

Frobenius manifolds, projective special geometry and Hitchin systems

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Abstract. We consider the construction of Frobenius manifolds associated to projective special geometry and analyse the dependence on choices involved. In particular, we prove that the underlying F -manifold is canonical. We then apply this construction to integrable systems of Hitchin type.

Introduction

One way of formulating the mirror symmetry conjecture is in terms of Frobenius manifolds. On the one hand (the A-side) it is well known that the quantum cohomology product gives rise to a natural Frobenius manifold. The other, B-side, is constructed from certain variations of Hodge structures. In the case of Calabi–Yau threefolds, this B-side of the story is perhaps less well-known to mathematicians and appears implicitly in [5], [4], [3], [17], [24].

One of the purposes of this paper is to give an elementary description of this B-side Frobenius geometry and to specify the dependence on choices one has to make. The starting point for this will be a projective special geometry, or, in other words, an abstract variation of Hodge structures of weight 3 of the type considered in [7]. There are two choices involved in the construction: one is a generator for the degree three subspace in the Hodge filtration, i.e., a volume form in the case of Calabi–Yau threefolds, second, and, more importantly, a choice of an *opposite filtration*. Some of these choices have a natural interpretation in terms of this special geometry as choices of affine coordinate patches. Other choices lie outside the realm of special geometry. Remarkably we find that the underlying F -manifold, cf. [24], is independent of all choices. Also remark that in the geometric context of Calabi–Yau threefolds, the choice of an opposite filtration can be eliminated by going to the large complex structure limit [10].

In the second part of the paper, we apply these results to integrable systems of Hitchin type [27]. Namely, first of all, we show that the special geometry on the base cf. [18] can be made projective. Equivalently, the variation of polarized Hodge structures of

weight one refines to a variation of Hodge like filtrations of weight three as in [7]. Underlying this construction is a certain family of cameral curves and a Seiberg–Witten differential that is constructed in terms of the natural \mathbb{C}^* -action on the total space of the integrable system. This is closely related to [11], [12] where in the cases of ADE-groups a family of Calabi–Yau threefolds was constructed whose variation of—a priori mixed—Hodge structures coincides with that of the Hitchin systems.

This brings us to the starting data of the first part of the paper, and gives us constructions of the associated Frobenius manifolds. In this example all choices, abstractly defined in the first part, have a natural interpretation in the elementary geometry of curves. We believe this Frobenius manifold to be of interest for the following reasons: first of all, as shown in [19] the Hitchin integrable systems associated to Langlands dual groups are SYZ-mirror to one another. Second, in the geometric transition conjecture for ADE-fibered Calabi–Yau manifolds [11], one of the two—conjecturally equal—string theories is captured entirely by the underlying special geometry.

Outline

A Kuranishi family with base manifold B_0 of three-dimensional Calabi–Yau manifolds gives rise to a variation of polarized Hodge structures of weight 3 on the primitive part of the middle cohomology bundle with some distinguished properties. Such a VPHS induces on the one hand projective special Kähler geometry on a manifold B with $\dim B = \dim B_0 + 1$, on the other hand it induces Frobenius manifold structures on a manifold M with $\dim M = 2 \dim B_0 + 2$. Both geometries contain some flat structures and potentials and both depend on additional choices.

The purpose of the first five sections is to review the (well known) constructions, to discuss the dependence on choices and to give a comparison. Section 1 gives definitions, section 2 treats Frobenius manifolds, section 3 shows that the underlying F-manifold structure is independent of choices, section 4 treats a part of projective special Kähler geometry, and section 5 compares them.

Section 6 reviews the Hitchin system, section 7 introduces the Seiberg–Witten differential and section 8 gives the variation of Hodge like filtrations of weight three on the base of the Hitchin system. Finally, the results of sections 1–5 are applied to the Hitchin system in section 9.

1. Some definitions

In the next five sections B_0 will be a small neighborhood of a base point 0 in a complex manifold of dimension n . When necessary, the size will be decreased, so essentially the germ $(B_0, 0)$ is considered, but we will not emphasize this.

1.1. Variation of polarized Hodge structures (VPHS). A VPHS of weight $w \in \mathbb{Z}$ on B_0 consists of data $(B_0, \mathcal{V}, \nabla, \mathcal{V}_{\mathbb{R}}, S, F^\bullet, w)$. Here \mathcal{V} is a holomorphic vector bundle with a flat connection ∇ and a real ∇ -flat subbundle $\mathcal{V}_{\mathbb{R}}$ such that $\mathcal{V} = \mathcal{V}_{\mathbb{R}} \otimes \mathbb{C}$, S is a $(-1)^w$ -

symmetric ∇ -flat nondegenerate pairing on \mathcal{V} with real values on $\mathcal{V}_{\mathbb{R}}$, the decreasing Hodge filtration F^\bullet is a filtration of holomorphic subbundles with

$$(1.1) \quad \nabla : \mathcal{O}(F^p) \rightarrow \mathcal{O}(F^{p-1}) \otimes \Omega_{B_0}^1 \quad (\text{Griffiths transversality}),$$

$$(1.2) \quad \mathcal{V} = F^p \oplus \overline{F^{w+1-p}} \quad (\text{Hodge structure}),$$

equivalent:

$$(1.3) \quad \mathcal{V} = \bigoplus_p H^{p,w-p} \quad \text{where } H^{p,w-p} := F^p \cap \overline{F^{w-p}},$$

$$(1.4) \quad S(F^p, F^{w+1-p}) = 0 \quad (\text{part of the polarization}),$$

$$(1.5) \quad i^{2p-w} S(v, \bar{v}) > 0 \quad \text{for } v \in H^{p,w-p} - \{0\} \quad (\text{rest of the polarization}).$$

In this chapter the real structure and the conditions (1.2), (1.3) and (1.5) will play no role. Data $(B_0, \mathcal{V}, \nabla, S, F^\bullet, w)$ as above with (1.1) and (1.4) only will be called *variation of Hodge like filtrations with pairing*.

The connection ∇ induces a Higgs field C on $\bigoplus_p F^p/F^{p+1}$,

$$(1.6) \quad C = [\nabla] : \mathcal{O}(F^p/F^{p+1}) \rightarrow \mathcal{O}(F^{p-1}/F^p) \otimes \Omega_{B_0}^1$$

with $C_X C_Y = C_Y C_X$ for $X, Y \in T_{B_0}$. From section 2 on, we consider only data which satisfy $w = 3$ and the two conditions:

$$(1.7) \quad \begin{cases} F^{w+1} = 0, & \text{rank } F^w = 1, & \text{rank } F^{w-1}/F^w = n, \\ \text{and thus } \text{rank } F^1/F^2 = n, & \text{rank } F^0/F^1 = 1, & F^0 = \mathcal{V}. \end{cases}$$

$$(1.8) \quad \begin{cases} \text{For any } \lambda_0 \in F_0^w - \{0\} & \text{the map} \\ C \cdot \lambda_0 : T_0 B_0 \rightarrow F_0^{w-1}/F_0^w & \text{is an isomorphism.} \end{cases}$$

As B_0 is small, (1.8) extends from $0 \in B_0$ to all points in B_0 . The conditions (1.7) and (1.8) together are called *CY-condition*. This condition was discussed in [7] and is weaker than the so-called H^2 -generating condition considered in [24], ch. 5.

1.2. Opposite filtrations. An *opposite filtration* U_\bullet is defined to be an increasing ∇ -flat filtration with

$$(1.9) \quad \begin{cases} \mathcal{V} = \bigoplus_p F^p \cap U_p & \text{or equivalently} \\ \mathcal{V} = F^p \oplus U_{p-1}, \end{cases}$$

$$(1.10) \quad S(U_p, U_{w-1-p}) = 0.$$

As it is ∇ -flat, U_\bullet is determined by $U_\bullet \mathcal{V}_0$ and will be identified with that filtration. The splitting in (1.9) is holomorphic, the one on (1.3) is only real analytic. Both are S -orthogonal in the sense

$$(1.11) \quad S(F^p \cap U_p, F^q \cap U_q) = 0 = S(H^{p,w-p}, H^{q,w-q}) \quad \text{if } p + q \neq w.$$

S and U_\bullet induce a symmetric and nondegenerate pairing g^U on \mathcal{V} by

$$(1.12) \quad g^U(a, b) := (-1)^p S(a, b) \quad \text{for } a \in \mathcal{O}(F^p \cap U_p), b \in \mathcal{O}(\mathcal{V}).$$

The splitting in (1.9) is also g^U -orthogonal in the sense of (1.11).

Now the connection ∇ decomposes into $\nabla = \nabla^U + C^U$, where ∇^U is a connection on each subbundle $F^p \cap U_p$ and C^U is \mathcal{O}_{B_0} -linear and maps $F^p \cap U_p$ to $(F^{p-1} \cap U_{p-1}) \otimes \Omega_{B_0}^1$. The flatness of ∇ is equivalent to ∇^U being flat, C^U being a Higgs field and the potentiality condition $\nabla^U(C^U) = 0$, more explicitly:

$$(1.13) \quad \nabla_X^U(C_Y^U) - \nabla_Y^U(C_X^U) - C_{[X, Y]}^U = 0 \quad \text{for } X, Y \in T_{B_0}.$$

Because of (1.11) and the ∇ -flatness of S , both S and g^U are ∇^U -flat, and g^U satisfies

$$(1.14) \quad g^U(C_X^U a, b) = g^U(a, C_X^U b) \quad \text{for } X \in T_{B_0}, a, b \in \mathcal{O}(\mathcal{V}).$$

That is: C^U is selfadjoint with respect to g^U . Define the endomorphism

$$\hat{V}^U : \mathcal{V} \rightarrow \mathcal{V}, \quad \hat{V}^U := \sum_p \left(p - \frac{w}{2} \right) \text{id}_{|F^p \cap U_p}.$$

Then \hat{V}^U is ∇^U -flat, and $(\hat{V}^U)^* = -\hat{V}^U$ where $*$ denotes the adjoint with respect to g^U , and $[C^U, \hat{V}^U] = C^U$.

In the case $w = 3$, the combination of the CY-condition (1.7) and (1.8) and of the choice of an opposite filtration U_\bullet leads to a Frobenius manifold, and the combination of the CY-condition and of the part U_1 of an opposite filtration leads to (a part of) projective special Kähler geometry. This will be discussed in the sections 2 and 4.

1.3. Frobenius manifolds and F-manifolds. An *F-manifold* (M, \circ, e, E) [23], [21] is a complex manifold of dimension ≥ 1 with a commutative and associative multiplication on the holomorphic tangent bundle T_M , a unit field $e \in T_M$ and an Euler field $E \in T_M$ with the following two properties: $\text{Lie}_E(\circ) = 0$ and

$$(1.15) \quad \text{Lie}_{X \circ Y}(\circ) = X \circ \text{Lie}_Y(\circ) + Y \circ \text{Lie}_X(\circ).$$

(1.15) implies $\text{Lie}_e(\circ) = 0$.

A *Frobenius manifold* (M, \circ, e, E, g) [15] is an F-manifold together with a symmetric nondegenerate \mathcal{O}_M -bilinear pairing g on T_M with the following properties: its Levi-Civita connection ∇^g is flat; there is a potential $\Phi \in \mathcal{O}_M$ such that for ∇^g -flat vector fields X, Y, Z

$$(1.16) \quad g(X \circ Y, Z) = XYZ(\Phi);$$

the unit field e is ∇^g -flat; the Euler field E satisfies $\text{Lie}_E(g) = (2 - w) \cdot g$ for some $w \in \mathbb{C}$.

In fact, the potentiality condition and the flatness imply (1.15), cf. [23], [21]. They also imply that the metric is multiplication invariant,

$$(1.17) \quad g(X \circ Y, Z) = g(X, Y \circ Z) \quad \text{for } X, Y, Z \in T_M.$$

It turns out that $\nabla_{\bullet}^g E$ is a flat endomorphism of the tangent bundle.

A Frobenius manifold M with a base point $0 \in M$ is called *semihomogeneous* if $w \in \mathbb{N}$ and if there are integers $0 = p_1 \leq p_2 \leq \dots \leq p_{\dim M - 1} \leq p_{\dim M} = w$ and flat coordinates t_i centered at 0 such that $e = \frac{\partial}{\partial t_1}$ and

$$(1.18) \quad E = \sum_{i=1}^{\dim M} (1 - p_i) t_i \frac{\partial}{\partial t_i}.$$

Then the numbers p_i are unique, because the numbers $p_i - 1$ are the eigenvalues of $\nabla_{\bullet}^g E$. The coordinates t_i are called semihomogeneous.

2. Frobenius manifolds from VPHS of weight 3

Throughout the whole section 2 except lemma 2.3, a variation

$$((B_0, 0), \mathcal{V}, \nabla, S, F^{\bullet}, w = 3)$$

of Hodge like filtrations with pairing (see 1.1 for this notion) of weight 3 with CY-condition (1.7) and (1.8) and $n = \dim B_0$ is fixed.

Theorem 2.2(a) gives a construction of Frobenius manifolds from it and an additional choice. This construction is well known, but usually hidden within much richer structures [5], [4], [3], [17], [24]. We will give a proof which will make the comparison with projective special Kähler geometry easy.

Theorem 2.2(b) shows that the underlying F-manifold with Euler field is independent of the additional choice, contrary to the flat structure and the metric. This result is new. It will be proved in section 3.6 (see also remark 2.3).

The section starts with lemma 2.1 which discusses the geometry of the variation of Hodge like filtrations of weight 3 with CY-condition and an opposite filtration. It yields coordinates t_2, \dots, t_{n+1} and a prepotential $\Psi \in \mathcal{O}_{B_0}$. It is complemented by lemma 2.4 which constructs the initial data of lemma 2.1 out of coordinates t_2, \dots, t_{n+1} on B_0 and an arbitrary function $\Psi \in \mathcal{O}_{B_0}$. It shows that the prepotential Ψ in lemma 2.1 is not subject to any hidden conditions. Finally, proposition 2.5 discusses the automorphisms of the F-manifold which underlies the Frobenius manifolds in theorem 2.2.

Lemma 2.1. *Additionally to the variation of Hodge like filtrations of weight 3 with CY-condition fixed above, choose the following data:*

(1) *An opposite filtration U_{\bullet} . By section (1.2) it induces a flat connection ∇^U on each subbundle of the splitting $\mathcal{V} = \bigoplus_{p=0}^3 F^p \cap U_p$; and the pairing S is ∇ -flat and ∇^U -flat.*

(2) A ∇^U -flat basis v_1, \dots, v_{2n+2} of \mathcal{V} which is compatible with the splitting of \mathcal{V} and the pairing S , and

$$(2.1) \quad \begin{cases} v_1 \in F^3; & v_2, \dots, v_{n+1} \in F^2 \cap U_2; \\ v_{n+2}, \dots, v_{2n+1} \in F^1 \cap U_1; & v_{2n+2} \in U_0; \end{cases}$$

$$(2.2) \quad S(v_1, v_{2n+2}) = -1; \quad S(v_k, v_l) = \delta_{k+n, l} \quad \text{for } 2 \leq k \leq n+1.$$

Let v_1^0, \dots, v_{2n+2}^0 be the ∇ -flat (here ∇ , not ∇^U) extension of $v_1(0), \dots, v_{2n+2}(0) \in \mathcal{V}_0$. Then there are unique coordinates t_2, \dots, t_{n+1} on B_0 and there is a unique function $\Psi \in \mathcal{O}_{B_0}$ which satisfy $\Psi(0) = 0$ and

$$(2.3) \quad v_1 = v_1^0 + \sum_{i=2}^{n+1} t_i v_i^0 + \sum_{i=2}^{n+1} \frac{\partial \Psi}{\partial t_i} \cdot v_{n+i}^0 + \left(\sum_{k=2}^{n+1} t_k \frac{\partial}{\partial t_k} - 2 \right) (\Psi) \cdot v_{2n+2}^0.$$

They also satisfy for $i, j, k \in \{2, \dots, n+1\}$ and $a \in \{n+2, \dots, 2n+1\}$

$$(2.4) \quad \nabla_{\frac{\partial}{\partial t_i}} v_1 = v_i,$$

$$(2.5) \quad v_i = v_i^0 + \sum_j \frac{\partial^2 \Psi}{\partial t_i \partial t_j} v_{n+j}^0 + \left(\sum_k t_k \frac{\partial}{\partial t_k} - 1 \right) \left(\frac{\partial \Psi}{\partial t_i} \right) \cdot v_{2n+2}^0,$$

$$(2.6) \quad \nabla_{\frac{\partial}{\partial t_i}} v_j = \sum_k \frac{\partial^3 \Psi}{\partial t_i \partial t_j \partial t_k} v_{n+k},$$

$$(2.7) \quad v_a = v_a^0 + t_{a-n} \cdot v_{2n+2}^0,$$

$$(2.8) \quad \nabla_{\frac{\partial}{\partial t_i}} v_a = \delta_{i+n, a} \cdot v_{2n+2},$$

$$(2.9) \quad v_{2n+2} = v_{2n+2}^0,$$

$$(2.10) \quad \nabla_{\frac{\partial}{\partial t_i}} v_{2n+2} = 0.$$

The function Ψ and the flat structure on B_0 from the coordinates t_2, \dots, t_{n+1} depend only on the choice of U_\bullet and $v_1(0)$.

Proof. Here and later the following convention for indices will be used:

$$(2.11) \quad i, j, k \in \{2, \dots, n+1\}, \quad a, b \in \{n+2, \dots, 2n+1\}, \quad \alpha \in \{1, \dots, 2n+2\}.$$

Define $p_1 = 3$, $p_i = 2$, $p_a = 1$, $p_{2n+2} = 0$, then $v_\alpha \in \mathcal{O}(F^{p_\alpha} \cap U_{p_\alpha})$. The sections v_α and v_α^0 satisfy the following properties:

$$v_\alpha^0 \in \mathcal{O}(U_{p_\alpha}) \quad (\text{because } U_{p_\alpha} \text{ is } \nabla\text{-flat}),$$

$$\nabla^U v_\alpha^0 = -C^U v_\alpha^0 \in \mathcal{O}(U_{p_\alpha-1}) \otimes \Omega_{B_0}^1,$$

$$(2.12) \quad v_\alpha \equiv v_\alpha^0 \pmod{\mathcal{O}(U_{p_\alpha-1})},$$

$$(2.13) \quad \nabla v_\alpha = C^U v_\alpha \in \mathcal{O}(F^{p_\alpha-1} \cap U_{p_\alpha-1}) \otimes \Omega_{B_0}^1.$$

(2.12) gives (2.9) and (2.10). There are unique functions $t_i, \kappa_i, \kappa_{2n+2} \in \mathcal{O}_{B_0}$ such that

$$(2.14) \quad v_1 = v_1^0 + \sum_{i=2}^{n+1} t_i v_i^0 + \sum_{i=2}^{n+1} \kappa_i \cdot v_{n+i}^0 + \kappa_{2n+2} \cdot v_{2n+2}^0.$$

The CY-condition (1.7) and (1.8) shows that t_2, \dots, t_{n+1} are coordinates on B_0 . They are centered at 0 because $v_1(0) = v_1^0$. From now on denote $\partial_i := \frac{\partial}{\partial t_i}$. Derivation of (2.14) gives

$$(2.15) \quad \begin{aligned} \nabla_{\partial_i} v_1 &= v_i^0 + \sum_j \partial_i \kappa_j \cdot v_{n+j}^0 + \partial_i \kappa_{2n+2} \cdot v_{2n+2}^0 \\ &\equiv v_i^0 \equiv v_i \pmod{\mathcal{O}(U_1)}. \end{aligned}$$

This and (2.13) show (2.4).

Now we will use two times the pairing S , first for (2.7) and (2.8), second for the existence of the function Ψ . Equation (2.13) shows $\nabla_{\partial_i} v_a \in \mathcal{O}_{B_0} \cdot v_{2n+2}$. The coefficient $\delta_{i+n,a}$ in (2.8) is determined by

$$\begin{aligned} 0 &= \partial_i(0) = \partial_i S(v_1, v_a) = S(\nabla_{\partial_i} v_1, v_a) + S(v_1, \nabla_{\partial_i} v_a) \\ &= S(v_i, v_a) + S(v_1, \nabla_{\partial_i} v_a) = \delta_{i+n,a} + S(v_1, \nabla_{\partial_i} v_a) \end{aligned}$$

and $S(v_1, v_{2n+2}) = -1$. This shows (2.7) and (2.8). Condition (2.2) also holds with v_x replaced by v_x^0 because S is ∇ -flat and $v_x^0(0) = v_x(0)$. This, together with (2.4) and (2.15) give

$$0 = S(v_i, v_j) = \partial_j \kappa_i \cdot S(v_i^0, v_{n+i}^0) + \partial_i \kappa_j \cdot S(v_{n+j}^0, v_j^0) = \partial_j \kappa_i - \partial_i \kappa_j.$$

Therefore there is a unique function $\Psi \in \mathcal{O}_{B_0}$ with $\Psi(0) = 0$ and

$$(2.16) \quad \partial_i \Psi = \kappa_i.$$

Derivation by ∂_i of (2.15) and (2.4) gives

$$(2.17) \quad \begin{aligned} \nabla_{\partial_i} v_j &= \sum_k \partial_i \partial_j \partial_k \Psi \cdot v_{n+k}^0 + \partial_i \partial_j \kappa_{2n+2} \cdot v_{2n+2}^0 \\ &\equiv \sum_k \partial_i \partial_j \partial_k \Psi \cdot v_{n+k}^0 \equiv \sum_k \partial_i \partial_j \partial_k \Psi \cdot v_{n+k} \pmod{\mathcal{O}(U_0)}. \end{aligned}$$

This and (2.13) show (2.6). Using (2.7) this gives

$$\begin{aligned} \nabla_{\partial_i} v_j &= \sum_k \partial_i \partial_j \partial_k \Psi \cdot v_{n+k} \\ &= \sum_k \partial_i \partial_j \partial_k \Psi \cdot v_{n+k}^0 + \sum_k \partial_i \partial_j \partial_k \Psi \cdot t_k \cdot v_{2n+2}^0. \end{aligned}$$

With (2.17) it implies

$$\begin{aligned}\partial_i \partial_j \kappa_{2n+2} &= \sum_k t_k \cdot \partial_i \partial_j \partial_k \Psi \\ &= \left(\sum_k t_k \partial_k \right) \partial_i \partial_j \Psi = \partial_i \partial_j \left(\sum_k t_k \partial_k - 2 \right) \Psi.\end{aligned}$$

We can conclude

$$(2.18) \quad \kappa_{2n+2} = \left(\sum_k t_k \partial_k - 2 \right) \Psi$$

because we know

$$\begin{aligned}(\partial_i \kappa_{2n+2})(0) &= 0 \quad [\Leftarrow v_i(0) = v_i^0 \text{ and (2.15)}], \\ \left(\partial_i \left(\sum_k t_k \partial_k - 2 \right) \Psi \right)(0) &= \left(\left(\sum_k t_k \partial_k - 1 \right) \partial_i \Psi \right)(0) \\ &= -\kappa_i(0) = 0 \quad [\Leftarrow v_1(0) = v_1^0], \\ \kappa_{2n+2}(0) &= 0 \quad [\Leftarrow v_1(0) = v_1^0], \\ \left(\left(\sum_k t_k \partial_k - 2 \right) \Psi \right)(0) &= -2\Psi(0) = 0.\end{aligned}$$

Now all equations (2.3)–(2.10) are proved.

In order to show that the function Ψ and the flat structure on B_0 from t_2, \dots, t_{n+1} are independent of the choice of v_x except $v_1(0)$, we suppose that a second choice \tilde{v}_x^0 is made with $\tilde{v}_1(0) = v_1(0)$. All its data are denoted using a tilde. The base change from the base (v_x) to the base (\tilde{v}_x) is constant, because both bases are flat with respect to ∇^U . Now $\tilde{v}_1 = v_1$ because both are ∇^U -flat extensions of $\tilde{v}_1(0) = v_1(0)$. With

$$S(v_1, v_{2n+2}) = -1 = S(\tilde{v}_1, \tilde{v}_{2n+2})$$

we obtain also $\tilde{v}_{2n+2} = v_{2n+2}$. Suppose $(\tilde{v}_i) = (v_i) \cdot A$, where $i \in \{2, \dots, n+1\}$ and $(\tilde{v}_i), (v_i)$ are row vectors, and $A \in \text{Gl}(n, \mathbb{C})$. Then $(\tilde{t}_i) = (\tilde{t}_i) \cdot A^t$, $(\tilde{\partial}_i) = (\partial_i) \cdot A$, $(\tilde{\kappa}_i) = (\kappa_i) \cdot A$ and thus $\tilde{\Psi} = \Psi$. This shows the desired independencies of the flat structure and Ψ .

In fact, rescaling of $v_1(0) \in F_0^3 - \{0\}$ leads to a rescaling of Ψ , but it does not affect the flat structure on B_0 . That depends only on U_\bullet . \square

Theorem 2.2. *A variation $((B_0, 0), \mathcal{V}, \nabla, S, F^\bullet)$ of Hodge like filtrations of weight 3 with pairing and with CY-condition (1.7) and (1.8) and $n = \dim B_0$ is fixed.*

(a) *Any choice of an opposite filtration U_\bullet and a generator $\lambda \in F_0^3$ leads in a canonical way (described below in the proof) to a Frobenius manifold $M^{U, \lambda} \supset \mathbb{C} \times B_0$ of dimension*

$2n + 2$. It is semihomogeneous with integers $(p_1, \dots, p_{2n+2}) = (0, 1, \dots, 1, 2, \dots, 2, 3)$ (1 and 2 each n times).

(b) The manifold $M^{U,\lambda}$ is canonically isomorphic to the manifold $M = \mathbb{C} \times B_2$ which is constructed in section 3.6. All Frobenius manifolds induce the same unit field e , Euler field E and multiplication \circ on M , so the same F-manifold structure. But in the general the metrics and flat structures differ.

The unit field is $e = \frac{\partial}{\partial t_1}$ if t_1 is the coordinate on \mathbb{C} of a coordinate system which respects the product $M = \mathbb{C} \times B_2$. The manifold B_2 comes equipped with a projection $p_2 : B_2 \rightarrow B_0$, the fibers are isomorphic to \mathbb{C}^{n+1} as affine algebraic manifolds. The Euler field induces a good \mathbb{C}^* -action on the fibers with weights $(1, \dots, 1, 2)$.

Part (b) will follow from the results in section 3.6.

Proof of part (a). All the data in lemma 2.1 will be used, and also the convention (2.11). The section v_1 is chosen such that $v_1(0) = \lambda$. Define $M^{U,\lambda} = \mathbb{C} \times B_0 \times \mathbb{C}^{n+1}$ with coordinates (t_1, \dots, t_{2n+2}) and the flat connection defined by these coordinates. Here (t_2, \dots, t_{n+1}) extend the coordinates on B_0 from lemma 2.1. Denote $\partial_\alpha = \frac{\partial}{\partial t_\alpha}$ for $\alpha = 1, \dots, 2n + 2$. Define a potential

$$(2.19) \quad \Phi(t_1, \dots, t_{2n+2}) := \Psi(t_2, \dots, t_{n+1}) + \frac{1}{2} t_1^2 t_{2n+2} + t_1 \sum_{i=2}^{n+1} t_i t_{n+i}.$$

Define a symmetric nondegenerate flat bilinear form g on $TM^{U,\lambda}$ by

$$(2.20) \quad \begin{aligned} g(\partial_1, \partial_\alpha) &= \delta_{\alpha, 2n+2}, & g(\partial_i, \partial_\alpha) &= \delta_{i+n, \alpha}, \\ g(\partial_{n+i}, \partial_\alpha) &= \delta_{i, \alpha}, & g(\partial_{2n+2}, \partial_\alpha) &= \delta_{\alpha 1}. \end{aligned}$$

Then Φ , g and formula (1.16) give the following multiplication \circ on $T_M^{U,\lambda}$.

$$(2.21) \quad \begin{aligned} \partial_1 \circ &= \text{id}; & \partial_i \circ \partial_j &= \sum_k \partial_i \partial_j \partial_k \Psi \cdot \partial_{n+k}, & \partial_i \circ \partial_a &= \delta_{i+n, a} \cdot \partial_{2n+2}, \\ \partial_i \circ \partial_{2n+2} &= 0, & \partial_a \circ \partial_b &= 0, & \partial_a \circ \partial_{2n+2} &= 0, & \partial_{2n+2} \circ \partial_{2n+2} &= 0. \end{aligned}$$

So, it respects the grading $\bigoplus_{p=0}^3 \left(\bigoplus_{p_\alpha=p} \mathcal{O}_M^{U,\lambda} \cdot \partial_\alpha \right)$ of $T_M^{U,\lambda}$, and the multiplication coefficients depend at most on t_2, \dots, t_{n+1} . We claim that it is commutative and associative. Commutativity of this multiplication is clear. The only nontrivial part of the associativity is given by

$$\begin{aligned} g((\partial_i \circ \partial_j) \circ \partial_k, \partial_1) &= g(\partial_i \circ \partial_j, \partial_k) = \Psi_{ijk} = \Psi_{jki} \\ &= g(\partial_j \circ \partial_k, \partial_i) = g(\partial_i \circ (\partial_j \circ \partial_k), \partial_1). \end{aligned}$$

Define the Euler field E as

$$(2.22) \quad E = \sum_{\alpha=1}^{2n+2} (1 - p_\alpha) \cdot t_\alpha \frac{\partial}{\partial t_\alpha}.$$

Then $E\Phi = E\Psi = 0$, $\text{Lie}_E \partial_x = (p_x - 1)\partial_x$, and $\text{Lie}_E(\circ) = \circ$ and $\text{Lie}_E(g) = (2 - 3)g$ follow immediately. Thus $(M^{U,\lambda}, \circ, e, E, g)$ is a semihomogeneous Frobenius manifold with semihomogeneous coordinates (t_1, \dots, t_{2n+2}) .

In order to show that this Frobenius manifold is independent of the choice of v_x^0 except $v_1(0) = \lambda$, we continue the argument from the end of the proof of lemma 2.1, with the same second choice \tilde{v}_x and the same matrix A . Define the isomorphism $\tilde{M} \rightarrow M^{U,\lambda}$ by $\tilde{t}_1 = t_1$, $\tilde{t}_{2n+2} = t_{2n+2}$, $(\tilde{t}_{n+i}) = (t_{n+i}) \cdot A$. Then $\tilde{g} = g$, $\tilde{\Phi} = \Phi$, $\tilde{E} = E$, so one obtains the same Frobenius manifold. \square

Remark 2.3. Different choices in lemma 2.1 and theorem 2.2 lead to different coordinate systems on B_0 and on M and different flat structures. The coordinate changes are very complicated when U_1 is changed. Trying to prove theorem 2.2(b) by controlling these coordinate changes looks hard.

But when U_1 is fixed and only U_0 and $U_2 = (U_0)^{\perp s}$ are changed, the coordinate changes are much simpler. This is addressed in section 5.2 which shows that all the coordinates and functions Ψ for fixed U_1 and varying U_0 and U_2 have a nice common origin from projective special geometry. We now sketch the common origin of the Frobenius algebra at the level of tangent spaces: consider

$$\text{Gr}(F_0^\bullet) = F_0^3 \oplus (F_0^2/F_0^3) \oplus (F_0^1/F_0^2) \oplus (F_0^0/F_0^1).$$

The pairing S on \mathcal{V} induces a bilinear form on $\text{Gr}(F_0^\bullet)$ whose symmetrization g (the construction is similar to g^U in (1.12)) gives the pairing of the Frobenius algebra. In order to define the multiplication we recall the definition of the Higgs field C and the isomorphism $T_0B_0 \cong F_0^2/F_0^3$ resulting from the Calabi-Yau condition together with a choice of nonzero $\lambda_0 \in F_0^3$.

We define the following commutative, associative, unital and graded multiplication \circ on $\text{Gr}(F_0^\bullet)$:

$$\begin{aligned} \lambda_0 \circ V &= V & \forall V \in \text{Gr}(F_0^\bullet), \\ C_X \lambda_0 \circ C_Y \lambda_0 &= C_X C_Y \lambda_0 & \forall X, Y \in T_0B_0, \\ C_X \lambda_0 \circ W &= C_X W & \forall X \in T_0B_0, W \in F_0^1/F_0^2. \end{aligned}$$

Other multiplications are zero unless they are required for commutativity. Associativity (and commutativity) of \circ follows immediately from the fact that the Higgs field gives commuting endomorphisms: for instance, associativity follows from

$$C_X(C_Y C_Z \lambda_0) = C_Z(C_X C_Y \lambda_0).$$

Together with g , this multiplication gives a Frobenius algebra which is to be compared with the one given in (2.20), (2.21). A different choice of λ_0 simply gives a rescaling of the multiplication.

Given the part U_1 of an opposite filtration, one finds

$$\text{Gr}(F_0^\bullet) \cong F_0^3 \oplus F_0^2/F_0^3 \oplus U_1$$

and the multiplication \circ and bilinear form g can be transferred to the right-hand side. It is possible to identify this space with the tangent space of a manifold, in the following way. Consider the holomorphic vector bundle $U_1 \rightarrow B_0$. Using the line bundle $\rho : F^3 \rightarrow B_0$ we can pull back this bundle to $\rho^*U_1 \rightarrow F^3$. This gives the isomorphism

$$T_{((0,c),v)}\rho^*U_1 \cong F_0^3 \oplus T_0B_0 \oplus U_1 \cong F_0^3 \oplus F_0^2/F_0^3 \oplus U_1$$

where $c \in F_0^3$, $v \in U_1$. So we can view $\text{Gr}(F_0^\bullet)$ as a tangent space to (the total space of) ρ^*U_1 . It is true that refining U_1 to a full opposite filtration U_\bullet allows one to use the bilinear form g^U together with \circ to define a Frobenius manifold structure on ρ^*U_1 . However, from these considerations it does not follow that all these manifolds are isomorphic as F-manifolds to one and the same M . This is the subject of section 3.

Lemma 2.4. *Let $(B_0, 0)$ be a germ of a manifold with coordinates t_2, \dots, t_{n+1} centered at 0, i.e. $t_2(0) = \dots = t_{n+1}(0)$, and let $\Psi \in \mathcal{O}_{B_0}$ be an arbitrary function with $\Psi(0) = 0$. Furthermore, let $\mathcal{V} \rightarrow B_0$ be a holomorphic vector bundle with two bases v_1, \dots, v_{2n+2} and v_1^0, \dots, v_{2n+2}^0 of sections which are related by (2.3), (2.5), (2.7) and (2.9).*

(a) *Let ∇ be the unique flat connection on \mathcal{V} with flat sections v_1^0, \dots, v_{2n+2}^0 . Then (2.4), (2.6), (2.8) and (2.10) hold.*

(b) *Define two filtrations F^\bullet and U_\bullet on \mathcal{V} by (2.1) and an antisymmetric pairing S by (2.2) and*

$$(2.23) \quad S(v_\alpha, v_\beta) = 0 \quad \text{for } (\alpha, \beta) \notin \{(1, 2n+2), (2n+2, 1)\} \\ \cup \{(i, i+n), (i+n, i) \mid i = 2, \dots, n+1\}.$$

Then $((B_0, 0), \mathcal{V}, \nabla, S, F^\bullet)$ is a Hodge like filtration with pairing of weight $w = 3$ and with CY-condition (1.7) and (1.8), and U_\bullet is an opposite filtration.

Proof. Again we use the convention (2.11) and write $\partial_i = \frac{\partial}{\partial t_i}$.

(a) (2.4), (2.8) and (2.10) are obvious, (2.6) follows from

$$\begin{aligned} \nabla_{\partial_i} v_j &= \sum_k \partial_i \partial_j \partial_k \Psi \cdot v_{n+k}^0 + \left(\sum_k t_k \partial_k \right) \partial_i \partial_j \Psi \cdot v_{2n+2}^0 \\ &= \sum_k \partial_i \partial_j \partial_k \Psi \cdot (v_{n+k}^0 + t_k \cdot v_{2n+2}^0) = \sum_k \partial_i \partial_j \partial_k \Psi \cdot v_{n+k}. \end{aligned}$$

(b) ∇ and F^\bullet satisfy Griffiths transversality (1.1) because of (2.4), (2.6), (2.8) and (2.10). For the same reason U_\bullet is ∇ -flat. By definition S satisfies (1.4) and (1.11).

The definition of S in (2.2) and (2.23) and the formulas (2.3), (2.5), (2.7) and (2.9) show (2.2) and (2.23) for v_α^0 instead of v_α . Therefore S is ∇ -flat.

Finally, the CY-conditions also hold, (1.7) is built-in, (1.8) follows from (2.4). \square

Proposition 2.5. *Consider the same data as in theorem 2.2(a) and the Frobenius manifold constructed there in its proof, including the additional choice of coordinates*

(t_1, \dots, t_{2n+2}) . Consider the group

$$\text{Aut}(M, B_0, \circ, e, E) := \{ \varphi : (M, 0) \rightarrow (M, 0) \text{ biholomorphic} \mid \varphi|_{B_0} = \text{id}|_{B_0}, \\ \varphi \text{ respects multiplication, unit field and Euler field} \}$$

of automorphisms of the underlying F-manifold which fix the submanifold B_0 . We use the same convention for the indices as in the proof of theorem 2.1(a):

$$i, j, k \in \{2, \dots, n+1\}, \quad a, b \in \{n+2, \dots, 2n+1\}, \quad \alpha \in \{1, \dots, 2n+2\}.$$

(a) For any automorphism $\varphi \in \text{Aut}(M, B_0, \circ, e, E)$ there exist $\beta \in \mathbb{C}^*$ and $\gamma_{ab} \in \mathcal{O}_{B_0}$ with

$$(2.24) \quad \varphi_1 = t_1, \quad \varphi_i = t_i, \quad \varphi_a = \beta \cdot t_a, \\ \varphi_{2n+2} = \beta \cdot t_{2n+2} + \sum_{a,b} \gamma_{ab}(t_2, \dots, t_{n+1}) \cdot t_a t_b$$

and

$$(2.25) \quad 0 = (\partial_i \circ \partial_j)(\varphi_{2n+2}) [= \left(\sum_k \partial_i \partial_j \partial_k \Psi \partial_{k+n} \right) (\varphi_{2n+2})].$$

(b) In the case when all $\partial_i \partial_j \partial_k \Psi = 0$ then (2.25) is empty, and β and the γ_{ab} can be chosen freely.

(c) If some $\partial_i \partial_j \partial_k \Psi \neq 0$ then $\beta = 1$, but the γ_{ab} are only subject to condition (2.25).

Proof. Consider an automorphism $\varphi : (M, 0) \rightarrow (M, 0)$. The three conditions $\varphi_*(e) = e$, $\varphi_*(E) = E$ and $\varphi|_{B_0} = \text{id}|_{B_0}$ are equivalent to the following:

$$(2.26) \quad \varphi_1 = t_1, \quad \varphi_i = t_i, \\ \varphi_a = \sum_b \beta_{ab}(t_2, \dots, t_{n+1}) \cdot t_b \quad \text{for some } \beta_{ab} \in \mathcal{O}_{B_0} \text{ with } \det(\beta_{ab}) \in \mathcal{O}_{B_0}^*, \\ \varphi_{2n+2} = \sum_{a,b} \gamma_{ab}(t_2, \dots, t_{n+1}) \cdot t_a t_b + \beta(t_2, \dots, t_{n+1}) \cdot t_{2n+2} \\ \text{for some } \gamma_{ab} \in \mathcal{O}_{B_0}, \beta \in \mathcal{O}_{B_0}^*.$$

Then the coordinate vector fields and their images under φ_* satisfy

$$(2.27) \quad \varphi_*(\partial_1) = \partial_1, \quad \varphi_*(\partial_{2n+2}) = \partial_{2n+2}(\varphi_{2n+2}) \cdot \partial_{2n+2} = \beta \cdot \partial_{2n+2}, \\ \varphi_*(\partial_a) = \sum_b \beta_{ba} \cdot \partial_b + \partial_a(\varphi_{2n+2}) \cdot \partial_{2n+2}, \\ \varphi_*(\partial_i) = \partial_i + \sum_a \partial_i(\varphi_a) \cdot \partial_a + \partial_i(\varphi_{2n+2}) \cdot \partial_{2n+2}.$$

The additional condition that φ respects the multiplication reduces in view of (2.21) and (2.27) to the conditions

$$(2.28) \quad \varphi_*(\partial_i) \circ \varphi_*(\partial_a) = \varphi_*(\partial_i \circ \partial_a) \quad \text{and} \quad \varphi_*(\partial_i) \circ \varphi_*(\partial_j) = \varphi_*(\partial_i \circ \partial_j).$$

The first one is equivalent to

$$\begin{aligned} \delta_{i+n,a} \cdot \beta \cdot \partial_{2n+2} &= \delta_{i+n,a} \cdot \varphi_*(\delta_{2n+2}) = \varphi_*(\partial_i \circ \partial_a) \\ &= \varphi_*(\partial_i) \circ \varphi_*(\partial_a) = \sum_b \beta_{ba} \cdot \delta_{i+n,b} \cdot \partial_{2n+2} = \beta_{i+n,a} \cdot \partial_{2n+2}. \end{aligned}$$

This is equivalent to $\beta_{ab} = \delta_{ab} \cdot \beta$ and to

$$(2.29) \quad \varphi_a = \beta \cdot t_a.$$

Taking this into account, the second equation in (2.28) becomes

$$\begin{aligned} \sum_k \partial_i \partial_j \partial_k \Psi \cdot (\beta \cdot \partial_{k+n} + \partial_{k+n}(\varphi_{2n+2}) \cdot \partial_{2n+2}) \\ &= \sum_k \partial_i \partial_j \partial_k \Psi \cdot \varphi_*(\partial_{k+n}) = \varphi_*(\partial_i \circ \partial_j) = \varphi_*(\partial_i) \circ \varphi_*(\partial_j) \\ &= \sum_k \partial_i \partial_j \partial_k \Psi \cdot \partial_{k+n} + \sum_a \partial_i \circ \partial_j(\beta) \cdot t_a \partial_a + \sum_a \partial_i(\beta) \cdot t_a \partial_a \circ \partial_j \\ &= \sum_k \partial_i \partial_j \partial_k \Psi \cdot \partial_{k+n} + [t_{i+n} \cdot \partial_j(\beta) + t_{j+n} \cdot \partial_i(\beta)] \cdot \partial_{2n+2}. \end{aligned}$$

This is equivalent to

$$(2.30) \quad \partial_i \partial_j \partial_k \Psi \cdot \beta = \partial_i \partial_j \partial_k \Psi \quad \text{for all } i, j, k$$

and

$$(2.31) \quad (\partial_i \circ \partial_j)(\varphi_{2n+2}) [= \sum_k \partial_i \partial_j \partial_k \Psi \partial_{k+n}(\varphi_{2n+2})] \\ = t_{i+n} \cdot \partial_j(\beta) + t_{j+n} \cdot \partial_i(\beta).$$

1st case, all $\partial_i \partial_j \partial_k \Psi = 0$. Then (2.30) is empty and (2.31) becomes $\partial_i(\beta) = 0$. In this case β is an arbitrary constant in \mathbb{C}^* and γ_{ab} are arbitrary.

2nd case, some $\partial_i \partial_j \partial_k \Psi \neq 0$. Then (2.30) says $\beta = 1$. Now (2.31) becomes $(\partial_i \circ \partial_j)(\varphi_{2n+2}) = 0$. \square

3. (TEP)-structures

3.1. Definitions. For the proof of theorem 2.2(b) we need a datum which is between the Frobenius manifold and its F-manifold, namely the (TEP)-structure on $(\pi^*TM)_{|\mathbb{C} \times \mathcal{M}}$

where $\pi : \mathbb{P}^1 \times M \rightarrow M$ is the projection. We will show that this structure does not depend on (U_\bullet, λ) . Then theorem 2.2(b) will follow easily.

A (TEP)-structure of weight $w \in \mathbb{Z}$ consists of data $(H \rightarrow \mathbb{C} \times M, \nabla, S)$. Here M is a complex manifold, $H \rightarrow \mathbb{C} \times M$ is a holomorphic vector bundle, ∇ is a flat connection on $H|_{\mathbb{C}^* \times M}$ with a pole of Poincaré rank 1 along $\{0\} \times M$, and P is a ∇ -flat $(-1)^w$ -symmetric nondegenerate pairing

$$P : H_{(z,t)} \times H_{(-z,t)} \rightarrow \mathbb{C} \quad \text{for } (z,t) \in \mathbb{C}^* \times M$$

which extends with $j : (z,t) \mapsto (-z,t)$ to a nondegenerate pairing

$$(3.1) \quad P : \mathcal{O}(H) \otimes j^* \mathcal{O}(H) \rightarrow z^w \mathcal{O}_{\mathbb{C} \times M}.$$

A (TLEP)-structure of weight $w \in \mathbb{Z}$ is an extension of the bundle $H \rightarrow \mathbb{C} \times M$ of a (TEP)-structure to a holomorphic vector bundle $\hat{H} \rightarrow \mathbb{P}^1 \times M$ such that the pole along $\{\infty\} \times M$ is logarithmic and such that P extends to an everywhere nondegenerate pairing from $\mathcal{O}(\hat{H}) \otimes j^* \mathcal{O}(\hat{H})$ to $z^w \mathcal{O}_{\mathbb{P}^1 \times M}$.

A (trTLEP)-structure is a (TLEP)-structure such that \hat{H} is a family of trivial bundles on \mathbb{P}^1 .

A (TEP)-structure induces a Higgs field

$$C = [z\nabla] : \mathcal{O}(H|_{\{0\} \times M}) \rightarrow \mathcal{O}(H|_{\{0\} \times M}) \otimes \Omega_M^1,$$

an endomorphism

$$U = [z^2 \nabla_{\partial_z}] : \mathcal{O}(H|_{\{0\} \times M}) \rightarrow \mathcal{O}(H|_{\{0\} \times M})$$

with $[C, U] = 0$ and a symmetric nondegenerate pairing

$$g = [z^{-w} P] : \mathcal{O}(H|_{\{0\} \times M}) \times \mathcal{O}(H|_{\{0\} \times M}) \rightarrow \mathcal{O}_M$$

with $C^* = C$ and $U^* = U$.

A (trTLEP)-structure is equivalent to a differential geometric structure on $H|_{\{0\} \times M}$ containing C and g and more data, which is called *Frobenius type structure* in [24], ch. 4, the equivalence is stated in [31], VI 7, [22], §5.2, and [24], §4.2.

There is a 1-1-correspondence between extensions of (TEP)-structures to (TLEP)-structures and monodromy invariant filtrations of the space

$$H^\infty := \{\text{global flat multivalued sections in } H|_{\mathbb{C}^* \times M}\}$$

(M is small and contractible), cf. [31], III.1.4, and [21], §8.2. So, obtaining extensions of (TEP)-structures to (TLEP)-structures is easy. There exist examples (already known by Birkhoff) of (TEP)-structures such that none of these extensions are (trTLEP)-structures. But the (TEP)-structures of interest for us have nice extensions to (trTLEP)-structures.

3.2. Two examples. For the constructions in this paper, the following two examples of (TEP)-structures play an important role:

(i) If M is a Frobenius manifold with $\text{Lie}_E(g) = (2 - d) \cdot g$, $d \in \mathbb{C}$, and $\pi : \mathbb{P}^1 \times M \rightarrow M$ is the projection, then π^*T_M is canonically equipped with a (trTLEP)-structure of (any) weight $w \in \mathbb{Z}$ (Dubrovin, Manin, e.g. [24], ch. 4):

$$(3.2) \quad P := z^w \cdot (\text{id}, j)^* g,$$

$$(3.3) \quad \nabla := \pi^* \nabla^g + \frac{1}{z} C + \left(-\frac{1}{z} E \circ -\nabla^g E + \frac{2-d+w}{2} \text{id} \right) \frac{dz}{z},$$

where C is the Higgs field on TM with $C_X = X \circ$.

(Compared to [24], ch. 4, here we changed the sign in $C_X = X \circ$ and used $-z$ instead of z in (3.3), in order to be compatible with section 2. The signs there are chosen to make the comparison with projective special geometry in section 5 smoother.)

(ii) Let $(M, \mathcal{V}, \nabla^\mathcal{V}, S, F^\bullet, w)$ be a variation of Hodge like filtrations with pairing (M is small and contractible). Let $\pi_{\mathbb{C}} : \mathbb{C} \times M \rightarrow M$ be the projection and $\pi_{\mathbb{C}}^* \nabla^\mathcal{V}$ be the flat connection on $\pi_{\mathbb{C}}^* \mathcal{V}$ whose flat sections are the pull backs of $\nabla^\mathcal{V}$ -flat sections in \mathcal{V} . Define a bundle $H \rightarrow \mathbb{C} \times M$ with $H|_{\mathbb{C}^* \times M} = \pi_{\mathbb{C}}^* \mathcal{V}|_{\mathbb{C}^* \times M}$ by

$$(3.4) \quad \mathcal{O}(H) := \sum_{p \in \mathbb{Z}} z^{w-p} \cdot \mathcal{O}(\pi_{\mathbb{C}}^* F^p),$$

a flat connection ∇ on $H|_{\mathbb{C}^* \times M}$ by

$$(3.5) \quad \nabla := \pi_{\mathbb{C}}^* \nabla^\mathcal{V}$$

and a pairing $P : (\pi_{\mathbb{C}}^* \mathcal{V})_{(z,t)} \times (\pi_{\mathbb{C}}^* \mathcal{V})_{(-z,t)} \rightarrow \mathbb{C}$ for $(z, t) \in \mathbb{C}^* \times M$ by

$$(3.6) \quad P(\pi_{\mathbb{C}}^* a, \pi_{\mathbb{C}}^* b) := \frac{1}{(2\pi i)^w} \cdot S(a, b).$$

Claim. Then $(H \rightarrow \mathbb{C} \times M, \nabla, P)$ is a (TEP)-structure of weight w .

On the one hand, this follows by unwinding the construction behind corollary 7.14(b) in [22] (an extra factor $1/(2\pi i)^w$ in (3.6) makes the definitions here compatible with [22]).

On the other hand, it can be seen directly as follows. (3.4), (3.5) and the Griffiths transversality (1.1) show $z \nabla_{\partial_z} \mathcal{O}(H) \subset \mathcal{O}(H)$ and $z \nabla_X \mathcal{O}(H) \subset \mathcal{O}(H)$ for $X \in T_M$. This gives the pole of Poincaré rank 1 along $\{0\} \times M$ (even $z^2 \nabla_{\partial_z} \mathcal{O}(H) \subset \mathcal{O}(H)$ would be sufficient).

The conditions (1.4), (3.4), (3.6) and the nondegenerateness of S show that P maps $\mathcal{O}(H) \otimes \mathcal{O}(H)$ to $z^w \mathcal{O}_{\mathbb{C} \times M}$ and that this map is nondegenerate. Obviously, P is $(-1)^w$ -symmetric and ∇ -flat.

3.3. F-manifolds from (TEP)-structures. There is a construction of Frobenius manifolds from meromorphic connections which goes back to the construction of Frobenius

manifolds in singularity theory by M. Saito [32] and K. Saito. It is formalized in [31], Théorème VII.3.6, [4], [1] and [24], theorems 4.2 and 4.5. In [24] the initial data are a (trTLEP)-structure with a distinguished section and an isomorphy condition. The following result gives the construction of a weaker datum, an F-manifold, from a weaker initial datum, a (TEP)-structure with an isomorphy condition. The proof relies on [22], 4.1.

Theorem 3.1. *Let $(H \rightarrow \mathbb{C} \times M, \nabla, P)$ be a (TEP)-structure (actually, the pairing P will not be used) with Higgs field $C = [z\nabla]$ and endomorphism $U = [z^2\nabla_{\partial_z}]$ on $\mathcal{O}(H_{|\{0\} \times M})$. Then $\mathcal{O}(H_{|\{0\} \times M})$ is a T_M -module. Suppose that the following isomorphy condition holds:*

$$(3.7) \quad \mathcal{O}(H_{|\{0\} \times M}) \text{ is a free } T_M\text{-module of rank 1.}$$

Then there is a unique multiplication \circ on T_M with $C_{X \circ Y} = C_X C_Y$ and a unique unit field e . The multiplication is commutative and associative. The unit field satisfies $C_e = \text{id}$. There is also a unique vector field E with $C_E = -U$. The tuple (M, \circ, e, E) is an F-manifold with Euler field.

Proof. The first part of the proof follows [22], lemma 4.1. Locally a section ξ in $H_{|\{0\} \times M}$ is chosen such that $C_\bullet \xi : TM \rightarrow H_{|\{0\} \times M}$ is an isomorphism. The multiplication \circ and the vector fields e and E are defined by

$$C_{X \circ Y} \xi = C_X C_Y \xi, \quad C_e \xi = \xi, \quad C_E \xi = -U \xi.$$

The multiplication is commutative and associative, and e is a unit field.

Because of

$$C_{X \circ Y} C_Z \xi = C_X C_Y C_Z \xi, \quad C_e C_Z \xi = C_Z \xi, \quad C_E C_Z \xi = -U C_Z \xi$$

the multiplication and the vector fields e and E are independent of the choice of ξ and satisfy

$$C_{X \circ Y} = C_X C_Y, \quad C_e = \text{id}, \quad C_E = -U.$$

The proof that they give an F-manifold with Euler field will use [22], lemma 4.3. In order to apply it, it would be nice to extend the (TEP)-structure to a (trTLEP)-structure. That is not always possible, but by [24], lemma 2.7, one can change and extend it (locally in M) to the following weaker structure: A holomorphic vector bundle $\hat{H} \rightarrow \mathbb{P}^1 \times M$ such that $\hat{H}_{|(\mathbb{C} - \{1\}) \times M} = H_{|(\mathbb{C} - \{1\}) \times M}$, such that the connection ∇ has logarithmic poles along $\{1\} \times M$ and $\{0\} \times M$ and such that \hat{H} is a family of trivial bundles on \mathbb{P}^1 (here M is supposed to be small). Because of the last condition

$$(3.8) \quad \mathcal{O}(\hat{H}_{|\{0\} \times M}) \cong \mathcal{O}(\hat{H}_{|\{\infty\} \times M}) \cong \pi_* \mathcal{O}(\hat{H})$$

and C and U on $\hat{H}_{|\{0\} \times M}$ as well as the residual connection ∇^{res} on $\hat{H}_{|\{\infty\} \times M}$ are shifted to the isomorphic sheaves. There are two further endomorphisms V and W (with $V + W = -\text{residue endomorphism on } H_{|\{\infty\} \times M}$) such that for fiberwise global sections $\sigma \in \pi_* \mathcal{O}(\hat{H})$

$$(3.9) \quad \nabla \sigma = \left(\nabla^{\text{res}} + \frac{1}{z} C + \left(\frac{1}{z} U + V + \frac{z}{z-1} W \right) \frac{dz}{z} \right) \sigma.$$

The flatness of ∇ yields $\nabla^{\text{res}}(C) = 0$ and $\nabla^{\text{res}}(U) - [C, V] + C = 0$. Therefore [22], lemma 4.3 applies and shows that (M, \circ, e, E) is an F-manifold with Euler field. \square

3.4. The classifying space \check{D}_{PHS} . For the rest of this section, a variation of Hodge like filtrations $((B_0, 0), \mathcal{V}, \nabla, S, F^\bullet)$ of weight $w = 3$ with pairing and CY-condition (1.7) and (1.8) is fixed. B_0 is a (sufficiently small) representative of a germ $(B_0, 0)$ of a manifold of dimension n . For $b \in B_0$ the filtration is denoted F_b^\bullet . By abuse of notation we also denote its ∇ -flat shift to the fiber \mathcal{V}_0 by F_b^\bullet .

There is a classifying space \check{D}_{PHS} for all Hodge like filtrations with the same discrete data as F_0^\bullet ,

$$(3.10) \quad \begin{aligned} \check{D}_{\text{PHS}} := \{ & \text{filtrations } F^\bullet \text{ on } \mathcal{V}_0 \mid S(F^p, F^{4-p}) = 0, \\ & 0 = F^4 \subset F^3 \subset F^2 \subset F^1 \subset F^0 = \mathcal{V}_0, \\ & \dim F^3 = 1 = \dim F^0/F^1, \\ & \dim F^2/F^3 = n = \dim F^1/F^2\}. \end{aligned}$$

It goes back to work of Griffiths and Schmid (see also [7]). It is a complex homogeneous space and a projective manifold. More concretely, it is a bundle over the Lagrangian Grassmannian

$$(3.11) \quad \check{D}_{\text{lag}} = \{F^2 \subset \mathcal{V}_0 \mid \dim F^2 = n + 1, S(F^2, F^2) = 0\}$$

with fibers $\mathbb{P}(F^2) \cong \mathbb{P}^n$. The base \check{D}_{lag} has dimension $n(n + 1)/2$, the fibers contain the possible choices of $F^3 \subset F^2$ and $F^1 = (F^3)^\perp$. The natural period map

$$(3.12) \quad \Pi : B_0 \rightarrow \check{D}_{\text{PHS}}, \quad b \mapsto F_b^\bullet,$$

is horizontal. Because of the CY-condition it is an embedding. It determines the variation of Hodge like filtrations.

3.5. The classifying space \check{D}_{BL} . There is a classifying space \check{D}_{BL} for certain (TEP)-structures with a natural projection $\pi_{\text{BL}} : \check{D}_{\text{BL}} \rightarrow \check{D}_{\text{PHS}}$. In a more general setting such spaces have been constructed in [20] and taken up again in [25], [26]. Here we restrict to the special case which we need. Before defining and discussing \check{D}_{BL} , some notations have to be established.

\mathcal{V}_0 is a $2n + 2$ dimensional complex vector space with antisymmetric and nondegenerate pairing S . The vector bundle $H' := \mathcal{V}_0 \times \mathbb{C}^*$ comes equipped with the trivial flat connection ∇ and a pairing

$$(3.13) \quad \begin{aligned} P : H'_z \times H'_{-z} & \rightarrow \mathbb{C} \quad \text{for } z \in \mathbb{C}^*, \\ (a, b) & \mapsto S(a, b), \quad \text{here } a, b \in H'_z = \mathcal{V}_0 = H'_{-z}. \end{aligned}$$

It is ∇ -flat, antisymmetric and nondegenerate. The space of global flat sections in H' is denoted C^0 . It is identified with \mathcal{V}_0 . For $\alpha \in \mathbb{Z}$ and $a \in \mathcal{V}_0$ the section $(z \mapsto z^\alpha \cdot a(z))$ is denoted $z^\alpha a$, the space of such sections is denoted $C^\alpha = z^\alpha \cdot C^0$.

The space $V^\alpha := \mathbb{C}\{z\} \cdot C^\alpha$ is the germ at 0 of the Deligne extension of $H' \rightarrow \mathbb{C}^*$ to a vector bundle on \mathbb{C} with logarithmic pole at 0 with α as the only eigenvalue of the residue endomorphism $[\nabla_{z\partial_z}]$. Together the spaces V^α , $\alpha \in \mathbb{Z}$, form the Kashiwara–Malgrange V -filtration. Of course $\text{Gr}_V^\alpha \cong C^\alpha$ canonically.

Any (TEP)-structure $(H \rightarrow \mathbb{C}, \nabla, P)$ with $H|_{\mathbb{C}^*} = H'$ is determined by the germ $\mathcal{H}_0 := \mathcal{O}(H)_0$ at 0. We are interested in the regular singular (TEP)-structures, i.e. those with $\mathcal{H}_0 \subset \sum_{\alpha} V^\alpha$. The spectrum of such a (TEP)-structure is the tuple $(\alpha_1, \dots, \alpha_{2n+2}) \in \mathbb{Z}^{2n+2}$ with $\alpha \leq \dots \leq \alpha_{2n+2}$ and

$$(3.14) \quad \#(i \mid \alpha_i = \alpha) = \dim \text{Gr}_V^\alpha \mathcal{H}_0 / \text{Gr}_V^\alpha z\mathcal{H}_0.$$

The (TEP)-structure induces a decreasing filtration $F^\bullet(H)$ on \mathcal{V}_0 by

$$(3.15) \quad F^p(H) = F^p(\mathcal{H}_0) := z^{p-3} \text{Gr}_V^{3-p} \mathcal{H}_0 \subset C^0 = \mathcal{V}_0.$$

The classifying space \check{D}_{BL} of (TEP)-structures relevant for us is

$$(3.16) \quad \check{D}_{\text{BL}} = \{ \text{regular singular (TEP)-structures } (H, \nabla, P) \text{ of weight 3 with } H|_{\mathbb{C}^*} = H' \text{ and spectrum } (\alpha_1, \dots, \alpha_{2n+2}) = (0, 1, \dots, 1, 2, \dots, 2, 3) \}$$

with 1 and 2 each n times.

Theorem 3.2. \check{D}_{BL} is an algebraic manifold and a bundle on \check{D}_{PHS} via

$$(3.17) \quad \pi_{\text{BL}} : \check{D}_{\text{BL}} \rightarrow \check{D}_{\text{PHS}}, \quad H \mapsto F^\bullet(H).$$

The fibers are isomorphic to \mathbb{C}^{n+1} as affine algebraic manifolds and carry a good \mathbb{C}^* -action with weights $(1, \dots, 1, 2)$. The corresponding zero section $\check{D}_{\text{PHS}} \hookrightarrow \check{D}_{\text{BL}}$ is given by the (TEP)-structures defined as in (3.4).

Proof. This theorem is a special case of [20], theorem 5.6, but here the proof simplifies. In the following we present the proof, as it provides useful explicit control on \check{D}_{BL} .

Lemma 3.3. $F^\bullet(H) \in \check{D}_{\text{PHS}}$ if $H \in \check{D}_{\text{BL}}$.

Proof. $F^\bullet(H)$ is decreasing because

$$\begin{aligned} F^{p+1}(H) &= z^{p+1-3} \text{Gr}_V^{3-(p+1)} \mathcal{H}_0 \\ &= z^{p-3} \text{Gr}_V^{3-p} z\mathcal{H}_0 \subset z^{p-3} \text{Gr}_V^{3-p} \mathcal{H}_0 = F^p(H). \end{aligned}$$

Because of the spectral numbers

$$(\dim F^p(H) \mid p = 3, 2, 1, 0) = (1, n + 1, 2n + 1, 2n + 2).$$

If $a_1 \in F^p(H)$ and $a_2 \in F^{4-p}(H)$ then there are sections

$$\sigma_1 \in \mathcal{H}_0 \cap (z^{3-p}a_1 + V^{4-p}) \quad \text{and} \quad \sigma_2 \in \mathcal{H}_0 \cap (z^{3-(4-p)}a_2 + V^{4-(4-p)}).$$

The z^2 -coefficient of $P(\sigma_1, \sigma_2) \in z^3\mathbb{C}\{z\}$ vanishes. This shows $S(a_1, a_2) = 0$. Therefore $S(F^p(H), F^{4-p}(H)) = 0$ and $F^\bullet(H) \in \tilde{D}_{\text{PHS}}$. \square

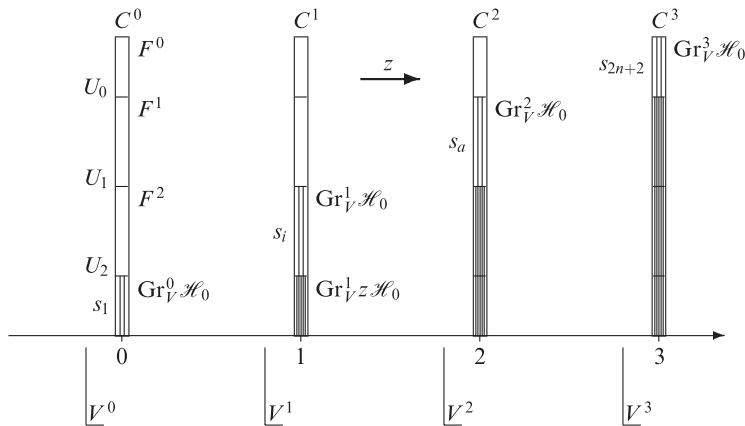
Now the fiber $\pi_{\text{BL}}^{-1}(F^\bullet)$ for an arbitrary F^\bullet shall be determined. The (TEP)-structures in this fiber will be described by certain distinguished sections in them. For that we make the same choices as in lemma 2.1, a filtration U_\bullet which is opposite to F^\bullet and a basis v_1, \dots, v_{2n+2} of \mathcal{V}_0 which satisfies (2.1) and (2.2). We use again the convention (2.11) for indices i, j, k, a, b, α . We define sections

$$(3.18) \quad \begin{cases} s_1 = v_1 \in C^0, & s_i = zv_i \in C^1, \\ s_a = z^2v_a \in C^2, & s_{2n+2} = z^3v_{2n+2} \in C^3, \end{cases}$$

and we define $(p_1, p_i, p_a, p_{2n+2}) = (3, 2, 1, 0)$ so that

$$v_\alpha \in F^{p_\alpha} \cap U_{p_\alpha} \quad \text{and} \quad s_\alpha \in z^{3-p_\alpha} \cdot F^{p_\alpha} \cap U_{p_\alpha}.$$

The following picture illustrates this and the next lemma.



Lemma 3.4. (a) For any $H \in \pi_{\text{BL}}^{-1}(F^\bullet)$ there exist unique sections $\sigma_\alpha \in \mathcal{H}_0$ with $\sigma_\alpha - s_\alpha \in \sum_{\beta > 3-p_\alpha} z^\beta \cdot U_{2-\beta}$. They form a $\mathbb{C}\{z\}$ -basis of \mathcal{H}_0 . Explicitly, they take the form

$$(3.19) \quad \begin{aligned} \sigma_1 &= s_1 + \sum_a y_a \cdot z^{-1} \cdot s_a + y_{2n+2} \cdot z^{-1} \cdot s_{2n+2}, \\ \sigma_i &= s_i + y_{n+i} \cdot z^{-1} \cdot s_{2n+2}, \\ \sigma_a &= s_a, \\ \sigma_{2n+2} &= s_{2n+2}, \end{aligned}$$

with some $y_a \in \mathbb{C}$, $y_{2n+2} \in \mathbb{C}$.

(b) The other way round, for any $y_a \in \mathbb{C}$ and $y_{2n+2} \in \mathbb{C}$, these sections generate over $\mathbb{C}\{z\}$ the germ \mathcal{H}_0 of a (TEP)-structure in $\pi_{\text{BL}}^{-1}(F^\bullet)$.

(c) Therefore $\pi_{\text{BL}}^{-1}(F^\bullet) \cong \mathbb{C}^{n+1}$ as an affine algebraic manifold, and

$$y_{n+2}, \dots, y_{2n+1}, y_{2n+2}$$

are coordinates on it.

Proof. (a) Because of the spectral numbers $\mathcal{H}_0 = (\mathcal{H}_0 \cap (C^0 + C^1 + C^2)) \oplus V^3$. Because of $F^\bullet(H) = F^\bullet$ there exist sections in $\mathcal{H}_0 \cap (s_x + V^{4-p_x})$. Existence and uniqueness of the sections σ_x is now an easy argument in linear algebra. It is also clear that they form a $\mathbb{C}\{z\}$ -basis of \mathcal{H}_0 . A priori they take the form

$$(3.20) \quad \sigma_1 = s_1 + \sum_a y_a \cdot z^{-1} \cdot s_a + x_{2n+2} \cdot z^{-2} \cdot s_{2n+2} + y_{2n+2} \cdot z^{-1} \cdot s_{2n+2},$$

$$\sigma_i = s_i + x_{n+i} \cdot z^{-1} \cdot s_{2n+2},$$

$$\sigma_a = s_a,$$

$$\sigma_{2n+2} = s_{2n+2},$$

with $y_a, x_{2n+2}, y_{2n+2}, x_a \in \mathbb{C}$. The germ \mathcal{H}_0 satisfies

$$(3.21) \quad z\nabla_{z\partial_z} \mathcal{H}_0 \subset \mathcal{H},$$

$$(3.22) \quad P : \mathcal{H}_0 \times \mathcal{H}_0 \rightarrow z^3\mathbb{C}\{z\} \quad \text{nondegenerate.}$$

On the other hand, the sections σ_x satisfy

$$(3.23) \quad z\nabla_{z\partial_z} \sigma_1 = \sum_a y_a \cdot \sigma_a + x_{2n+2} \cdot z^{-1} \cdot \sigma_{2n+2} + 2y_{2n+2} \cdot \sigma_{2n+2},$$

$$(3.24) \quad z\nabla_{z\partial_z} \sigma_i = z \cdot \sigma_i + x_{2n+2} \cdot \sigma_{2n+2},$$

$$(3.25) \quad z\nabla_{z\partial_z} \sigma_a = 2z \cdot \sigma_a,$$

$$(3.26) \quad z\nabla_{z\partial_z} \sigma_{2n+2} = 3z \cdot \sigma_{2n+2},$$

$$(3.27) \quad P(\sigma_\alpha, \sigma_\beta) = \begin{pmatrix} 2x_{2n+2} & z^2(y_{n+i} - x_{n+i}) & 0 & z^3 \\ z^2(x_{n+i} - y_{n+i}) & 0 & z^3 & 0 \\ 0 & z^3 & 0 & 0 \\ z^3 & 0 & 0 & 0 \end{pmatrix},$$

with $\alpha \in \{1, i, a, 2n+2\}$ and $\beta \in \{1, j, b, 2n+2\}$. Both (3.23) and (3.27) show $x_{2n+2} = 0$, (3.27) shows also $x_a = y_a$. This proves part (a).

(b) The sections σ_x generate over $\mathbb{C}\{z\}$ the germ \mathcal{H}_0 of the sections of a vector bundle $H \rightarrow \mathbb{C}$ which extends $H' \rightarrow \mathbb{C}^*$. Because of (3.23)–(3.27), \mathcal{H}_0 satisfies (3.21) and (3.22). Therefore (H, ∇, P) is a (TEP)-structure. It is in $\pi_{\text{BL}}^{-1}(F^\bullet)$.

(c) It is now also clear. \square

There is a natural \mathbb{C}^* -action on \check{D}_{BL} which respects the fibers of π_{BL} . It is defined (coordinate independently) as follows. For any $r \in \mathbb{C}^*$ define $\pi_r : \mathbb{C} \rightarrow \mathbb{C}, z \mapsto r \cdot z$. Then $(H, \nabla, P) \in \pi_{\text{BL}}^{-1}(F^\bullet)$ is mapped by $r \in \mathbb{C}^*$ via the \mathbb{C}^* -action to $\pi_r^*(H, \nabla, P) \in \pi_{\text{BL}}^{-1}(F^\bullet)$.

This action works as follows on the sections and coordinates in the last lemma. If $a \in C^0$ and $\alpha \in \mathbb{Z}$ then $\pi_r^*(z^\alpha \cdot a) = r^\alpha \cdot z^\alpha \cdot a$, so

$$\begin{aligned} \pi_r^* \sigma_1 &= s_1 + \sum_a r \cdot y_a \cdot z^{-1} \cdot s_a + r^2 \cdot y_{2n+2} \cdot z^{-1} \cdot s_{2n+2}, \\ \pi_r^* \sigma_i &= r \cdot (s_i + r \cdot y_{n+i} \cdot z^{-1} \cdot s_{2n+2}), \\ \pi_r^* \sigma_a &= r^2 \cdot \sigma_a, \\ \pi_r^* \sigma_{2n+2} &= r^3 \cdot \sigma_{2n+2}, \end{aligned}$$

and the \mathbb{C}^* -action on $\pi_{BL}^{-1}(F^\bullet)$ is given in the coordinates (y_a, y_{2n+2}) by

$$r \cdot (y_a, y_{2n+2}) = (r \cdot y_a, r^2 \cdot y_{2n+2}).$$

This finishes the proof of the theorem. \square

Remark 3.5. (i) Sections like the σ_α above were used first in [32], ch. 3.

(ii) The vector bundle $\mathcal{V}_0 \times \check{D}_{\text{PHS}}$ carries the trivial flat connection $\nabla^{\mathcal{V}_0 \times D}$ and the tautological filtration F^\bullet . The filtration is a family of Hodge like filtrations, but not a variation, because the Griffiths transversality is violated. Nevertheless, a filtration U_\bullet which is opposite to a reference filtration $F_\bullet^* \in \check{D}_{\text{PHS}}$ and then also to all filtrations nearby, induces a decomposition $\nabla^{\mathcal{V}_0 \times D} = \nabla^U + C^U$ into a flat connection

$$\nabla^U : \mathcal{O}(F^p \cap U_p) \rightarrow \mathcal{O}(F^p \cap U_p) \otimes \Omega_{(\text{nbhd of } F_\bullet^*)}^1$$

and a tensor $C^U : \mathcal{O}(F^p \cap U_p) \rightarrow \mathcal{O}(U_{p-1}) \otimes \Omega_{(\text{nbhd of } F_\bullet^*)}^1$.

A basis $v_1(*), \dots, v_{2n+2}(*)$ of vectors with (2.1) and (2.2) for F_\bullet^* extends to a $\nabla^{\mathcal{V}_0 \times D}$ -flat basis of sections of $\mathcal{V}_0 \times \check{D}_{\text{PHS}}$ with (2.1) and (2.2). The formulas (3.18) and (3.19) extend to these sections and yield a trivialization of the bundle $\pi_{BL} : \check{D}_{BL} \rightarrow \check{D}_{\text{PHS}}$ on $(\text{nbhd of } F_\bullet^*) \subset \check{D}_{\text{PHS}}$, with fiber coordinates y_a, y_{2n+2} .

(iii) The vector field on \check{D}_{BL} which generates the canonical \mathbb{C}^* -action is denoted E_{BL} . It is tangent to the fibers of π_{BL} . In local coordinates as in (ii) it is $E_{BL} = \sum_a y_a \frac{\partial}{\partial y_a} + 2y_{2n+2} \frac{\partial}{\partial y_{2n+2}}$. The zero section $\check{D}_{\text{PHS}} \hookrightarrow \check{D}_{BL}$ consists of the (TEP)-structures with $\mathcal{H}_0 = \sum_p \mathbb{C}\{z\} \cdot z^{3-p} \cdot F^p$, $F^\bullet \in \check{D}_{\text{PHS}}$ (as in (3.4)).

(iv) Any (TEP)-structure in \check{D}_{BL} is determined by $L := \mathcal{H}_0 \cap (C^0 + C^1 + C^2)$. The z^2 -coefficient of P restricts to a symplectic form $P^{(2)}$ on $C^0 + C^1 + C^2$. The multiplication by z restricts to a nilpotent endomorphism $\mu_z : C^0 + C^1 + C^2 \rightarrow C^1 + C^2$ with $C^0 \xrightarrow{\mu_z} C^1 \xrightarrow{\mu_z} C^2 \xrightarrow{\mu_z} 0$. We leave it to the reader to show that the classifying space \check{D}_{BL} can be identified with the following classifying space of certain lagrangian subspaces,

$$\begin{aligned} \check{D}_{BL} &= \{L \subset C^0 + C^1 + C^2 \mid \mu_z(L) \subset L, \mu_z(\nabla_{z\partial_z} L) \subset L, P^{(2)}(L, L) = 0, \\ &\dim L = 3n + 3, \dim L \cap (C^1 + C^2) = 3n + 2, \dim L \cap C^2 = 2n + 1\}. \end{aligned}$$

3.6. The canonical (TEP)-structure with isomorphy condition. As in section 3.4 a variation of Hodge like filtrations $((B_0, 0), \mathcal{V}, \nabla^\mathcal{V}, S, F^\bullet)$ of weight $w = 3$ with pairing and CY-condition (1.7) and (1.8) is fixed. The base space B_0 is identified with its image $\Pi(B_0) \subset \check{D}_{\text{PHS}}$ under the period map $\Pi : B_0 \rightarrow \check{D}_{\text{PHS}}$ in (3.12). Define

$$(3.28) \quad B_2 := \pi_{\text{BL}}^{-1}(\Pi(B_0)) \quad \text{and} \quad M := \mathbb{C} \times B_2.$$

The coordinate on the factor \mathbb{C} in $\mathbb{C} \times B_2$ is denoted y_1 . The tautological family of (TEP)-structures on \check{D}_{BL} restricts to a family of (TEP)-structures on B_2 . We extend it to a family $(H \rightarrow \mathbb{C} \times B_2, \nabla, P)$ of (TEP)-structures on M by twisting all sections with $e^{y_1/z}$.

Theorem 3.6. *With these definitions:*

(a) *This is a (TEP)-structure on M with isomorphy condition (3.7). Theorem 3.1 applies and gives M a canonical F-manifold structure. The unit field is $e = \frac{\partial}{\partial y_1}$, the Euler field is $E = y_1 \frac{\partial}{\partial y_1} - (E_{\text{BL}})_{|B_2}$ (E_{BL} is defined in remark 3.5(iii)).*

(b) *For any of the Frobenius manifolds in theorem 2.2, the underlying manifold $M^{U, \lambda}$ is canonically isomorphic to M . The isomorphism respects the F-manifold structure and the Euler field.*

Proof. It will be proved in several steps. For the rest of the section an opposite filtration U_\bullet and a vector $\lambda \in F_0^3 - \{0\}$ as in theorem 2.2 are chosen. Furthermore, sections v_1, \dots, v_{2n+2} on \mathcal{V} as in lemma 2.1 and with $v_1(0) = \lambda$ are chosen. Lemma 2.1 yields coordinates t_2, \dots, t_{n+1} on B_0 and a prepotential $\Psi \in \mathcal{O}_{B_0}$. Then the formulas (3.18) and (3.19) in section 3.5 provide sections σ_α which generate the tautological family of (TEP)-structures on $B_2 \subset \check{D}_{\text{BL}}$.

Lemma 3.7. (a) *For $i, j, k \in \{2, \dots, n + 1\}$ and $a \in \{n + 2, \dots, 2n + 1\}$*

$$(3.29) \quad z\nabla\sigma_1 = \sum_i \sigma_i dt_i + \sum_a \sigma_a dy_a + \sigma_{2n+2} dy_{2n+2} + \left(\sum_a y_a \sigma_a + 2y_{2n+2} \sigma_{2n+2} \right) \frac{dz}{z},$$

$$(3.30) \quad z\nabla\sigma_i = \sum_j \left(\sum_k \partial_i \partial_j \partial_k \Psi \cdot \sigma_{n+k} \right) \cdot dt_j + \sigma_{2n+2} \cdot dy_{n+i} + y_{n+i} \cdot \sigma_{2n+2} \cdot \frac{dz}{z} + \sigma_i dz,$$

$$(3.31) \quad z\nabla\sigma_a = \sigma_{2n+2} \cdot dt_{a-n} + 2\sigma_a \cdot dz,$$

$$(3.32) \quad z\nabla\sigma_{2n+2} = 3\sigma_{2n+2} \cdot dz.$$

(b) *The family of tautological (TEP)-structures has a pole of Poincaré rank 1 along $\{0\} \times M$ and is therefore a (TEP)-structure on B_2 . Its bundle is denoted $H^{B_2} \rightarrow \mathbb{C} \times B_2$.*

(c) *The sections σ_α define an extension to a (trTLEP)-structure on B_0 .*

Proof. (a) These formulas follow from (3.18), from the formulas in lemma 2.1, from (3.23)–(3.26) and from derivating the sections σ_α with $\nabla_{\frac{\partial}{\partial y_a}}$ and $\nabla_{\frac{\partial}{\partial y_{2n+2}}}$.

(b) Obvious.

(c) This follows from (a) and (3.27). \square

Lemma 3.8. *The bundle $H \rightarrow \mathbb{C} \times M$ whose sheaf is $\mathcal{O}(H) = e^{y_1/z} \cdot pr_2^* \mathcal{O}(H^{B_2})$ (where $pr_2 : \mathbb{C} \times B_2 \rightarrow B_2$ is the projection) is a (TEP)-structure with*

$$(3.33) \quad z\nabla(e^{y_1/z} \sigma_\alpha) = e^{y_1/z} \cdot z\nabla(\sigma_\alpha) + e^{y_1/z} \cdot \sigma_\alpha dy_1 - y_1 \cdot e^{y_1/z} \cdot \frac{dz}{z}.$$

It satisfies the isomorphy condition (3.7). The sections $e^{y_1/z} \cdot \sigma_\alpha$ define an extension to a (trTLEP)-structure.

Proof. (3.33) shows that the pole along $\{0\} \times M$ is of Poincaré rank 1. The pairing P satisfies

$$P(e^{y_1/z} \cdot \sigma_\alpha, e^{y_1/(-z)} \cdot \sigma_\beta) = P(\sigma_\alpha, \sigma_\beta) \in z^3 \cdot \mathbb{C},$$

so it is the pairing of a (TEP)-structure. By (3.29)–(3.33) the sections $e^{y_1/z} \cdot \sigma_\alpha$ define an extension to a (trTLEP)-structure. The Higgs field of the (TEP)-structure satisfies the isomorphy condition (3.7) because of (3.29) and (3.33). \square

Now theorem 3.1 applies and gives a canonical F-manifold structure. (3.33) shows $C_{\frac{\partial}{\partial y_1}} = \text{id}$, therefore $e = \frac{\partial}{\partial y_1}$. In the following calculation, $[\cdot]$ denotes the restriction to $H_{\{\{0\} \times M\}}^{\frac{\partial}{\partial y_1}}$,

$$\begin{aligned} & C_{y_1 \partial_1 - \sum_a y_a \partial_a - 2y_{2n+2} \partial_{2n+2}} [e^{y_1/z} \cdot \sigma_1] \\ &= y_1 [e^{y_1/z} \cdot \sigma_1] - \sum_a y_a [e^{y_1/z} \cdot \sigma_a] - 2y_{2n+2} [e^{y_1/z} \cdot \sigma_{2n+2}] \\ &= -U [e^{y_1/z} \cdot \sigma_1] := -[z\nabla_{z\partial_z} e^{y_1/z} \cdot \sigma_1]. \end{aligned}$$

Because $[e^{y_1/z} \cdot \sigma_1]$ generates $\mathcal{O}(H_{\{\{0\} \times M\}})$ as a T_M -module, this is sufficient to see $C_{y_1 \partial_1 - \sum_a y_a \partial_a - 2y_{2n+2} \partial_{2n+2}} = -U$ and $y_1 \frac{\partial}{\partial y_1} - (E_{BL})|_{B_2} = E$. Part (a) of theorem 3.6 is proved.

It rests to prove part (b). The choice of the sections v_1, \dots, v_{2n+2} yields coordinates (t_2, \dots, t_{n+1}) on B_0 , coordinates (t_1, \dots, t_{2n+2}) on $M^{U, \lambda}$ and coordinates $(y_1, t_2, \dots, t_{n+1}y_{n+2}, \dots, y_{2n+2})$ on M .

Of course, the most natural isomorphism between $M^{U, \lambda}$ and M is by identifying these coordinates. At the end of the proofs of lemma 2.1 and theorem 2.2 it was discussed how the coordinates (t_1, \dots, t_{2n+2}) change if (v_1, \dots, v_{2n+2}) are changed, but U_\bullet and λ are fixed.

One sees easily from (3.19) that the coordinates $(y_1, t_2, \dots, t_{n+1}, y_{n+2}, \dots, y_{2n+2})$ change in the same way. Therefore the isomorphism $M^{U,\lambda} \cong M$ above is canonical. Obviously it respects unit field and Euler field.

It also respects the multiplication. To see this, one chooses the section $\xi := [e^{y_1/z} \cdot \sigma_1]$ in $H|_{\{0\} \times M}$ and observes that the isomorphism $C_\bullet \xi : TM \rightarrow H|_{\{0\} \times M}$ maps e to ξ , ∂_i to $[e^{y_1/z} \cdot \sigma_i]$, $\frac{\partial}{\partial y_a}$ to $[e^{y_1/z} \cdot \sigma_a]$ and $\frac{\partial}{\partial y_{2n+2}}$ to $[e^{y_1/z} \cdot \sigma_{2n+2}]$. The Higgs field of the multiplication on TM is mapped to the Higgs field C on $H|_{\{0\} \times M}$. One can extract the Higgs field C from (3.29)–(3.33). Comparison with (2.21) shows that the multiplications coincide. This proves part (b) of theorem 3.6. \square

Remark 3.9. (i) The isomorphism $C_\bullet \xi : TM \rightarrow H|_{\{0\} \times M}$ above with $\xi := [e^{y_1/z} \cdot \sigma_1]$ lifts to an isomorphism from the (trTLEP)-structure on π^*TM in section 3.2(i) to the (trTLEP)-structure on M in the last lemma, with global sections $[e^{y_1/z} \cdot \sigma_\alpha]$.

(ii) In the beginning of section 3.3 a standard construction of Frobenius manifolds from meromorphic connections was mentioned. It can be applied to the Frobenius manifolds in theorem 2.2 and theorem 3.6. There it uses the (trTLEP)-structure with isomorphism condition constructed in the last lemma and the isomorphism $C_\bullet \xi : TM \rightarrow H|_{\{0\} \times M}$ with ξ as above.

4. Projective special (Kähler) geometry

This section presents some aspects of projective special geometry in a form which will make the comparison with Frobenius manifolds easy. It does not offer new results, and it neglects some aspects, for example the role of the pairing and an induced hermitian metric. Because of that we put the ‘‘Kähler’’ in brackets. Projective special (Kähler) geometry has a purely holomorphic part, the special coordinates, which are related to Frobenius manifolds, and a part involving the real structure, which is not related to Frobenius manifolds, but to tt^* -geometry [22]. We will touch the latter part only in the last part 4.4 of this section. More complete accounts, different aspects and motivation are provided in [18], [8], [1], [6].

4.1. The setting and two period maps. Let $(B_0, \mathcal{V}, \nabla, S, F^\bullet)$ be a variation of Hodge like filtrations with pairing of weight $w = 3$ which satisfies the CY-condition (1.7) and (1.8), with $n = \dim B_0$. As before, B_0 is a small neighborhood of a base point $0 \in B_0$. When necessary, the size of B_0 will be decreased, so essentially the germ $(B_0, 0)$ is considered.

The most important manifold in this section is $B = F^3 - \{\text{zero section}\}$, together with the natural projection $p : B \rightarrow B_0$. The fibers $F_b^3 - \{0\} \cong \mathbb{C}^*$ come equipped with a \mathbb{C}^* -action from the vector space structure, the corresponding vector field on B is denoted ε . Points in B are denoted (δ, b) where $b \in B_0$ and $\delta \in F_b^3 - \{0\}$. The pull back with p yields on B a variation of Hodge like filtrations with pairing $(p^*\mathcal{V}, p^*\nabla, p^*S, p^*F^\bullet)$ of weight 3. The bundle $p^*\mathcal{V}$ carries the tautological generating section σ_{taut} with $\sigma_{\text{taut}}(\delta, b) = \delta$. It satisfies

$$(4.1) \quad (p^*\nabla)_\varepsilon \sigma_{\text{taut}} = \sigma_{\text{taut}}.$$

There are two natural and related period maps:

$$(4.2) \quad P_1 : B \rightarrow \mathcal{V}_0, \\ (\delta, b) \mapsto \nabla\text{-flat shift of } \delta \in F_b^3 \subset \mathcal{V}_b \text{ to } \mathcal{V}_0,$$

$$(4.3) \quad P_2 : TB \rightarrow p^*F^2, \\ X \mapsto (p^*\nabla)_X \sigma_{\text{taut}}.$$

Here $TB = T^{1,0}B$ is the holomorphic tangent bundle. Only in the last section 4.4 also $T^{0,1}B$, $T^{\mathbb{C}}B = T^{1,0}B \oplus T^{0,1}B$ and $T^{\mathbb{R}}B$ will be used.

Because B_0 is small and B is a \mathbb{C}^* -bundle on B_0 , the flat connection ∇ induces the trivialization $\tau_1 : \mathcal{V} \xrightarrow{\cong} \mathcal{V}_0 \times B_0$ of the vector bundle \mathcal{V} , and $p^*\nabla$ induces the trivialization $\tau_2 : p^*\mathcal{V} \xrightarrow{\cong} \mathcal{V}_0 \times B$ of the vector bundle $p^*\mathcal{V}$.

Lemma 4.1. *The period maps P_1 and P_2 satisfy the following properties:*

(a) P_1 and P_2 are related by

$$(4.4) \quad \tau_2 \circ P_2 = (P_1)_* : TB \rightarrow P_1^*T\mathcal{V}_0 = P_1^*(\mathcal{V}_0 \times \mathcal{V}_0) = \mathcal{V}_0 \times B.$$

(b) P_1 is an embedding.

(c) P_2 is an embedding and thus locally (locally in B) an isomorphism of vector bundles.

Proof. (a) It follows from the definitions.

(b) The restriction of P_1 to $p^{-1}(0) = F_0^3 - \{0\} \subset B$ is the tautological embedding $F_0^3 - \{0\} \rightarrow F_0^3 \subset \mathcal{V}_0$. Because of this and because B_0 is small, for P_1 being an embedding it is sufficient to show that its differential $(P_1)_*$ is injective at points $(\delta, 0) \in p^{-1}(0)$. At such points $(P_1)_* = P_2 : T_{(\delta,0)}B \rightarrow F_0^2 \subset \mathcal{V}_0$. This map $P_2 : T_{(\delta,0)}B \rightarrow F_0^2$ is an isomorphism because of (4.1) and the CY-condition (1.7) and (1.8).

(c) Because B_0 is small and B is a \mathbb{C}^* -bundle on B_0 , also this follows from the fact that the map $P_2 : T_{(\delta,0)}B \rightarrow F_0^2$ is an isomorphism for any $(\delta, 0) \in p^{-1}(0)$. \square

4.2. Flat structure. The same situation as in section 4.1 is considered. Now additionally a ∇ -flat subbundle $U_1 \subset \mathcal{V}$ of rank $n + 1$ and with $S(U_1, U_1) = 0$ is chosen. It is called *opposite subbundle* if $F^2 + U_1 = \mathcal{V}$, equivalent: $F^2 + U_1 = F^2 \oplus U_1$, also equivalent: $F^2 \cap U_1 = \{\text{zero section}\}$. Because B_0 is small, these conditions are also equivalent to their restrictions to the zero fiber \mathcal{V}_0 .

Lemma 4.2. (a) *The following three conditions are equivalent:*

(i) U_1 is an opposite subbundle.

(ii) *The composition $pr_1 \circ P_1 : B \rightarrow \mathcal{V}_0 \rightarrow \mathcal{V}_0/(U_1)_0$ is an embedding, so locally an isomorphism. Here $pr_1 : \mathcal{V}_0 \rightarrow \mathcal{V}_0/(U_1)_0$ is the projection.*

(iii) The composition $pr_2 : p^*F^2 \rightarrow p^*\mathcal{V} \rightarrow p^*\mathcal{V}/p^*U_1$ of embedding and projection is an isomorphism of vector bundles. Then also $pr_2 \circ P_2 : TB \rightarrow p^*\mathcal{V}/p^*U_1$ is an isomorphism of vector bundles.

(b) Suppose that (i)–(iii) hold. The vector space structure on $\mathcal{V}_0/(U_1)_0$ induces a flat structure on B with flat and torsion free connection ∇^{U_1} . The flat connection $p^*\nabla$ on $p^*\mathcal{V}$ induces a flat connection on the quotient bundle $p^*\mathcal{V}/p^*U_1$ because p^*U_1 is a flat subbundle, and that connection induces via $(pr_2 \circ P_2)^*$ a flat connection ∇' on TB . Then $\nabla' = \nabla^{U_1}$.

Proof. (a) (i) \Leftrightarrow (iii) is trivial. The flat connection on $p^*\mathcal{V}/p^*U_1$ which is induced from $p^*\nabla$ on $p^*\mathcal{V}$, yields the trivialization $\tau_3 : p^*\mathcal{V}/p^*U_1 \xrightarrow{\cong} \mathcal{V}_0/(U_1)_0 \times B$ of the vector bundle $p^*\mathcal{V}/p^*U_1$. Then

$$(4.5) \quad \tau_3 \circ (pr_2 \circ P_2) = (pr_1 \circ P_1)_*$$

as maps from TB to $(pr_1 \circ P_1)^*T(\mathcal{V}_0/(U_1)_0) = \mathcal{V}_0/(U_1)_0 \times B$.

(ii) is equivalent to two conditions: First, that the restriction of $pr_1 \circ P_1$ to $p^{-1}(0)$, which is just the map

$$pr_1 \circ P_1 : F_0^3 - \{0\} \hookrightarrow F_0^3 \hookrightarrow \mathcal{V}_0 \rightarrow \mathcal{V}_0/(U_1)_0,$$

is an embedding, and second that the differential $(pr_1 \circ P_1)_*$ at points of $p^{-1}(0)$ is an isomorphism. The first condition is equivalent to $F_0^3 \cap (U_1)_0 = \{0\}$ which is part of (i), and the second condition is equivalent to (iii) and thus to (i), because of (4.5).

(b) This follows from (4.5). \square

4.3. Special coordinates. The same situation as in 4.1 is considered. The flat structure ∇^{U_1} on B from an opposite subbundle can be enriched by an additional choice, which leads to certain flat coordinates, the *special coordinates*.

Now $a_1, \dots, a_{n+1}, b_1, \dots, b_{n+1}$ are ∇ -flat sections of \mathcal{V} which form a symplectic basis everywhere. Then $U_1 = \langle b_1, \dots, b_{n+1} \rangle$ and $V_1 = \langle a_1, \dots, a_{n+1} \rangle$ are ∇ -flat subbundles of rank $n+1$ and with $S(U_1, U_1) = 0 = S(V_1, V_1)$.

By abuse of notation we write also a_i for p^*a_i and b_i for p^*b_i . There are unique functions $z_i, w_i \in \mathcal{O}_B$, $i = 1, \dots, n+1$, with

$$(4.6) \quad \sigma_{\text{taut}} = \sum_{i=1}^{n+1} z_i \cdot a_i + \sum_{i=1}^{n+1} w_i \cdot b_i.$$

Lemma 4.3. *The following properties hold true:*

(a) $\varepsilon(z_i) = z_i$, $\varepsilon(w_i) = w_i$. If z_1, \dots, z_{n+1} are coordinates on B , they are ∇^{U_1} -flat. Then $\varepsilon = \sum_{i=1}^{n+1} z_i \partial_{z_i}$. Furthermore,

$$(4.7) \quad z_1, \dots, z_{n+1} \text{ are coordinates on } B \Leftrightarrow U_1 \text{ is an opposite subbundle,}$$

$$(4.8) \quad w_1, \dots, w_{n+1} \text{ are coordinates on } B \Leftrightarrow V_1 \text{ is an opposite subbundle.}$$

If U_1 is an opposite subbundle then z_1, \dots, z_{n+1} are called special coordinates. If additionally V_1 is an opposite subbundle then w_1, \dots, w_{n+1} are called adjoint special coordinates.

(b) Suppose that U_1 is an opposite subbundle. Then there is a unique function $\Psi^{U_1, V_1} \in \mathcal{O}_B$ with

$$(4.9) \quad \frac{\partial \Psi^{U_1, V_1}}{\partial z_i} = w_i \quad \text{for } i = 1, \dots, n+1,$$

and

$$(4.10) \quad \varepsilon(\Psi^{U_1, V_1}) = 2 \cdot \Psi^{U_1, V_1}.$$

It depends only on U_1 and V_1 , not on the symplectic basis. It is called a prepotential.

(c) Suppose that U_1 is an opposite subbundle and that $V_1^0 \in \mathcal{L}(\mathcal{V}, U_1)$ where

$$\mathcal{L}(\mathcal{V}, U_1) := \{V_1 \subset \mathcal{V} \mid \nabla\text{-flat subbundle of rank } n+1, \\ S(V_1, V_1) = 0, \mathcal{V} = U_1 \oplus V_1\}.$$

$\mathbb{C}[z_1, \dots, z_{n+1}]_2$ denotes the polynomials homogeneous of degree 2. Then

$$\Psi^{U_1, V_1} \in \Psi^{U_1, V_1^0} + \mathbb{C}[z_1, \dots, z_{n+1}]_2,$$

and the map

$$\mathcal{L}(\mathcal{V}, U_1) \rightarrow \Psi^{U_1, V_1^0} + \mathbb{C}[z_1, \dots, z_{n+1}]_2, \quad V_1 \mapsto \Psi^{U_1, V_1},$$

is a bijection.

(d) The class $\Psi^{U_1, V_1} + \mathbb{C}[z_1, \dots, z_{n+1}]_2$ of prepotentials in (c) is characterized by the third derivatives $XYZ\Psi^{U_1, V_1}$ where $X, Y, Z \in \bigoplus_{i=1}^{n+1} \mathbb{C} \cdot \partial_{z_i}$ are flat vector fields, and these third derivatives are given by

$$(4.11) \quad -S(\sigma_{\text{taut}}, \nabla_X \nabla_Y \nabla_Z \sigma_{\text{taut}}) = XYZ\Psi^{U_1, V_1}.$$

This is a coordinate free characterization of this class of prepotentials. The class of prepotentials depends only on the flat structure ∇^{U_1} on B .

Proof. (a) (4.1) gives $\varepsilon(z_i) = z_i$, $\varepsilon(w_i) = w_i$. The map $pr_1 \circ P_1$ from section 4.2 is now explicitly

$$(4.12) \quad pr_1 \circ P_1 : B \rightarrow \mathcal{V}_0 / (U_1)_0, \quad (\delta, b) \mapsto \sum_{i=1}^{n+1} z_i \cdot [a_i].$$

It is an embedding iff z_1, \dots, z_{n+1} are coordinates on B . Lemma 4.2 applies and gives (4.7).

If z_1, \dots, z_{n+1} are coordinates, they are ∇^{U_1} -flat because of (4.12). In that case $\varepsilon = \sum_{i=1}^{n+1} \varepsilon(z_i) \partial_{z_i} = \sum_{i=1}^{n+1} z_i \partial_{z_i}$. (4.8) is analogous to (4.7).

(b) $\nabla_{\partial_{z_i}} \sigma_{\text{taut}} = a_i + \sum_{j=1}^{n+1} \frac{\partial w_j}{\partial z_i} \cdot b_j$ is a section in p^*F^2 , and $S(F^2, F^2) = 0$, so

$$0 = S(\nabla_{\partial_{z_i}} \sigma_{\text{taut}}, \nabla_{\partial_{z_j}} \sigma_{\text{taut}}) = \frac{\partial w_i}{\partial z_j} S(a_i, b_j) + \frac{\partial w_j}{\partial z_i} S(b_j, a_i) = \frac{\partial w_i}{\partial z_j} - \frac{\partial w_j}{\partial z_i}.$$

There exists a function $\Psi \in \mathcal{O}_B$ with $\frac{\partial \Psi}{\partial z_i} = w_i$. It is unique up to addition of a constant. It is claimed that there is exactly one function Ψ^{U_1, V_1} in this class with $\varepsilon(\Psi^{U_1, V_1}) = 2 \cdot \Psi^{U_1, V_1}$. Obviously there exists at most one such function. For the existence observe

$$\partial_{z_i} \varepsilon(\Psi) = [\partial_{z_i}, \varepsilon](\Psi) + \varepsilon \partial_{z_i}(\Psi) = \partial_{z_i}(\Psi) + \varepsilon(w_i) = 2w_i.$$

Therefore $\frac{1}{2} \varepsilon(\Psi)$ is also in the class. Because of $\frac{1}{2} \varepsilon(\Psi) = \Psi + \text{constant}$, $\varepsilon\left(\frac{1}{2} \varepsilon(\Psi)\right) = \varepsilon(\Psi)$, so $\frac{1}{2} \varepsilon(\Psi)$ is the desired function Ψ^{U_1, V_1} .

For the independence of the symplectic basis, consider a symplectic base change which fixes U_1 and V_1 ,

$$\begin{aligned} (a'_1, \dots, a'_{n+1}) &= (a_1, \dots, a_{n+1}) \cdot A, \\ (b'_1, \dots, b'_{n+1}) &= (b_1, \dots, b_{n+1}) \cdot (A^{\text{tr}})^{-1} \quad \text{with } A \in \text{GL}(n+1, \mathbb{C}). \end{aligned}$$

Then $(z'_1, \dots, z'_{n+1}) = (z_1, \dots, z_{n+1}) \cdot (A^{\text{tr}})^{-1}$, $(w'_1, \dots, w'_{n+1}) = (w_1, \dots, w_{n+1}) \cdot A$,

$$(\partial_{z'_1}, \dots, \partial_{z'_{n+1}}) = (\partial_{z_1}, \dots, \partial_{z_{n+1}}) \cdot A,$$

and thus $(\partial_{z'_1}, \dots, \partial_{z'_{n+1}})(\Psi) = (w'_1, \dots, w'_{n+1})$, so $\Psi' = \Psi$. Therefore Ψ^{U_1, V_1} depends only on U_1 and V_1 , not on the symplectic basis.

(c) Suppose that

$$a_1, \dots, a_{n+1}, \quad b_1, \dots, b_{n+1}, \quad U_1 = \langle b_1, \dots, b_{n+1} \rangle \quad \text{and} \quad V_1 = \langle a_1, \dots, a_{n+1} \rangle$$

are given, with U_1 an opposite subbundle. For any $V'_1 \in \mathcal{L}(\mathcal{V}, U_1)$ there are unique $a'_1, \dots, a'_{n+1} \in V'_1$ such that $a'_1, \dots, a'_{n+1}, b_1, \dots, b_{n+1}$ are a symplectic basis and

$$(a'_1, \dots, a'_{n+1}) = (a_1, \dots, a_{n+1}) + (b_1, \dots, b_{n+1}) \cdot A.$$

Then $A = A^{\text{tr}}$.

The corresponding map

$$\{A \in M((n+1) \times (n+1), \mathbb{C}) \mid A = A^{\text{tr}}\} \rightarrow \mathcal{V}$$

is a bijection. This and the following formulas give the claimed 1-1 correspondence,

$$\begin{aligned} \sigma_{\text{taut}} &= \sum_{i=1}^{n+1} z_i a_i + \sum_{i=1}^{n+1} w_i b_i = \sum_{i=1}^{n+1} z_i a'_i + \sum_{i=1}^{n+1} \left(w_i - \sum_{j=1}^{n+1} A_{ji} z_j \right) b_i, \\ w'_i &= w_i - \sum_{j=1}^{n+1} A_{ji} z_j, \\ \Psi^{U_1, V'_1} &= \Psi^{U_1, V_1} - \frac{1}{2} \sum_{i,j} A_{ij} z_i z_j. \end{aligned}$$

(d) Derivation of $0 = S(\sigma_{\text{taut}}, \nabla_{\partial_{z_j}} \nabla_{\partial_{z_k}} \sigma_{\text{taut}})$ by ∂_{z_i} gives

$$\begin{aligned} -S(\sigma_{\text{taut}}, \nabla_{\partial_{z_i}} \nabla_{\partial_{z_j}} \nabla_{\partial_{z_k}} \sigma_{\text{taut}}) &= S(\nabla_{\partial_{z_i}} \sigma_{\text{taut}}, \nabla_{\partial_{z_j}} \nabla_{\partial_{z_k}} \sigma_{\text{taut}}) \\ &= S\left(a_i + \sum_{m=1}^{n+1} (\partial_{z_i} \partial_{z_m} \Psi) \cdot b_m, \sum_{l=1}^{n+1} (\partial_{z_j} \partial_{z_k} \partial_{z_l} \Psi) \cdot b_l\right) \\ &= \partial_{z_i} \partial_{z_j} \partial_{z_k} \Psi. \end{aligned}$$

This completes the proof. \square

4.4. Data involving the real structure. Now let $(B_0, \mathcal{V}, \nabla, \mathcal{V}_{\mathbb{R}}, S, F^{\bullet})$ be a VPHS of weight $w = 3$ which satisfies the CY-condition (1.7) and (1.8). In the sections 4.1 to 4.3 we concentrated on one purely holomorphic aspect of projective special geometry, the flat structure and special coordinates after choosing U_1 and $a_1, \dots, a_{n+1}, b_1, \dots, b_{n+1}$.

For the sake of completeness here we discuss another datum, a connection ∇^{psg} on $T^{\mathbb{C}}B$ which involves the real structure. A third aspect, a hermitian pairing from the polarization will not be discussed here. In the following $TB = T^{1,0}B$, $T^{\mathbb{C}}B = T^{1,0}B \oplus T^{0,1}B$ and $T^{\mathbb{R}}B$ will be used.

The period map P_2 and the real structure $\mathcal{V}_{\mathbb{R}}$ induce an extended period map

$$(4.13) \quad \begin{aligned} P_3 : T^{\mathbb{C}} &\rightarrow p^* \mathcal{V}, \\ P_3 = P_2 : T^{1,0}B &\rightarrow p^* F^2, \\ P_3 = \overline{P_2} : T^{0,1}B &\rightarrow \overline{p^* F^2}, \quad X \mapsto \overline{P_2(\overline{X})}. \end{aligned}$$

Lemma 4.4. *For this period map we have:*

(a) P_3 is an isomorphism of \mathbb{C} -vector bundles. It respects the real structures, i.e. it maps $T^{\mathbb{R}}B$ to $p^* \mathcal{V}_{\mathbb{R}}$.

(b) Let ∇^{psg} be the connection on $T^{\mathbb{C}}B$ induced by $p^* \nabla$ via P_3 . It is flat and thus gives $T^{\mathbb{C}}B$ the structure of a holomorphic vector bundle. Of course, the subbundles $P_3^*(p^* F^3)$, $P_3^*(p^* F^2) = T^{1,0}B$ and $P_3^*(p^* F^1)$ of $T^{\mathbb{C}}B$ are holomorphic subbundles with respect to this holomorphic structure. The connection ∇^{psg} is torsion free.

Proof. Part (a) is obvious after lemma 4.1(c), in part (b) only the torsion freeness of ∇^{psg} is nontrivial. As it is classical and we will not use it, we leave the proof to the reader. \square

Remark 4.5. Let $J : T^{\mathbb{R}}B \rightarrow T^{\mathbb{R}}B$ with $J^2 = -\text{id}$ give the complex structure on B . The condition that $T^{1,0}B \subset T^{\mathbb{C}}B$ is a holomorphic subbundle with respect to the holomorphic structure on $T^{\mathbb{C}}B$ from ∇^{psg} is equivalent to

$$(4.14) \quad (\nabla_X^{\text{psg}}J)(Y) = (\nabla_Y^{\text{psg}}J)(X) \quad \text{for } X, Y \in T_B^{\mathbb{C}}$$

[22], Lemma 3.6. The condition (4.14) is often used as defining condition for affine special geometry. Thus affine special geometry on a manifold M means that there is a torsion free and flat connection which together with the (Hodge) decomposition $T^{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$ and the real subbundle $T^{\mathbb{R}}M$ yields a variation of Hodge structures of weight 1 on the complex tangent bundle $T^{\mathbb{C}}M$ [22], Proposition 3.7. Of course, in the present situation this holds, projective special geometry includes affine special geometry. The Hitchin system on the other hand exhibits the opposite behavior: we will show that the affine special geometry refines to a projective one.

5. Comparison

Let $(B_0, \mathcal{V}, \nabla, S, F^\bullet)$ be a variation of Hodge like filtrations with pairing of weight 3 which satisfies the CY-condition (1.7) and (1.8), with $n = \dim B_0$. As always, B_0 is supposed to be small, a germ of a manifold at a base point $0 \in B_0$.

In sections 2 and 3 we discussed a manifold $M \supset B_0$ of dimension $2n + 2$ and Frobenius manifold structures on it depending on a choice (U_\bullet, λ) , where U_\bullet is an opposite filtration and $\lambda \in F_0^3 - \{0\}$.

In section 4 we discussed a manifold B of dimension $n + 1$ which is a \mathbb{C}^* -bundle on B_0 , and a holomorphic aspect of projective special geometry, a flat structure (and special coordinates) depending on a choice of an opposite subbundle U_1 .

Now the constructions and data will be compared.

5.1. Choice of U_0 and U_2 . In the first lemma we start with B and a choice of the subbundles U_0 and U_2 of an opposite filtration U_\bullet . In the second lemma U_1 will be added.

Lemma 5.1. *Let U_0 and U_2 be flat subbundles of \mathcal{V} with*

$$(5.1) \quad U_0 = (U_2)^{\perp_s}, \quad U_2 = (U_0)^{\perp_s}, \quad \text{rank } U_0 = 1, \quad \text{rank } U_2 = 2n + 1.$$

(a) *Then $F^3 + U_2 = \mathcal{V} \Leftrightarrow F^1 + U_0 = \mathcal{V}$.*

(b) *Suppose that $F^3 + U_2 = \mathcal{V}$. The flat connection on the quotient bundle \mathcal{V}/U_2 and the isomorphism $F^3 \hookrightarrow \mathcal{V} \rightarrow \mathcal{V}/U_2$ yield a flat connection on F^3 and a trivialization*

$\tau_4 : F^3 \rightarrow F_0^3 \times B_0$. This restricts to a trivialization

$$(5.2) \quad P^{U_2} : B \rightarrow (F_0^3 - \{0\}) \times B_0$$

of the \mathbb{C}^* -bundle $B = F^3 - \{\text{zero section}\}$.

(c) The additional choice $\lambda \in F_0^3 - \{0\}$ distinguishes a hypersurface

$$(P^{U_2})^{-1}(\{\lambda\} \times B_0) \cong B_0 \quad \text{in } B.$$

Proof. For (a) remark $(F^3 \cap U_2)^{\perp s} = (F^3)^{\perp s} + U_2^{\perp s} = F^1 + U_0$ and

$$F^3 + U_2 = \mathcal{V} \quad \Leftrightarrow \quad F^3 \cap U_2 = \{\text{zero section}\}.$$

(b) and (c) are clear. \square

Lemma 5.2. *Let $(B_0, \mathcal{V}, \nabla, S, F^\bullet)$ be a variation of Hodge like filtrations with pairing of weight 3 satisfying the CY-condition.*

(a) *The choice of U_0 and U_2 with (5.1) and $F^3 + U_2 = \mathcal{V}$ and the choice of an opposite subbundle U_1 are together just the choice of an opposite filtration U_\bullet .*

(b) *Suppose that such a choice is made. Then the hypersurfaces*

$$(P^{U_2})^{-1}(\{\lambda\} \times B_0) \subset B, \quad \lambda \in F_0^3 - \{0\},$$

are ∇^{U_1} -flat hyperplanes of B , and they all induce the same flat structure on B_0 .

Proof. (a) is trivial. (b) By the embedding $pr_1 \circ P_1 : B \rightarrow \mathcal{V}_0/(U_1)_0$ in lemma 4.2 (a)(ii), the fibration of B by hyperplanes $(P^{U_2})^{-1}(\{\lambda\} \times B_0)$, $\lambda \in F_0^3 - \{0\}$, is mapped to the fibration of $\mathcal{V}_0/(U_1)_0$ by the affine hyperplanes $[\lambda] + (U_2)_0/(U_1)_0$. \square

5.2. Flat structures and (pre)potentials. In the proof of theorem 2.2(a), the choice (U_\bullet, λ) with U_\bullet an opposite filtration and $\lambda \in F_0^3 - \{0\}$ led to a Frobenius manifold structure on $M \supset B_0$ with potential $\Phi = \Psi + \dots$ as in (2.19) and $\Psi \in \mathcal{O}_{B_0}$. The additional choice of v_1^0, \dots, v_{2n+2}^0 leads to flat coordinates t_1, \dots, t_{2n+2} with t_2, \dots, t_{n+1} flat coordinates on $B_0 \subset M$.

In lemma 4.3 the choice of an opposite subbundle U_1 and another subbundle V_1 led to a prepotential $\Psi^{U_1, V_1} \in \mathcal{O}_B$ and a flat structure on B . The additional choice of a symplectic basis $a_1, \dots, a_{n+1}, b_1, \dots, b_{n+1}$ with $U_1 = \langle b_1, \dots, b_{n+1} \rangle$ and $V_1 = \langle a_1, \dots, a_{n+1} \rangle$ led to flat special coordinates z_1, \dots, z_{n+1} on B . These data will be compared now.

Theorem 5.3. *Choose (U_\bullet, λ) as above. Choose v_1^0, \dots, v_{2n+2}^0 as in lemma 2.1, with $v_1^0 = \lambda$. Choose $a_i = v_i^0$ ($i = 1, \dots, n + 1$) and $b_i = v_{n+i}^0$ ($i = 2, \dots, n + 1$) and $b_1 = -v_{2n+2}^0$. Then $a_1, \dots, a_{n+1}, b_1, \dots, b_{n+1}$ are a symplectic basis, and $V_1 = F_0^2$.*

(a) Then

$$(5.3) \quad (P^{U_2})^{-1}(\{\lambda\} \times B_0) = \{z_1 = 1\} \subset B,$$

and B_0 is embedded into B as this hyperplane. The flat structure on B_0 from the Frobenius manifold coincides with the flat structure which B_0 inherits from B by this embedding.

(b) The following equalities hold true:

$$(5.4) \quad t_i = z_i|_{\{z_1=1\}} \quad \text{for } i = 2, \dots, n + 1,$$

$$(5.5) \quad \Psi = \Psi^{U_1, F_0^2}|_{\{z_1=1\}}.$$

(c) The potential Φ of the Frobenius manifold can be changed by adding any element of $\mathbb{C}[t_1, \dots, t_{2n+2}]_{\leq 2}$ (where the index means degree ≤ 2) without changing the Frobenius manifold.

All the prepotentials in the class $\Psi^{U_1, F_0^2} + \mathbb{C}[z_1, \dots, z_{n+1}]_2$ from lemma 4.3(d) give via (5.5) and (2.19) ($\Phi = \Psi + \dots$) all the Frobenius manifold potentials in the class $\Phi + \mathbb{C}[t_2, \dots, t_{n+1}]_{\leq 2}$.

Proof. Compare (2.3) and (4.6),

$$(5.6) \quad v_1 = v_1^0 + \sum_{i=2}^{n+1} t_i \cdot v_i^0 + \sum_{i=2}^{n+1} \partial_i \Psi \cdot v_{n+i}^0 + \left(\left(\sum_{k=2}^{n+1} t_k \partial_k - 2 \right) \Psi \right) \cdot v_{2n+2}^0,$$

$$(5.7) \quad \sigma_{\text{taut}} = z_1 \cdot v_1^0 + \sum_{i=2}^{n+1} z_i \cdot v_i^0 + \sum_{i=2}^{n+1} w_i \cdot v_{n+i}^0 - w_1 \cdot v_{2n+2}^0.$$

On the hyperplane $\{z_1 = 1\}$ the section σ_{taut} restricts to v_1 , with

$$\begin{aligned} t_i &= z_i|_{\{z_1=1\}}, \\ \partial_i \Psi &= w_i|_{\{z_1=1\}} \quad \text{for } i = 2, \dots, n + 1, \\ \left(\sum_{k=2}^{n+1} t_k \partial_k - 2 \right) \Psi &= -w_1|_{\{z_1=1\}}. \end{aligned}$$

The equations $t_i = z_i|_{\{z_1=1\}}$ show part (a) and (5.4). The equations $\partial_i \Psi = w_i|_{\{z_1=1\}}$ for $i = 2, \dots, n + 1$ give

$$\frac{\partial}{\partial t_i} \left((\Psi^{U_1, F_0^2})|_{\{z_1=1\}} \right) = \left(\frac{\partial}{\partial z_i} \Psi^{U_1, F_0^2} \right)|_{\{z_1=1\}} = w_i|_{\{z_1=1\}} = \frac{\partial}{\partial t_i} \Psi.$$

This shows $(\Psi^{U_1, F_0^2})|_{\{z_1=1\}} = \Psi + \text{constant}$. In order to see that this constant is 0, we use $\left(\sum_{k=2}^{n+1} t_k \partial_k - 2 \right) \Psi = -w_1|_{\{z_1=1\}}$ and $v_1(0) = v_1^0$, which gives the first equality in the follow-

ing equations,

$$\begin{aligned} 0 &= -\left(\left(\sum_{k=2}^{n+1} t_k \partial_k - 2\right)\Psi\right)(0) = w_1(z_1 = 1, z_i = 0) \quad (i = 2, \dots, n + 1) \\ &= \left(\frac{\partial \Psi^{U_1, F_0^2}}{\partial z_1}\right) \quad (z_1 = 1, z_i = 0) \\ &= (\varepsilon \Psi^{U_1, F_0^2}) \quad (z_1 = 1, z_i = 0) \\ &= 2 \cdot \Psi^{U_1, F_0^2} \quad (z_1 = 1, z_i = 0). \end{aligned}$$

As $\Psi(0) = 0$, this shows (5.5). Part (c) is clear. \square

Remark 5.4. The theorem says that the Frobenius manifold structures on M with choices (U_\bullet, λ) with fixed U_1 , but varying (U_0, U_2, λ) have a nice common geometric origin. The flat structures on B_0 come from different embeddings of B_0 as affine hyperplanes in the flat manifold B . The parts Ψ of the Frobenius manifold potentials $\Phi = \Psi + \dots$ arise via restriction of the same prepotential Ψ^{U_1, F_0^2} .

6. Hitchin systems

The remainder of this paper is devoted to the application of the theory developed thus far to certain integrable systems as constructed by [27]. These are examples of so-called *algebraically completely integrable systems*, which in turn are known to give variations of Hodge structures of weight one on their base space. We will show that this can be refined in a natural way to a variation of Hodge like filtrations of weight three as described in the first part of the paper, which allows us to apply the results formulated there. We begin with a brief review of these integrable systems.

6.1. The moduli space of Higgs bundles. Let C be a complex curve of genus $g(C) \geq 2$, and fix a complex reductive group G with Lie algebra \mathfrak{g} . A *principal Higgs bundle* is a pair (P, Φ) , where $P \rightarrow C$ is a holomorphic principal G -bundle over C , and Φ —called the Higgs field—is an element of $H^0(C, \text{ad}(P) \otimes K_C)$, that is, a holomorphic one-form with values in the adjoint bundle $\text{ad}(P)$ of P .

Recall that a principal G -bundle P is said to be *stable* if the adjoint bundle is a stable vector bundle, i.e., for every proper subbundle $F \subset \text{ad}(P)$, we have

$$\text{deg}(F)/\text{rk}(F) < \text{deg}(\text{ad}(P))/\text{rk}(\text{ad}(P)).$$

As proved in [30], the moduli space \mathcal{M} of stable principal G -bundles is a smooth quasi-projective complex variety of dimension $\dim \mathcal{M} = \dim G(g(C) - 1) + \dim Z(G)$, where $Z(G)$ is the center of G . Its tangent space is given by

$$T_{[P]}\mathcal{M} \cong H^1(C, \text{ad}(P)),$$

so by Serre-duality, a Higgs bundle whose underlying principal bundle is stable determines a unique point in $T^*\mathcal{M}$.

The complex manifold $\mathfrak{X} := T^*\mathcal{M}$ forms an open dense subspace of the full moduli space of Higgs bundles. As a cotangent bundle, it carries a canonical holomorphic symplectic form ω_{can} : the tangent space to \mathfrak{X} at the point $[P, \Phi]$ fits into an exact sequence

$$0 \rightarrow H^0(C, \text{ad } P \otimes K_C) \rightarrow T_{[P, \Phi]}\mathfrak{X} \rightarrow H^1(C, \text{ad } P) \rightarrow 0.$$

The symplectic form is just the antisymmetrized version of the pairing between the first and third entry as induced by Serre-duality. Alternatively, there is a gauge-theoretical construction of this moduli space [27] which also explains the *hyperkähler* nature of \mathfrak{X} . We shall not be concerned in this paper with this enriched structure except for the existence of a Kähler form ω_K on \mathfrak{X} which is of type $(1, 1)$ with respect to the canonical complex structure as a cotangent bundle to a complex manifold.

We will now describe Hitchin’s fibration

$$p : \mathfrak{X} \rightarrow \tilde{B} := \bigoplus_{i=1}^k H^0(C, K_C^{\otimes d_i}),$$

for certain degrees $d_i \in \mathbb{N}$, and where $k = \text{rank}(\mathfrak{g})$. Choose a basis of invariant polynomials $p_1, \dots, p_k \in \mathbb{C}[\mathfrak{g}]^G$, where p_i has degree d_i . Each of these p_i defines a map

$$p_i : H^0(C, \text{ad}(P) \otimes K_C) \rightarrow H^0(C, K_C^{\otimes d_i}).$$

Now p is simply induced by the map $p(P, \Phi) := \sum_{i=1}^k p_i(\Phi)$. The fundamental theorem of Hitchin [27] states that the map p defines an algebraic integrable system on \mathfrak{X} . This means that:

(i) p is the restriction of a proper holomorphic map to an open dense subspace whose generic fibers are Lagrangian with respect to the holomorphic symplectic form ω_{can} .

(ii) The Kähler form ω_K restricts to each fiber to define a positive polarization.

6.2. Cameral curves and abelianization. Let $\Delta \subset \tilde{B}$ be the discriminant of the map p above and define $B := \tilde{B} \setminus \Delta$. By Hitchin’s result stated above, the fiber $\mathfrak{X}_b := p^{-1}(b) \subset \mathfrak{X}$ is a dense open subset of a compact polarized abelian variety of dimension $\dim G(\mathfrak{g}(C) - 1) + \dim Z(G)$ for each $b \in B$. It can be identified as a generalized Prym variety of a branched cover C_b of C , called the *cameral cover*.

Fix a maximal torus $T \subseteq G$ with Lie algebra $\mathfrak{t} \subseteq \mathfrak{g}$, a Borel subgroup H of G which contains T , and denote the associated Weyl group by W . By Chevalley’s theorem, restriction of polynomials induces an isomorphism $\mathbb{C}[\mathfrak{g}]^G \cong \mathbb{C}[\mathfrak{t}]^W$. Consider now the quotient map $\mathfrak{t} \rightarrow \mathfrak{t}/W$. Twisted with the canonical bundle K_C this defines a Galois covering

$\mathfrak{t} \otimes K_C \rightarrow (\mathfrak{t} \otimes K_C)/W$, and observe that $(\mathfrak{t} \otimes K_C)/W \cong \bigoplus_{i=1}^k K_C^{\otimes d_i}$. With this, the cameral cover for $b \in B$ is defined as

$$C_b := b^*(\mathfrak{t} \otimes K_C) \subset \mathfrak{t} \otimes K_C.$$

The projection of the bundle $\mathfrak{t} \otimes K_C$ to the base C induces a projection $\pi_b : C_b \rightarrow C$. By construction, this defines a W -Galois covering of C , where the Weyl group acts by the restriction of the action on \mathfrak{t} .

Remark 6.1. For the classical groups, it is sometimes more convenient to use the smaller *spectral covers*. Let us explain this for the case $G = \mathrm{GL}(n, \mathbb{C})$. In this case the underlying moduli space \mathcal{M} is of course simply the moduli space of stable vector bundles of rank n . Let $\lambda \in \Lambda$ be the weight of the fundamental representation of $\mathrm{GL}(n, \mathbb{C})$ on \mathbb{C}^n , and denote its stabilizer under action of the Weyl group by W_λ . The spectral cover is defined as the quotient C_b/W_λ .

The abelianization procedure is the following: for any principal G -bundle P over C , the structure group of the pull-back π_b^*P has a canonical reduction to H . The T -bundle associated to the projection $H \rightarrow T$ may not be T -invariant, but choosing a theta-divisor on C gives a canonical twist to a W -invariant T -bundle [33]. With this one proves:

Theorem 6.2 (Abelianization, see [14], [16], [27], [33]). (i) *Locally around a point $(P, \Phi) \in \mathfrak{X}_b$, the moduli space of Higgs bundles \mathfrak{X} is isomorphic to the moduli space of pairs (\tilde{C}, \tilde{P}) , where \tilde{C} is a W -invariant deformation of the cameral cover C_b , and \tilde{P} is a W -invariant T -bundle over it.*

(ii) *With this isomorphism, the projection $(\tilde{C}, \tilde{P}) \rightarrow \tilde{C}$ defines a Lagrangian foliation of an open subset of \mathfrak{X} .*

Weyl group invariant infinitesimal deformations of C_b in $\mathfrak{t} \otimes K_C$ are given by elements in $H^0(C_b, N_{C_b})^W$, where $N_{C_b} \rightarrow C_b$ is the normal bundle to $C_b \hookrightarrow \mathfrak{t} \otimes K_C$. The symplectic form on K_C defines an isomorphism $N_{C_b} \cong \mathfrak{t} \otimes K_{C_b}$ so that (ii) above gives the exact sequence

$$(6.8) \quad 0 \rightarrow H^1(C_b, \mathfrak{t} \otimes \mathcal{O}_{C_b})^W \rightarrow T_{(P, \Phi)}\mathfrak{X} \rightarrow H^0(C_b, \mathfrak{t} \otimes K_{C_b})^W \rightarrow 0.$$

7. The Seiberg–Witten differential

In this section we will define the Seiberg–Witten differential on the cameral curves associated to the Hitchin system and study its properties. In particular, we will relate the differential to the \mathbb{C}^* -action on the moduli space of Higgs bundles.

7.1. The \mathbb{C}^* -action. Let (P, Φ) be a Higgs bundle over the curve C with P a stable G -bundle. For $\xi \in \mathbb{C}^*$, we can scale the Higgs field to $\xi\Phi$ to obtain another Higgs bundle and this induces a holomorphic action $\varphi_\xi(P, \Phi) := (P, \xi\Phi)$ on the moduli space \mathfrak{X} . Of

course, this is simply the canonical action of \mathbb{C}^* on the cotangent bundle $\mathfrak{X} = T^*\mathcal{M}$, from which one immediately deduces that

$$\varphi_\zeta^* \omega_{\text{can}} = \zeta \omega_{\text{can}},$$

i.e., the canonical symplectic form is *conformal* with respect to the \mathbb{C}^* -action. Let E be the generating (holomorphic) vector field of this action, and define the Liouville form as $\alpha := \iota_E \omega_{\text{can}}$. By the conformal property of the symplectic form above we have $\text{Lie}_E \omega_{\text{can}} = \omega_{\text{can}}$ and therefore $d\alpha = \omega_{\text{can}}$.

Let $b \in B$ and consider the restriction $\alpha_b := \alpha|_{p^{-1}(b)}$, a holomorphic one-form on the fiber $p^{-1}(b)$. Recall that Hitchin’s result stated in section 6.1 identified this fiber as an Abelian variety.

Lemma 7.1. *The holomorphic one-form α_b is translation invariant.*

Proof. As above, let (p_1, \dots, p_k) denote the components of the Hitchin map $p : \mathfrak{X} \rightarrow B$. Standard symplectic geometry shows that the Hamiltonian vector fields X_i of p_i for $i = 1, \dots, k$ are tangential to the fibers of p and precisely generate the affine symmetry the fiber $p^{-1}(b)$ exhibits as an Abelian variety. Let $i_b : p^{-1}(b) \hookrightarrow \mathfrak{X}$ be the canonical inclusion. Then we have

$$\begin{aligned} \text{Lie}_{X_i} \alpha_b &= (d\iota_{X_i} + \iota_{X_i} d) i_b^* \alpha \\ &= i_b^* d\iota_{X_i} \iota_E \omega_{\text{can}} + \iota_{X_i} i_b^* d\alpha \\ &= -i_b^* d\iota_E dp_i + \iota_{X_i} i_b^* \omega_{\text{can}} \\ &= -d_i i_b^*(dp_i) \\ &= 0. \end{aligned}$$

Here we have used that the fibration $p : \mathfrak{X} \rightarrow B$ is Lagrangian, i.e., $i_b^* \omega_{\text{can}} = 0$ and that the p_i are homogeneous of degree d_i . \square

Introduce the following \mathbb{C}^* -action on the base B of the Hitchin system:

$$\zeta \cdot (b_1, \dots, b_k) = (\zeta^{d_1} b_1, \dots, \zeta^{d_k} b_k),$$

where $\zeta \in \mathbb{C}^*$ and $b = (b_1, \dots, b_k) \in B$ with $b_i \in H^0(C, K_C^{\otimes d_i})$. Obviously, equipped with this action, the Hitchin map $p : \mathfrak{X} \rightarrow B$ is \mathbb{C}^* -equivariant. In the following, we denote the generating vector field of this action on B by \mathcal{E} .

7.2. Definition and properties. A translation invariant one-form on an Abelian variety determines a unique element in the linear dual of the tangent space at a generic point. Consulting the short exact sequence (6.8), this means an element in $H^0(C_b, \mathfrak{t} \otimes K_{C_b})^W$ for the case at hand, viz. the fiber $\mathfrak{X}_b := p^{-1}(b)$ of the Hitchin map:

Definition 7.2. The Seiberg–Witten differential on the cameral curve

$$\lambda_{\text{SW}} \in H^0(C_b, \mathfrak{t} \otimes K_{C_b})^W$$

is the holomorphic one-form determined by the translation invariant one-form α_b , the restriction of the Liouville form to the fiber \mathfrak{X}_b .

There is an alternative definition of this differential as follows: Recall that the cameral curve C_b is canonically embedded in the total space of the vector bundle $\mathfrak{t} \otimes K_C$. There is a holomorphic action of \mathbb{C}^* by scaling along the fibers of this bundle. As a holomorphic cotangent bundle, K_C carries a canonical holomorphic symplectic form. On the tensor product $\mathfrak{t} \otimes K_C$, this can be interpreted as an \mathfrak{t} -valued symplectic form, denoted ω_{K_C} . Let ∂_ξ be the generator of the \mathbb{C}^* -action. Once again, the contraction $\theta := \iota_{\partial_\xi} \omega_{K_C}$, called the Liouville form, is a potential for this symplectic form.

Proposition 7.3. *The Seiberg–Witten form is equal to the restriction of the Liouville form:*

$$\lambda_{\text{SW}} = \theta|_{C_b}.$$

Proof. This is a consequence of the abelianization of Higgs bundles as described in section 6.2. Recall that the Hitchin map $p : \mathfrak{X} \rightarrow B$ is \mathbb{C}^* -equivariant, and projects the generating vector field E to \mathcal{E} . Let $(P, \Phi) \in \mathfrak{X}_b$. Because the Hitchin map defines a Lagrangian fibration, and the Seiberg–Witten differential λ_{SW} is defined by restricting $\iota_E \omega_{\text{can}}$ to the fiber \mathfrak{X}_b over $b \in B$, it follows from the exact sequence (6.8) that it is given by

$$\lambda_{\text{SW}} = \mathcal{E}(b) \in T_b B = H^0(C_b, \mathfrak{t} \otimes K_{C_b})^W.$$

Let $\xi \in \mathbb{C}^*$. It is an easy consequence of the definitions that

$$C_{\mu \cdot b} = \xi \cdot \{C_b\},$$

where on the right-hand side we use the canonical \mathbb{C}^* -action on K_C and the embedding $C_b \hookrightarrow \mathfrak{t} \otimes K_{C_b}$. The generator ∂_ξ of this action therefore defines a W -invariant deformation of C_b in $\mathfrak{t} \otimes K_{C_b}$ which corresponds to \mathcal{E} using the isomorphism

$$H^0(C_b, N_{C_b})^W \cong H^0(C_b, \mathfrak{t} \otimes K_{C_b})^W.$$

As explained below Theorem 6.2, this isomorphism is induced by contracting with the symplectic form on K_C . But the Liouville form is precisely defined as $\iota_{\partial_\xi} \omega_{K_C}$, so the result now follows. \square

Some of the information about the cameral cover is conveniently encoded in the zero divisor $D_{\lambda_{\text{SW}}}$ of λ_{SW} . The previous proposition clarifies where these zeroes are: using the fact that ω_{K_C} is nondegenerate one finds for any vector field v that $\iota_v \lambda_{\text{SW}}(p) = 0$ for $p \in C_b$ if and only if $\theta(p) = 0$ or $v(p) = c \cdot \partial_\xi(p)$ for some constant c . The first set of points are the intersections of C_b with C while the second set consists of the branch points of the covering map $\pi_b : C_b \rightarrow C$. We split $D_{\lambda_{\text{SW}}} = D_{\text{int}} + D_{\text{br}}$ into the intersection and branch points accordingly and calculate their degrees, cf. [29]. The map π_b has degree $|W|$, the order of the Weyl group. By definition, $\deg(D_{\text{int}}) = \deg(C_b \cap s_0)$ with s_0 the zero section of $K_C \rightarrow C$. This is the same as the intersection degree with any other section $s \in H^0(C, K_C)$

$$\deg(D_{\text{int}}) = \deg(C_b \cap s) = \deg(\pi) \deg(s) = |W| \cdot |K_C|.$$

We now turn our attention to the branch points. Since the cameral cover is the pull-back via b of the W -Galois cover $t \rightarrow t/W$, we are interested in the branch points of the latter. If σ_α denotes the reflection in the root α , then

$$\sigma_\alpha h = h \iff \alpha(h) = 0.$$

The map $t \mapsto t/W$ has branch points exactly on the zero divisor of the map $h \rightarrow \prod_\alpha \alpha(h)$.

This gives a degree Δ hypersurface $H \subset t \otimes K_C$, where Δ denotes the number of roots of \mathfrak{g} . The branch divisor of the cameral cover is the intersection divisor of b with H and therefore has degree $|\Delta| \cdot |K_C|$ where $|K_C| = 2g(C) - 2$ is the degree of the canonical divisor. This immediately gives

$$\deg(D_{\text{br}}) = |\Delta| \cdot |K_C|.$$

This is consistent with the Riemann–Hurwitz formula, which in this case reads

$$g(C_b) = \frac{|W| \cdot |K_C|}{2} + \frac{|K_C| \cdot |\Delta|}{2} + 1$$

so that indeed

$$\deg(D_{\lambda_{\text{sw}}}) = 2g(C_b) - 2 = |K_C|(|W| + |\Delta|) = \deg(D_{\text{int}}) + \deg(D_{\text{br}}).$$

The multiplicities of the points in $D_{\lambda_{\text{sw}}}$ may depend on the point in the base B , but in the generic situation $C_b \cap s_0$ consists of transversal intersections (giving first order zeroes of λ_{sw}) and the branch points are all of second order (giving second order zeroes). From now on, we will assume to be in the generic situation.

8. Variations of Hodge structures from Cameral curves

In this section we study a variation of Hodge structures associated to the family of cameral curves of the Hitchin system. A priori, this is a variation of weight one; in physics terminology the base is a *rigid* special Kähler manifold, in mathematical terms it is called *affine* special Kähler (cf. [18], [1], [22]). However, a careful analysis of the Seiberg–Witten differential in this variation shows that there exists a canonical refinement to a variation of weight three. In the physics literature this is called a *local* special Kähler manifold, in the mathematics literature one refers to this situation as *projective* special Kähler.

8.1. The variation of weight one. We review the variation of Hodge structures of weight $w = 1$ over B using the setup as in [9]. The family of cameral curves $f : \mathcal{C} \rightarrow B$ is defined such that $\mathcal{C}_b := f^{-1}(b) \cong C_b$. Recall that \mathcal{C} is equipped with an action of the Weyl group which preserves the fibers of f . Consider now the direct image functor of f in the category of W -equivariant sheaves

$$f_* : \text{Sh}_W(\mathcal{C}) \rightarrow \text{Sh}(B),$$

which assigns to $\mathcal{S} \in \text{Sh}_W(\mathcal{C})$ the sheaf

$$U \mapsto \mathcal{S}(f^{-1}(U))^W.$$

Its derived functors are denoted by R^*f_* . Let Λ be the root lattice of G and denote by $\underline{\Lambda}$ the associated locally constant sheaf on \mathcal{C} equipped with the canonical W -action. Homotopy invariance of cohomology implies that the sheaf of \mathbb{Z} -modules

$$\mathcal{V}_{\mathbb{Z}} := R^1f_*\underline{\Lambda} \in \text{Sh}(B)$$

forms a local system on B whose stalk at $b \in B$ equals $(\mathcal{V}_{\mathbb{Z}})_b = H^1(C_b, \underline{\Lambda})^W$. Next we consider the tensor product

$$\mathcal{V} := \mathcal{V}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{O}_B,$$

a coherent sheaf of holomorphic sections of a vector bundle over B . Because $\underline{\Lambda} \otimes_{\mathbb{Z}} \mathbb{C} \cong \mathfrak{t}$, its fiber at $b \in B$ is given by $\mathcal{V}_b = H^1(C_b, \mathfrak{t})^W$. Obviously, the map f is proper and therefore we have isomorphisms

$$\mathcal{V} \cong R^1f_*(\mathfrak{t} \otimes_{\mathbb{C}} f^*\mathcal{O}_B) \cong \mathcal{H}^1(f_*(\mathfrak{t} \otimes \Omega_{\mathcal{C}/B}^{\bullet})).$$

Here the relative differentials are defined through the following short exact sequence of coherent sheaves on \mathcal{C} :

$$(8.9) \quad 0 \rightarrow f^*\Omega_B^{\bullet} \rightarrow \Omega_{\mathcal{C}}^{\bullet} \rightarrow \Omega_{\mathcal{C}/B}^{\bullet} \rightarrow 0.$$

The middle term carries a natural decreasing filtration via

$$F^k = \text{image}[f^*\Omega_B^k \otimes_{\mathcal{O}_{\mathcal{C}}} \Omega_{\mathcal{C}}^{\bullet-k} \rightarrow \Omega_{\mathcal{C}}^{\bullet}].$$

The associated spectral sequence degenerates and leads to a filtration on $(\mathfrak{t} \otimes \Omega_{\mathcal{C}/B}^{\bullet}, d)$, the Hodge filtration. For the case at hand, this filtration has weight one; $F^1 \subset F^0 = \mathcal{V}$, with $F^1 = f_*(\mathfrak{t} \otimes \Omega_{\mathcal{C}/B}^1)$, i.e., $F_b^1 = H^0(C_b, \mathfrak{t} \otimes K_{C_b})^W \subset H^1(C_b, \mathfrak{t})^W$. The differential

$$\nabla : E_1^{0,1} \cong \mathcal{V} \rightarrow E_1^{1,1} \cong f^*\Omega_B^1 \otimes \mathcal{V}$$

is a flat connection on \mathcal{V} , called the Gauss–Manin connection, whose flat sections are given by $\mathcal{V}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$. Finally, there is a polarization $S : \mathcal{V} \times \mathcal{V} \rightarrow \mathcal{O}_B$ given by

$$(8.10) \quad S_b(\alpha, \beta) = \langle \alpha \cup \beta, [C_b] \rangle,$$

where the cup-product includes taking the inner product of two elements in \mathfrak{t} . Since we work with the first derived functor, it is antisymmetric: $S(\alpha, \beta) = -S(\beta, \alpha)$. Furthermore, it is ∇ -flat:

$$(8.11) \quad dS(\alpha, \beta) = S(\nabla\alpha, \beta) + S(\alpha, \nabla\beta).$$

The total of these data $(B, \mathcal{V}, \nabla, \mathcal{V}_{\mathbb{Z}}, S, F^{\bullet})$ define a variation of polarized Hodge structures of weight $w = 1$, cf. section 1.1.

8.2. The derivative of the Seiberg–Witten differential. Consider the variation of polarized Hodge structures $(B, \mathcal{V}, \nabla, \mathcal{V}_{\mathbb{Z}}, S, F^{\bullet})$ associated to the family of cameral curves $f : \mathcal{C} \rightarrow B$ constructed in the previous section. By definition, the universal curve \mathcal{C} comes

equipped with an embedding $\mathcal{C} \hookrightarrow \mathfrak{t} \otimes K_C \times B$. Pulling back the \mathfrak{t} -valued Liouville form θ on $\mathfrak{t} \otimes K_C$, one obtains a holomorphic one-form λ on \mathcal{C} which restricts to the Seiberg–Witten differential λ_{SW} on each fiber C_b . In the following we write λ_b for this restriction. By definition of the relative differential forms, the one-form λ_{SW} defines a section of $\mathfrak{t} \otimes \Omega^1_{\mathcal{C}}$ which, under the projection to $\mathfrak{t} \otimes \Omega^1_{\mathcal{C}/B}$ and the direct image f_* , defines a section $\lambda_{\text{SW}} \in F^1 \subset \mathcal{V}$ and restricts to the Seiberg–Witten differential on each fiber:

$$\lambda_b \in F_b^1 = H^0(C_b, \mathfrak{t} \otimes K_{C_b})^W.$$

8.2.1. The Čech–de Rham resolution. To compute the derivative of the Seiberg–Witten differential under the Gauss–Manin connection, we use a Čech-resolution of the relative de Rham complex $(\Omega^{\bullet}_{\mathcal{C}/B}, d)$ and calculate the hypercohomology following [9]. Define

$$U = \{x \in \mathcal{C} \mid d\pi_{f(x)} \neq 0\},$$

i.e., the complement of the branch points of the cameral cover or equivalently the complement of the second order zeroes of the Seiberg–Witten differential. We choose $V \subset \mathcal{C}$ such that $V \cap C_b$ consists of a disjoint union of small disks V_1, \dots, V_Z around the second order zeroes $p_1, \dots, p_Z \in C_b$ of λ_b . Here $Z = |\Delta| |K_C|$ denotes the number of branch points, i.e., second order zeroes of λ_{SW} . For any W -equivariant sheaf $\mathcal{S} \in \text{Sh}_W(\mathcal{C})$, write $f_*^U \mathcal{S} \in \text{Sh}(B)$ short for the composition $f_*(i_U)_* \mathcal{S}|_U$, where $i_U : U \hookrightarrow \mathcal{C}$ is the inclusion, and similarly for f_*^V and $f_*^{U \cap V}$.

In the relative Čech–de Rham complex, to compute $R^1 f_*$ we need the following part of the double complex of coherent sheaves on B :

$$(8.12) \quad \begin{array}{ccccc} & & (L_{X_U}, L_{X_V}) & & \\ & & \curvearrowright & & \\ f_* \Omega^1_{\mathcal{C}/B}(\mathcal{C}) & \longrightarrow & f_*^U \Omega^1_{\mathcal{C}/B} \oplus f_*^V \Omega^1_{\mathcal{C}/B} & \xrightarrow{\delta} & f_*^{U \cap V} \Omega^1_{\mathcal{C}/B} \\ & & \uparrow d_{\mathcal{C}/B} & \searrow \iota_{X_V - X_U} & \uparrow L_{X_U} \\ & & f_*^U \Omega^0_{\mathcal{C}/B} \oplus f_*^V \Omega^0_{\mathcal{C}/B} & \xrightarrow{\delta} & f_*^{U \cap V} \Omega^0_{\mathcal{C}/B} \end{array}$$

The vertical map $d_{\mathcal{C}/B}$ is the relative de Rham differential and δ denotes the Čech differential. The notation X_U, X_V will be explained below. With this resolution, elements in $R^1 f_*$ will be represented as cocycles in

$$(8.13) \quad \underbrace{(\Omega^1_{\mathcal{C}/B}(U) \oplus \Omega^1_{\mathcal{C}/B}(V))}_{(I)} \oplus \underbrace{\Omega^0_{\mathcal{C}/B}(U \cap V)}_{(II)}$$

so a relative differential α is represented by a triple

$$(\alpha_U, \alpha_V, g_{\alpha}^{U \cap V})$$

satisfying

$$d_{\mathcal{C}/B} g_{\alpha}^{U \cap V} = \delta(\alpha_V, \alpha_U).$$

In terms of this complex, the Hodge filtration is given by (8.13) and the polarization S is given by a trace-residue pairing

$$(8.14) \quad S_b(\alpha, \beta) = \sum_{k=1}^Z \text{Res}_{V_k} \langle g_\alpha^k dg_\beta^k \rangle$$

where g_α^k denotes the restriction of $g_\alpha^{U \cap V}$ to $U \cap V_k$ and $\langle \dots \rangle$ indicates the use of a pairing on \mathfrak{t} .

We now describe the Gauss–Manin connection. Over U and V , one can choose splittings of the exact sequence of sheaves

$$(8.15) \quad 0 \rightarrow f_* \Theta_{\mathcal{C}/B} \rightarrow f_* \Theta_{\mathcal{C}} \rightarrow \Theta_B \rightarrow 0.$$

This provides lifts X_U, X_V of holomorphic vector fields X on B . Conversely, such lifts define a splitting. The Gauss–Manin connection now has an explicit description:

$$(8.16) \quad \nabla_X(\alpha_U, \alpha_V, g_\alpha^{U \cap V}) = (L_{X_U} \alpha_U, L_{X_V} \alpha_V, L_{X_U} g_\alpha^{U \cap V} + \iota_{X_U - X_V} \alpha_V).$$

It is well-known that the part of the Gauss–Manin connection that actually shifts degree in the Hodge filtration, i.e., the Higgs field

$$C_X : F^1 \rightarrow \mathcal{V}/F^1,$$

equals taking the cup-product with the Kodaira–Spencer class: $C_X(\alpha) = \alpha \cup \kappa(X)$. Here $\alpha \in H^0(C_b, \mathfrak{t} \otimes K_{C_b})^W$, $\kappa : T_b B \rightarrow H^1(C_b, \Theta_{C_b})$ is the Kodaira–Spencer map and the notation stands short for the natural pairing

$$H^0(C_b, \mathfrak{t} \otimes K_{C_b})^W \times H^1(C_b, \Theta_{C_b}) \rightarrow H^1(C_b, \mathfrak{t} \otimes \mathcal{O}_{C_b})^W.$$

In the relative Čech–de Rham complex, if α is represented by $(\alpha_U, \alpha_V, 0)$ then C_X is given by the interior product $\iota_{X_U - X_V} \alpha_V$, which maps (I) to (II) in (8.13).

8.2.2. Derivatives of λ_{SW} . We now have the machinery to start the computation:

Lemma 8.1. $\nabla_X \lambda_{\text{SW}} \in F_b^1$ for all $X \in T_b B$.

Proof. We have to show that the composition

$$C_X : F^1 \xrightarrow{\nabla_X} \mathcal{V} \rightarrow \mathcal{V}/F^1,$$

applied to λ_{SW} , is zero. Since λ is naturally defined on \mathcal{C} , λ_{SW} can be represented as a differential on \mathcal{C} by $(\lambda|_U, \lambda|_V, 0)$. One finds on $U \cap V$ that

$$\iota_{X_U - X_V} \lambda|_V = \delta(\iota_{X_U} \lambda|_U, \iota_{X_V} \lambda|_V).$$

Since this is exact, it follows that $C_X \lambda_{\text{SW}} = 0$. \square

Recall, cf. (6.8), that $T_bB \cong H^0(C_b, \mathfrak{t} \otimes K_{C_b})^W$. For $X \in T_bB$, we write α_X for the holomorphic differential associated to X by this isomorphism.

Proposition 8.2. *For all $X \in T_bB$, we have*

$$\nabla_X \lambda_{\text{SW}} = \alpha_X.$$

Proof. We have already seen in the previous lemma that the part of $\nabla_X \lambda$ which maps from (I) to (II) in the Čech–de Rham complex (8.13), is exact. The remaining part, mapping (I) to (I), is given by taking the Lie derivatives L_{X_U}, L_{X_V} with respect to holomorphic lifts of X to U and V . By Cartan’s formula

$$L_{X_U} \lambda|_U = (d\iota_{X_U} + \iota_{X_U} d)\lambda|_U.$$

From this we see that

$$(L_{X_U} \lambda_U, L_{X_V} \lambda_V) - d(\iota_{X_U} \lambda_U, \iota_{X_V} \lambda_V) = (\iota_{X_U} \omega_{K_C}, \iota_{X_V} \omega_{K_C}),$$

where we have used that $\iota_{X_U} d\lambda|_U = \iota_{X_U} \omega|_U \in f_*^U \Omega_{\mathcal{C}/B}^1$. Recall that the second term on the left-hand side is exactly the derivative of the cocycle needed in the proof of the previous lemma to make $C_X \lambda_{\text{SW}}$ equal to zero. We now claim that on $U \cap V$ we have

$$\iota_{X_U} \omega_{K_C} - \iota_{X_V} \omega_{K_C} = 0$$

in $f_*^{U \cap V} \Omega_{\mathcal{C}/B}^1$. Indeed, the difference $X_U - X_V$ is a section of $\ker f_* \subset \Theta_{\mathcal{C}}$, that is, is tangent to each fiber C_b of $f : \mathcal{C} \rightarrow B$. But ω_{K_C} is a (\mathfrak{t} -valued) symplectic form, so $\iota_{X_U - X_V} \omega_{K_C} = 0$ as a relative differential form. It follows that $\iota_{X_U} \omega_{K_C}|_{\mathcal{C}}$ and $\iota_{X_V} \omega_{K_C}|_{\mathcal{C}}$ are the restrictions of an element of $f_* \Omega_{\mathcal{C}/B}^1$ which is by definition α_X . \square

As a corollary one finds the following rather obvious fact:

Corollary 8.3. *For the generator \mathcal{E} of the \mathbb{C}^* -action on B , we have*

$$\nabla_{\mathcal{E}} \lambda_{\text{SW}} = \lambda_{\text{SW}}.$$

8.3. The variation of weight three. Consider the variation of Hodge structures of weight one constructed in section 8.1. With the result of the previous section, we can now refine the filtration to obtain a variation of Hodge like filtrations of weight 3: Introduce

$$\begin{aligned} \mathcal{F}^3 &:= \mathcal{O}_B \cdot \lambda_{\text{SW}}, \\ \mathcal{F}^2 &:= R^1 f_* \Omega_{\mathcal{C}/B}^1, \\ \mathcal{F}^1 &:= (\mathcal{F}^3)^{\perp S}, \\ \mathcal{F}^0 &:= \mathcal{V}, \end{aligned}$$

and note that $\mathcal{F}^2 = F^1$. We introduce the projectivization $p : B \rightarrow \mathbf{P}(B)$ with respect to the \mathbb{C}^* -action and obtain

Theorem 8.4. *The data $(\mathbf{P}(B), \mathcal{V}, \nabla, \mathcal{V}_{\mathbb{Z}}, S, \mathcal{F}^\bullet)$ define a variation of Hodge like filtrations of weight 3 satisfying the CY-condition.*

Proof. Clearly, \mathcal{F}_b^\bullet defines a decreasing filtration of weight 3 on the fiber \mathcal{V}_b over $b \in B$. Therefore, the only thing left to check is that the filtration satisfies Griffiths transversality with respect to the Gauss–Manin connection, i.e.,

$$\nabla \mathcal{F}^\bullet \subset \mathcal{F}^{\bullet-1}.$$

In degree 3, this property is equivalent to lemma 8.1. In degree 2, let $\alpha \in f_*\Omega_{\mathcal{C}/B}^1$ and compute

$$\begin{aligned} S_b(\nabla_X \alpha, \lambda_{\text{SW}}) &= \int_{C_b} \langle \nabla_X \alpha \wedge \lambda_{\text{SW}} \rangle \\ &= \int_{C_b} \langle \alpha \wedge \nabla_X \lambda_{\text{SW}} \rangle - d \left(\int_{C^{X(b)}} \langle \alpha \wedge \lambda_{\text{SW}} \rangle \right) \\ &= 0, \end{aligned}$$

because both λ_{SW} , as well as its derivatives $\nabla_X \lambda_{\text{SW}}$ are holomorphic differentials. Here $\langle \dots \rangle$ indicates that the pairing on \mathfrak{t} has been used. Since \mathcal{F}^1 is defined as the symplectic complement of λ_{SW} , this proves that $\nabla \mathcal{F}^2 \subset \mathcal{F}^1$. This completes the proof of Griffiths transversality.

Finally, the CY-condition says that $\nabla \mathcal{F}^3$ should generate \mathcal{F}^2 . But this is clearly implied by proposition 8.2. \square

Remark 8.5. The polarization S has the wrong signature for a full VPHS of weight 3. Since this signature is not used in sections 4 and 5 we can endow the base of the Hitchin system with a projective special (Kähler) geometry and apply the results stated there.

8.4. The derivative of the period map. We give two expressions for the derivative of the period map corresponding to the family of cameral covers $f : \mathcal{C} \rightarrow B$. One of them (theorem 8.6) is inspired by the fact that the variation of Hodge structure of weight 1 can be refined in a natural way to a variation of Hodge structure of weight 3, which is reminiscent of a family of Calabi-Yau threefolds. The other expression (theorem 8.7) is a residue formula originally due to Balduzzi [2], who generalized a formula of Pantev. Similar formulas are known for matrix models, see e.g. [28].

Given a base curve and a complex reductive group, consider the family of cameral curves $f : \mathcal{C} \rightarrow B$ with central fiber C_{b_0} . Associated to this family is a *period map* cf. (3.11)

$$\Pi : B \rightarrow \check{D}_{\text{lag}}$$

which is given by the embedding $\mathcal{F}^2 \subset \mathcal{V}_b$ composed with parallel transport using the Gauss–Manin connection. Recall from (6.8) that this is a Lagrangian embedding with respect to the natural symplectic pairing on \mathcal{V}_{b_0} . We are interested in the derivative of the period map

$$d\Pi_b : (T_B)_b \rightarrow \text{Hom}(\mathcal{F}_b^2, (\mathcal{F}_b^2)^*).$$

In terms of the Kodaira–Spencer map and the Gauss–Manin connection a theorem of Griffiths gives

$$d\Pi_b(X)(\alpha, \beta) = S_b(\alpha, C_X\beta).$$

Using the natural isomorphism $(T_B)_b \cong \mathcal{F}_b^2$ given by $X \rightarrow \nabla_X\lambda_b$ the derivative $d\Pi$ becomes a tensor on B :

$$d\Pi_b : (T_B)_b \rightarrow ((T_B^*)_b)^{\otimes 2}$$

which is given by

$$(8.17) \quad d\Pi_b(X)(Y, Z) = S_b(\nabla_Y\lambda_b, \nabla_X\nabla_Z\lambda_b).$$

Integration by parts combined with the ∇ -flatness of S_b shows that (this is one of Riemann’s bilinear relations)

$$d\Pi_b : (T_B)_b \rightarrow \text{Sym}^2(T_B^*)_b.$$

It is well-known [13] that integrable systems give special period maps in the sense that $d\Pi$ is a cubic

$$d\Pi \in H^0(B, \text{Sym}^3(T_B^*)).$$

In the case of the Hitchin system, we can use the variation of weight 3 given in the previous section together with flatness of ∇ to conclude that $d\Pi(X, Y, Z)$ is indeed symmetric in its first and last arguments:

$$d\Pi_b(X, Y, Z) - d\Pi_b(Z, Y, X) = S_b(\nabla_Y\lambda_b, \nabla_{[X, Z]}\lambda_b) = 0.$$

We now arrive at a formula for $d\Pi$ which is reminiscent of a family of Calabi–Yau threefolds, with λ_b playing the role of the holomorphic three-form.

Theorem 8.6. *The derivative of the period map is given by (compare with (4.11))*

$$d\Pi_b(X, Y, Z) = - \int_{C_b} \langle \lambda_b \wedge \nabla_X \nabla_Y \nabla_Z \lambda_b \rangle.$$

Proof. Use integration by parts with respect to Z in (8.17), the ∇ -flatness of S_b and the symmetry in X, Y, Z . \square

In [12] a family of noncompact CY-threefolds was constructed in the case of ADE groups whose variation of mixed Hodge structure of weight 3 turns out to be pure, and in fact a Tate twist of a variation of Hodge structure of weight 1, which is compatible with the fact that S defines an indefinite polarization in weight 3. The authors of [12] have shown that the Yukawa cubic of this family of threefolds corresponds to the cubic above.

The expression in theorem 8.6 is not manifestly symmetric in its arguments. There is another, more symmetric, formula due to Balduzzi [2] who generalized a result for $G = \text{SL}_2$

by Pantev. We will give a different derivation of his result here, which uses the Čech–de Rham complex as described in section 8.2. We will choose coordinates on $U \cap V$ suggested by the cameral cover $\pi : C_b \rightarrow C$: one can pull back an affine coordinate on C via π to serve as a local coordinate z_U on U . For a generic point $b \in B$ the cover has second order branch points, which we will view as maps

$$p : B \xrightarrow{r} V \xrightarrow{z_U} \mathbb{C}.$$

Given the branch point p , a suitable holomorphic coordinate on the component of V containing $r(b)$ is given by

$$z_V = \sqrt{z_U - p(b)}.$$

The Seiberg–Witten differential has a second order zero at each of the branch points and can be represented in the Čech–de Rham complex by

$$\lambda_b = (fz_V^2 dz_V|_U, fz_V^2 dz_V|_V, 0)$$

where f is a \mathbb{t} -valued holomorphic function on V with $f \circ r(b) \neq 0$. The horizontal lifts X_U, X_V of a vector field X on B are determined by the chosen coordinates via

$$X_U(z_U) = 0, \quad X_V(z_V) = 0.$$

We are now ready to compute the contribution to $d\Pi$ coming from the component of V containing p . A straightforward computation using (8.16) and the fact that

$$\iota_{X_U} dz_V = \frac{-L_X(p)(b)}{2z_V}$$

now gives

$$\nabla_Y \lambda_b = \left(*, *, -\frac{z_V f}{2} L_Y(p)(b) \right)$$

which has a first order zero at the branch point. The first two terms will not contribute to (8.17), so we omit them here. Acting with ∇_X gives

$$\begin{aligned} \nabla_X \nabla_Y \lambda_b &= \left(*, *, * - L_{X_U} \left(\frac{z_V f}{2} L_Y(p)(b) \right) \right) \\ &= \left(*, *, * + \frac{f}{4z_V} L_X(p)(b) L_Y(p)(b) \right). \end{aligned}$$

Only the term containing a pole at the branch point is displayed and terms which are irrelevant for (8.17) are omitted. Using (8.14), we now arrive at the following result:

$$d\Pi(X, Y, Z) = \sum_{p \in D_{\text{br}}} \frac{L_X(p)L_Y(p)L_Z(p)}{8} \langle f \circ r, f \circ r \rangle$$

where $\langle \cdot, \cdot \rangle$ denotes the pairing between two elements of \mathfrak{t} . For semi-simple Lie groups there is only one Weyl-invariant pairing up to a scalar, which is the Killing form. It gives an isomorphism $\mathfrak{t} \cong \mathfrak{t}^*$ and the pairing can be expressed in terms of the root system R as

$$(8.18) \quad \langle h_1, h_2 \rangle = \sum_{\alpha \in R} \langle h_1, \alpha \rangle \langle \alpha, h_2 \rangle.$$

The quadratic residue Res_p^2 of a quadratic differential at a point p is defined as the coefficient of $z^{-2} dz \otimes dz$ in a Laurent expansion in terms of a coordinate z centered at p , and is independent of z .

Theorem 8.7 (Balduzzi, cf. [2]). *For semi-simple Lie groups one has*

$$(8.19) \quad d\Pi_b(X, Y, Z) = \sum_{p \in D_{\text{br}}} \sum_{\alpha \in R} \text{Res}_p^2 \left[\frac{\langle \alpha, \nabla_X \lambda_b \rangle \otimes \langle \alpha, \nabla_Y \lambda_b \rangle \otimes \langle \alpha, \nabla_Z \lambda_b \rangle}{\langle \alpha, \lambda_b \rangle} \right].$$

Proof. The quotient of two holomorphic differentials is a meromorphic function, so the term in brackets is a meromorphic quadratic differential. From the computation of the Gauss–Manin derivatives in the Čech–de Rham complex given above one finds that the derivatives of λ have an expansion around the branch points in terms of z_V :

$$\nabla_X \lambda = \left[L_X(p) \frac{f}{2} + \mathcal{O}(1) \right] dz_V.$$

Similarly

$$\frac{\langle \alpha, \nabla_X \lambda \rangle}{\langle \alpha, \lambda \rangle} = \frac{1}{2z_V^2} \left[\frac{L_X(p) \langle \alpha, f \rangle}{\langle \alpha, f \rangle} + \mathcal{O}(1) \right] = \frac{1}{2z_V^2} [L_X(p) + \mathcal{O}(1)].$$

Taking the quadratic residue and using (8.18) directly gives the desired result. \square

Remark 8.8. Replacing the root system by an orthonormal basis for the dual pairing gives an analogous expression for $d\Pi$ in the case of reductive non-semisimple groups.

9. The Frobenius manifold

The results of the previous section show that the Hitchin system gives rise to projective special geometry as in section 4 on B_0 . We illustrate in this case the choices necessary to define the Frobenius manifold structure: a natural generator $\zeta_0 \in F^3$ is provided by the Seiberg–Witten differential, and a choice of opposite filtration U_\bullet is described geometrically in terms of a choice of cycles on the cameral curve.

9.1. The opposite filtration. Recall the discussion of opposite filtrations in section 1.2. There is a natural procedure to define an opposite filtration on \mathcal{V} , viewed as a VHS of weight 3, as follows: fix $b \in B$, and consider $(\mathcal{V}_{\mathbb{Z}})_b^* = H_1(C_b, \Lambda)^W$. Combining the inner

product on Λ with the intersection form on $H_1(C_b, \mathbb{Z})$ defines a symplectic form \mathcal{I} , the dual of S , on the lattice $H_1(C_b, \Lambda)^W$:

$$\mathcal{I}(c_1, c_2) := \langle c_1 \cdot c_2 \rangle$$

for $c_1, c_2 \in H_1(C_b, \Lambda)^W$.

Now we choose a Lagrangian subspace $L^2 \subset H_1(C_b, \Lambda)^W$ and a one-dimensional subspace $L^3 \subset L^2$ of it, subject to the condition

$$(9.20) \quad L^3 \not\subset \ker \lambda_b.$$

We will also need the complement $L^1 = (L^3)^{\perp \mathcal{I}}$ of L^3 with respect to \mathcal{I} . With this we define

$$\begin{aligned} (U_0)_b &:= \{v \in \mathcal{V}_b \mid L^1 \subset \ker v\}, \\ (U_1)_b &:= \{v \in \mathcal{V}_b \mid L^2 \subset \ker v\}, \\ (U_2)_b &:= \{v \in \mathcal{V}_b \mid L^3 \subset \ker v\}, \\ (U_3)_b &:= \mathcal{V}_b. \end{aligned}$$

We extend these subspaces by parallel transport to ∇ -flat subbundles U_\bullet in a small neighborhood of $b \in B$.

Proposition 9.1. *U_\bullet defines an opposite filtration for the variation of Hodge-like filtrations on \mathcal{V} .*

Proof. By construction, the subbundles U_\bullet are ∇ -flat. Next, let us check that

$$\mathcal{V}_b = F_b^p \oplus (U_{p-1})_b,$$

for all $p = 0, \dots, 3$. For this, first observe that $\text{rk}(\mathcal{V}) = \text{rk}(F^p) + \text{rk}(U_{p-1})$, so we just have to verify that $F_b^p \cap (U_{p-1})_b = \{0\}$. Since F_b^3 is spanned by λ_b , this follows for $p = 3$ from condition (9.20). For $p = 2$, we have that $\alpha \in F^2 \cap U_1$ implies that $\alpha \in H^0(C_b, \mathfrak{t} \otimes K_{C_b})^W$, i.e., α is a holomorphic differential, and $L^2 \subset \ker \alpha$. By Abel's theorem, this implies that α has to be zero. An element $\alpha \in F^1 \cap U_0$ satisfies by definition

$$\alpha \in (F^3)^{\perp S} \quad \text{and} \quad (L^3)^{\perp \mathcal{I}} \subset \ker \alpha.$$

But the fact that S and \mathcal{I} are dual implies that (9.20) is equivalent to

$$(L^3)^{\perp \mathcal{I}} \not\subset \ker(\lambda_b^{\perp S}).$$

Finally, the condition that $S(U_p, U_{2-p}) = 0$ follows from the fact that \mathcal{I} is the dual of S and $\mathcal{I}(L^p, L^{4-p}) = 0$. \square

9.2. Special coordinates and the prepotential. Now we choose a symplectic basis $(a_1, \dots, a_n, b_1, \dots, b_n)$ of $H_1(C_b, \Lambda)^W$ with the sets of cycles $\{a_1\}$, $\{a_1, \dots, a_n\}$,

$\{a_1, \dots, a_n, b_2, \dots, b_n\}$ providing bases for L^3, L^2, L^1 respectively. An alternative proof of proposition 9.1 can be given by using this basis and the fact that

$$S(\alpha, \beta) = \sum_{k=1}^{n+1} \left(\int_{a_k} \alpha \int_{b_k} \beta - \int_{b_k} \alpha \int_{a_k} \beta \right).$$

The choice of cycles gives rise to the special coordinates

$$z_i = \oint_{a_i} \lambda$$

and adjoint coordinates

$$\frac{\partial \Psi}{\partial z_i} = \oint_{b_i} \lambda$$

on the Hitchin base B , where $\Psi \in \mathcal{O}_B$ denotes the prepotential. The choice of U_0 (or L^3) determines the hyperplane

$$B_0 := \{z_1 = 1\} \subset B$$

and the coordinates $t_i = z_i|_{z_1=1}$ on it. The germ of a Frobenius manifold structure on $\mathbb{C} \times B \times \mathbb{C}^{n+1}$ is completely specified by coordinates t_1, \dots, t_{2n+2} which include the coordinates on B just defined, together with the potential $\Phi(t)$ in (2.19) and the Euler vector field E in (2.22).

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