\mathfrak{R} \models^r \phi$ implies $\mathfrak{P} \models^r \phi$. We leave the details to the reader.

Thus, as both the earlier counterexample and the differing behaviour with respect to substructures show, in moving from the square semantics to the relativised semantics we have weakened the logic. What could motivate such a move?

Consider what using relativised models means in terms of Groenendijk and Stokhof's DPL semantics: it means that instead of using the entire product space of the first order valuations, we may use instead any non-empty projection closed subspace thereof. To put it another way, instead of taking as our semantic entities a first order model $\mathbf{M}$ together with (implicitly) all pairs of first order valuations over $\mathbf{M}$, we have moved to a setting where the fundamental semantic entities are pairs $\langle \mathbf{M}, \mathcal{V} \rangle$, where $\mathcal{V}$ is some chosen non-empty subset of pairs of valuations.

Looked at this way the move is quite familiar: it's what underlies Henkin's use of generalised models for second order logic (see Henkin (1950) and van Benthem and Doets (1984)) or Thomason's introduction of generalised frames to modal logic (see Thomason (1972)). More recently the idea has also been applied to first order logic by Németi (see Németi (1992)). In all three cases the move can be viewed can be viewed as an attempt to find a semantic setting which improves the metatheoretic properties of the language in question. Moreover, this move is a principled one (as Németi and Andréka have recently argued[4]) because it separates out a 'logical core' from extraneous combinatorial complexities that arise because we only consider the entire set of possible valuations. In fact, although we cannot go into this point in detail here, the research programme for first order logic initiated by Németi and Andréka has philosophical as well as technical interest. It can very naturally be viewed as accepting the standard Quinean argument (namely, that set theory needs to be carefully separated from logic) and using it against the standard semantics of Quine's favoured bastion of logical purity, namely first order logic itself. Under this view,

[4]Invited talks given at the Applied Logic Conference, Amsterdam, 1992.

standard first order logic, with it's reliance on the full powerset of valuations, has needlessly committed itself to assumptions that are essentially set-theoretic in nature.

Thus there are general reasons for thinking the move to the relativised semantics an interesting one. However it is arguable that some such move is particularly pressing if one wants to model the ideas underlying dynamic interpretation. The point is this. While in DPL validity is defined in terms of entire (square) state spaces, it is by no means clear that this is the best way to view more recent proposals in dynamic semantics. For example, building on Zeevat's (1991) algebraic analysis of DRT, Visser (1992a, 1992b) and Vermeulen (1991, 1992) have attempted to analyse the actual process of building up representations. These approaches locate the heart of dynamism in the process of merging DRSs, and give various mathematical models of this process. Here the appropriate perspective seems to be the relativised semantics: the fundamental semantic entities of interest are of models together with the set of valuations actually built up in the course of analysing the discourse. In short, while the square semantics best fits standard DPL, we suspect that it may be too strong to capture more recent ideas. What seems to be needed to classify the various dynamic proposals is a hierarchy of two dimensional logics. The square semantics would yield strong logics in such a hierarchy, the relativised semantics much weaker ones.

It's worth stressing that there are many options to explore in setting up such a hierarchy; the simple two way split we have proposed here is probably far too coarse. For a start, we have insisted that relativised models be closed under projections to the diagonal. This is mathematically pleasant, but may well be too strong. For example, we might imagine that while processing a sentence an error is encountered that blocks further processing; we become 'trapped' off the diagonal. If we are going to model such intuitions we need a broader class of partial squares: it might be fruitful to look at what happens when we only insist on closure under the first projection, or drop the idea of projection closure altogether.

Other dimensions of variation readily suggest themselves. Why should we consider *all* valuations over relativised or square models? If anything, our preceeding discussion strongly suggests we should consider working with 'generalised two dimensional models'. Lastly, Fernando (1992) forcefully argues that dynamic ideas should be modelled in terms of *partial* assignments. Perhaps partial two dimensional logics (which to the best of our knowledge are presently *terra incognita*) should be investigated.

These remarks lead us well beyond the scope of the present paper. The two way split we have introduced is all that will be explored here, and indeed we are now ready to consider the axiomatisations of these two notions. However before doing so, we want to make a simple but suggestive notational change to L. This will lead us in the direction of Boolean modal logic.

4 Boolean Modal Logic

In this (short) section we will make a notational change in our language, reflecting a new perspective on the system. To introduce this view, let us make two observations. First, as all formulas of the form $\phi \rhd \psi$ are *tests*, off the diagonal there is not much action going on; and in fact, for any model $\mathfrak{M}$, it will be fruitful to view the extension sets of formulas (that is, sets of the form $\{(i,j) \in \mathfrak{M} \mid \mathfrak{M}, (i,j) \models \phi\}$) as forming a *Boolean Algebra*. Second, suppose we travel from (i,i) to (k,k) via a pair (i,k) where ϕ happens to be true. One might say that in such a situation, (k,k) is *ϕ-accessible* from (i,i). Note that in this terminology, in any model $\mathfrak{M}$ we have that (i,i) is ϕ-accessible from itself iff $\mathfrak{M}, (i,i) \models \phi$.

Putting these observations together, it is as if we are dealing with a Kripke semantics where the possible worlds are the diagonal points, connected by accessibility relations forming a Boolean Algebra. This perspective brings our system close to some formalisms known in the literature on modal logic, notably from a line of papers by a group of Bulgarian logicians — we mention Gargov, Passy and Tinchev (1987). They develop a system called *Boolean Modal Logic* (BML), where a typical formula is of the form $\langle\alpha\rangle\phi$, α being a Boolean construct over a set of atomic constants referring to binary relations. In other words, in this modal formalism the diamonds themselves form a Boolean Algebra.

As will turn out in the sequel, this perspective on our logic is so useful that we stress the similarities between BML and our Dynamic Implication Logic (DIL) by changing our notation:

Definition 4.1 *Let* ATOM *be a set of atomic propositional variables.* WFF *the set of wffs over* ATOM*, is the least set containing* ATOM *such that if $\phi, \psi \in$ WFF then $\neg\phi$, $\phi \wedge \psi$ and $\langle\phi\rangle\psi \in$ WFF. We define δ to be $\langle\top\rangle\top$, and we define $[\phi]\psi$ to be $\neg\langle\phi\rangle\neg\psi$. We make the important remark that:*

$$\langle\phi\rangle\psi \text{ is just a notational variant of } \phi \rhd \psi.$$

Thus the semantics of this language has already been discussed in detail. □

Note that now indeed $\langle\phi\rangle\psi$ holds at a world (i,j) iff $i = j$ and (informally), there is a ψ-world (k,k) that is ϕ-accessible from (i,i); our system has something of the flavour of BML. There are, however, a number of differences: first, we consider relativized models whereas Gargov, Passy and Tinchev do not; and second, in BML the programs and the formulas form different sorts, whereas in our system the formulas themselves index the diamonds. Thus, our system is in a sense 'hybrid'. The formulas lead a double life: they are both 'propositions' and 'programs'.

In the above paragraph we used the word 'program'. This brings us to another important connection — one that is in fact taking us home. For, the idea of having an 'algebra of diamonds' is a main characteristic of Propositional Dynamic Logic (PDL) itself, one of the main sources of inspiration for the work of Groenendijk and Stokhof.[5] In PDL, of course, the relational operations are not static (that is, Boolean) as in BML, but dynamic: union, composition and iteration of programs are among the important 'diamond constructors' in PDL.

A system that in a sense combines the ideas of PDL and BML has already been considered in the literature. This system is the Dynamic Modal Logic (DML) devised by van Benthem (1991a) and developed by de Rijke (1992a, 1992b). The DML algebra of diamonds has *both* static and dynamic operators (though it lacks the iteration constructor). Like our Dynamic Implication Logic, DML is a hybrid system; formulas lead a double life. The most important difference between DML and the Dynamic Implication Logic discussed here is that DML is a one dimensional, not a two dimensional, modal logic. We will later consider the various tradeoffs and relationships existing between the systems just mentioned in more detail.[6]

To close this section we will make two remarks on the link of our (reformulated) Dynamic Implication Logic with systems developed in the literature on dynamic semantics for natural language. First, our notational change reflects notation adopted in later developments of dynamic semantics by Groenendijk and Stokhof, namely their use of *state switcher* notation in Dynamic Montague Grammar (see Groenendijk and Stokhof (1990)). Second, a return to the setting of the types of dynamic logic pioneered in the computational literature has

[5] For a detailed survey of PDL, see Harel (1984).

[6] In passing, as Frank Veltman remarked, there may be connections to be made between DIL and the technical literature on conditional logic, though we cannot explore this possibility here.

recently been advocated by Muskens (1991). Muskens observes that intensional logic underlying Montague Semantics is rich enough to encode dynamic notions, and shows that only relatively modest changes to the translation function from natural language are needed to capitalise on this.

5 Boolean completeness

With our new notation to hand, in this section we will show that for both the relativized and the full square semantics, the set of valid formulas allows a concise description in the form of a complete axiomatization. We will prove two completeness theorems; first one for the relativized semantics, and then, on top of that, a result for the full square frames. Let us start by defining the axiom systems:

Definition 5.1 *RAX (Relativized Axiom System) is the axiom system containing the following axioms:*

($A0$) *Any complete set of axioms for propositional logic*
($A1$) $([\phi]\chi \wedge [\psi]\chi) \rightarrow [\phi \vee \psi]\chi$
($A2$) $[\phi](\psi \rightarrow \psi') \rightarrow ([\phi]\psi \rightarrow [\phi]\psi')$
($A3$) $[\phi]\psi \rightarrow \delta$
($A4$) $[\phi]\psi \rightarrow (\phi \rightarrow \psi)$
($A5$) $(\delta \wedge (\phi \rightarrow \psi)) \rightarrow [\delta \wedge \phi]\psi$
($A6$) $\delta \rightarrow [\phi]\delta$
(D) $\langle\phi\rangle\psi \leftrightarrow (\delta \wedge \neg[\phi]\neg\psi)$

Its rules of inference are:
(MP) *From* $\vdash \phi$ *and* $\vdash \phi \rightarrow \psi$ *conclude* $\vdash \psi$
(N) *From* $\vdash \psi$ *conclude* $\vdash \delta \rightarrow [\phi]\psi$
(MO) *From* $\vdash \phi \rightarrow \phi'$ *conclude* $\vdash [\phi']\psi \rightarrow [\phi]\psi$

FAX (Full Square Axiom System) is obtained by adding the following axioms to those of RAX:
($A7$) $\langle\top\rangle[\top]\phi \rightarrow [\top]\phi$
($A8$) $(\langle\phi\rangle\psi \wedge \langle\neg\phi\rangle\psi) \rightarrow [\top]\langle\neg\delta\rangle\psi$

The intuitive meaning of the axioms and rules will be discussed in the soundness proof below.

Definition 5.2 *Concerning axiom systems, we define the following notions: a* **derivation** *is a sequence of formulas* $\phi_1, \ldots, \phi_n$ *such that every item on the list* ϕ_i *is either an axiom, or it arises out of earlier items* ϕ_j *by application of a rule. A* **theorem** *of the system is a formula that can occur as the last item of a derivation. A formula* ϕ *is* **derivable from a set** Σ, *if there are formulas* $\sigma_1, \ldots, \sigma_n$ *in* Σ *such that the formula* $(\sigma_1 \wedge \ldots \wedge \sigma_n) \rightarrow \phi$ *is a theorem. A set of formulas is* **consistent** *if* $\perp$ *is not derivable from it,* **maximal consistent** *if it is consistent itself, but no extension of it (in the same language) is consistent. We will use the abbreviation* **MCS** *for 'maximal consistent set'.*

RAX/FAX-derivability of formulas will be denoted as follows: $\vdash^r \phi$ *if* ϕ *is a RAX-theorem,* $\vdash^s \phi$ *if* ϕ *is a FAX-theorem.* □

Convention 5.3 *We will often suppress explicit reference to the model* $\mathfrak{M}$ *in the notation for satisfaction, writing* $(i,j) \models \xi$ *for* $\mathfrak{M},(i,j) \models \xi$. □

The theorems that we will prove are the following:

Theorem 5.4 (SOUNDNESS AND COMPLETENESS.)
(i) RAX is sound and complete for relativized square validity, i.e. $\vdash^r \phi$ *iff* $\models^r \phi$.
(ii) FAX is sound and complete for full square validity, i.e. $\vdash^s \phi$ *iff* $\models^s \phi$.

Proof of soundness.
First we will indicate why the RAX axioms are valid: assume that we are in some relativized model. For (A1), suppose that $(i,j) \models [\phi]\chi \wedge [\psi]\chi$, then we have to show that $(i,j) \models [\phi \vee \psi]\chi$. Now if $(i,k) \models \phi \vee \psi$, then either $(i,k) \models \phi$ or $(i,k) \models \psi$. In both cases we find $(k,k) \models \chi$. (A2) states that $[\phi]$ behaves like a modal $\Box$ in that it distributes over $\rightarrow$; its validity is clear.

By (A3), formulas of the form $[\phi]\psi$ are false off the diagonal (which is what we want) , while (A4) and (A5) concern the situation where the ϕ-part in $[\phi]\psi$ is true at the diagonal itself. Essentially, the effect of $[\phi]\psi$ is then like the implication $\phi \rightarrow \psi$. (A6) states that the second argument of $[\cdot]\cdot$ refers to diagonal points.

(D) can be seen as a definition of $\langle\phi\rangle\psi$ in terms of δ and $[\phi]\psi$, namely: on the diagonal $\langle\phi\rangle$ is the dual of $[\phi]$, off the diagonal $\langle\phi\rangle\psi$ is false.

Concerning the rules of inference, note that (N) is like the modal rule of necessitation, but not quite the same: we need the antecedent (δ) in the conclusion $\delta \rightarrow [\phi]\psi$, because $[\phi]\psi$ is false off the diagonal. (MO) expresses anti-monotonicity of the dyadic operator, $[\cdot]\cdot$, in its first argument. To show its soundness: assume $(i,j) \models [\phi']\psi$ in some relativised model, and let k be such that $(i,k) \models \phi$. Then by $\vdash^r \phi \rightarrow \phi'$ and the (implicit) induction hypothesis that therefore $\phi \rightarrow \phi'$ is r-valid, $(i,k) \models \phi'$; but then $(k,k) \models \psi$. So $(i,j) \models [\phi]\psi$.

The remaining two axioms are valid in full squares only. First consider (A7): let (i,j) be a world in a full square model such that $(i,j) \models \langle\top\rangle[\top]\phi$, then $i = j$ and there is a k with $(k,k) \models [\top]\phi$. To show that $(i,i) \models [\top]\phi$, let l be such that $(i,l) \models \top$. Note that now (k,l) exists, as our model is full. Therefore $(l,l) \models \phi$, as $(k,k) \models [\top]\phi$.

The last and most interesting axiom of FAX is (A8), which is closely connected to the so-called Henkin equation in cylindric algebra; see Henkin, Monk and Tarski (1971). To show its validity, let $\langle\phi\rangle\psi \wedge \langle\neg\phi\rangle\psi$ be true in a world (i,j) of some full square model. Then $i = j$ and there are k, k' with $(i,k) \models \phi$, $(i,k') \models \neg\phi$, $(k,k) \models \psi$ and $(k',k') \models \psi$. The important thing to notice is that k and k' are *different*, as ϕ is true at (i,k) and false at (i,k'). This implies that there are at least *two* different points on the diagonal where ψ holds. But then from any arbitrary point (l,l) on the diagonal, there will always be a different point where ψ is true (note that in this step we need the fact that both (l,k) and (l,k') exist), so $(l,l) \models \langle\neg\delta\rangle\psi$. As l was arbitrary, this gives $(i,i) \models [\top]\langle\neg\delta\rangle\psi$. Thus we have proved the required soundness results. $\Box$

The proof of the completeness part of the theorem is spread out over the next two subsections. We will need the basic properties that are listed below (without proof):

Proposition 5.5 *(i) Let Δ be a MCS. Then:*

$\phi \wedge \psi \in \Delta$	*iff*	$\phi \in \Delta$ *and* $\psi \in \Delta$
$\phi \vee \psi \in \Delta$	*iff*	$\phi \in \Delta$ *or* $\psi \in \Delta$
$\neg\phi \in \Delta$	*iff*	$\phi \notin \Delta$,
$\phi \rightarrow \psi \in \Delta, \phi \in \Delta$	*implies*	$\psi \in \Delta$.

(ii) Every consistent set is contained in a maximal consistent set. That is, Lindenbaum's Lemma holds. $\Box$

Proposition 5.6 *Some theorems and derived rules of the system:*

$(T1)$	$\langle\phi\rangle\psi \rightarrow (\langle\phi\wedge\chi\rangle\psi \vee \langle\phi\wedge\neg\chi\rangle\psi)$
$(T2)$	$\langle\phi\wedge\delta\rangle\psi \rightarrow (\phi\wedge\psi)$
$(T3)$	$\langle\phi\rangle(\psi\vee\chi) \rightarrow (\langle\phi\rangle\psi \vee \langle\phi\rangle\chi)$
$(DR1)$	*From* $\vdash \phi\rightarrow\psi$ *conclude* $\vdash \langle\phi\rangle\chi \rightarrow \langle\psi\rangle\chi$
$(DR2)$	*From* $\vdash \phi\rightarrow\psi$ *conclude* $\vdash \langle\chi\rangle\phi \rightarrow \langle\chi\rangle\psi$

□

5.1 Completeness: Relativized Case

The idea underlying the completeness proof is the following: we start with a MCS Σ, and we want to find a relativised model $\mathfrak{M}$ for it. This model $\mathfrak{M}$ will be built up in stages. At stage 0, we know everything about one point (pair) of the model, namely that all $\sigma\in\Sigma$ should hold at it. At the final stage, we know everything about all points. In the successive intermediate steps we know everything about more and more points.

To formalize this plan, we define:

Definition 5.7 *A* **matrix** *is a triple* $\lambda = (l, R, \Lambda)$ *with* l *a natural number or* ω, $R\subseteq l\times l$ *(thta is,* R *consists of objects of the form* (i,j) *with* $i,j<l$*), and* Λ *is a function, assigning an MCS to every element of* R. *The image* $\Lambda(i,j)$ *will sometimes be called the* **label** *of* (i,j). □

Matrices determine valuations, and thus models, in an obvious way:

Definition 5.8 *Let* $\lambda = (l,R,\Lambda)$ *be a matrix. The model determined by* λ *(notation:* $\mathfrak{M}^\lambda$*), is given as* $\mathfrak{M}^\lambda = (R, V^\lambda)$ *with* R *from* λ, *and* V^λ *given by*

$$(*)\qquad (i,j)\in V^\lambda(p) \text{ iff } p\in\Lambda(i,j).$$

In other words, atoms are true at a point (i,j) *iff they belong to the label of* (i,j). □

Now to come back to the completeness issue, the aim of the proof is to find a construction of a matrix such that $(*)$ does not hold for atomic formulas only, but for *every* formula. We will come up with a matrix λ having the following nice property:

$$(TL)\qquad \text{For all } \phi\text{: } \mathfrak{M}^\lambda,(i,j)\models\phi \text{ iff } \phi\in\Lambda(i,j).$$

After establishing (TL), we are virtually finished, at least if our MCS Σ is the label of some pair (i_0,j_0) in the domain of Λ — which will be the case by construction. Indeed, (TL) will then guarantee that all Σ-formulas are true at the point (i_0,j_0), which is precisely what we want.

So, in order to find a construction yielding a matrix satisfying (TL), let us first give some necessary (and sufficient) conditions. The basic idea here is that if *truth* of a formula at a pair will correspond to its *membership* in the label of the pair, then the labelling must satisfy conditions resembling the inductive definition of truth:

Definition 5.9 *Let* $\lambda = (l,R,\Lambda)$ *be a matrix. We call* λ **coherent** *if it satisfies (CH1) and (CH2) below, and* **saturated** *if it meets requirement (S):*

(CH1)	*For all* $(i,j)\in R$: $\delta\in\Lambda(i,j)$ *iff* $i=j$
(CH2)	*For all* $(i,j)\in R$: $[\phi]\psi\in\Lambda(i,i)$ *and* $\phi\in\Lambda(i,j)$ *implies* $\psi\in\Lambda(j,j)$
(S)	*For all* $(i,i)\in R$: $\langle\phi\rangle\psi\in\Lambda(i,i)$ *implies* $\exists j(\phi\in\Lambda(i,j)$ & $\psi\in\Lambda(j,j))$

λ *is* **perfect** *if it is both coherent and saturated.* □

Proposition 5.10 *For any perfect matrix we can prove a truth lemma:*

$$(TL) \qquad \text{For all } \phi\text{: } \mathfrak{M}^\lambda, (i,j) \models \phi \text{ iff } \phi \in \Lambda(i,j)$$

Proof.
The proof is by an induction on ϕ, of which we only present one of the modal cases, viz. where ϕ is of the form $[\psi]\chi$. Following our convention, in the proof that follows we suppress reference to the induced model in the notation for satisfaction, writing $(i,j) \models \xi$ for $\mathfrak{M}^\lambda, (i,j) \models \xi$.

(*Left to right.*) Let $(i,j) \models [\psi]\chi$. By the truth definition, $i = j$, so by (CH1), $\delta \in \Lambda(i,j)$. To arrive at a contradiction, suppose that $[\psi]\chi \notin \Lambda(i,j)$. By (D) then $\langle\psi\rangle\neg\chi$ is in $\Lambda(i,i)$. By saturation, there is a k with $\psi \in \Lambda(i,k)$ and $\neg\chi \in \Lambda(k,k)$. The inductive hypothesis now gives $(i,k) \models \psi$ and $(k,k) \models \neg\chi$, so using the truth definition we indeed contradict $(i,j) \models [\psi]\chi$.

(*Right to left.*) Let $[\psi]\chi \in \Lambda(i,j)$, then we have to show that $(i,j) \models [\psi]\chi$. First of all, by (A3), $\delta \in \Lambda(i,j)$, so by (CH1), $i = j$. Now let k be such that $(i,k) \models \psi$. We have to establish that $(k,k) \models \chi$. Note that by the induction hypothesis this boils down to proving $\chi \in \Lambda(k,k)$ from $\psi \in \Lambda(i,k)$. But this is immediate by (CH2) and the assumption $[\psi]\chi \in \Lambda(i,j)$. □

It will be clear from the proof that the conditions (CH) and (S) are designed so as to let the inductive step of the above proof go through. It is convenient for later use to introduce some terminology for expressing condition (CH2).

Definition 5.11 *Let Γ, Φ and Θ be MCSs. The notation $arrow(\Gamma, \Phi, \Theta)$ means that $\delta \in \Gamma$, and for all formulas ϕ and θ: $[\phi]\theta \in \Gamma$ and $\phi \in \Phi$ imply $\theta \in \Theta$.* □

In words: $arrow(\Gamma, \Phi, \Theta)$ expresses that nothing in our logic forbids the MCSs Γ and Θ to be 'connected by' Φ. Note that condition (CH2) is equivalent to:

$$\text{For all } (i,j) \in R: \ arrow(\Lambda(i,i), \Lambda(i,j), \Lambda(j,j)).$$

A useful fact about the concept just introduced:

Proposition 5.12 *Let Γ, Φ and Θ be MCSs such that $arrow(\Gamma, \Phi, \Theta)$. Then $\delta \in \Phi$ implies $\Gamma = \Phi = \Theta$.*

Proof.
First we show that $\Gamma \subseteq \Theta$: let γ be an arbitrary element of Γ. By (A5), $[\delta]\gamma \in \Gamma$, so by $arrow(\Gamma, \Phi, \Theta)$ we have $\gamma \in \Theta$.

Second, we establish that $\Phi \subseteq \Theta$: take an arbitrary ϕ in Φ, then $\phi \wedge \delta \in \Phi$. As $(\phi \wedge \delta) \rightarrow \phi$ is an axiom and thus in Γ, we can use (A5) to find $[\delta \wedge \phi]\phi \in \Gamma$. Then by $arrow(\Gamma, \Phi, \Theta)$ we get $\phi \in \Theta$.

The proposition now follows from the fact that any two MCSs are identical if one is contained in the other. □

We are now able to go into some more detail concerning our construction of a perfect matrix λ. This λ will be built up in countably many stages. In every stage we are dealing with a finite, coherent *approximation* of λ. Such an approximation will in general not be perfect, but the principle of the construction is that any of its defects can, and eventually will, be repaired (in a sense made precise below). By a standard combinatorial trick we can make sure that in the limit step of the construction we have created a coherent matrix without defects.

Convention 5.13 *Assume that we have a fixed set of propositional variables.* □

Definition 5.14 *Let $\lambda = (l, R, \Lambda)$ be a matrix. A* **defect** *of λ is a pair $d = (\phi, i)$ where ϕ is of the form $\langle\psi\rangle\chi$ and (i, i) is in R such that*

$$\langle\psi\rangle\chi \in \Lambda(i, i), \quad \textit{but there is no } j \textit{ with } \psi \in \Lambda(i, j) \textit{ and } \chi \in \Lambda(j, j).$$

Assume that we have an enumeration of all objects of the form $(\langle\phi\rangle\psi, n)$ with $\langle\phi\rangle\psi$ a formula and n a natural number. □

Note that intuitively, defects are witnesses that a matrix does not satisfy condition (S).

Definition 5.15 *Let $\lambda = (l, R, \Lambda)$ and $\mu = (m, P, M)$ be matrices. λ is a* **submatrix** *of μ, or μ an* **extension** *of λ, notation: $\lambda \subset \mu$, if (i) $l < m$, (ii) $R \subset P$, and (iii) for all $(i, j) \in R$, $\Lambda(i, j) = M(i, j)$.*

Let $\lambda_1, \lambda_2, \ldots$ be a sequence of matrices such that $\lambda_1 \subset \lambda_2 \subset \ldots$. The **union** *$\lambda = \bigcup_{n\in\omega} \lambda_n$ is given as $\lambda = (l, R, \Lambda)$ where $l = \sup\{l_n \mid n \in \omega\}$, $R = \bigcup_{n\in\omega} R_n$ and $\Lambda = \bigcup_{n\in\omega} \Lambda_n$.* □

Proposition 5.16 *Let $\lambda_1 \subset \lambda_2 \subset \ldots$ be a sequence of coherent matrices. Then its union $\lambda = \bigcup_{n\in\omega} \lambda_n$ is coherent too.*

Proof.
We only check (CH2), leaving (CH1) to the reader. Let $[\phi]\psi$ be a formula in $\Lambda(i, i)$ such that $\phi \in \Lambda(i, j)$. By definition of λ, there is an n such that $[\phi]\psi \in \Lambda_n(i, i)$ and $\phi \in \Lambda_n(i, j)$. As λ_n is coherent, this implies $\psi \in \Lambda_n(j, j)$; but then, using the definition of λ again, we have that $\psi \in \Lambda(j, j)$. □

The main lemma of the completeness proof is the following, which states that every defect of a matrix can be repaired:

Proposition 5.17 (Repair Lemma) *Let $\lambda = (l, R, \Lambda)$ be a coherent matrix with a defect d. Then λ has a coherent extension $\lambda' = (l + 1, R', \Lambda')$ lacking this defect.*

The proof of this lemma will come later; first we want to show *why* this lemma is the crucial one in the construction. The reason is: by successive applications of the Repair Lemma, we can extend any coherent matrix to a perfect one; this is the content of the Extension Lemma:

Proposition 5.18 (Extension Lemma) *Let λ be a coherent matrix. Then there is a perfect matrix λ' extending λ.*

Proof.
Let $\lambda = (l, R, \Lambda)$ be a coherent matrix. We will define a sequence $\lambda = \lambda_0 \subset \lambda_1 \subset \ldots$ such that the union λ_ω is the required perfect matrix.

For any arbitrary matrix μ, we define μ^+ as follows: if μ is perfect, then $\mu^+ = \mu$. Otherwise, let d be the first defect of μ (with respect to the enumeration given in convention 5.14), and define μ^+ as the result of repairing d in μ. (Note that this repair is possible by the Repair Lemma.)

Now set

$$\begin{aligned} \lambda_0 &= \lambda \\ \lambda_{n+1} &= \lambda_n^+ \\ \lambda_\omega &= \bigcup_{n\in\omega} \lambda_n \end{aligned}$$

By the Repair Lemma, every λ_n is coherent, so by proposition 5.16, λ_ω is coherent. Now suppose that λ_ω is not saturated; let $d = (\langle\phi\rangle\psi, i)$ be its *first* defect. There must be a stage n in the construction such that d is also the first defect of λ_n. But then d is repaired at this stage, so that it is not a defect of $\lambda_{n+1} = \lambda_n^+$ anymore. In other words, there is a $j \leq l_{n+1}$ such that $\phi \in \Lambda_{n+1}(i,j)$ and $\psi \in \Lambda_{n+1}(j,j)$ (in fact, by construction it follows that $j = l_n + 1$). Now the definition of a union tells us that this is in contradiction with the fact that d is a defect of λ_ω. □

Further, with the Extension Lemma proved, it is now straightforward to show:

Proof of Relativized Completeness.
Let Σ be a consistent set. Extend Σ to a maximal consistent set Σ'. Distinguish two cases:

(I) $\delta \in \Sigma'$. In this case, let λ be the matrix $(1, \{(0,0)\}, \Lambda)$ with $\Lambda(0,0) = \Sigma'$.

(II) $\neg\delta \in \Sigma'$. We leave it to the reader to verify that there are MCSs Γ, Δ with $arrow(\Gamma, \Sigma', \Delta)$. Let λ be the matrix $(2, R, \Lambda)$, where $R = \{(0,0), (0,1), (1,0)\}$ and $\Lambda(0,0) = \Gamma$, $\Lambda(0,1) = \Sigma'$ and $\Lambda(1,1) = \Delta$.

In both cases we are dealing with a coherent matrix. Let μ be any perfect extension of λ, then by Proposition 5.10, there is a model $(\mathfrak{M}^\mu)$ with a world where all Σ' and a fortiori, all Σ-formulas are true. Thus we have found a model for Σ. □

Thus the crucial step in the argument really is the Repair Lemma, so let us now turn to its proof:

Proof of Repair Lemma.
Let d be a defect of the coherent matrix $\lambda = (l, R, \Lambda)$. Without loss of generality we may assume that d is of the form $(\langle\phi\rangle\psi, 0)$. Our aim is to find MCSs Φ, Ψ such that if we extend λ with the pairs $(0, l)$ and (l, l) having labels Φ resp. Ψ, the resulting λ' is the matrix we are looking for. We will go one level deeper in the proof nesting: here is a sublemma providing suitable Φ and Ψ:

Proposition 5.19 (Arrow Lemma) *Let $\langle\phi\rangle\psi$ a formula of the MCS Γ. Then there are MCSs Φ, Ψ such that $\phi \in \Phi$, $\psi \in \Psi$ and $arrow(\Gamma, \Phi, \Psi)$.*

Proof.
By (T1), $\langle\phi\rangle\psi \to (\langle\phi \wedge \delta\rangle\psi \vee \langle\phi \wedge \neg\delta\rangle\psi)$ is in Γ, (cf. 5.5), so at least one of $\langle\phi \wedge \delta\rangle\psi$ or $\langle\phi \wedge \neg\delta\rangle\psi$ is in Γ. The first case is easy: take $\Phi = \Psi = \Gamma$ (for, by (T2) both ϕ and ψ are in Γ.)

The other case requires more effort. Φ and Ψ will be constructed by a simultaneous induction. Set $\Phi_0 := \{\phi \wedge \neg\delta\}$ and $\Psi_0 := \{\psi\}$.

Let there be given an enumeration $\alpha_0, \alpha_1, \ldots$ of the set of all formulas. In the inductive step $n+1$ of the proof, we will decide, for both Φ_{n+1} and Ψ_{n+1}, whether to put α_n or $\neg\alpha_n$ in. Write, for brevity, α instead of α_n.

We assume that Φ_n and Ψ_n are given, and finite, and that ϕ_n (ψ_n) is defined as the conjunction of all formulas in Φ_n (Ψ_n). As an induction hypothesis we also assume that $\langle\phi_n\rangle\psi_n$ is in Γ.

First we consider Φ_{n+1}: by proposition 5.6 we have $\vdash \langle\phi_n\rangle\psi_n \rightarrow (\langle(\phi_n \wedge \alpha)\rangle\psi_n \vee (\langle(\phi_n \wedge \neg\alpha)\rangle\psi_n)$, so by the maximal consistency of Γ, we find one of $\langle(\phi_n \wedge \alpha)\rangle\psi_n$ or $\langle(\phi_n \wedge \neg\alpha)\rangle\psi_n$ in Γ.

In the first case, set $\Phi_{n+1} := \Phi_n \cup \{\alpha\}$, otherwise $\Phi_{n+1} := \Phi_n \cup \{\neg\alpha\}$. Note that now $\langle\phi_{n+1}\rangle\psi_n$ is in Γ.

For Ψ_{n+1} our action is similar: using (DR2), (T3) and some propositional logic, we show that either $\langle\phi_{n+1}\rangle(\psi \wedge \alpha)$ or $\langle\phi_{n+1}\rangle(\psi \wedge \neg\alpha)$ is in Γ. We define $\Psi_{n+1} := \Psi_n \cup \{\alpha\}$ or $\Psi_{n+1} := \Psi_n \cup \{\neg\alpha\}$, depending on which is the case.

Note that both Φ_{n+1} and Ψ_{n+1} are finite, and that by definition of Ψ_{n+1}, $\langle\phi_{n+1}\rangle\psi_{n+1}$ is in Γ, so we are safe for the next inductive step.

We set $\Phi := \bigcup_{n\in\omega}\Phi_n$, $\Psi := \bigcup_{n\in\omega}\Psi_n$. It follows by construction that Φ and Ψ are MCSs, and that $\phi \in \Phi$ and $\psi \in \Psi$.

To show that $arrow(\Gamma, \Phi, \Psi)$, let $[\eta]\theta$ be in Γ and η in Φ. Suppose, to arrive at a contradiction, that $\theta \notin \Psi$. As Ψ is a MCS, this implies $\neg\theta \in \Psi$. By definition of Φ and Ψ, there is a natural number n such that $\eta \in \Phi_n$, $\neg\theta \in \Psi_n$. By definition of ϕ_n and some obvious propositional logic we find: $\vdash \phi_n \rightarrow \eta$, $\vdash \psi_n \rightarrow \neg\theta$. By construction we have $\langle\phi_n\rangle\psi_n \in \Gamma$, so by (DR1), (DR2), $\langle\eta\rangle\neg\theta \in \Gamma$. This gives the desired contradiction, as δ is in Γ. Thus the Arrow Lemma is proved.

With the Arrow Lemma established, the proof of the Repair Lemma is now straightforward. The Arrow Lemma gives us Φ and Ψ with $\phi \in \Phi$, $\psi \in \Psi$ and $arrow(\Lambda(0,0), \Phi, \Psi)$. We define λ' by

$$\begin{aligned} R' &= \{R \cup \{(0,l),(l,l)\} \\ \Lambda' &= \Lambda \cup \{((0,l),\Phi),((l,l),\Psi)\}. \end{aligned}$$

To check that λ' is coherent, let us first consider condition (CH1). The only thing to worry about is whether $\neg\delta$ is in $\Lambda(0,l) = \Phi$. Now if this were not the case, then by proposition 5.12 we have $\Lambda(0,0) = \Phi = \Psi$. But now we would find both ϕ and ψ in $\Lambda(0,0)$, contradicting the fact that $(\langle\phi\rangle\psi, 0)$ is a *defect* of λ.

It is straightforward to verify that λ' meets (CH2), and as clearly $\lambda' \supset \lambda$, we have found an extension of λ meeting all the required constraints. □

5.2 Completeness: Full Square Case

The completeness proof with respect to the full square semantics uses the same idea used in the relativized square case: starting with a MCS Σ, we build a model $\mathfrak{M}$ for Σ by approximating it with matrices. The difference is that $\mathfrak{M}$ must be based on a full square frame, and as a consequence, we want all approximating matrices to be *full* as well:

Definition 5.20 *A matrix $\lambda = (l, S, \Lambda)$ is* **full** *if $S = l \times l$. Such a full matrix will be represented as a pair $\lambda = (l, \Lambda)$.* □

The reader is invited to check that, *except for the Repair Lemma*, all of the definitions, constructions and lemmas of the previous subsection go through with the obvious adaptations: the union of a sequence of coherent and full matrices is full and coherent, a full matrix determines a full square model, and so on. Thus it only remains to prove the full matrix version of the Repair Lemma. In the proof of this, the following technical lemma will play a crucial rôle:

Proposition 5.21 (Fill Lemma) *Let $\Gamma, \Phi, \Theta, \Phi'$ and Θ' be MCSs such that $arrow(\Gamma, \Phi, \Theta)$ and $arrow(\Gamma, \Phi', \Theta')$. Then there is a MCS Π with $arrow(\Theta, \Pi, \Theta')$ such that $\neg\delta \in \Pi$ if $\Phi \neq \Phi'$ or $\Theta \neq \Theta'$.*

Proof.
If $\Phi = \Phi'$ and $\Theta = \Theta'$, take $\Pi = \Theta$.

In the other case, Π will be built up in stages, and as in the proof of the Arrow Lemma given in the previous subsection, every approximation Π_n of Π is a finite set of formulas, the conjunction of which is denoted by π_n. As an inductive hypothesis of our construction, we take

$$(IH) \qquad\qquad arrow(\Theta, \Pi_n, \Theta'),$$

where we stretch the definition of the arrow-relation (cf. Definition 5.11) to arbitrary consistent sets.

First we define $\Pi_0 := \{\neg\delta\}$.

Claim 1: (IH) holds for $n = 0$.
Proof: Assume that $[\neg\delta]\psi \in \Theta$. We have to show $\psi \in \Theta'$. Distinguish the following two cases:

(I) $\Theta \neq \Theta'$. In this case, there is a θ' with $\theta' \in \Theta'$, $\neg\theta' \in \Theta$. By $[\neg\delta]\psi \in \Theta$, it easily follows that $[\neg\delta](\theta' \rightarrow \psi) \in \Theta$; $\theta' \notin \Theta$ implies $\theta' \rightarrow \psi \in \Theta$, so we also have $[\delta](\theta' \rightarrow \psi) \in \Theta$. By (A1) then, $[\top](\theta' \rightarrow \psi) \in \Theta$. We leave it to the reader to verify that with $arrow(\Gamma, \Phi, \Theta)$, this gives $\langle\top\rangle[\top](\theta' \rightarrow \psi) \in \Gamma$. Now we need the new axiom (A7), giving $[\top](\theta' \rightarrow \psi) \in \Gamma$. So by $arrow(\Gamma, \Phi', \Theta')$we find $\theta' \rightarrow \psi \in \Theta'$, and as $\theta' \in \Theta'$, $\psi \in \Theta'$.

(II) $\Theta = \Theta'$; by assumption this implies $\Phi \neq \Phi'$, so there is a ϕ with $\phi \in \Phi$, $\neg\phi \in \Phi'$. Suppose that $\psi \notin \Theta'$, so $\neg\psi \in \Theta' = \Theta$. It is easy to check that this gives $\langle\phi\rangle\neg\psi \in \Gamma$ and $\langle\neg\phi\rangle\neg\psi \in \Gamma$. So by the new axiom (A8), we get $[\top]\langle\neg\delta\rangle\neg\psi \in \Gamma$. Then our assumption $arrow(\Gamma, \Phi, \Theta)$ gives $\langle\neg\delta\rangle\neg\psi \in \Theta$, contradicting the fact that $\delta \wedge [\neg\delta]\psi \in \Theta$. This concludes the proof of Claim 1.

Let there be given an enumeration $\alpha_0, \alpha_1, \ldots$ of the set of all formulas. In the inductive step $n+1$, of the proof, we will decide to put either α_n or $\neg\alpha_n$ in Π_{n+1}. Write α instead of α_n, and define $\Pi^+ := \Pi_n \cup \{\alpha\}$, $\Pi^- := \Pi_n \cup \{\neg\alpha\}$.

Claim 2: At least one of the following holds: $arrow(\Theta, \Pi^+, \Theta')$ or $arrow(\Theta, \Pi^-, \Theta')$.
Proof: Suppose neither is true, then there are ξ^+ and ξ^- in Θ' such that $[\pi_n \wedge \alpha]\neg\xi^+ \in \Theta$ and $[\pi_n \wedge \neg\alpha]\neg\xi^- \in \Theta$. Let ξ be the formula $\xi^+ \wedge \xi^-$, then $\xi \in \Theta'$ and $[\pi_n \wedge \alpha]\neg\xi \wedge [\pi_n \wedge \neg\alpha]\neg\xi \in \Theta$. By axiom (A1) this gives $[\pi_n]\neg\xi \in \Theta$, so by (IH) $\neg\xi \in \Theta'$, by which we arrive at a contradiction. This concludes the proof of claim 2.

By taking $\Pi_{n+1} := \Pi^+$ or Π^- according to whether $arrow(\Theta, \Pi^+, \Theta')$ or $arrow(\Theta, \Pi^-, \Theta')$, it is clear that we find a new approximation satisfying the induction hypothesis for $n+1$.

Finally, set $\Pi := \bigcup_{n \in \omega} \Pi_n$. We leave it to the reader to verify that Π is maximal consistent and that $arrow(\Theta, \Pi, \Theta')$. □

Proposition 5.22 (Full Repair Lemma) *Let $\lambda = (l, \Lambda)$ be a coherent full matrix with a defect d. Then there is a coherent full matrix $\lambda' = (l+1, \Lambda') \supset \lambda$ lacking this defect.*

Proof.
Without loss of generality we may assume that the defect is of the form $(\langle\phi \wedge \neg\delta\rangle\psi, 0)$. The Arrow Lemma 5.19 gives us labels for $(0, l)$ and (l, l) such that $\phi \wedge \neg\delta \in \Lambda(0, l)$, $\psi \in \Lambda(l, l)$ and $arrow(\Lambda(0,0), \Lambda(0,l), \Lambda(l,l))$ (cf. the proof of the old Repair Lemma). Pictorially, the situation is as follows:

$$\begin{array}{ccccccc}
\Lambda(0,l) & ? & \ldots & ? & \ldots & ? & \Lambda(l,l) \\
\\
\Lambda(0,l-1) & \Lambda(1,l-1) & \ldots & \Lambda(i,l-1) & \ldots & \Lambda(l-1,l-1) & ? \\
\\
\vdots & \vdots & & \vdots & & \vdots & \vdots \\
\\
\Lambda(0,i) & \Lambda(1,i) & \ldots & \Lambda(i,i) & \ldots & \Lambda(l-1,i) & ? \\
\\
\vdots & \vdots & & \vdots & & \vdots & \vdots \\
\\
\Lambda(0,1) & \Lambda(1,1) & \ldots & \Lambda(i,1) & \ldots & \Lambda(l-1,1) & ? \\
\\
\Lambda(0,0) & \Lambda(1,0) & \ldots & \Lambda(i,0) & \ldots & \Lambda(l-1,0) & ?
\end{array}$$

We still have to find suitable images for the pairs indicated by the question marks:

(I) Labels for the pairs (i, l) with $0 < i < l$.
We can consider the procedure for each $\Lambda(i, l)$ separately. Consider the Fill Lemma above, with $\Gamma = \Lambda(0,0)$, $\Phi = \Lambda(0, i)$, $\Theta = \Lambda(i, i)$, $\Phi' = \Lambda'(0, l)$ and $\Theta' = \Lambda(l, l)$. The lemma gives us MCSs Π_i with $arrow(\Lambda(i,i), \Pi_i, \Lambda(l,l))$ and $\neg\delta \in \Pi_i$. (For the last assumption, note that we have $\Lambda(i,i) \neq \Lambda(l,l)$ or $\Lambda(0,i) \neq \Lambda(0,l)$ — otherwise there would have been no defect). So, by taking

$$\Lambda'(i,l) := \Pi_i,$$

we keep the matrix coherent.

(II) Labels for the pairs (l, i) with $0 \leq i < l$.
Again, for each $\Lambda'(l, i)$ separately, we apply the Fill Lemma, now with $\Gamma = \Phi' = \Theta' = \Lambda(0,0)$, $\Phi = \Lambda'(i, l)$ and $\Theta = \Lambda'(l, l)$. Note that $\Gamma \neq \Lambda'(i, l)$ as $\delta \in \Gamma$ and $\neg\delta \in \Lambda'(i, l)$. So, the Fill Lemma provides MCSs Ψ_i satisfying $arrow(\Lambda(l,l), \Psi_i, \Lambda(i,i))$ and $\neg\delta \in \Psi_i$. We set

$$\Lambda'(l,i) := \Psi_i$$

and again we have not destroyed the coherency of the matrix.

We leave it to the reader to make a final check that the newly defined matrix Λ' satisfies all the required conditions. □

We have thus proved the crucial result needed to establish:

Proof of Full Square Completeness.
A straightforward refinement of the proof for the relativized case that makes use of the Full Repair Lemma. □

6 A Boolean free fragment

In this section we examine the logic of $\Rightarrow$ in the absence of the Boolean operators. As has already been explained, this seems a sensible path to explore: the use of dynamic semantics is often seen as a replacement of, not as an accompaniment to, static notions. For example, in Groenendijks and Stokhof's system of DPL only dynamic operators are present. The Boolean free system we examine here has reasonable claim to the title 'minimal': the system has only $\Rightarrow$ and $\perp$ present as logical operators. (Actually, we will also include δ as a primitive symbol, for the diagonal will play a prominent role in the sequel. However one of the first things we will note is that δ is definable in terms of $\Rightarrow$ and $\perp$, thus we could dispense with it if we wished.)

To be more precise, we are going to work with the following language L'. It's primitive symbols are a denumerably infinite collection ATOM of atomic symbols, which we write as p, q, r and so on; two constant symbols, $\perp$ and δ; a two place modality $\Rightarrow$; and the punctuation symbols) and (. The set of wffs of our language WFF is the smallest set containing $\perp$, δ and all the elements of ATOM such that if $\phi, \psi \in$ WFF, then $(\phi \Rightarrow \psi) \in$ WFF. The following syntactic definitions will prove useful. We define the *degree* of a wff (that is, $deg(\phi)$) to be the number of occurrences of $\Rightarrow$ it contains. If ϕ contains at least one occurrence of $\Rightarrow$ then ϕ is called a *complex wff.*

The definition of model, and the distinction between relativised models and square models are unchanged from previous sections. Let $\mathfrak{M}$ $(= (\mathfrak{T}, V))$ be any model, and let $(i, j) \in \mathfrak{T}$. Then, for all L' wffs ϕ, we define the satisfaction relation $\mathfrak{M}, (i, j) \models \phi$ as follows:

$$
\begin{array}{lll}
\mathfrak{M}, (i,j) \not\models \perp & & \\
\mathfrak{M}, (i,j) \models \delta & \text{iff} & i = j \\
\mathfrak{M}, (i,j) \models q & \text{iff} & (i,j) \in V(q), \text{ for all propositional variables } q \\
\mathfrak{M}, (i,j) \models \phi \Rightarrow \psi & \text{iff} & i = j \ \textit{and} \\
 & & (\forall k)(\mathfrak{M}, (i,k) \models \phi \ \ \textit{implies} \ \ (\exists l)(\mathfrak{M}, (k,l) \models \psi)))
\end{array}
$$

In short, in effect we have stepped back to the beginning of this paper and are working directly with the original notion of dynamic implication with its 'three step' evaluation pattern. Because we don't have full Boolean expressivity at our disposal anymore, we can't work with the simpler evaluation pattern of $\rhd$ anymore, but will have to tackle the logic of $\Rightarrow$ directly. Note that because we have chosen a different stock of primitives, L' is not, strictly speaking, a fragment of L; but it will be clear from our earlier discussion how to truthfully embed L' in L, and in what follows we shall simply regard L' as a Boolean free fragment of L.

Before defining a suitable notion of validity, let's note some basic properties of L'. First of all, note that in one very obvious respect our language is essentially simple: not much can happen off the diagonal. More accurately, in any model $\mathfrak{M}$ at any pair (i, j) such that $i \neq j$, the only wffs that can be satisfied are propositional variables. All the other wffs in the language are tests.

In spite of this limitation L' is capable of some interesting things. First, as has already been mentioned, note that we need not have included δ as a primitive operator: as a simple check shows, given any wff ϕ, any model $\mathfrak{M}$, and any pair (i, j) in $\mathfrak{M}$, we have that

$\mathfrak{M}, (i,j) \models (\perp \Rightarrow \phi)$ iff $i = j$. (Note that the choice of ϕ is completely irrelevant to the satisfiability of such wffs.)

Next, note that we can define the 'test version' Φ^t of the two dimensional cylindric operator Φ. We define $\Phi^t \phi$ to be $\delta \Rightarrow \phi$. It is easy to see that in any model $\mathfrak{M}$ at any diagonal pair (i,i), we have that $\mathfrak{M}, (i,i) \models \Phi^t \phi$ iff there is a k such that $\mathfrak{M}, (i,k) \models \phi$. Incidentally, this operator was considered by Groenendijk and Stokhof (1991). They write it as $\Diamond$ and call it the closure (or assertion) operator.

Next, note that we can define the 'dynamic negation' introduced by Groenendijk and Stokhof. We define $-\phi$ to be $\phi \Rightarrow \perp$. A simple check shows that $\phi \Rightarrow \perp$ is satisfied in any model $\mathfrak{M}$ at a diagonal pair (i,i) iff there is no k such that $\mathfrak{M}, (i,k) \models \phi$, and this is precisely how dynamic negation is defined in DPL. Note (as Groenendijk and Stokhof point out) that $- - \phi$ is just $\Phi^t \phi$.

The next thing to notice is that L' is strong enough to simulate classical propositional calculus; indeed propositional calculus can be embedded in it quite transparently.[7] The following definition isolates a fragment that is a simple image of propositional calculus:

Definition 6.1 *Let PC be the smallest set of L' wffs containing $\perp$, Φq (for all propositional variables q), and such that if $\phi, \psi \in$ PC, $(\phi \Rightarrow \psi) \in$ PC.* □

Note that all the wffs in PC are tests. This yields:

Proposition 6.2 *For all $\phi \in$ PC, ϕ is satisfiable in a model iff ϕ is satisfiable in a singleton model.*

Proof.
The right to left direction is trivial. For the converse, suppose $\mathfrak{R}, (i,j) \models \phi$, where $\mathfrak{R} = (\mathfrak{R}, V)$. As all wffs in PC are tests, $i = j$. Let $\mathfrak{R}' = \{(i,i)\}$ and let V' be the valuation on $\mathfrak{R}'$ defined by $(i,i) \in V'(q)$ iff there is a k such that $(i,k) \in V(q)$, for all atoms q. Clearly $(\mathfrak{R}', V')$ is a singleton relativised model. The fact that it satisfies ϕ follows by induction on $deg(\phi)$. □

Now it should be clear how to use PC to simulate ordinary propositional calculus. Given any wff φ of propositional calculus (which without loss of generality we can regard as constructed out of our propositional variables using material implication $\rightarrow$ and $\perp$) we turn it into a PC wff by the simple expedient of replacing every $\rightarrow$ by a $\Rightarrow$, and every propositional variable q by $\Phi^t q$. More precisely, we translate φ into a PC wff ϕ as follows: $T(q) = \Phi q$; $T(\perp) = \perp$; and $T(\phi \rightarrow \pi) = T(\phi) \Rightarrow T(\pi)$. This translation gives us all we need: φ is a satisfiable wff (in the usual sense of propositional calculus) iff $T(\varphi)$ is a satisfiable wff of L'. To see this we need merely observe that if φ is a satisfiable wff of propositional calculus, then the satisfying truth assignment gives rise to a singleton L' model in the obvious way. Conversely, if $T(\varphi)$ is a satisfiable L' wff, then as $T(\varphi) \in$ PC, by the previous proposition it is satisfiable in a singleton model. The valuation in this singleton model can be regarded as a truth assignment in the sense of propositional calculus, and this assignment satisfies φ.

But while L' is capable of a suprising amount, the lack of the Booleans lead to obvious expressive limitations. For example, we *cannot* stand on the diagonal and insist that a piece of information ϕ is true everywhere on the same longitude. That is, we cannot define a universal operator $\boxminus^t$ to match Φ^t. We will see later why this is so.

With these preliminaries to hand, let's discuss the notion of *validity*. Obviously it would be foolish to work with the notion we worked with in earlier sections: because the only wffs

[7] Kees Vermeulen and Albert Visser have also pointed this out, and essentially the same point is made by Groenendijk and Stokhof.

in L' that can be true off the diagonal are atomic symbols, and because no atomic symbol is true in all models, this would be a vacuous notion. Instead we take Groenendijk and Stokhof's definition as our starting point. Recall from section 3 that they define a wff ϕ to be valid iff for all models $\mathfrak{M}$, and for all inputs i, there an output j such that $\mathfrak{M}, (i,j) \models \phi$. That is, a wff is valid iff it is always possible to process it, no matter what the input is. However a moments thought shows that when working with L' this concept is equivalent to the simpler notion of *diagonal validity* discussed earlier, and we shall take this latter notion as fundamental:

Definition 6.3 (dr-validity and ds-validity) *If $\mathfrak{M}^r$ is a relativised model and ϕ an L' wff then ϕ is dr-valid on $\mathfrak{M}^r$ (notation: $\mathfrak{M}^r \models^{dr} \phi$) iff for all diagonal pairs (i,i) in $\mathfrak{M}^r$ we have $\mathfrak{M}^r, (i,i) \models \phi$. A wff ϕ is* dr-valid *(notation: $\models^{dr} \phi$) iff for all relativised models $\mathfrak{M}^r$ we have $\mathfrak{M}^r \models \phi$. Similarly, if $\mathfrak{M}^s$ is a square model and ϕ an L' wff then ϕ is ds-valid on $\mathfrak{M}^s$ (notation: $\mathfrak{M}^s \models^{ds} \phi$) iff for all diagonal pairs (i,i) in $\mathfrak{M}^s$ we have $\mathfrak{M}^s, (i,i) \models \phi$. A wff ϕ is* ds-valid *(notation: $\models^{ds} \phi$) iff for all square models $\mathfrak{M}^s$ we have $\mathfrak{M}^s \models \phi$.* □

Let's take a look closer at the relationship between these two notions. First we prove the following result. This shows that a certain a 'monotonicity' property holds: we can expand relativised models into square models.

Proposition 6.4 (Monotonicity lemma) *Let $\mathfrak{M}^r$ $(= (\mathfrak{R}, V))$ be a relativised model, let $\mathfrak{M}^s$ $(= (\mathfrak{S}, V'))$ be a square model, and let $diag(\mathfrak{S}) = \{(i,j) \in \mathfrak{S} : i = j\}$. Suppose that $diag(\mathfrak{S}) \subseteq \mathfrak{R} \subseteq \mathfrak{S}$, and that for all atoms q, $V(q) = V'(q)$. Then for all $(i,j) \in \mathfrak{R}$ and all wffs ϕ, we have that $\mathfrak{M}^r, (i,j) \models \phi$ iff $\mathfrak{M}^s, (i,j) \models \phi$.*

Proof.
We prove the result by induction on $deg(\phi)$. The base case is clear: since V and V' assign the same subset of $\mathfrak{R}$ to each atom q, the models agree on the propositional variables; and trivially both models agree on $\perp$ and δ. So suppose the desired result holds for all wffs θ such that $0 \leq deg(\theta) < n$. Let $deg(\phi \Rightarrow \psi) = n$, and $(i,j) \in \mathfrak{R}$. We want to show that $\mathfrak{M}^r, (i,j) \models \phi \Rightarrow \psi$ iff $\mathfrak{M}^s, (i,j) \models \phi \Rightarrow \psi$.

(*Right to left.*) We show the contrapositive. Suppose $\mathfrak{M}^r, (i,j) \not\models \phi \Rightarrow \psi$. Now if $i \neq j$ it is trivial that $\mathfrak{M}^s, (i,j) \not\models \phi \Rightarrow \psi$ also. So suppose $i = j$. Then we have:

$$(\exists (i,k) \in \mathfrak{R})(\mathfrak{M}^r, (i,k) \models \phi \ \ and \ \ (\forall (k,l) \in \mathfrak{R})(\mathfrak{M}^r, (k,l) \not\models \psi).$$

By the induction hypothesis this means:

$$(\exists (i,k) \in \mathfrak{R})(\mathfrak{M}^s, (i,k) \models \phi \ \ and \ \ (\forall (k,l) \in \mathfrak{R})(\mathfrak{M}^s, (k,l) \not\models \psi).$$

As $\mathfrak{R} \subseteq \mathfrak{S}$, we have $(i,k) \in \mathfrak{S}$. Moreover note that all pairs $(k,u) \in \mathfrak{S} \backslash \mathfrak{R}$ are off the diagonal, so the only wffs that could be satisfied at these pairs are propositional variables. However this possibility is excluded by the definition of V', hence we conclude that *all* wffs are false at these points. In particular, this means that $(\forall (k,u) \in \mathfrak{S} \backslash \mathfrak{R})(\mathfrak{M}^s, (k,u) \not\models \psi)$, and hence:

$$(\exists (i,k) \in \mathfrak{S})(\mathfrak{M}^s, (i,k) \models \phi \ \ and \ \ (\forall (k,l) \in \mathfrak{S})(\mathfrak{M}^s, (k,l) \not\models \psi)).$$

Hence, as $(i,j) \in \mathfrak{S}$, we have $\mathfrak{M}^s, (i,j) \not\models \phi \Rightarrow \psi$, and we have shown the contrapositive of the desired result.

(*Left to right.*) Suppose $\mathfrak{M}^r, (i,j) \models \phi \Rightarrow \psi$. This happens iff $i = j$ and:

$$(\forall (i,k) \in \mathfrak{R})(\mathfrak{M}^r, (i,k) \models \phi \ \ implies \ \ (\exists (k,l) \in \mathfrak{R})(\mathfrak{M}^r, (k,l) \models \psi)).$$

By the inductive hypothesis we thus have:

$$(\forall (i,k) \in \mathfrak{R})(\mathfrak{M}^s, (i,k) \models \phi \ \ implies \ \ (\exists (k,l) \in \mathfrak{R})(\mathfrak{M}^s, (k,l) \models \psi)).$$

Now $\mathfrak{R} \subseteq \mathfrak{S}$ so we have:

$$(\forall (i,k) \in \mathfrak{R})(\mathfrak{M}^s, (i,k) \models \phi \ \ implies \ \ (\exists (k,l) \in \mathfrak{S})(\mathfrak{M}^s, (k,l) \models \psi)).$$

Further, as we saw above, we have $(\forall (i,u) \in \mathfrak{S}\backslash\mathfrak{R})(\mathfrak{M}^s, (i,u) \not\models \phi)$, thus:

$$(\forall (i,v) \in \mathfrak{S})(\mathfrak{M}^s, (i,v) \models \phi \ \ implies \ \ (\exists (v,l) \in \mathfrak{S})(\mathfrak{M}^s, (v,l) \models \psi)).$$

As $(i,j) \in \mathfrak{S}$ and $i = j$ this means $\mathfrak{M}^s, (i,j) \models \phi \Rightarrow \psi$, and we have our desired result. □

The Monotonicity Lemma makes it clear why $\boxdot^t$, the universal companions of $\diamond^t$, is not definable. For suppose for the sake of a contradiction that such an operator was definable. That is suppose we could define an operator $\boxdot^t$ such that in any relativised model $\mathfrak{M}^r$ at any pair (i,j):

$$\mathfrak{M}^r, (i,j) \models \boxdot^t \phi \ \text{ iff } \ i = j \ \ and \ \ (\forall k)(\mathfrak{M}^r, (i,k) \models \phi).$$

But this is absurd. Consider any model $\mathfrak{M}$ of cardinality greater than one consisting of only diagonal points such that q is true at all points. Then $\boxdot^t q$ is true at all points. Unfortunately, the Monotonicity Lemma tells us that we can expand $\mathfrak{M}$ to a square model $\mathfrak{M}'$, setting all propositional variables (including q) to false off the diagonal, and preserve the truth of all formulas (including $\boxdot^t q$) in the process. But clearly $\boxdot^t q$ would be false in any such expanded model, so we conclude that $\boxdot^t$ is not definable.

However, more usefully, the Monotonicity Lemma that ds-validity and dr-validity coincide:

Corollary 6.5 *For all wffs ϕ, $\models^{dr} \phi$ iff $\models^{ds} \phi$.*

Proof.
The left to right implication is trivial. For the converse we show the contrapositive. Suppose a relativised model $\mathfrak{M}^r$ $(= (\mathfrak{R}, V))$ falsifies ϕ at (i,j). Let $\mathfrak{S}$ be $diag(\mathfrak{R}) \times diag(\mathfrak{R})$; let V' be the valuation on $\mathfrak{S}$ defined by $(k,l) \in V'(q)$ iff $(k,l) \in V(q)$, for all atoms q and all $(k,l) \in \mathfrak{S}$; and let $\mathfrak{M}^s$ be $(\mathfrak{S}, V')$. $\mathfrak{M}^r$ and $\mathfrak{M}^s$ satisfy the conditions of the Monotonicity Lemma, thus $\mathfrak{M}^s, (i,j) \not\models \phi$. □

The most obvious consequence of this result is that it will simplify our proof theory: as ds-validity and dr-validity coincide for L', we need develop only one proof system. Let us call this single notion of validity simply 'validity' from now on, and write $\models \phi$ to mean that ϕ is valid. We turn now to the problem of finding a proof system that will generate all and only the valid wffs.

6.1 A tableaux system

Because we lack the Boolean operators it is difficult to pin down the logical behaviour of $\Rightarrow$ using axioms. On the other hand, it is quite simple to visualise geometrically what is involved in building a satisfying (or falsifying) model for an L' wff. If we make precise the ideas underlying the geometrical intuition, we are lead quite naturally to the idea of defining a special kind of tableaux system.

Recall that a tableaux system is essentially a system for building models. Tableaux are trees, and each branch of a tree records what happens during one particular attempt to build a model. If inconsistent information is found in the course of exploring one option, that branch is said to *close*. Branches that have not closed are called *open*, and any open branches remaining when the tableaux construction algorithm halts supply the information needed to build models. Although essentially devices for building models, tableaux systems can also be regarded as proof systems: if our tableaux system cannot build a falsifying model for a wff ϕ (that is, if every branch closes when we try to build a model falsifying ϕ) then we say ϕ is provable. That is, closed tableaux are proofs.

So how do we adapt these ideas to the present setting? Let's begin informally. Suppose we are standing at a diagonal point, say $(0,0)$, and we wish to falsify $\theta \Rightarrow \theta'$. Obviously, if θ and θ' are both tests, all we have to do is try to make θ true at $(0,0)$, and θ' false at $(0,0)$. Similarly, faced with the task of making $\theta \Rightarrow \theta'$ true at $(0,0)$, we should either try to make θ false at $(0,0)$, or we should attempt to verify both θ and θ' at $(0,0)$. All this is familiar: it's nothing but the familiar tableaux rules for classical material implication.

Now, as long as we are dealing with dynamic implications between complex formulas, we are (in effect) in the domain of ordinary propositional calculus: we just keep breaking these formulas down in accordance with the familiar tableaux rules. However when we eventually reach atomic level matters change. Because propositional variables are not tests, it does not suffice merely to make them true or false at $(0,0)$; in general we will need to look at other input/output pairs. Indeed, sometimes we may need to ***add*** new input/output pairs to the pairs we already have. For example, suppose we need to falsify a wff of the form $p \Rightarrow \phi$ at $(0,0)$, and we already know that both p and ϕ are satisfied at $(0,0)$. Then we must add two new pairs (say $(0,1)$ and $(1,1)$) and make p true at $(0,1)$ and ϕ false at $(1,1)$. In short, we need we need develop a tableaux system that can keep track of which wffs need to be verified and which falsified at the various coordinates, and which can also introduce new input/output pairs when needed.

A natural way to meet these requirements is to use *prefixed (signed) tableaux systems*. These were developed by Fitting (1983) to provide easy-to-use proof systems for various one dimensional modal logics (for example S5). The most important idea of such tableaux system is that they manipulate (signed) formulas that are preceded by a *prefix*. In Fitting's systems these prefixes are thought of as names for possible worlds. Our prefixes will be slightly different: they will be ordered pairs, and can be thought of as the coordinates of input/output pairs. Having the coordinates of input/output pairs available for manipulation gives us all the structure we need to define a proof system. Parenthetically, because of the burden carried by the prefixes, it's perhaps natural to think of the system defined below as a *labelled deduction system* in the sense of Gabbay (1991), however the link with the work of Fitting is more direct.

Definition 6.6 (Prefixed signed wffs) *Given a non-empty set of* **indices**, *an ordered triple of the form $((n,m),S,\phi)$, where both n and m are indices, $S \in \{T,F\}$, and ϕ is a wff of L', is called a* **prefixed signed wff**. *The ordered pair (n,m) is called the* **prefix** *of ϕ, and S is called the* sign *of ϕ.* □

Although this is not essential, it will be convenient in what follows to fix our set of indices to be the non-negative integers. When writing prefixed signed wffs we invariably drop the outer brackets and commas, thus typical prefixed signed wffs are written as $(0,0)F(p \Rightarrow q)$ and $(0,1)T(\delta \Rightarrow (q \Rightarrow \bot))$. The function of the sign is to keep track of whether a wff ϕ is to be verified ($T\phi$) or falsified ($F\phi$). The function of the prefix, is tell us at which input/output pair the verification or falsification of ϕ must take place.

Definition 6.7 *Given a finite set of prefixed signed wffs Σ, by the* **prefix set** *of Σ is meant the set containing all and only the prefixes occurring in some prefixed signed wff $\sigma \in \Sigma$. (More formally, it's the set of first projections of elements of Σ.) By the* **index set** *of Σ is meant the set of indices that occur in some prefix in the prefix set of Σ. (More formally, it's the set containing precisely the first and second projections of all the prefixes in the prefix set of Σ.) Note that the index set of Σ is a finite set of non-negative integers. By* $\mathbf{max}_{\Sigma}$ *is meant the maximum of this set.* □

We are now ready to define our tableaux rules. The rules have an 'updating' function: their task is to take as input a prefixed signed wff (or: a pair of prefixed signed wffs) and to return another wff (or indeed, as many as three wffs). We introduce the following conventions for stating the rules:

Convention 6.8 θ and θ' are metavariables over tests, and ν and ν' are metavariables over propositional variables. (In addition S is a metavariable over T and F, and ϕ is a metavariable over arbitrary wffs, but these metavariables are used only in one rule, namely *R9*.)

In reading the following rules it is important to keep these conventions in mind.

R1	$(x,x)T(\theta \Rightarrow \theta')$	$\rightsquigarrow$	$(x,x)F\theta$ *or* $(x,x)T\theta$ *and* $(x,x)T\theta'$
R2	$(x,x)F(\theta \Rightarrow \theta')$	$\rightsquigarrow$	$(x,x)T\theta$ *and* $(x,x)F\theta'$
R3	$(x,x)T(\nu \Rightarrow \theta)$ *and* $(x,y)T\nu$	$\rightsquigarrow$	$(y,y)T\theta$
R4	$(x,x)F(\nu \Rightarrow \theta)$	$\rightsquigarrow$	$(x,y)T\nu$ *and* $(y,y)F\theta$ *where* $y = max_\Sigma + 1$
R5	$(x,x)T(\theta \Rightarrow \nu)$	$\rightsquigarrow$	$(x,x)F\theta$ *or* $(x,x)T\theta$ *and* $(x,y)T\nu$ *and* $(y,y)T\delta$ *where* $y = max_\Sigma + 1$
R6	$(x,x)F(\theta \Rightarrow \nu)$	$\rightsquigarrow$	$(x,x)T\theta$ *and* $(x,x)F(\delta \Rightarrow \nu)$
R7	$(x,x)T(\nu \Rightarrow \nu')$ *and* $(x,y)T\nu$	$\rightsquigarrow$	$(y,y)T(\delta \Rightarrow \nu')$
R8	$(x,x)F(\nu \Rightarrow \nu')$	$\rightsquigarrow$	$(x,y)T\nu$ *and* $(y,y)F(\delta \Rightarrow \nu')$ *where* $y = max_\Sigma + 1$
R9	$(x,x)F(\delta \Rightarrow \nu)$ *and* $(x,y)S\phi$	$\rightsquigarrow$	$(x,y)F\nu$

We call the expression to the left of the $\rightsquigarrow$ the *antecedents(s)* or *argument(s)* of the rule, and the expression to the right the *consequent(s)* or *result(s)* of the rule.

The first thing to note about these rules is that they are of many different kinds. Some of the most obvious distinctions are that while six of them take only one prefixed signed wff as an argument (such rules are called ***unary***) three of them (namely *R3*, *R7* and *R9*) are ***binary***. Note that in the binary rules *R3* and *R7*, exactly one of the antecedents is a complex (prefixed signed) wff;[8] this antecedent is called the *major premiss* of the rule, and the other antecedent (a prefixed signed propositional variable) is called the minor premiss. In *R9* the wff of the form $(x,x)F(\delta) \Rightarrow \nu$ is called the major premiss and the other wff (whatever it is) is called the minor premiss. Note also that some of rules can yield more than one wff as output, while others yield only one. In addition, two of the rules (namely *R1* and *R5*) are *disjunctive*; they offer a choice of outputs on a given input; or, to put it another way, they are non-deterministic updates.

However perhaps the most important class of rules to draw attention to comprises *R4*, *R5* and *R8*, which we sometimes call *prefix creating rules.* These rules are special because

[8] In the sequel we shall sometimes refer to prefixed signed wffs simply as wffs.

while all the other rules can only manipulate prefixes already in use, these rules are allowed to introduce a new prefixes containing a brand new index. This ability is not unrestricted: the purpose of mysterious looking side condition '*where* $y = max_\Sigma + 1$' is precisely to control this ability. The side condition means that when applying this rule to some prefixed signed wff on a branch Σ of a tableaux, y can be instantiated to the number obtained by adding one to the highest index appearing on the branch. (These rules will only ever be used one finite sets, so max_Σ will be well defined. Also, recall that we are using the non-negative integers as indices.) It is these three rules, and the way they interact with the other rules (and in particular *R3*, *R7* and *R9*) that give the system its flavour.

Before beginning the formal definition and study of this system, it may be helpful to consider a simple example of how we intend to use these rules. So, let's construct a simple tableaux. As our example, let's consider a wff that is *not* valid, namely $p \Rightarrow p$. That this wff is not valid is shown by the following three point countermodel:

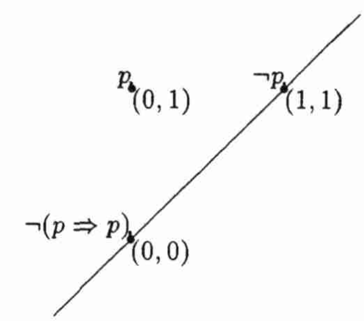

Figure 4.

Now, we will attempt to falsify $p \Rightarrow p$ using a tableaux construction. As $p \Rightarrow p$ is not valid we do *not* want a closed tableaux to result in this case; in fact we want the tableaux to halt with at least one open branch, and we hope that this open branch will give us all the information we need to construct a countermodel. Let's see what happens.

The first step of our construction is to place the top node of the tableaux in place. Doing so yields:

$$(0,0)F(p \Rightarrow p)$$

(Note that as we are attempting to falsify it, we placed the sign F in front of the formula. The choice of $(0,0)$ as starting prefix is conventional, and we shall abide by this convention in the remainder of this paper.) Now we have a rather trivial tableaux: it has one point, one branch. The next thing to do is extend it if we can, so we look for an applicable rule. In fact only *R8* is applicable. We apply it obtaining:

$$\begin{array}{l} (0,0)F(p \Rightarrow p) \\ (0,1)Tp \\ (1,1)F(\delta \Rightarrow p) \end{array}$$

Next we note that *R9* is applicable to the final item in our tableaux. (Note the way that this final item is used as both the major and minor premiss for this application of *R9*.) We make the application and obtain:

$$(0,0)F(p \Rightarrow p)$$
$$(0,1)Tp$$
$$(1,1)F(\delta \Rightarrow p)$$
$$(1,1)Fp$$

Inspection of the tableaux reveals that no more rules are applicable, thus the tableaux construction halts. We can 'read off' a countermodel in an obvious way. The relativised square underlying the model consists of the indices on this branch (namely $(0,0)$, $(0,1)$ and $(1,1)$) and we define a valuation V by insisting that p is true at precisely those pairs (i,j) such that $(i,j)Tp$ occurs on the branch. So the valuation here will make p true only at $(0,1)$. In short, the tableaux has built the simple countermodel displayed earlier.

This example should make clear the basic intuitions of the system, and what we mean by a tableaux. Actually, it's going to be some time before we discuss tableaux again. Instead, we are going to consider some properties of these rules from a slightly more abstract perspective. In particular we're going to prove a certain *correctness* result, a *model existence* result, and a *termination* result. Thus, when we finally return to the topic of tableaux systems, we will have virtually everything we need to give a proof of soundness and completeness.

Before we set about these tasks we must make some basic notions clear. Being completely rigorous about every detail concerning the instantiation and application of rules would take a lot of space (and wouldn't be much fun either). Thus, for these basic ideas we rely on a certain amount of co-operation on the part of the reader. The reader in need of more should consult Fitting (1983); the technical ideas underlying this discussion are largely based on this source.

First of all we need to understand what it means to *instantiate* a rule in a finite set of prefixed signed wffs Σ. Informally, it means to find prefixed signed wffs in this set that fill all the available slots in the rule's antecedent(s) and consequent(s) in in a 'matching' or 'consistent' fashion. More formally, we can regard our rules as statements about Σ; the metavariables are to be regarded as universally quantified. Then to instantiate a rule in such a finite set Σ is simply to perform a successful universal instantiation in that finite universe. (Note that we pay no attention to the side conditions on *R4*, *R5* and *R8* here. Their role will soon become apparent.)

Definition 6.9 *Let Σ be a finite set of prefixed signed wffs. We say that a unary rule is* **inapplicable** *to a prefixed signed wff σ iff when the antecedent of the rule is instantiated in σ, all the prefixed signed wffs needed to instantiate (one of the disjuncts of) the consequent in matching fashion are elements of Σ. A unary rule is* **applicable** *to a prefixed signed wff σ iff it is not inapplicable. The notions of inapplicability and applicability for binary rules are defined analogously in terms of pairs of prefixed signed wffs σ and σ'.* □

Intuitively, if a rule is applicable to a prefixed signed wff in some set Σ this means that some prefixed signed wffs are 'missing' from Σ. This leads to the following concept:

> *To apply an applicable rule (or to 'update' with an applicable rule) is to add those missing wffs (and no more) to Σ. That is, the result of applying a rule to a wffs in Σ is a new set Σ' such that $\Sigma \subset \Sigma'$, and $\Sigma' \backslash \Sigma$ contains precisely the 'missing items'.*

It is clear that when apply a rule we never have to add more than three items to Σ. Note that because two rules (namely *R1* and *R5*) are disjunctive (that is, there is more than one way to correctly instantiate the consequent for a given choice of antecedent) such minimal additions are not necessarily unique.

Note that there is an important syntactic relations holding between the antecedent and consequent of any instantiation of any rule. Suppose ϕ is a formula. We define a *extended subformula* of ϕ to be a subformula of ϕ, or a wff of the form $\delta \Rightarrow p$, where p is a propositional variable occurring in ϕ. It is clear that given any instantiation of any rule, the wff in the consequent(s) is an extended subformula of the wff in the major premiss.

With these basic ideas to hand we can get down to business. Our first task is to prove that our rules are in some sense 'correct', or 'sound'. Essentially we will show that given a (finite) set of prefixed signed wffs which has been successfully 'interpreted' in some model, then if we update Σ using one of the rules, the resulting set Σ' is still interpretable. The first task is to give a precise notion of 'interpretation':

Definition 6.10 *Let Σ be a set of prefixed signed formulas, let $\mathfrak{M} = (\mathfrak{P}, V)$ be a model, and let $p(\mathfrak{P}) = \{m : (m, m) \in \mathfrak{P}\}$. By an* **interpretation** *of Σ in $\mathfrak{M}$ is meant a function I from the index set of Σ to $p(\mathfrak{P})$. We say that Σ is* **satisfied under the interpretation** *I in $\mathfrak{M}$ iff:*

1. *$(i, j)T\phi \in \Sigma$ implies $\mathfrak{M}, (I(i), I(j)) \models \phi$;*
2. *$(i, j)F\phi \in \Sigma$ implies $\mathfrak{M}, (I(i), I(j)) \not\models \phi$.*

Further, we say that a set of prefixed signed wffs Σ is **satisfiable** *iff Σ is satisfied under some interpretation I in some model $\mathfrak{M}$.* □

In short, if we have found an interpretation for a set of prefixed signed wffs, it means we have found a model that assigns truth values to all the wffs in a way that is consistent with the signs, at a collection of pairs that agree with the demands made by the prefixes.

Theorem 6.11 (Correctness) *Suppose Σ is a satisfiable set of prefixed signed wffs, and suppose a rule R is applicable to some prefixed signed wff (or: pair of prefixed signed wffs) in Σ. Then at least one of the updates Σ' of Σ obtained by applying R to this prefixed signed wff (or: pair of prefixed signed wffs) is satisfiable.*

Proof.
If the rule in question cannot introduce new prefixes (that is, the rule is one of *R1*, *R2*, *R3*, *R6*, *R7* or*R9*) the result is straightforward. For by assumption Σ is satisfiable; that is, Σ is satisfiable under some interpretation I in some model $\mathfrak{M}$. Now, for any of the rules just mentioned except *R1* there is only one possible way of updating Σ. (That is, none of the other rules are disjunctive.) Call the unique set that can result Σ'; it is immediate from the truth definition that Σ' is satisfied in $\mathfrak{M}$ under the interpretation I. On the other hand, if the rule in question is *R1* there are two possible sets, Σ' and Σ'' say, that can result from the updating process; and it is immediate from the truth definition that one of these two sets must be satisfied in $\mathfrak{M}$ under I.

So, let's consider what happens if a prefix creating rule (that is, one of *R4*, *R5* or *R8*) is used. We treat the case for *R8*. Suppose that *R8* is applied to $(a, a)F(p \Rightarrow q)$, producing $(a, b)Tp$ and $(b, b)F(\delta \Rightarrow q)$, where b is $max_\Sigma + 1$. Let Σ' be $\Sigma \cup \{(a, b)Tp, (b, b)F(\delta \Rightarrow q)\}$; that is, Σ' is the result of updating Σ in this manner with *R8*. We wish to show that Σ' is satisfiable.

Now as $(a, a)F(p \Rightarrow q) \in \Sigma$, then as Σ is satisfiable under the interpretation I in $\mathfrak{M}$ we have $\mathfrak{M}, (I(a), I(a)) \not\models (p \Rightarrow q)$. This means:

$$(\exists k)(\mathfrak{M}, (I(a), k) \models p \;\; and \;\; (\forall j)(\mathfrak{M}, (k, j) \not\models q))$$

which in turn is equivalent to:

$$(\exists k)(\mathfrak{M},(I(a),k)\models p \;\; and \;\; \mathfrak{M},(k,k)\not\models(\delta\Rightarrow q)).$$

Define a new interpretation I' extending I as follows: I' equals $I\cup\{(b,k)\}$, where b is $max_\Sigma+1$. We note that as b is $max_\Sigma+1$, it is not in the domain of I, thus I' is a well defined function. Clearly Σ' is satisfied in $\mathfrak{M}$ under the interpretation I'.

The cases for *R4* and *R5* are similar. □

With our correctness result to hand, let's turn to the problem of proving a 'model existence theorem' or 'truth lemma'. This theorem will be the semantic cornerstone of the tableaux completeness result. Essentially it says that if a set of prefixed signed wffs is 'coherent' and 'closed under all rules' it has a model. These two conditions are essentially the usual conditions of 'consistency' and 'downward closedness' required of Hintikka sets, and the result is an analogue in this Boolean free setting of Hintikka's model existence result.

Definition 6.12 *Let H be a set of prefixed signed wffs. We say that H is* **coherent** *iff for all indices i, j and all wffs ϕ we have:*

1. $(i,j)T\bot\notin H$;
2. $(i,j)T\phi\in H$ *implies* $(i,j)F\phi\notin H$;
3. $(i,j)T\theta\in H$ *implies* $i=j$, *for all tests* θ;
4. $(i,j)F\delta\in H$ *implies* $i\neq j$.

We say that H is **closed under all rules** *(or* **rule saturated***) iff no rule is applicable to any wff $\sigma\in\Sigma$.* □

Definition 6.13 *Let H be a set of prefixed signed wffs whose prefix set is closed under projections to the diagonal. By the* **model** $\mathfrak{H}$ **induced by** H *is meant the triple $\langle\mathbf{P},V\rangle$, where $\mathbf{P}$ is the prefix set of H, and $V:\mathrm{ATOM}\longrightarrow Pow(\mathbf{P})$ is defined by $(i,j)\in V(p)$ iff $(i,j)Tp\in H$. V is called the* **induced valuation**. □

We are now ready to prove the result:

Theorem 6.14 (Model existence) *Let H be a set of prefixed signed wffs whose prefix set is closed under projections to the diagonal that is both coherent and rule saturated. Let $\mathfrak{H}$ be the model induced by H. Then:*

$$\begin{array}{ll} 1. & (i,j)T\phi\in H \;\; implies \;\; \mathfrak{H},(i,j)\models\phi \\ 2. & (i,j)F\phi\in H \;\; implies \;\; \mathfrak{H},(i,j)\not\models\phi \end{array}$$

Moreover, for all propositional variables p, the converse of the first condition holds. That is, $\mathfrak{H},(i,j)\models p$ implies $(i,j)Tp\in H$.

Proof.
We prove that the two conditions hold by induction on $deg(\phi)$.

Step 0: $deg(\phi)=0$. As $deg(\phi)=0$, the only prefixed signed wffs we have to consider are of the form $(i,j)S\phi\in H$, where ϕ is either a propositional variable p, or one of the constants δ or $\bot$. It follows immediately from the definition of valuation and the fact that H is coherent, that the conditions hold in all three cases. It is also immediate from the definition of the induced valuation that the converse of the first condition holds for all propositional variables.

This completes step 0 of the argument. As all subsequent steps involve wffs ϕ such that $deg(\phi) > 0$ (that is, tests) we have that if $(i,j)T\phi \in H$ then $i = j$ (this is one of the coherency conditions enjoyed by H). Further, we have that if $(i,j)F\phi \in H$ and $i \neq j$ then $\mathfrak{H},(i,j) \not\models \phi$. Thus in all subsequent steps it suffices to consider prefixed signed wffs with prefixs (i,j) such that $i = j$.

Step 1: $deg(\phi) = 1$. We distinguish four cases.

Case 1. ϕ has the form $\theta \Rightarrow \theta'$, where $\theta, \theta' \in \{\delta, \bot\}$. As H is closed under *R1* and *R2*, using results from step 0 it is clear that the two desired conditions hold.

Case 2. ϕ has the form $p \Rightarrow \theta$, where $\theta \in \{\delta, \bot\}$. Suppose $(i,i)T(p \Rightarrow \theta) \in H$. Either there is a j such that $(i,j)Tp \in H$ or there is no such j. Now, if there is no such j, then using the fact (proved in step 0) that $\mathfrak{H},(i,j) \models p$ at precisely those pairs (i,j) such that $(i,j)Tp \in H$, for all j we have that $\mathfrak{H},(i,j) \not\models p$; thus trivially $\mathfrak{H},(i,j) \models p \Rightarrow \theta$. So suppose there is at least one such j. Then for any such j, because H is closed under *R3* we have $(j,j)T\theta \in H$. By results from step 0 we have $\mathfrak{H},(i,j) \models p$ and $\mathfrak{H},(j,j) \models \theta$. As the pairs (i,j) such that $(i,j)Tp \in H$ are precisely the pairs at which $\mathfrak{H},(i,j) \models p$, we thus have $\mathfrak{H},(i,i) \models p \Rightarrow \theta$ as required.

Suppose $(i,i)F(p \Rightarrow \theta) \in H$. Then by closure under *R4* we have that for some j, $(i,j)Fp$ and $(j,j)T\theta \in H$. By results from step 0 we thus have $\mathfrak{H},(i,j) \not\models p$ and $\mathfrak{H},(j,j) \models \theta$; thus $\mathfrak{H},(i,i) \not\models p \Rightarrow \theta$ as required.

Case 3. ϕ has the form $\theta \Rightarrow p$, where $\theta \in \{\delta, \bot\}$. Suppose $(i,i)T(\theta \Rightarrow p) \in H$. Then, by closure under *R5*, *either* $(i,i)F\theta \in H$ *or* for some j, $(i,i)T\theta$, $(i,j)Tp$ and $(j,j)T\delta \in H$. If the first alternative holds, then using results from step 0 we have $\mathfrak{H},(i,i) \not\models \theta$, and as θ is a test for all j we thus have $\mathfrak{H},(i,j) \not\models \theta$; thus trivially $\mathfrak{H},(i,i) \models \theta \Rightarrow p$ as required. On the other hand, if the second alternative holds, then using results from step 0, we have $\mathfrak{H},(i,i) \models \theta$ and $\mathfrak{H},(i,j) \models p$. Moreover, as θ is a test for all $k \neq i$ we have $\mathfrak{H},(i,k) \not\models \theta$. Thus $\mathfrak{H},(i,i) \models \theta \Rightarrow p$ as required. (Note that we don't make use of the fact that $(j,j)T\delta \in H$ in driving through this case. Adding this third wff is simply a convenient way of ensuring closure under projections to the diagonal during the update process.)

So suppose $(i,i)F(\theta \Rightarrow p) \in H$. Then, by closure under *R6* we have that $(i,i)T\theta \in H$ and $(i,i)F(\delta \Rightarrow p) \in H$. First, by the results from step 0, we have $\mathfrak{H},(i,i) \models \theta$. Further, by closure under *R9*, for all indices k such that there is a prefixed signed wff in H whose prefix is (i,k), we have $(i,k)Fp \in H$. Thus, again using a result from step 0, for all k we have $\mathfrak{H},(i,k) \not\models p$. As $\mathfrak{H},(i,i) \models \theta$ we thus have that $\mathfrak{H},(i,i) \not\models \theta \Rightarrow p$.

Case 4. Suppose $(i,i)T(p \Rightarrow q) \in H$. Then either there is a j such that $(i,j)Tp \in H$ or this is not the case. So suppose that this is not the case. Then as $\mathfrak{H},(i,j) \models p$ at precisely those pairs (i,j) such that $(i,j)Tp \in H$, for all j we have $\mathfrak{H},(i,j) \not\models p$; thus trivially $\mathfrak{H},(i,i) \models p \Rightarrow q$, as required. So suppose there is at least one such j. Then for every such j, by closure under *R7* we have $(j,j)T(\delta \Rightarrow q)$. But our results for case 3 tell us that $\mathfrak{H},(j,j) \models \delta \Rightarrow q$. That is, for every such j there is a k such that $\mathfrak{H},(j,k) \models q$. Now the j such that $\mathfrak{H},(i,j) \models p$ are precisely those j such that $(i,j)Tp \in H$. Thus then $\mathfrak{H},(i,i) \models p \Rightarrow q$.

So suppose $(i,i)F(p \Rightarrow q) \in H$. Then by closure under *R8*, for some j we have $(i,j)Tp \in H$ and $(j,j)F(\delta \Rightarrow q) \in H$. Thus we have $\mathfrak{H},(i,j) \models p$, and further, by closure under *R9*, for all k we have $\mathfrak{H},(j,k) \not\models q$. Thus $\mathfrak{H},(i,i) \not\models p \Rightarrow q$.

Inductive step. Assume the required conditions hold of all wffs ϕ such that $deg(\phi) \leq n$. The arguments that show that they must then also hold for wffs of degree $n+1$ are essentially identical to those used in cases 1-3 of step 1 and are left to the reader. The required result thus follows by induction on $deg(\phi)$. □

We turn now to the problem of ***termination***. We intend to use the rules as follows. We are given some wff ϕ and we want to see whether it is possible to falsify it. So we form a singleton set $\Sigma = \{(0,0)F\phi\}$ and begin applying applicable rules to Σ. This produces a strictly increasing sequence of sets $\Sigma = \Sigma_0 \subset \Sigma_1 \subset \Sigma_2 \subset \cdots$ where each item in the sequence is the result of updating the previous item in the sequence by some applicable rule. We stop if we produce an set that is not coherent or if no more rules are applicable. Such a sequence is called an ***update sequence***, and we want to show that all update sequences are finite.

Now, we have already observed that applying a rule to a wff (or wffs) produces only extended subformulas of the rule in question. Moreover, the complexity of the wffs produced is lower. The complexity measure in question is ***dpc***, which is defined as follows. If ϕ is a propositional variable, $\perp$ or δ then $dpc(\phi)$ is zero. If ϕ contains at least one occurrence of $\Rightarrow$ then $dpc(\phi)$ is the number of occurrences of $\Rightarrow$ in ϕ plus the number of occurrences of propositional variables in ϕ. Note that with one exception, all rule applications strictly lower *dpc*. To be more precise, with one exception, the sum of the *dpc* values of all the consequents of a rule application is strictly less than the *dpc* of the antecedent (in case the rule in question is a unary rule) or the major premiss (in case the rule in question is a binary rule). The one exception is a special instance of *R6*, namely when the wff in the antecedent is $\perp \Rightarrow \nu$. However note that such an application can cause no problems. One of its consequents is $(x,x)T\perp$ which introduces an incoherency and thus immediately aborts the update sequence.

With these observations to hand it is straightforward to prove:

Proposition 6.15 (Termination lemma) *All update sequences are finite.*

Proof.
Left to the reader. □

With these results to hand, we are now ready to discuss tableaux, define a notion of provability, and prove a completeness result.

We won't give a formal definition of what a tableaux is. The intuitive picture given earlier — namely that a tableaux is tree shaped record of attempts to build a model for some prefixed signed wff — should be good enough. To be a little more precise, the branches of the tree are just records of particular update sequences, and a branch is closed iff the update sequence is incoherent, and open otherwise. A tableaux is said to be closed iff all its branches are closed, and open iff it contains at least one open branch. The reader who wants more details, or who finds any of our terminology mystifying, is once more advised to consult Fitting (1983).

Given that the notion of a tableaux is clear, we can now define our concept of provability.

Definition 6.16 *We say that a wff ϕ is provable (written $\vdash \phi$) iff there is a tableaux beginning with $(0,0)F\phi$ that is closed. That is, a closed tableaux is a proof of the (F-signed) formula at its root node.* □

What we want to show is a soundness and completeness result for this notion of provability. That is, we want to show that $\vdash \phi$ iff $\models \phi$. Actually, most of the work has already be done. Just one things remains. Following Fitting, we still need to discuss the notion of a *systematic construction* of tableaux. We turn to this matter now.

The systematic construction is defined inductively in stages as follows.

Stage 1. Draw the tree consisting of a single node decorated with with $(0,0)F\phi$. This concludes the first stage.

Suppose n stages of the construction have been completed. If the tableaux we have constructed is closed, stop. Likewise, if no rule is applicable to any formula(s), stop. Otherwise we go on to:

Stage n+1. Choose an occurrence of prefixed signed formula as high up in the tree as possible to which a rule is applicable. (Break any deadlocks by choosing the leftmost such occurrence.) Apply the rule. If the rule in question is not disjunctive, add the consequents of the rule to end of every branch that the formula in question is a member of. If the rule is question is disjunctive (that is, one of *R1* or *R5*), split the end of all branches through the formula in question, and add wffs resulting from one disjunctive choice to the first branch, and the wffs resulting from the other choice to the second. This completes stage $n+1$.

Note that this procedure terminates yielding a finite tree; all we are doing is systematically building a record of a number of update sequences, but by Lemma 6.15 all such sequences are finite. We are now ready for our result.

Theorem 6.17 (Soundness and completeness) $\vdash \phi$ *iff* $\models \phi$.

Proof.
(*Left to right.*). We show the contrapositive. Suppose $\not\models \phi$. Then there is a model $\mathfrak{M}$ and a pair (i,i) in $\mathfrak{M}$ such that $\mathfrak{M},(i,i) \not\models \phi$. But now consider the singleton set of prefixed signed wffs $\Sigma = \{(0,0)F\phi\}$. Clearly the interpretation I that maps 0 to i satisfies Σ in $\mathfrak{M}$. But by theorem 6.11 (correctness) this means that at least one branch of any tableaux starting with $(0,0)F\phi$ must be satisfiable. A satisfiable branch cannot be closed, so any such tableaux must contain an open branch. But this means $\not\vdash \phi$ and we have shown the contrapositive.

(*Right to left.*) We show the contrapositive. Suppose $\not\vdash \phi$. Then there is no closed tableaux with $(0,0)F\phi$ at its root node. In particular, then, the *systematic tableaux construction* does not produce a closed tableaux; the systematic construction algorithm terminates leaving at least one open branch, say B. Now, it is clear that the prefix set of B is closed under projections to the diagonal (our original stage 1 tableaux had this property, and the property is preserved by the rules), and moreover it is clear from the systematic nature of the algorithm that B is rule saturated. Thus the conditions of theorem 6.14 (model existence) are met, and the model $\mathfrak{B}$ induced by B falsifies ϕ at $(0,0)$. Thus $\not\models \phi$, and we have shown the contrapositive as desired. □

A general comment is in order. With this completeness result we have captured the logic of L'; moreover, because tableaux construction is (intuitively) a dynamic process, we have done so in a seemly natural manner. However further thought reveals that matters are not quite so clear cut.

Intuitively, in dropping the Booleans from the language, we have chosen to work with 'purely dynamical' system. However it might well be argued that in introducing the apparatus needed to define the tableaux system (namely prefixes and signs) we have, in effect, smuggled static notions back into the language. Although our *language* is 'purely dynamic', or *proof system* (arguably) is not.

We feel that there's some truth in this objection.[9] Indeed this seems to be a general state of affairs: the other attempts to capture the logic of dynamism that we are familiar with seem to trade on the availability of some level of static expressivity. A particularly

[9]Raised by Albert Visser and Kees Vermeulen.

interesting example is the way that van Eijck and de Vries (1992) use Hoare logic to analyse an enrichment of DPL. They remark that "intuitions about static meaning seem to be much better developed than intuitions about dynamic meaning" (op. cit. page 20), and then capture these static intuitions using Hoare logical pre- and post-conditions. Essentially these 'freeze out' static state descriptions that enable rules of inference to be stated.

But while this is an interesting (and potentially important) objection, at present it is far from clear what a 'truly dynamic' inference system might be.[10] There do seems to be interesting intuitions about dynamical reasoning that are not captured by our tableaux system; and there does seem to be more to 'going dynamic' than simply dropping the Booleans from the language — but what sort of object is a dynamic proof? Vermeulen (1993), building on earlier work by Zeinstra (1990), suggests that it may be possibly to develop a genuine dynamic proof theory by taking seriously the idea that proofs are *texts*. These are interesting proposals, but at present it is unclear whether they can be combined with the approach of the present paper.

7 In the landscape of transition logics

Now that we understand some of the technicalities of Dynamic Implication Logic (DIL) let us examine its position in the landscape of similar, 'transitional' dynamic logics. Here we understand the term 'transitional' to mean that (at least) part of the syntax of the language gets interpreted as a set of transitions. The systems we are particularly interested in have already been mentioned: they include the Boolean Modal Logic (BML) of Gargov, Passy and Tinchev (1987); the Dynamic Modal Logic (DML) of van Benthem (1991a) and de Rijke (1992a, 1992b); Propositional Dynamic Logic (PDL) and Arrow Logic (see Marx *et. al.* (1992)).

Perhaps the best way to identify DIL among its neighbours is the following: let us consider the various choices one has to make when devising a formalism of transitional dynamic logic, and then see how these parameters are set in the systems just named. So, we look at the following aspects of dynamic logics: the *ontological status of transitions* in the various systems, whether or not there is *evaluation at transitions*, and which *choices of connectives* have been made.

Ontological status of transitions Perhaps the most basic decision one has to make is whether the transitions themselves are the basic entities, or whether they are defined as 'real' pairs of states as in DIL.[11] In this paper we have only given attention to this second, two dimensional approach, and the majority of the above systems (notably BML, DML and PDL) do the same.

However in van Benthem (1991a) it is argued that more abstract logics of Arrows yielded by the following the first approach may provide a suitable level of abstraction for many dynamic phenomena, and the literature on Arrow Logic (see for example Marx *et. al.* (1992) and Gyuris *et. al.* (1992)) has classified many such logics. It seems to be straightforward to develop a version of DIL along these lines, but we won't do so here.

[10] It's important to realise that it is *proof systems* (that is, syntactic entities) and not consequence relations, that are being discussed here. There has been a great deal of discussion of various notions of dynamic consequence (see van Benthem (1991a) for example), but, apart from the references noted below, almost nothing on what a genuinely dynamic notion of *proof* might be.

[11] Note that the models of the second approach can always be viewed as forming a subclass of (isomorphic copies of) special models of the first kind.

Evaluation at transitions This is quite a subtle point: recall that we are talking about systems that have syntactic items referring to sets of transitions, (that is, binary relations in the two-dimensional approach). Now the question is whether the system allows for *explicit* evaluation of these items at transitions. An example will make clear what is at stake: in PDL, it is the programs that refer to relations, but *programs only occur within diamonds.* This means that one cannot express the *identity* of two programs in the language, one can only insist that the *effects* of executing two programs π_1 and π_2 are the same ($\langle\pi_1\rangle\phi \leftrightarrow \langle\pi_2\rangle\phi$ expresses this).

Now, although BML, DML and PDL share DIL's ontology, they use it differently. In these formalisms formulas are evaluated at at points, not pairs as in DIL. In effect, they treat the algebra of diamonds as of secondary importance. On the other hand, the idea of evaluating at transitions is fundamental to Arrow Logic -- though, as has already been mentioned, some versions of Arrow Logic work with a more abstract notion of transition than DIL does.

Choice of (transition) connectives Here the alternatives are abundant: the basic options seem to be whether to have Boolean transition constructors (disjunction, conjunction, negation, falsum) as in BML; or 'dynamic' constructors (composition, diagonal, dynamic implication, converse, iteration) as in PDL; or both.

The various existing systems differ widely here: Boolean Modal Logic gets its name from the fact that it has all and only the Boolean operators on relations; our DIL has only the diagonal and the not-too-dynamic dynamic implication besides the Booleans; DPL has union, composition and iteration; and Arrow Logic has all the Booleans and composition, diagonal and converse. A distinctive aspect of DML is that it has got operations going from states to transitions and vice versa. That is, it can make propositions out of diamonds and vice versa.

When looked at in these terms, an obvious feature of DIL is that does *not* have a transitional operator referring to relational composition ($\circ$). The semantics of this operator is as follows:

$$\mathfrak{M}, (i,j) \models \phi \circ \psi \text{ iff } (\exists k)(\mathfrak{M}, (i,k) \models \phi \;\; and \;\; \mathfrak{M}, (k,j) \models \psi).$$

Geometrically this amounts to the following:

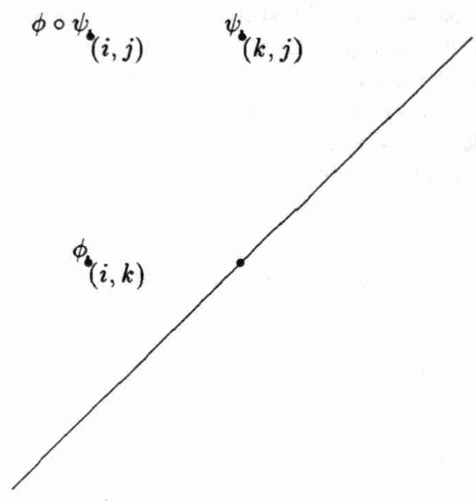

Figure 5.

There are two main differences between the evaluation patterns of $\circ$ and dynamic implication. The first is that $\circ$ is not a test. The second is that while $\circ$ exhibits a 'three step search' similar to that of dynamic implication, there is a crucial difference. When using $\circ$ we must finish on the same latitude we started at. (Geometrically we must 'complete the square', as the above picture shows.)

Now, composition is an important operator in dynamic semantics. For example, is used in Groenendijk and Stokhof's DPL, where it called is dynamic conjunction. What happens if we add it to DIL?

Composition is a very powerful operator. When both the Booleans and δ are present, it can define all the operators of cylindric modal logic and hence (as discussed in section 3) dynamic implication. In fact, in terms of the full square semantics, under these conditions we are working in a very rich fragment of the relational modal logic of Venema (1991), and in terms of the relativised semantics we are very close to various versions of Arrow Logic, which is by now one of the better known sites of the transition logic landscapes.

However when $\circ$ is added when the Boolean operators are absent, matters are unclear. Given the presence of $\circ$ in Groenendijk and Stokhof's system (and the absence there of Boolean operators), it seems a natural next project to look at the Boolean free system in $\circ$, $\Rightarrow$, and $\perp$; that is, L' strengthened by $\circ$. (It would also be natural from a logical perspective to go on and add the iteration operator $*$ to this system.)

To sum up, DIL is like most transition logics (except some versions of Arrow logic) in that it treats transitions as 'real' pairs of states. It is unlike most transition logic (except Arrow logic) in that it allows explicit valuations at states. In its choice of operators it is closest to BML, and if strengthened with $\circ$ it will collapse into either a subsystem of relational modal logic, or some variant of Arrow Logic, depending on whether the square or relativised semantics is employed.

8 Conclusion

In this paper we have examined the logic of dynamic implication from the perspective of two dimensional modal logic. We conclude with some general remarks and suggestions for further work.

In our view, if the idea of dynamic semantics for natural language is to advance further, it is important to gain some mathematical insight into the underlying ideas. While DRT and its various dynamical offspring have given rise to interesting work in formal semantics over the last decade, understanding of the logical issues they raise has lagged far behind. At present, there is no generally accepted mathematical picture of what dynamic semantics actually is; so to speak, no counterpart of Montague's "Universal Grammar" exists that does for the word 'dynamic' what Montague did for 'compositional'. If 'dynamic semantics' is to be more than the activity of defining ever more complex systems, unifying perspectives must be found.

It would be premature to claim that the two dimensional perspective explored in this paper provides such a unifying perspective; we've merely discussed one fairly simple idea (dynamic implication) in some detail. However, for reasons we hope are already clear, we feel the perspective is a promising one. Essentially, what the two dimensional perspective offers is a seemingly natural level of abstraction for viewing the various proposals that have been made in the formal semantics literature. It indicates the common core of these proposals in an easily visualised manner, and links recent dynamic ideas with pioneering work in formal semantics. Moreover, it draws attention to the possible relevance of recent work for formal semantics of recent work on logics of transitions.

For these reasons we feel the perspective deserves further exploration. Two matters seem particularly pressing. For a start, stronger systems at the propositional level need to be investigated. As has already been mentioned, we think it would be interesting to explore (Boolean free) logics in the connectives $\circ$, $\Rightarrow$ and $\perp$, possibly augmented by $*$. But the real challenge is to extend two dimensional ideas to the first order level. How this is best to be done we leave for further research.

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