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# Killing spinor spacetimes and constant-eigenvalue Killing tensors

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## Abstract

A class of Petrov type D Killing spinor spacetimes is presented having the peculiar property that their conformal representants can only admit Killing tensors with constant eigenvalues.

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## 1. Introduction

In [8] ‘KS spacetimes’ were defined as non-conformally flat spacetimes admitting a *non-null* valence two-spinor  $X_{AB}$ , satisfying the conformally invariant Killing spinor equation

$$\nabla_{A'(A} X_{BC)} = 0. \quad (1)$$

The associated two-form  $D_{ab} = X_{AB}\epsilon_{A'B'} + \bar{X}_{A'B'}\epsilon_{AB}$  is a conformal Killing–Yano (CKY) tensor and, being non-null, KS spacetimes form the subclass of Petrov type D CKY spacetimes. Their repeated principal Weyl spinors are aligned with the principal spinors of  $X_{AB}$  [11] and define geodesic shear-free null congruences.

The square  $P_{ab} = D_a{}^c D_{cb}$  of a CKY tensor  $D$  is a conformal Killing tensor of Segre type [(11)(11)] and hence Killing spinors allow the construction of constants of motion along null geodesics; they have the additional significance [5] that they are the geometric objects from which one may construct symmetry operators for the massless Dirac equation. KS spacetimes necessarily include all spacetimes which are conformally related to Petrov type D Killing–Yano spacetimes: the inclusion is strict, a counterexample being given by the Kinnersley case III metrics [1, 2, 7].

A KS spacetime always admits a conformal representant in which the trace of the associated conformal Killing tensor  $P_{ab}$  is constant. In this representant, the conformal

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Killing tensor becomes a Killing tensor but has two constant eigenvalues. One may ask whether different conformal representants exist, admitting Killing tensors with non-constant eigenvalues: in the affirmative case this greatly simplifies the construction of the canonical line elements of KS spacetimes (see [4] for the case where the two eigenvalues provide independent functions of the coordinates and [8, 12] for the case where one of the eigenvalues is constant). In this paper we present a first example of KS spacetimes in which any associated Killing tensor always has two constant eigenvalues.

As in [12] the Geroch–Held–Penrose formalism [3] is used, but to ease comparison with the literature (particularly with regard to a possible interpretation of the energy–momentum tensor) we follow the notation and sign conventions of [9]: the tetrad basis vectors are taken as  $\mathbf{k}, \ell, \mathbf{m}, \bar{\mathbf{m}}$  with  $-k^a \ell_a = 1 = m^a \bar{m}_a$ . The correspondence with the Newman–Penrose operators and the basis one-forms is taken as  $(m^a, \bar{m}^a, \ell^a, k^a) \leftrightarrow (\delta, \bar{\delta}, \Delta, D)$  and  $(\bar{m}_a, m_a, -k_a, -\ell_a) \leftrightarrow (\omega^1, \omega^2, \omega^3, \omega^4)$  (this has the effect of changing the sign of the trace  $K$  in [12]). For completeness we repeat in section 2 the construction of the main equations, omitting however all details.

## 2. Preliminaries

Writing the Killing spinor as  $X_{AB} = X o_{(A} t_{B)}$ , the components of (1) imply

$$\kappa = \sigma = 0 \quad (2)$$

and

$$\flat X = -\rho X, \quad (3)$$

$$\delta X = -\tau X, \quad (4)$$

together with their ‘primed versions’ and  $X' = X$ . It follows that  $\Psi_2$  is the only non-vanishing component of the Weyl spinor; the spin coefficients  $\rho, \rho', \tau$  and  $\tau'$  are assumed to be non-zero (otherwise we re-obtain the metrics of [4, 8]). A conformal representant  $(\mathcal{M}, \hat{g})$  is fixed by imposing  $|\hat{X}| = 1$ . In the manifold  $(\mathcal{M}, \hat{g})$  one has  $\hat{\rho} + \bar{\hat{\rho}} = \hat{\rho}' + \bar{\hat{\rho}}' = \hat{\tau} + \bar{\hat{\tau}}' = 0$ , while in the  $(\mathcal{M}, g)$  manifold with  $g = \Omega^2 \hat{g}$  one has  $\Omega^2 = X \bar{X}$ .

Defining the trace-free conformal Killing tensor

$$P_{ab} = X_{AB} \bar{X}_{A'B'} = \frac{\Omega^2}{2} (m_{(a} \bar{m}_{b)} + \ell_{(a} n_{b)}), \quad (5)$$

one can show that  $P_{ab}$  will be the trace-free part of a Killing tensor  $K_{ab} = P_{ab} + \frac{1}{4} K g_{ab}$  if  $\nabla_b P^b_a + \frac{3}{4} \nabla_a K = 0$  or in components

$$\flat K + \Omega^2 (\rho + \bar{\rho}) = 0 = \delta K - \Omega^2 (\tau + \bar{\tau}'). \quad (6)$$

In terms of the eigenvalues  $a = (\Omega^2 + K)/4$  and  $b = (\Omega^2 - K)/4$  of

$$K_{cd} = 2a m_{(c} \bar{m}_{d)} + 2b \ell_{(c} n_{d)} \quad (7)$$

this means, using (3, 4),

$$\delta(a) = 0, \quad \flat(b) = 0, \quad (8)$$

$$\flat(a) = -(a + b)(\rho + \bar{\rho}), \quad (9)$$

$$\delta(b) = -(a + b)(\tau + \bar{\tau}'), \quad (10)$$

which can alternatively be written as

$$da = -(a + b)[(\rho + \bar{\rho})\omega^4 + (\rho' + \bar{\rho}')\omega^3] \quad (11)$$

$$db = -(a+b)[(\tau + \bar{\tau}')\omega^1 + (\bar{\tau} + \tau')\omega^2]. \quad (12)$$

KS spacetimes therefore admit at least one conformal representant,  $(\mathcal{M}, \hat{g})$ , in which a Killing tensor exists, which however has constant eigenvalues. Insisting on the existence of a conformal representant in which the eigenvalues are *not* both constants, extra integrability conditions result from the equations  $dda = ddb = 0$ .

It is preferable to manipulate all ensuing equations in the  $(\mathcal{M}, \hat{g})$  manifold, where we drop the  $\hat{\phantom{x}}$  symbol from here onward: the remaining spin coefficients are then  $\rho = -\bar{\rho}$ ,  $\rho' = -\bar{\rho}'$  and  $\tau = -\bar{\tau}'$  and the integrability conditions for the system (3), (4) simplify to

$$\flat\rho - \flat\rho' = 0, \quad (13)$$

$$\delta\tau' - \delta'\tau = 0, \quad (14)$$

$$\flat\tau' - \delta'\rho = 0. \quad (15)$$

The GHP equations reduce then to the system

$$\flat\rho = 0, \quad (16)$$

$$\delta\rho = 2\rho\tau + \Phi_{01}, \quad (17)$$

$$\flat\tau = 2\rho\tau + \Phi_{01}, \quad (18)$$

$$\delta\tau = 0, \quad (19)$$

$$\flat\rho' - \delta\tau' = -\rho\rho' - \tau\bar{\tau} - \Psi_2 - \frac{1}{12}R \quad (20)$$

and impose the following restrictions on the curvature:

$$\Phi_{00} = -\rho^2, \quad \Phi_{02} = -\tau^2, \quad (21)$$

$$E = -\frac{R}{12} - \rho\rho' + \tau\tau', \quad (22)$$

where  $E$  is the real part of  $\Psi_2 = E + iH$ .

Introducing 0-weighted quantities  $u, v$  (both real and with  $u' = u, v' = v$ ) and  $\phi, \phi'$  (complex) by

$$R = 8(u - v) - 16\rho\rho', \quad (23)$$

$$\Phi_{11} = u + v - 2\rho\rho', \quad (24)$$

$$\Phi_{01} = -3\rho\tau - 2\frac{\rho}{\tau'}\phi, \quad (25)$$

one can show [12] that a conformal representant admitting a Killing tensor with non-constant eigenvalues can only exist when one of the following conditions hold<sup>4</sup>:

$$\exists \text{ non-constant } b \iff \phi + \bar{\phi} = \phi' + \bar{\phi}' = 0, \quad (26)$$

$$\exists \text{ non-constant } a \iff \phi + \bar{\phi}' = 0. \quad (27)$$

The corresponding spacetimes were called  $KS_1$  or  $KS_2$ , respectively. All  $KS_1 \cap KS_2$  spacetimes were discussed in [4], while the remaining spacetimes belonging to  $KS_1 \cup KS_2$  were dealt with in [8, 12]. It was left undecided however whether KS spacetimes existed which did not

<sup>4</sup> Note the print error in equation (40) of [12].

belong to  $KS_1 \cup KS_2$ . In the next paragraph we show that the answer is affirmative: KS spacetimes do exist in which both conditions (26) and (27) are violated.

Introducing 0-weighted extension variables  $U$  (real) and  $V$  (complex) by

$$\flat\rho' = \flat'\rho = -iU, \quad (28)$$

$$\flat'\phi = \rho' \left( 2 \frac{\phi\phi'}{|\tau|^2} + V \right) + i \frac{U\phi}{\rho}, \quad (29)$$

the Bianchi identities and the ‘first level’ integrability conditions on  $\rho, \rho', \tau, \tau'$  provide expressions for the directional derivatives of  $\phi, u, v, U, V$  and  $H$ . Constructing the ‘second level’ integrability conditions, by applying the commutator operators to the latter variables, leads to an over-determined system of equations, the general solution of which so far has not been obtained. The system can however be greatly simplified by making the technical assumption that the function  $\phi' - \phi$  is real:

$$\bar{\phi}' - \bar{\phi} = \phi' - \phi. \quad (30)$$

Defining 0-weighted real variables  $r > 0$  and  $m > 0$  by

$$r^2 = Q\rho\rho', \quad m = |\tau| \quad (Q = \pm 1), \quad (31)$$

one can show that this implies

$$\phi' = \phi, \quad (32)$$

$$V = \frac{2}{m^2}(m^4 - 2\phi\bar{\phi}), \quad (33)$$

$$U = 2iQ\frac{r^2}{m}(\phi - \bar{\phi}), \quad (34)$$

$$H = \frac{2i}{m^2}(\phi - \bar{\phi})(Qr^2 + m^2), \quad (35)$$

$$u = \frac{1}{2m^2}(3m^4 + 6Qr^2m^2 - 2(\phi + \bar{\phi})Qr^2 - 4\phi\bar{\phi}), \quad (36)$$

$$v = -\frac{1}{2m^4}(2m^6 + 3Qr^2m^4 - 2(\phi + \bar{\phi})m^4 - 4Qr^2\phi\bar{\phi}). \quad (37)$$

In terms of the curvature components, this gives

$$R = -\frac{4}{m^2}[(4\phi\bar{\phi} + 2m^2(\phi + \bar{\phi}) - 5m^4)(Qr^2 + m^2)], \quad (38)$$

$$\Psi_2 = \frac{4}{3m^4}(\phi + 2m^2)(\bar{\phi} - m^2)(Qr^2 + m^2), \quad (39)$$

$$\Phi_{11} = \frac{1}{2m^4}(-4\phi\bar{\phi} + 2(\phi + \bar{\phi})m^2 + m^4)(m^2 - Qr^2). \quad (40)$$

Herewith the differential equations for the remaining 0-weighted variables can be succinctly written as

$$dX = X(-\tau\omega^1 + m^2\tau^{-1}\omega^2 - Qr^2\rho^{-1}\omega^3 - \rho\omega^4), \quad (41)$$

$$dr = \frac{r(\phi - \bar{\phi})}{m^2X} dX, \quad (42)$$

$$dm = \frac{\phi - \bar{\phi}}{mX} dX, \quad (43)$$

$$d\phi = \frac{2(\phi\bar{\phi} - m^4)}{m^2X} dX, \quad (44)$$

while the GHP-derivatives of the weighted quantities  $\rho$  and  $\tau$  are given by

$$\begin{aligned} \delta\rho &= \frac{\rho\tau}{m^2}(2\phi - m^2), & \delta'\rho &= \frac{\rho}{\tau}(2\bar{\phi} - m^2), & \flat'\rho &= \frac{2Qr^2}{m^2}(\phi - \bar{\phi}), & \flat\rho &= 0, \\ \delta\tau &= 0, & \delta'\tau &= 2(\bar{\phi} - \phi), & \flat'\tau &= \frac{Qr^2\tau}{m^2\rho}(m^2 - 2\bar{\phi}), & \flat\tau &= \frac{\rho\tau}{m^2}(2\phi - m^2). \end{aligned}$$

One can easily verify that the integrability conditions for this system are identically satisfied, such that corresponding solutions exist, and that  $m/r$  and  $\Re\phi$  are constants, with  $\Re\phi \neq 0$  as otherwise both conditions (26) and (27) would hold. Since  $\rho\tau \neq 0$  the tetrad can be invariantly fixed (up to interchange of  $k$  and  $l$ ), and one invokes from (21), (25) and (38)–(44) that the components of the Riemann tensor and its covariant derivatives w.r.t. such a tetrad contain at most one functionally independent function. Thus, the corresponding spacetimes admit a three- or four-dimensional maximal group of isometries [6, 9]. The first possibility corresponds to the non-constant  $r$  and will be discussed elsewhere. In the next paragraph we will discuss the class of solutions corresponding to the last possibility, which can be characterized by one of the requirements in the following lemma, the proof of which is immediate from the above.

**Lemma.** *For non-conformally flat spacetimes which violate both (26) and (27), and which satisfy condition (30), the following requirements are equivalent.*

- *The spacetime is purely electric, i.e. the function  $\Psi_2 \neq 0$  is real.*
- *The function  $\phi$  is real.*
- *The (real and positive) function  $r$  is constant.*
- *The (real and positive) function  $m$  is constant.*
- *The spacetime is homogeneous (allows for a four-dimensional group of motions).*

*In this case all invariantly defined functions (in particular,  $\phi$ ,  $\Psi_2$ ,  $R$  and  $\Phi_{11}$ ) are constant.*

By (44) this furthermore implies that  $\phi = \pm m^2$ , where  $\phi = m^2$  leads to conformally flat solutions, cf (39). For  $\phi = -m^2$ , however, we get

$$\Psi_2 = -\frac{8}{3}(m^2 + Qr^2), \quad R = 20(m^2 + Qr^2). \quad (45)$$

Note that when  $Q = -1$  we obtain conformally flat solutions (with  $R = 0$ ) if  $m = r$ , and we will therefore restrict to  $m \neq r$  in this case.

### 3. Homogeneous solutions

In order to integrate the above system under conditions (45), we switch to the Newman–Penrose formalism [10] and fix a boost and rotation such that  $\tau = m$  and  $\rho = iQr$ . For the spin coefficients  $\alpha, \beta, \epsilon, \gamma$  this implies

$$\epsilon = \frac{3}{2}iQr, \quad \gamma = \frac{3}{2}ir, \quad \alpha = \beta = \frac{3}{2}m. \quad (46)$$

Introducing new basis one-forms by

$$\Omega^1 = (\omega^1 - \omega^2), \quad \Omega^2 = \omega^1 + \omega^2, \quad \Omega^3 = \omega^3 + Q\omega^4, \quad \Omega^4 = \omega^3 - Q\omega^4, \quad (47)$$

such that the line element reads

$$ds^2 = 2(\Omega^{12} + \Omega^{22} - Q\Omega^{32} + Q\Omega^{42}), \quad (48)$$

the Cartan equations become

$$\begin{aligned} d\Omega^1 &= 2r\Omega^3 \wedge \Omega^2, \\ d\Omega^2 &= -2\Omega^3 \wedge (r\Omega^1 + mQ\Omega^4), \\ d\Omega^3 &= 2Q\Omega^2 \wedge (r\Omega^1 + mQ\Omega^4), \\ d\Omega^4 &= -2m\Omega^3 \wedge \Omega^2. \end{aligned} \quad (49)$$

It follows that  $m\Omega^1 + r\Omega^4$  is exact and that (the dual vector field of)  $\sqrt{-Q}\Omega^2 + \Omega^3$  is hypersurface-orthogonal.

### 3.1. $Q = +1$

When  $Q = 1$  one can immediately introduce coordinates  $t, x$  and  $y$  such that

$$\Omega^1 = dt - \frac{r}{m}\Omega^4, \quad (50)$$

$$\Omega^3 + i\Omega^2 = e^{u+iv} d(x + iy). \quad (51)$$

The cases  $m = r$  and  $m \neq r$  must now be treated separately. When  $m \neq r$  equations (49) integrate to

$$\Omega^4 = \frac{m}{2(m^2 - r^2)}(dv - 2r dt + u_y dx - u_x dy), \quad (52)$$

with  $u(x, y)$  a solution of the Liouville equation

$$u_{xx} + u_{yy} = 4(m^2 - r^2) e^{2u}. \quad (53)$$

This equation can be solved analytically, yielding solutions involving a free analytic function  $F(x + iy)$ , but this result is not needed here. Since all spin coefficients are constant and independent of  $F$ , all components of the Riemann tensor with respect to the fixed tetrad are constant and independent of  $F$ . This implies that line elements involving different  $F$  are equivalent [6], and we can choose the particular solution

$$e^u = \frac{1}{\mathcal{K}(x, y)}, \quad \mathcal{K}(x, y) = 1 - (m^2 - r^2)(x^2 + y^2). \quad (54)$$

Hence the line element is given by (48), where

$$\begin{aligned} \Omega^1 &= \frac{1}{2(m^2 - r^2)}(2m^2 dt - r dv) - \frac{r}{\mathcal{K}(x, y)}(y dx - x dy), \\ \Omega^2 &= \frac{1}{\mathcal{K}(x, y)}(\sin v dx + \cos v dy), \\ \Omega^3 &= \frac{1}{\mathcal{K}(x, y)}(\cos v dx - \sin v dy), \\ \Omega^4 &= \frac{m}{2(m^2 - r^2)}(-2r dt + dv) + \frac{m}{\mathcal{K}(x, y)}(y dx - x dy). \end{aligned} \quad (55)$$

When  $m = r$  the integration of the Cartan equations is straightforward and leads to the line element (48) with

$$\begin{aligned} \Omega^1 &= d(v - u) + 2mx dy, \\ \Omega^2 &= \sin 2mv dx + \cos 2mv dy, \\ \Omega^3 &= \cos 2mv dx - \sin 2mv dy, \\ \Omega^4 &= du - 2mx dy. \end{aligned} \quad (56)$$

This metric can be obtained as a singular limit of (55): performing the coordinate transform

$$v \rightarrow 2(m^2 - r^2)(v/m - xy) + 2rt \quad (57)$$

in (55), taking the limit  $r \rightarrow m$  and renaming  $(t, v) \rightarrow (v, u)$ , one precisely arrives at (56).

### 3.2. $Q = -1$ ( $m \neq r$ )

It is convenient now to define coordinates  $t, x$  and  $y$  by

$$\begin{aligned} \Omega^1 &= \frac{r}{r^2 + m^2} (dt - dy) - \frac{r}{m} \Omega^4, \\ \Omega^2 &= \frac{1}{r^2 + m^2} \mathcal{E}^{-1} dx - \Omega^3, \end{aligned} \quad (58)$$

with

$$\mathcal{E} = \exp \left( 2 \frac{r^2 y + m^2 t}{r^2 + m^2} \right). \quad (59)$$

The second Cartan equation implies the existence of a new independent function  $z$  such that

$$\Omega^4 = \frac{m}{m^2 + r^2} (dt + z dx), \quad (60)$$

after which the remaining Cartan equations integrate to

$$\Omega^3 = \frac{1}{2} \mathcal{E} \left[ -dz + \left( z^2 + F(x) + \frac{1}{m^2 + r^2} \mathcal{E}^{-2} \right) dx \right], \quad (61)$$

with  $F$  a free function. After a coordinate transformation

$$y \leftarrow m^2 t + \frac{m^2 + r^2}{2} \log y \quad (62)$$

the line element becomes (up to a constant re-scaling)

$$ds^2 = -m^2 (dt + z dx)^2 + r^2 (dy + z dx)^2 + \frac{1}{2} \mathcal{E}^{-2} dx^2 + \frac{1}{2} (r^2 + m^2)^2 \mathcal{E}^2 [(z^2 + F) dx - dz]^2, \quad (63)$$

where  $\mathcal{E}$  is still defined by (59), in terms of  $t$  and the new coordinate  $y$ . Under a coordinate transform

$$t \rightarrow t + \xi(x), \quad y \rightarrow y + \xi(x), \quad dx \rightarrow e^{2\xi} dx, \quad z \rightarrow e^{-2\xi} (z - \xi_x), \quad (64)$$

the function  $F(x)$  can be made to vanish by choosing for  $\xi(x)$  any local solution of the Riccati equation  $\frac{d^2 \xi}{dx^2} - \frac{d\xi}{dx}^2 = F$ . A Lorentz transformation of the original null tetrad, defined by

$$\begin{aligned} \omega^1 &= \frac{1}{2} [-e^{i\pi/4} \sigma^1 - e^{-i\pi/4} \sigma^2 + i\sigma^3 - i\sigma^4], \\ \omega^3 &= \frac{1}{2} [e^{-i\pi/4} \sigma^1 + e^{i\pi/4} \sigma^2 + \sigma^3 + \sigma^4], \\ \omega^4 &= \frac{1}{2} [-e^{-i\pi/4} \sigma^1 - e^{i\pi/4} \sigma^2 + \sigma^3 + \sigma^4], \end{aligned} \quad (65)$$

allows one to write the line element as  $ds^2 = 2(\sigma^1 \sigma^2 - \sigma^3 \sigma^4)$  with the following simple expressions for the basis one-forms:

$$\begin{aligned} \sigma^1 &= \frac{1}{2\sqrt{2}} \left[ (\mathcal{E} z^2 - \frac{i}{m^2 + r^2} \mathcal{E}^{-1}) dx - \mathcal{E} dz \right], \\ \sigma^3 &= \frac{1}{2(m^2 + r^2)} [m dt + (m + r)z dx + r dy], \\ \sigma^4 &= \frac{1}{2(m^2 + r^2)} [m dt + (m - r)z dx - r dy]. \end{aligned} \quad (66)$$



#### 4. Energy–momentum tensor

We investigate whether there exists pure radiation, Einstein–Maxwell or perfect fluid spacetimes in the conformal classes with representants (55), (56) and (66). The equations will be tackled in the Newman–Penrose formalism, fixing boost and rotation as in the previous section. Let us first outline the general scheme to be followed. The constant spin coefficients of the chosen tetrad  $\mathcal{B} := (m^a, \bar{m}^a, \ell^a, k^a) \leftrightarrow (\delta, \bar{\delta}, \Delta, D)$  in the original spacetimes are given by (46),  $\tau = \pi = m$  and  $\mu = Q\rho = ir$ . Herewith the components of the trace-free Ricci tensor, as calculated from the Newman–Penrose equations, are

$$\Phi_{rs} = \begin{pmatrix} r^2 & -5Qrmi & -m^2 \\ 5Qrmi & \frac{7}{2}(Qr^2 - m^2) & 5mri \\ -m^2 & -5mri & r^2 \end{pmatrix}, \quad (67)$$

while the Ricci scalar and the only non-zero Weyl scalar  $\Psi_2$  are given by (45). As in the above discussion, we are interested in spacetimes with  $m > 0$ ,  $r > 0$  and  $\Psi_2 \neq 0$ . We now perform a conformal transformation  $ds^2 \rightarrow ds'^2 = \Omega^{-2}ds^2$ , and take  $\mathcal{B}' := (m'^a, \bar{m}'^a, \ell'^a, k'^a) = (\Omega m^a, \Omega \bar{m}^a, \Omega \ell^a, \Omega k^a)$  as the NP null tetrad for  $ds'^2$ . The spin coefficients of this tetrad are

$$\begin{aligned} \kappa' &= \nu' = \sigma' = \lambda' = 0, & \tau' &= \Omega m + \delta\Omega, & \pi' &= \Omega m - \bar{\delta}\Omega, \\ \beta' &= \frac{3}{2}\Omega m - \frac{1}{2}\delta\Omega, & \alpha' &= \frac{3}{2}\Omega m + \frac{1}{2}\bar{\delta}\Omega, & \rho' &= i\Omega Qr + D\Omega, \\ \mu' &= i\Omega r - \Delta\Omega, & \epsilon' &= \frac{3}{2}i\Omega Qr - \frac{1}{2}D\Omega, & \gamma' &= \frac{3}{2}i\Omega r + \frac{1}{2}\Delta\Omega, \end{aligned}$$

the appearing directional derivative operators still being the vectors of the tetrad  $\mathcal{B}$ . Substituting this in the Newman–Penrose equations one obtains

$$\Psi'_0 = \Psi'_1 = \Psi'_3 = \Psi'_4 = 0, \quad \Psi'_2 = -\frac{8\Omega^2}{3}(m^2 + Qr^2), \quad (68)$$

in accordance with the conformal transformation properties of the Weyl tensor, and

$$\Phi'_{00} = \Omega^2 r^2 + \Omega D(D(\Omega)), \quad (69a)$$

$$\Phi'_{01} = -5\Omega^2 Qrmi - \Omega Qri\delta(\Omega) - 3\Omega m D(\Omega) + \Omega\delta(D(\Omega)), \quad (69b)$$

$$\Phi'_{02} = -\Omega^2 m^2 + \Omega\delta(\delta(\Omega)), \quad (69c)$$

$$\begin{aligned} \Phi'_{11} &= \frac{\Omega}{2}(riQ\Delta(\Omega) + riD(\Omega) + m\bar{\delta}(\Omega) + m\delta(\Omega)) \\ &\quad + \frac{7}{2}\Omega^2(Qr^2 - m^2) + \frac{\Omega}{2}\Delta(D(\Omega)) + \frac{\Omega}{2}\bar{\delta}(\delta(\Omega)), \end{aligned} \quad (69d)$$

$$\Phi'_{12} = 5\Omega^2 mri + \Omega\delta(\Delta(\Omega)) + 3\Omega m \Delta(\Omega) - \Omega ri\delta(\Omega), \quad (69e)$$

$$\Phi'_{22} = \Omega^2 r^2 + \Omega\Delta(\Delta(\Omega)), \quad (69f)$$

$$\begin{aligned} R' &= 20\Omega^2(Qr^2 + m^2) + 24\Delta(\Omega)D(\Omega) - 24\delta(\Omega)\bar{\delta}(\Omega) \\ &\quad + 6\Omega(\bar{\delta}(\delta(\Omega)) + \delta(\bar{\delta}(\Omega)) - \Delta(D(\Omega)) - D(\Delta(\Omega))). \end{aligned} \quad (69g)$$

Conditions on the energy–momentum of  $ds'^2$  lead via Einstein's equations to conditions on  $\Phi'_{rs}$  and  $R'$ . Equations (69) and their complex conjugates form ten real PDEs, and together with the six NP commutator relations applied to  $\Omega$ , these allow one to solve for all second-order derivatives of  $\Omega$ . The resulting integrability conditions, which are equivalent to the 20 NP Bianchi equations for  $ds'^2$ , form a set of first-order PDEs which will be analyzed in this section.

#### 4.1. Pure radiation

The spacetime with metric  $ds^2$  is a pure radiation spacetime iff its energy–momentum tensor is given by  $T_{ab} = \Phi n_a n_b$ , where  $n^a$  is a null vector:

$$n_1 n_2 - n_3 n_4 = 0. \quad (70)$$

Using Einstein's equations with the cosmological constant  $\Lambda$ ,  $G_{ab} = T_{ab} - \Lambda g_{ab}$ , this translates into

$$\Phi'_{rs} = \frac{\Phi}{2} \begin{pmatrix} n_4^2 & n_1 n_4 & n_1^2 \\ n_2 n_4 & \frac{1}{2}(n_1 n_2 + n_3 n_4) & n_1 n_3 \\ n_2^2 & n_2 n_3 & n_3^2 \end{pmatrix}, \quad (71)$$

$$R' = 4\Lambda. \quad (72)$$

After solving for all second-order derivatives of  $\Omega$  as discussed above, one should analyze the integrability conditions. For aligned solutions  $n_1 = n_2 = 0$ , these imply

$$Qr^2 + m^2 = 0. \quad (73)$$

Hence, for  $Q = 1$  there are no solutions, whereas for  $Q = -1$ , the solutions are conformally flat.

For non-aligned solutions one can eliminate  $n_2$  using (70), and the integrability conditions yield linear first-order differential equations for  $n_1, n_3, n_4, \Omega$  and  $\Phi$ . Elimination of the first-order derivatives yields the same scalar equation (73); hence, there are no non-conformally flat solutions.

#### 4.2. Einstein–Maxwell fields

The gravitational field is an Einstein–Maxwell field iff with respect to  $\mathcal{B}'$  one has

$$\Phi'_{rs} = \begin{pmatrix} F_0 \bar{F}_0 & F_0 \bar{F}_1 & F_0 \bar{F}_2 \\ F_1 \bar{F}_0 & F_1 \bar{F}_1 & F_1 \bar{F}_2 \\ F_2 \bar{F}_0 & F_2 \bar{F}_1 & F_2 \bar{F}_2 \end{pmatrix}, \quad (74)$$

$$R' = 4\Lambda, \quad (75)$$

where  $\Lambda$  is a possible cosmological constant, and the complex fields  $F_0, F_1$  and  $F_2$  moreover satisfy the Maxwell equations

$$\begin{aligned} D(F_1) - \bar{\delta}(F_0) &= F_1[2Qr + D(\ln(\Omega^2))] - F_0[2m + \bar{\delta}(\ln(\Omega^2))], \\ \Delta(F_1) - \delta(F_2) &= F_1[-2ri + \Delta(\ln(\Omega^2))] + F_2[2m - \delta(\ln(\Omega^2))], \\ \delta(F_1) - \delta(F_0) &= F_1[2m + \delta(\ln(\Omega^2))] - F_0[2ri + \Delta(\ln(\Omega^2))], \\ \bar{\delta}(F_1) - D(F_2) &= F_1[-2m + \bar{\delta}(\ln(\Omega^2))] + F_2[2Qri - D(\ln(\Omega^2))]. \end{aligned} \quad (76)$$

As the trace-free Ricci tensor of a null Maxwell field ( $F_0 F_2 = F_1^2$ ) has the algebraic structure (71) [9], this case is excluded by the result of the previous paragraph. Regarding potential non-null fields, one can distinguish between the cases where at least one of its null eigendirections is aligned with a principal null direction of the Weyl tensor ( $F_0 = 0 \neq F_1$  and/or  $F_2 = 0 \neq F_1$ ) or not ( $F_0 F_2 \neq 0$ ). In the latter case an overdetermined system of integrability conditions arises, but we have been unable to decide on the (non-)existence and ampleness of solutions. The aligned case turns out to be excluded: taking  $k$  as an aligned null vector ( $F_0 = 0 \neq F_1$ ), and combining the  $[\Delta, D]$  and  $[\delta, \bar{\delta}]$  commutator relations applied to  $D(\Omega)$  (also making use of (76),  $D(\Lambda) = 0$  and  $\Psi'_2 \neq 0$ ), one finds  $D(\Omega) = 0$ , in contradiction with (69a). This confirms the statement in [8] that doubly aligned Petrov type D electrovac have conformal representants admitting a Killing spinor with non-constant eigenvalues.

### 4.3. Perfect fluid

The spacetime with metric  $ds^2$  is a perfect fluid (PF) spacetime iff its energy–momentum is given by  $T_{ab} = Su_a u_b + pg_{ab}$ , where  $u^a$  is a unit timelike vector,

$$u_1 u_2 - u_3 u_4 = -\frac{1}{2}. \quad (77)$$

Using Einstein's equations, this translates into

$$\Phi'_{rs} = \frac{S}{2} \begin{pmatrix} u_4^2 & u_1 u_4 & u_1^2 \\ u_2 u_4 & \frac{1}{2}(u_1 u_2 + u_3 u_4) & u_1 u_3 \\ u_2^2 & u_2 u_3 & u_3^2 \end{pmatrix}, \quad (78)$$

$$R' = S - 4p. \quad (79)$$

One can again solve these equations together with the NP commutation relations for all second-order derivative operators in  $\Omega$ , in terms of  $p$ ,  $S$ ,  $u_a$ ,  $\Omega$  and first-order derivatives of  $\Omega$ . For aligned solutions  $u_1 = u_2 = 0$ ,  $u_3 = 1/(2u_4)$  the integrability conditions imply  $Qr^2 + m^2 = 0$ , such that there are no non-conformally flat solutions.

For non-aligned solutions, one can eliminate  $u_2$  using equation (77). The integrability conditions form a set of first-order PDEs for  $\Omega$ ,  $u_1$ ,  $u_3$ ,  $u_4$ ,  $S$  and  $p$ . Interpreting these as linear equations for the derivatives of the variables, these are consistent iff

$$u_3 = Qu_4 \quad (80)$$

$$u_4^2 = u_1^2 + 1/2. \quad (81)$$

$u_a$  cannot satisfy equation (77) if  $u_3 = -u_4$ , as  $u_2 = \overline{u_1}$ , which implies that there are no PF solutions for  $Q = -1$ . If on the other hand  $Q = 1$ , substitution of equations (80) and (81) in the integrability conditions yields a new set of differential equations linear in the derivatives of  $\Omega$ ,  $u_1$ ,  $S$ ,  $p$ . Elimination of the first-order derivatives yields the scalar equation

$$u_1^2 = -\frac{1}{4}. \quad (82)$$

This is however in contradiction with equations (77), (80), (81), which imply  $u_1 = u_2 = \overline{u_1}$ . The constructed class of KS spacetimes therefore contains no PF solutions.

## 5. Conclusion

In [12] it was left as an open question whether spacetimes which violate both equation (26) and equation (27) exist. While the analysis of the general class so far has not been completed, one can proceed by assuming condition (30). We showed that such spacetimes do exist, and that the conformal representant with  $|X| = 1$  admits a three- or four-dimensional maximal group of isometries. The line element for the latter spacetimes, which are purely electric, was constructed, and it was shown that their respective conformal classes do not contain any PF or pure radiation members. It remains an open question whether these classes contain a necessarily non-aligned, Einstein–Maxwell spacetime. This is an intriguing problem, as almost all Petrov type D Einstein–Maxwell solutions known to date belong to the aligned family. The integration of the spacetimes admitting a three-dimensional maximal group of isometries will be discussed elsewhere.

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