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Extension of Poincaré's program for integrability, chaos and bifurcations

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Abstract. We will review the achievements of Henri Poincaré in the theory of dynamical systems and will add a number of extensions and generalizations of his results. It is pointed out that the attention given to two degrees-of-freedom Hamiltonian systems is rather deceptive as near stable equilibrium such systems play a special part. We illustrate Poincaré's theory of critical exponents for the Hamiltonian (1 : 2 : 2)-resonance. To assess the measures of chaos, asymptotic estimates in terms of magnitude and timescales can be given. Another of Poincaré's topic, bifurcations, is briefly reviewed.
Keywords: Henri Poincaré, Critical exponents, Asymptotic integrability, Bifurcations.

1 Key results of Poincaré

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We restrict ourselves to a discussion of the 'Méthodes nouvelles de la mécanique célèste' (based on [17]), leaving aside for instance the interesting thesis and the Mémoire on differential equations. The following key results are in the field of dynamical systems, the chapter indication refers to the 'Méthodes nouvelles':

- Poincaré-expansion with respect to a small parameter around a particular solution of a differential equation (chapter 2).
- The Poincaré-Lindstedt expansion method (chapter 3) as continuation method and as bifurcation method for periodic solutions.
- Characteristic exponents and expansion of exponents in the presence of a small parameter; exponents when first integrals exist (chapter 4).
- The famous proof that in general for time-independent Hamiltonian systems no other first integrals exist besides the energy (chapter 5).
- The idea of 'asymptotic series' as opposed to convergent series (chapters 7 and 8).
- The divergence of series expansions in celestial mechanics (chapters 9 and 13).
- The Poincaré-domain to characterise resonance in normal forms (chapter 13 and in his thesis).
- The notion of 'asymptotic invariant manifold' (chapter 25).
- The recurrence theorem (chapter 26).
- The Poincaré-map as a tool for dynamical systems (chapter 27).
- Homoclinic (doubly asymptotic) and heteroclinic solutions; the image of the corresponding orbit structure.

The term 'New methods' contrasts with the old methods of Lagrange, Laplace, Delaunay, Jacobi that are correct and classical, but leaving a great many unsolved problems. This holds in particular for integrability questions, convergence of series which is related to Poisson-stability and bifurcation theory.

2 The deception of two degrees-of-freedom

A time-independent Hamiltonian produces equations of motion that have in general only one first integral, the energy. So, for integrability of a two d-o-f Hamiltonian system, a second independent integral is needed, but near stable equilibrium both numerics and analytic approximation suggests integrability in this case. Why?

The reason is, that the measure of chaos in two d-o-f near stable equilibrium is exponentially small. We will discuss this in more detail in section four. A famous example is the Hénon-Heiles problem [6] that was published in 1964. For small values of the energy it looks integrable and many futile expansions were computed to pinpoint this apparent second integral. The 'proofs'were futile, but as we shall see, such expansions are not useless. For small values of the energy they describe the KAM-tori that abound near stable equilibrium of near-integrable systems.

The dynamics of a time-independent Hamiltonian system corresponds with a two-dimensional area-preserving Poincaré-map. We can turn this around:

a two-dimensional area-preserving map has a suspension that is Hamiltonian. Consider as an example a two-dimensional area-preserving map T_H studied by Igor Hoveijn [9]:

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \sin x \begin{pmatrix} -\sin\alpha \\ \cos\alpha \end{pmatrix}. \quad (1)$$

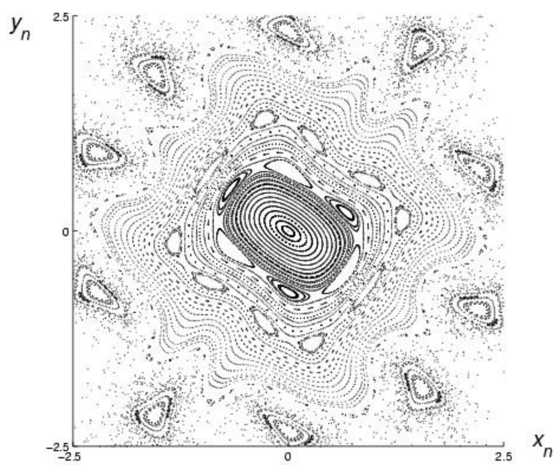

Fig. 1. The area-preserving map T_H produced by eq. (1) for $\alpha = 3\pi/5$. In the centre of the plane there is a dominant family of closed KAM-curves. In between the curves there are again stable and unstable periodic solutions but they can not be observed at this level of precision. Outside this family of closed curves one finds stable periodic solutions associated with unstable periodic solutions. Moving out one observes a stable and unstable 10-periodic solution and further on another pair of 10-periodic solutions. The unstable solutions have stable and unstable manifolds that intersect an infinite number of times producing chaotic behaviour. The dots correspond with orbits returning chaotically in the plane when applying the map repeatedly. Poincaré described the folding process, see [17], that can be seen dynamically by considering a small square in the plane and following its subsequent mappings (figure courtesy Igor Hoveijn).

In fig. 1 we took $\alpha = 3\pi/5$. The closed KAM-curves around the centre suggest that for small values of x and y the map is nearly integrable. For larger values of x and y the chaotic nature of the map becomes more transparent.

3 Critical exponents, the (1 : 2 : 2)-resonance

Chapter four of the 'Méthodes nouvelles' introduces characteristic exponents. Consider an n-dimensional autonomous equation of the form

$$\dot{x} = X(x),$$

and suppose we know a particular solution $x = \phi(t)$. We call this a generating solution. When studying neighbouring solutions of $\phi(t)$ we put

$$x = \phi(t) + \xi.$$

The variational equations of $\phi(t)$ are obtained by substituting $x = \phi(t) + \xi$ into the differential equation and linearising for small ξ to obtain

$$\dot{\xi} = \frac{\partial X}{\partial x}|_{x=\phi(t)}\xi.$$

If $\phi(t)$ is a periodic solution, the variational equations are a Floquet system.

Using the variational equations we can obtain a linear system of equations of which the characteristic eigenvalue equation produces the characteristic exponents. There are some important cases:

- It is clear from the linear system determining the characteristic exponents that if $X(x,t)$ does not depend explicitly on t, the autonomous case, $\dot{\phi}(t)$ is a solution, so one of the characteristic exponents is zero.
- If the vector field is time-dependent ($\dot{x} = X(x,t)$) and contains a small parameter μ, can be expanded with respect to this parameter and admits a T-periodic solution $\phi(t)$ for $\mu = 0$, a periodic solution for small nonzero values of μ exists if all the characteristic exponents of $\phi(t)$ are nonzero.
- If the vector field X is autonomous, has a periodic solution and we have one and only one zero characteristic exponent, the same conclusion for the existence of a periodic solution holds.
- If we have a T-periodic equation of $\dot{x} = X(x,t)$ with T-periodic solution $\phi(t)$ and in addition an analytic first integral $F(x) =$ constant, at least one of the characteristic exponents of $\phi(t)$ is zero; the rather exceptional case for this result is if all the partial derivatives $\partial F/\partial x$ vanish for $x = \phi(t)$.
- If the vector field X is autonomous and we have p independent first integrals, $p < n$, we have at least $p + 1$ characteristic exponents zero.

A number of special results hold in the case that our nonlinear system of differential equations is Hamiltonian and autonomous. Poincaré proves, that in this case the $2n$ characteristic exponents of a periodic solution, emerge in pairs $\lambda_i, -\lambda_i$, equal in size and of opposite sign. In addition, the energy integral produces two characteristic exponents zero; if there exist p other independent first integrals we have either $2p + 2$ characteristic exponents zero or, in the exceptional case, the functional determinants of the integrals restricted to the periodic solution vanish. For the proof, Poincaré uses Poisson brackets and the theory of independent solutions of linear systems.

If the time-independent Hamiltonian system has a periodic solution with more than two zero characteristic exponents, this can be caused by the presence of another first integral besides the energy or it may be the exceptional case.

Examples of more than two zero characteristic exponents are found in the normal forms of three degrees-of-freedom systems in $1:2:n$-resonance with $n > 4$, where normalization to H_3 produces two families of periodic solutions on the energy manifold. The normal form truncated to cubic terms is integrable. The families break up when adding higher order normal form terms; see [13]. The technical problems connected with drawing conclusions from the presence of more than two zero characteristic exponents, have probably prevented its use in research of conservative dynamics, but the statement “a continuous family of periodic solutions on the energy manifold is a non-generic phenomenon” is one of the remaining features in the literature. Nowadays the analysis is easier by the use of numerical continuation methods.

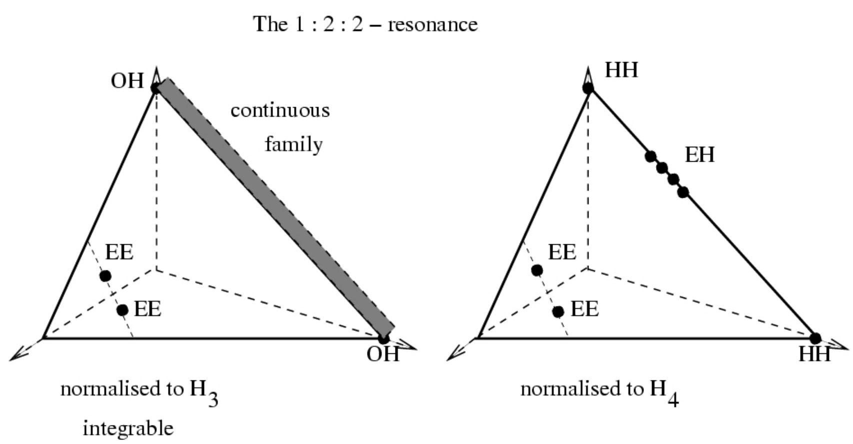

Fig. 2. The three d-o-f (1 : 2 : 2)-Hamiltonian resonance, left the periodic solutions in an action simplex of the Hamiltonian normlised to H_3, right normalization to H_4 (figure courtesy Springer).

Regarding the complications it is of interest to look at the general (1 : 2 : 2)-resonance, see fig. 2. The general Hamiltonian has 56 cubic terms, normalization leaves three cubic terms. Remarkably enough, the cubic normal form is integrable with energy integral, a quadratic integral and a cubic integral. However, on the energy manifold we find one continuous family of periodic solutions and two isolated solutions. So we have here an exceptional case as described by Poincaré. It turns out that the cubic normal form displays a hidden symmetry that vanishes at higher order. The interpretation is that the phase-flow shows this symmetry with accuracy $O(\varepsilon)$ on the timescale $1/\varepsilon$, the integrability is asymptotic with error estimate $O(\varepsilon^2 t)$ (see the next section for the error estimates). In fig. 2 the action simplex of the normal form to H_4 (before normalization 126 terms) shows the break-up of the continuous family of periodic solutions into six periodic solutions on the energy manifold.

4 Measures of chaos

Most Hamiltonian systems are not integrable. However, as we shall see, this is a very deceptive statement although it is mathematically correct. To get

this in the right perspective, we shall start by outlining suitable approximation methods. These are canonical normal form methods, sometimes called after Birkhoff-Gustavson, and averaging performed in a canonical way. The methods admit precise error estimates and enable us therefore to determine local measures of regularity and chaos. The methods also permit us to locate normal modes and other short-periodic solutions.
We recall that two d-o-f time-independent Hamiltonian systems near stable equilibrium can be normalized and that the normal form is always integrable to any order. The integrals are the Hamiltonian and its quadratic part. The motion on the KAM-tori dominates phase-space and this result expresses that the amount of chaos near stable equilibrium is exponentially small. Explicitly: near stable equilibrium, the measure of chaos is $O(\varepsilon^a \exp(-1/\varepsilon^b)$ for suitable positive constants a, b where the energy $E = O(\varepsilon^2)$. An illustrative example is studied in [7].

4.1 Approximations and normal forms

Consider the n degrees of freedom time-independent Hamiltonian

$$H(p,q) = \frac{1}{2}\sum_{i=1}^{n} \omega_i \left(p_i^2 + q_i^2\right) + H_3 + H_4 + \cdots . \tag{2}$$

with H_k, $k \geq 3$ a homogeneous polynomial of degree k and positive frequencies ω_i. We introduce a small parameter ε into the system by rescaling the variables by $q_i = \varepsilon \overline{q_i}, p_i = \varepsilon \overline{p_i}, i = 1, \cdots, n$ and dividing the Hamiltonian by ε^2. This implies that we localize near stable equilibrium with energy $O(\varepsilon^2)$.
We can define successive, nonlinear coordinate (or *near-identity*) transformations that will bring the Hamiltonian into the so-called Birkhoff normal form; see [3] and [14] for details and references. For a general dynamical systems reference see [1,4], for symmetry in the context of Hamiltonian systems see [4,10]. A stimulating text on chaos and resonance is [5]. In action-angle variables τ, ϕ, a Hamiltonian H is said to be in Birkhoff normal form of degree $2k$ if it can be written as

$$H = \sum_{i=1}^{n} \omega_i \tau_i + \varepsilon^2 P_2(\tau) + \varepsilon^4 P_3(\tau) + \cdots + \varepsilon^{2k-2} P_k(\tau),$$

where $\tau = (\tau_1, \cdots, \tau_n)$ and $P_i(\tau)$ is a homogeneous polynomial of degree i in $\tau_i = \frac{1}{2}({p_i}^2 + {q_i}^2), i = 1, \cdots, n$. The variables τ_i are called actions; note that if Birkhoff normalization is possible, the angles have been eliminated. If a Hamiltonian can be transformed into Birkhoff normal form, the dynamics is fairly regular. The system is integrable with integral manifolds which are tori described by taking τ_i constant. The flow on the tori is quasi-periodic.
Suppose a Hamiltonian is in Birkhoff normal form to degree m, but the frequencies are satisfying a resonance relation of order $m + 1$. This means that H_{m+1}, H_{m+2} etc. may contain resonant terms which can not be transformed away. The procedure is now to split H_{m+1}, H_{m+2} etc. in resonant terms and

terms to which the Birkhoff normalization process can be applied. The resulting normal form will generally contain resonant terms and is called Birkhoff-Gustavson normal form. It contains terms dependent on the action τ and on resonant combination angles of the form $\chi_i = k_1\phi_1 + \cdots + k_n\phi_n$. In practice we have to consider a truncation of the Birkhoff-Gustavson normal form $\bar{H}$ at some degree $p \geq m$:

$$\bar{H} = H_2 + \varepsilon\bar{H}_3 + \varepsilon^2\bar{H}_4 + \cdots + \varepsilon^{p-2}\bar{H}_p. \tag{3}$$

Because of the construction we have the following results:

- $\bar{H}$ is conserved for the original Hamiltonian system (2) with error $O(\varepsilon^{p-1})$ for all time.
- H_2 is conserved for the original Hamiltonian system (2) with error $O(\varepsilon)$ for all time. So the normal form has at least two integrals. Symmetry can enhance the regularity, see [13].
- If we find other integrals of the Birkhoff-Gustavson normal form, we have slightly weaker error estimates. Explicitly, suppose that $F(p,q)$ is an independent integral of the truncated Hamiltonian system (3), we have for the solutions of the original Hamiltonian system (2) the estimate

$$F(p,q) - F(p(0), q(0)) = O(\varepsilon^{p-1}t).$$

An important consequence is the following statement: if the phaseflow induced by the truncated Hamiltonian (3) is completely integrable, the flow of the original Hamiltonian (2) is approximately integrable or asymptotically integrable in the sense described above. In this case the original system is called *formally integrable*. This implies that the irregular, chaotic component in the flow of the original Hamiltonian is limited by the given error estimates and must be a small-scale phenomenon on a long timescale. For details see [14] and [13].

4.2 Normal modes and short-periodic solutions

Liapunov proved that if the frequencies ω_i satisfy no resonance relation, the normal modes, obtained by linearization, can be continued for the full, nonlinear Hamiltonian system (2), resulting in at least n short-periodic solutions with periods ε-close to $2\pi/\omega_i$.

Weinstein [18] proved that even in the case of resonance, there exist at least n short-periodic solutions of Hamiltonian system (2). Note, that these periodic solutions are not necessarily continuations of the linear modes, the term 'normal modes' in this context can be confusing. Another important point is that n short-periodic solutions is really the minimum number. For instance in the case of two degrees of freedom, 2 short-periodic solutions are guaranteed to exist by the Weinstein theorem. But in the $1:2$ resonance case one finds generically 3 short-periodic solutions for each (small) value of the energy. One of these is a continuation of a linear normal mode, the other two are not. For higher-order resonances like $3:7$ or $2:11$, there exist for an open set of parameters four short-periodic solutions of which two are continuations of the normal modes.

Of course symmetry and special Hamiltonian examples may change this picture drastically.

4.3 Three degrees of freedom

The question of asymptotic integrability is different for more than two degrees of freedom. First we consider the genuine first order resonances of three d-o-f systems.

First-order resonances

As we have seen, the cubic normal form of the $(1:2:2)$-resonance is integrable; this is caused by a hidden symmetry which reveals itself by normalization. The $(1:2:1)$-resonance and the $(1:2:3)$-resonance on the other hand are not integrable for an open set of parameters of the Hamiltonian. The results are illustrated for the four first-order resonances in the table from [13].

If three independent integrals of the normalized system can be found, the normalized system is integrable. The integrability depends in principle on how far the normalization is carried out ($\overline{H}_k$ represents the normal form of H_k, the homogeneous part of the Hamiltonian of degree k). The formal integrals have a precise asymptotic meaning as discussed in section 4.1. We use the following abbreviations: *no cubic integral* for no quadratic or cubic third integral; *discr. symm.* q_i for discrete (or mirror) symmetry in the p_i, q_i-degree of freedom; *2 subsystems at* $\overline{H}_k$ for the case that the normalized system decouples into a one and a two degrees of freedom subsystem upon normalizing to H_k. In the second and third column one finds the number of known integrals when normalizing to $\overline{H}_3$ respectively $\overline{H}_4$.

The remarks which have been added to the table reflect some of the results known on the non-existence of third integrals. Note that the results presented here are for the general Hamiltonian and that additional assumptions, in particular involving symmetry, may change the results. In this respect it is interesting that in a number of applications, chaotic dynamics appears to be of relatively small size. An example is the dynamics of elliptical galaxies that display three-axial symmetry. Astrophysical observations suggest highly nonlinear but integrable motion. The statements above with indication 'Assumptions': 'general', are for Hamiltonian systems in general form near stable equilibrium.

Example: the $(1:2:3)$*-resonance*

This resonance was analyzed in [8] and [15]. We will summarize some results and formulate some open problems. When normalizing to H_4 one finds 7 short-periodic (families of) solutions. One of them is for an open set of parameters complex unstable (for the complementary set it is unstable of saddle type). This complex instability is a source of chaotic behaviour. Using Šilnikov-Devaney theory, it is shown in [8] that a horseshoe map exists in the normal form to H_4 which makes the normal form chaotic.

Numerics indicate that the normal form $\bar{H} = H_2 + \bar{H}_3$ is already chaotic, but a proof is missing. Also the dynamics of the case where the periodic solution is unstable, but of saddle type, has still to be characterized.

Discrete symmetry in either the first or the last degree of freedom makes the normal form to H_4 integrable.

Table 1. Integrability of the normal forms of the four genuine first order resonances.

Resonance	Assumptions	H_3	H_4	Remarks
1:2:1	general	2	2	no analytic third integral
	discr.symm. q_1	2	2	no analytic third integral
	discr.symm. q_2	3	3	$\overline{H}_3 = 0$; 2 subsystems at $\overline{H}_4$
	discr.symm. q_3	2	2	no analytic third integral
1:2:2	general	3	2	no cubic third integral at $\overline{H}_4$
	discr.symm. q_2 and q_3	3	3	$\overline{H}_3 = 0$; 2 subsystems at $\overline{H}_4$
1:2:3	general	2	2	no analytic third integral
	discr.symm. q_1	3	3	2 subsystems at $\overline{H}_3$ and $\overline{H}_4$
	discr.symm. q_2	3	3	$\overline{H}_3 = 0$
	discr.symm. q_3	3	3	2 subsystems at $\overline{H}_3$ and $\overline{H}_4$
1:2:4	general	2	2	no cubic third integral
	discr.symm. q_1	2	2	no cubic third integral
	discr.symm. q_2 or q_3	3	3	2 subsystems at $\overline{H}_3$ and $\overline{H}_4$

Higher-order resonances

Higher order resonances abound in applications. The results discussed thus far are mostly general, but, with regards to applications, it is very important to look again at the part played by symmetries. This will be illustrated for the $(1:3:7)$-resonance and will be discussed in some detail. This also serves as an example that resonances with odd resonance numbers are particularly sensitive to symmetries.

Example: the $(1:3:7)$*-resonance*

We start with the general Hamiltonian with this resonance in H_2:

$$H_2 = \tau_1 + 3\tau_2 + 7\tau_3.$$

At H_3 level there is no resonance and we find after normalization, $\bar{H}_3 = 0$. There are two combination angles active at H_4 level:

$$\chi_1 = 3\phi_1 - \phi_2 \text{ and } \chi_2 = \phi_1 + 2\phi_2 - \phi_3.$$

At H_5 level no combination angles are added, H_5 can be brought in Birkhoff normal form. We list the consequences of mirror symmetry in each respective degree of freedom:

- In the first d-o-f: χ_1 and χ_2 not active; formal integrability until $\bar{H}_5$, chaotic dynamics has measure $O(\varepsilon^4 t)$.
- In the second d-o-f: χ_1 not active; formal integrability until $\bar{H}_7$, chaotic dynamics has measure $O(\varepsilon^6 t)$.
- In the third d-o-f: χ_2 not active; formal integrability until $\bar{H}_7$, chaotic dynamics has measure $O(\varepsilon^6 t)$.
- The case of mirror symmetry in all three d-o-f. is discussed below.

One can continue the analysis to higher order normal forms to obtain more precise estimates of the remaining chaotic dynamics. We discuss an example.

Three-axial elliptical galaxies in $(1:3:7)$*-resonance*
In this case we have discrete (mirror) symmetry in three degrees of freedom. Until H_7 the system can be brought into Birkhoff normal form, chaotic dynamics has measure $O(\varepsilon^6 t)$ which predicts regular behaviour on a long timescale. The Birkhoff-Gustavson normal form $\bar{H}_8$ contains the combination angles $6\phi_1 - 2\phi_2$ and $2\phi_1 + 4\phi_2 - 2\phi_3$.
The situation needs a very high degree of normalization as becomes clear when considering the analysis of periodic solutions. Because of the discrete symmetry $\tau_i = 0, i = 1, 2, 3$ each corresponds with a two d-o-f submanifold of the original (symmetric) Hamiltonian. The normal modes are exact periodic solutions of the normal form and the original Hamiltonian. The normal forms in these 4-dimensional submanifolds are all integrable (section 4.1) and chaotic behaviour takes place in exponentially small sets. Consider the question of how far we have at least to normalize the flow in these submanifolds.
Case $\tau_1 = 0$. This is the worst case, as it involves the $3:7$-resonance. In the symmetric case this system has to be normalized to H_{20} to characterize the periodic solutions.
Case $\tau_2 = 0$ involving the $1:7$-resonance. The system has to be normalized to H_{16} to characterize the periodic solutions.
Case $\tau_3 = 0$ involving the $1:3$-resonance. This relatively well-known system has to be normalized to H_8 to characterize the periodic solutions. In [13] it is described how to deal with such higher-order cases.

4.4 A remark on chains of oscillators

Our knowledge of chains of oscillators is still restricted. Remarkably enough the normal form of the $1:2:\cdots:2$-resonance with n degrees of freedom is integrable. Consider the Hamiltonian

$$H(p,q) = \frac{1}{2}(p_1^2 + q_1^2) + \sum_{i=2}^{n}(p_i^2 + q_i^2) + H_3 + \cdots,$$

where $H_3 + \cdots$ represents the general cubic and higher order terms. The Hamiltonian is formally integrable and the proof runs along the lines of the analysis of the $(1:2:2)$-resonance, displaying again hidden symmetry.
A spectacular result arises for the classical Fermi-Pasta-Ulam problem which is a chain of identical oscillators coupled by nearest neighbour interaction. At low energy levels the chain shows recurrence and no chaos. Recently it was shown in [12] by normal form methods and symmetry considerations, that a nearby integrable system exists which make the KAM-theorem applicable. This solves the recurrence phenomenon at low energy.

5 Bifurcations

The analysis of periodic solutions is based on the implicit function theorem. If the conditions of the theorem are not satisfied we have a *bifurcation*.

The treatment in the 'Méthodes nouvelles' is very general, it applies to nonlinear ODEs, including dissipative systems. The bifurcations discussed have a universal character:

- Hopf bifurcation (continuation near an equilibrium).
- Transcritical bifurcation, exchange of stability.
- Emergence and vanishing of periodic solutions in pairs.

A bifurcation that plays a very prominent role in nonlinear dynamics is the Hopf bifurcation, also referred to as Poincaré-Andronov-Hopf bifurcation. It may happen that, when an equilibrium point of which the eigenvalues depend on parameters, will have two purely imaginary eigenvalues if one of the parameters (μ) assumes a critical value, say μ_0. In this case, depending on the nonlinearities, there may exist a nearby periodic solution. If the periodic solution emerges for $\mu < \mu_0$ it is called subcritical, for $\mu > \mu_0$ it is supercritical.

For periodic solutions and fixed points of a map, there are analogous results, where one usually refers to generalized Hopf, Hopf-Hopf or Neimark-Sacker bifurcation.

The first place where the Hopf bifurcation arises in the literature is in the Méthodes nouvelles. Poincaré considers an equilibrium of an autonomous equation in R^n

$$\dot{x} = X(x)$$

and views this equilibrium as a periodic solution with arbitrary period. Suppose that there is a parameter μ in the equation and that $x_1 = x_2 = \cdots = x_n = 0$ is an equilibrium for any value of μ. We will look for a periodic solution near the origin $x = 0$ for $\mu = 0$, with initial value $x(0) = \beta$ and $x(T) = \psi + \beta$. If we can determine T with $\psi = 0$ and non-trivial β, we have found a periodic solution. Poincaré finds from the determinant of the Jacobian J:

$$J = \frac{\partial X}{\partial x}|_{\mu=0,x=0},$$

that if $|J| \neq 0$, we will have only the trivial solution $\beta = 0$, corresponding with the equilibrium solution $x = 0$. The condition $|J| = 0$ to obtain a nontrivial solution corresponds with (at least) two eigenvalues to be purely imaginary and conjugate. This condition makes the existence of a small periodic solution branching off equilibrium $x = 0$ possible, but we still have to consider the nonlinear terms to see whether a periodic solution actually emerges.

The eigenvalues λ_i will depend on μ. Adding the condition that at the critical value $\mu_0 = 0$, we have two conjugate imaginary eigenvalues $\lambda_{i,j}$ with $d\lambda_{i,j}/d\mu \neq 0$, we will call such a bifurcation value of μ a Hopf point.

Poincaré considers in the 'Méthodes nouvelles' as an example the equations formulated by Hill for the motion of the Moon, two second-order equations with one nontrivial equilibrium. The equilibrium corresponds with the Moon being in constant conjunction or opposition at constant distance of the Earth. The eigenvalues of the Jacobian as formulated above, have two real values and two conjugate imaginary ones. The conclusion is that a periodic solution exists near this equilibrium in near-opposition or near-conjunction with an amplitude that grows with the small parameter $\sqrt{\mu}$. As two conjugate eigenvalues are real, it will be unstable.

The classical example of the Van der Pol-equation is easier to analyse. Poincaré's interest in wireless telegraphy induced him to use periodic solutions obtained by this type of bifurcation, see [17].

5.1 Tori created by Neimark-Sacker bifurcation

Another important scenario to create a torus, arises from the Neimark-Sacker bifurcation. For an instructive and detailed introduction see Kuznetsov (2004) [11]. Suppose that we have obtained an averaged equation $\dot{x} = \varepsilon f(x, a)$, with dimension 3 or higher, by variation of constants and subsequent averaging; a is a parameter or a set of parameters. It is well-known that if this equation contains a hyperbolic critical point, the original equation contains a periodic solution. The first order approximation of this periodic solution is characterized by the time variables t and εt.
Suppose now that by varying the parameter a a pair of eigenvalues of the critical point becomes purely imaginary. For this value of a the averaged equation undergoes a Hopf bifurcation producing a periodic solution of the averaged equation; the typical time variable of this periodic solution is εt and so the period will be $O(1/\varepsilon)$. As it branches off an existing periodic solution in the original equation, it will produce a torus; it is associated with a Hopf bifurcation of the corresponding Poincaré map and the bifurcation has a different name: Neimark-Sacker bifurcation. The result will be a two-dimensional torus which contains two-frequency oscillations, one on a timescale of order 1 and the other with timescale $O(1/\varepsilon)$.

A special case of a system studied by Bakri et al. (2004) [2] is:

$$\begin{aligned}\ddot{x} + \varepsilon\kappa\dot{x} + (1 + \varepsilon\cos 2t)x + \varepsilon xy &= 0,\\ \ddot{y} + \varepsilon\dot{y} + 4(1+\varepsilon)y - \varepsilon x^2 &= 0.\end{aligned}$$

This is a system with parametric excitation and nonlinear coupling; κ is a positive damping coefficient which is independent of ε. Away from the coordinate planes we may use amplitude-phase variables by $x = r_1\cos(t+\psi_1), \dot{x} = -r_1\sin(t+\psi_1), y = r_2\cos(2t+\psi_2), \dot{y} = -2r_2\sin(2t+\psi_1)$; after first order averaging we find, omitting the subscripts a, the system

$$\begin{aligned}\dot{r}_1 &= \varepsilon r_1\left(\frac{r_2}{4}\sin(2\psi_1-\psi_2) + \frac{1}{4}\sin 2\psi_1 - \frac{1}{2}\kappa\right),\\ \dot{\psi}_1 &= \varepsilon\left(\frac{r_2}{4}\cos(2\psi_1-\psi_2) + \frac{1}{4}\cos 2\psi_1\right),\\ \dot{r}_2 &= \varepsilon\frac{r_2}{2}\left(\frac{r_1^2}{4r_2}\sin(2\psi_1-\psi_2) - 1\right),\\ \dot{\psi}_2 &= \frac{\varepsilon}{2}\left(-\frac{r_1^2}{4r_2}\cos(2\psi_1-\psi_2) + 2\right).\end{aligned}$$

Putting the righthand sides equal to zero produces a nontrivial critical point corresponding with a periodic solution of the system for the amplitudes and phases and so a quasi-periodic solution of the original coupled system in x and y. We find for this critical point the relations

$$r_1^2 = 4\sqrt{5}r_2, \cos(2\psi_1 - \psi_2) = \frac{2}{\sqrt{5}}, \sin(2\psi_1 - \psi_2) = \frac{1}{\sqrt{5}}, r_1 = 2\sqrt{2\kappa + \sqrt{5 - 16\kappa^2}}.$$

This periodic solution exists if the damping coefficient is not too large: $0 \leq \kappa < \frac{\sqrt{5}}{4}$. Linearization of the averaged equations at the critical point while using these relations produces a (4×4) matrix A.

A condition for the existence of the periodic solution is that the critical point is hyperbolic, i.e. the eigenvalues of the matrix A have no real part zero. It is possible to express the eigenvalues explicitly in terms of κ by using a software package like MATHEMATICA. However, the expressions are cumbersome. Hyperbolicity is the case if we start with values of κ just below $\frac{\sqrt{5}}{4} = 0.559$. Diminishing κ we find that, when $\kappa = 0.546$, the real part of two eigenvalues vanishes. This value corresponds with a Hopf bifurcation producing a nonconstant periodic solution of the averaged equations. This in its turn corresponds with a torus in the original equations (in x and y) by a Neimark-Sacker bifurcation. As stated before, the result will be a two-dimensional torus which contains two-frequency oscillations, one on a timescale of order 1 and the other with timescale $O(1/\varepsilon)$.

6 Breakdown and bifurcations of tori

Complementary to the emergence of tori, their breakdown is of great theoretical and practical interest. In particular we would like to have a general idea of how two-dimensional invariant tori break down and how nontrivial limit sets are created when certain parameters are varied. To obtain insight the analysis of maps can be very helpful as the phenomena governed by differential equations are much more implicit.

A common feature is the presence of stable and unstable periodic solutions in p/q-resonance on a torus. Breakup can be triggered by heteroclinic tangencies, arising when a parameter is varied. This leads rather quickly to strange, chaotic behaviour. There are other scenarios producing strange behaviour where the normal hyperbolicity of the torus decreases more gradually. For an introduction and references see [16].

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