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$$\text{InertiaTensor} := \sum_{k=1}^n m_k \left(\|\vec{r}_k\|^2 \mathbf{1} - \vec{r}_k \vec{r}_k^T \right)$$
$$\Gamma_{2,2}^1 = \frac{(1 + 2\frac{\partial}{\partial r})}{\partial r}$$

Relative entropy as a measure of inhomogeneity in general relativity

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We introduce the notion of relative volume entropy for two spacetimes with preferred compact spacelike foliations. This is accomplished by applying the notion of Kullback-Leibler divergence to the volume elements induced on spacelike slices. The resulting quantity gives a lower bound on the number of bits which are necessary to describe one metric given the other. For illustration, we study some examples, in particular gravitational waves, and conclude that the relative volume entropy is a suitable device for quantitative comparison of the inhomogeneity of two spacetimes.

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I. INTRODUCTION

How much information is required to describe one solution to the Einstein equations in terms of another? More detailed versions of this question are: Just how much more complex is a solution of the Einstein equation in the presence of matter than a vacuum solution? How “complicated” is a gravitational wave? In this paper, we provide a possible answer to these questions in terms of the relative entropy of the volume elements associated with the metric tensors of two spacetimes with a fixed manifold structure.

Our approach is inspired by two well-known facts:

- In mathematical information theory, relative information is measured by the Kullback-Leibler divergence, or relative entropy, of probability measures.¹
- In order for our answer to make sense, we want it to be a diffeomorphism invariant of the ordered pair of metrics on the given manifold. (The manifold structure needs to be fixed; in this sense, our entropy is similar to the “Lipschitz distance” between metrics on a given manifold, rather than between different metrics on different manifolds, compare Sec. 7.2 in Ref. 2.) A natural covariant of a pseudo-Riemannian metric is the Radon-Nikodym derivative of its associated measure on spacelike slices. (Recall that the Radon-Nikodym derivative $dv/d\mu$ of two measures ν and μ , with ν absolutely continuous with respect to μ , is a function ϕ such that $\int f d\nu = \int f \phi d\mu$. Here, ϕ is unique up to a set of μ -measure zero, compare Ref. 3, Chap. 19. Intuitively, we can think of the Radon-Nikodym derivative as a Jacobian for measures.)

We therefore propose to compute the relative entropy of two continuous probability distributions associated with our metrics, namely the normalized volume densities in spacelike slices.

For this method to make sense without further complications, we restrict ourselves to the following situation. Suppose we are given a topological space

$$M = \mathbb{R}_t \times X,$$

where X is compact, and two metrics

$$g_{\mu\nu}^{(k)} dx^\mu dx^\nu = -dt^2 + \gamma_{ij}^{(k)} dx^i dx^j \quad (k = 1, 2; i, j = 1, \dots, d)$$

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on M . Define the normalized volume densities by

$$\tilde{\rho}_k = \sqrt{\det \gamma_{ij}^{(k)}} / \text{vol}_k, \quad (1)$$

where

$$\text{vol}_k = \int d^d x \sqrt{\det \gamma_{ij}^{(k)}}. \quad (2)$$

Then

$$\tilde{\rho}_k d^d x$$

is proportional to the physical volume in a (natural) spacelike slice corresponding to the coordinate volume $d^d x$.

Mathematically, the normalized volume densities can be interpreted as probability measures on X (though we are still dealing with entirely deterministic physics). In a spacelike slice, they are absolutely continuous with respect to each other, and the Radon-Nikodym derivative of the measures $\tilde{\rho}_1 d^d x$ and $\tilde{\rho}_2 d^d x$ is equal to $\tilde{\rho}_1 / \tilde{\rho}_2$.

We now introduce the Kullback-Leibler divergence (relative entropy) of our volume densities, the *relative volume entropy*, as

$$E = E(\tilde{\rho}_2 \| \tilde{\rho}_1) = \int d^d x \tilde{\rho}_2 \log(\tilde{\rho}_2 / \tilde{\rho}_1).$$

The well-known information theoretic meaning of this entropy is that it measures the number of “nats” (or bits if we took the logarithm to the base 2) that are necessary to describe the volume density $\tilde{\rho}_2$ when given $\tilde{\rho}_1$.

For our problem of comparing spacetimes, this implies the following result: *At least $E \cdot \log_2 e$ bits will be necessary to describe $(M, g_{\mu\nu}^{(2)})$ in terms of $(M, g_{\mu\nu}^{(1)})$.*

The relative entropy is known to be strictly positive for $\tilde{\rho}_1 \neq \tilde{\rho}_2$, and invariant under coordinate changes on spacelike slices. (It is a basic irritant in information theory that for continuous distributions the usual “differential entropy” $\int d^d x \tilde{\rho}_2 \log(\tilde{\rho}_2)$ *does not* have these properties and one needs to fix a “background distribution” $\tilde{\rho}_1$. This is very much in line with Jaynes’ viewpoint that entropy depends on a choice of reference frame.⁴) For example, isometric metrics have vanishing relative entropy, since the volume elements transform by a trivial Jacobian.

Of course, our proposed information measure neglects some relative information by focussing solely on the volume densities—on the other hand, this has the advantage to make it more easily computable. This is illustrated by the sample computations in this paper. They will also show that, although the measure is based on local volume density fluctuations, it is also a good measure for the difference in *inhomogeneity* between the metrics.

To illustrate this last statement: A Kasner solution in any dimension $D > 3$ has zero relative entropy over a Minkowski background, since its normalized volume density is independent of the parameters of the solution, cf. Sec. II A. On the other hand, cosmological solutions in $2 + 1$ -dimensions, whose volume densities have a “lumpy,” soliton-like behavior over a given vacuum background, have a rather interesting relative entropy, cf. Sec. II B. (Similar spacetimes were studied by Deser–Jackiw–’t Hooft,^{5,6} and later in Ref. 7.) For example, when space is spherical ($X = S^2$), and we place $n - 1$ particles on the sphere, so that the resulting volume density is $\tilde{\rho}_n$, we obtain a relation of the form

$$E(\tilde{\rho}_n \| \tilde{\rho}_0) \sim \log n \quad (n \rightarrow \infty).$$

This certainly is a very satisfying result: *The more particle sources are in the energy-momentum tensor, the more information is necessary to describe spacetime over a vacuum background.* By numerical integration, the very same result appears to hold true also for $X = T^2$, so we see that the relative entropy has little to do with cosmic topology, but rather with the amount of local fluctuation in volume.

Finally, we remark that the definition is readily generalized from the compact setting to the locally compact setting, by choosing a cut-off function for the entropy and computing the limit as the cut-off vanishes.

It is a very challenging question to relate information theoretic entropy of a spacetime to gravitational entropy proposals (such as for example Bekenstein–Hawking black hole entropy,⁸ Brandenberger–Mukhanov–Prokopec entropy,⁹ or Zalaletdinov’s proposal based on almost Killing symmetry¹⁰). Our object of study in this paper is much more modest: we give a cruder but readily understandable information-theoretic entropy that compares different solutions, which *a posteriori* is seen to be a measure of inhomogeneity.

A complementary version of relative entropy for spacelike slices, based directly on inhomogeneity as input, was considered by Buchert *et al.* in Refs. 11–13: we base our entropy on the metric tensor, whereas Buchert *et al.* base it on the energy-momentum tensor. The complementarity of the two approaches is particularly palpable in the case of dust, where the energy density scales like $1/\bar{\rho}$.

We now proceed to work out some examples in order to illustrate how the relative volume entropy behaves for different classes of spacetimes. The examples are chosen as easy as possible, so that we can isolate intrinsic aspects of the entropy: we choose a homogeneous Kasner solution (Sec. II A), to study the influence of homogeneity; then we choose a 2 + 1-dimensional spherical model with “naked” particles (Sec. II B 1), to study the influence of point-like inhomogeneities; then we study the influence of topology by switching to a toroidal model (Sec. II B 2), and finally we study a gravitational wave as a case with non-point-like structure (Sec. II C).

II. EXAMPLES OF RELATIVE VOLUME ENTROPIES

A. Kasner metrics vs. Minkowski space

Let

$$M = \mathbb{R}_t \times T^d \quad (d > 2),$$

where T^d is the d -torus.

The Kasner metric on this space is (see e.g., Ref. 14)

$$ds_{\text{Kasner}}^2 = -dt^2 + \sum_{i=1}^d t^{2p_i} (dx^i)^2,$$

where the parameters p_i satisfy

$$\sum_{i=1}^d p_i = 1, \quad \sum_{i=1}^d p_i^2 = 1, \quad (3)$$

and the coordinates on the torus are periodic with period-length 1.

Since

$$\gamma_{ij}^{(\text{Kasner})} = t^{2p_i} \delta_{ij} \quad (\text{no sum on } i),$$

using the first relation in (3), we find that the volume (2) of Kasner is

$$\text{vol}_{\text{Kasner}} = t,$$

so that the *normalized* volume density (1) reads

$$\tilde{\rho}_{\text{Kasner}} = 1;$$

just the same as for the Minkowski metric:

$$\tilde{\rho}_{\text{Kasner}} = \tilde{\rho}_{\text{Minkowski}}.$$

Consequently,

$$\mathbb{E}(\tilde{\rho}_{\text{Kasner}} \parallel \tilde{\rho}_{\text{Minkowski}}) = 0.$$

At face value, this result is hardly surprising, as the Kasner metric is just “expanding Minkowski space.” As mentioned in the Introduction, the relative volume entropy detects the difference in inhomogeneity between the metrics.

B. Cosmology in 2 + 1 dimensions with $\Lambda > 0$

A more interesting relative volume entropy arises for certain 2 + 1-dimensional “cosmological” spacetimes with cosmological constant $\Lambda > 0$ and pointlike matter.

Let us begin by reviewing the basic features of these spacetimes.^{5–7} The spacetime topology is

$$M = \mathbb{R}_t \times X^2,$$

with metric

$$ds^2 = -dt^2 + a(t)^2 \gamma_{ij} dx^i dx^j,$$

where X^2 is a compact surface. We choose isothermal coordinates x, y on X^2 and write the metric as

$$ds^2 = -dt^2 + a(t)^2 e^\phi (dx^2 + dy^2), \quad (4)$$

and the Einstein equation as

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu},$$

where, following, Ref. 5 we take $T_{\mu\nu}$ to be of the form appropriate for a cloud of non-interacting point particles i of mass m_i sitting at *fixed* coordinates $\vec{x}_i = (x_i, y_i)$,

$$\begin{cases} T_{00} = a(t)^{-2} e^{-\phi} \sum_i m_i \delta^{(2)}(\vec{x} - \vec{x}_i), \\ T_{\mu\nu} = 0 \quad \text{for } (\mu, \nu) \neq (0, 0). \end{cases}$$

By distributing the point particles in a suitable manner over the manifold X^2 , one can obtain a matter distribution which satisfies the cosmological principles of homogeneity and isotropy (possibly modulo some subtle mathematical constraints. For the example of three particles, see Ref. 15).

Plugging in our metric ansatz into the Einstein equation, we find that the only non-trivial components are (using the sign convention $\Delta = \partial_x^2 + \partial_y^2$),

$$\begin{cases} \left(\frac{\dot{a}}{a} \right)^2 - \frac{e^{-\phi}}{2a^2} \Delta \phi - \Lambda = a^{-2} e^{-\phi} \sum_i m_i \delta^{(2)}(\vec{x} - \vec{x}_i), \\ -e^\phi a \ddot{a} + \Lambda a^2 e^\phi = 0. \end{cases} \quad (5)$$

The second equation in (5) is readily integrated to give,

$$a(t) = c_1 \cosh \sqrt{\Lambda} t + c_2 \sinh \sqrt{\Lambda} t.$$

For convenience, let us pick the solution where $c_1 = 1$ and $c_2 = 0$, such that

$$a(t) = \cosh \sqrt{\Lambda} t.$$

Then, after a change of variables $e^\phi = \rho$, the first equation in (5) becomes

$$\Delta \log \sqrt{\rho} + \Lambda \rho = - \sum_i m_i \delta^{(2)}(\vec{x} - \vec{x}_i), \quad (6)$$

the well-known Liouville equation with point sources. From the equation it follows that, outside the sources, the metric has constant curvature Λ . From Eqs. (4) and (2), we find that at time t , the universe has volume

$$\text{vol}(t) = \left(\int \rho d^2x \right) \cosh^2(\sqrt{\Lambda} t),$$

that is, we have a bouncing solution with initial volume

$$\text{vol}(t = 0) = \int \rho d^2x.$$

We note that the normalized volume density (1) now reads

$$\tilde{\rho} = \frac{\rho}{\int \rho d^2x}.$$

Observe how any dependence on time has disappeared.

We now specialize to the case where $X^2 = S^2$ is a sphere.

1. The sphere

In the absence of matter ($m_i = 0$), we find a unique solution to the Liouville equation (6), corresponding to the round metric of radius $1/\sqrt{\Lambda}$ (initial volume $4\pi/\Lambda$), which we write as

$$ds_1^2 = \rho_1(dx^2 + dy^2),$$

where

$$\rho_1 = \frac{4}{\Lambda} \frac{1}{(1 + |z|^2)^2} \quad (z \equiv x + yi).$$

The introduction of point sources gives rise to different, non-round metrics on S^2 . For example, if

$$f(z) = \frac{P(z)}{Q(z)}$$

is a rational function of the complex variable $z = x + iy$, then

$$\rho_f = \frac{4}{\Lambda} \frac{|f'(z)|^2}{(1 + |f(z)|^2)^2}$$

gives a solution to Eq. (6), for a suitable choice of particle positions and masses. (These solutions were already studied in connection with the Jackiw-Pi model, for example by Horváthy and Yéras.¹⁶) It follows from Ref. 16 that in this case, the initial volume is

$$\text{vol}(t = 0) = \frac{4\pi}{\Lambda} \text{ord}_\infty(f), \tag{7}$$

where $\text{ord}_\infty(f)$ is the number of poles of $f(z)$.

For $\text{deg}(f) \neq 0, 1$, this solution has a different character from the everywhere constant curvature solution ρ_1 (which arises for $f(z) = z$). Indeed, at the zeros of

$$P'(z)Q(z) - P(z)Q'(z),$$

the metric $\rho_f(dx^2 + dy^2)$ has a conical singularity.

Figure 1 shows a plot of ρ_f where $f(z) = 1/z + 1/(z - 1)$, and also of ρ_1 .

The striking feature of the relative entropy of $\tilde{\rho}_f$ with respect to ρ_1 , i.e.,

$$E(\tilde{\rho}_f \| \tilde{\rho}_1),$$

is that it is independent of our isothermal coordinates and is sensitive to the lumps we see in the plots.

In the example of a radially symmetric solution $\rho_n = \rho_{z^n}$ corresponding to the power $f(z) = z^n$, it is not too difficult to see that

$$E(\tilde{\rho}_n \| \tilde{\rho}_1) = \int_{\mathbb{R}^2} \tilde{\rho}_n \log \left(\frac{\tilde{\rho}_n}{\tilde{\rho}_1} \right) d^2x = \log n + 2(J_n - 1),$$

where

$$J_n = \int_0^\infty \frac{dx}{(1+x)(1+x^n)} \rightarrow \log 2 \quad (n \rightarrow \infty).$$

Physically, the solution ρ_n arises when there are $n - 1$ particles of mass $m = -2\pi$ at the north pole of S^2 and $n - 1$ particles of the same kind at the south pole. (The appearance of negative

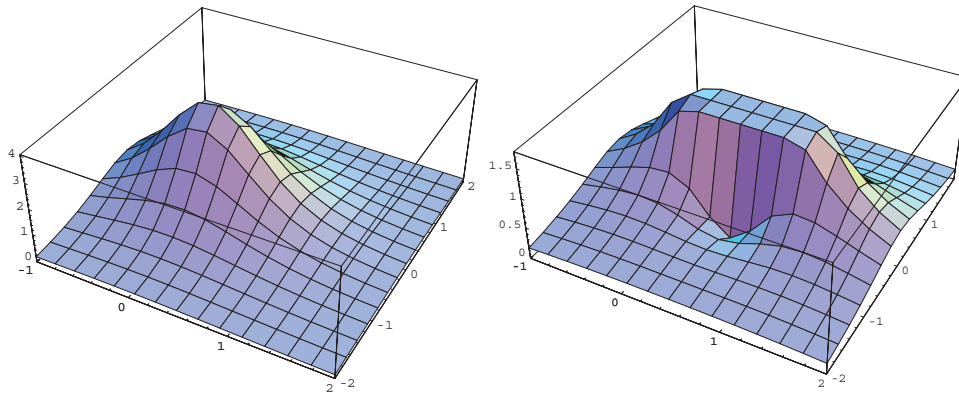


FIG. 1. (Color online) Two solutions of the Liouville equation for $X^2 = S^2$. Left is ρ_1 , that is, the solution of the Liouville equation corresponding to zero particles ($f(z) = 1/z$). Right is the solution of the Liouville equation associated with the function $f(z) = 1/z + 1/(z - 1)$.

masses need not disturb us here, as long as we are talking about a toy model.) Thus, here the relative volume entropy encodes the number of particles n . Alternatively, we can think of two particles of mass $m = -2\pi(n - 1)$ each, one sitting at the north pole, the other at the south pole, and in this interpretation, n need not be an integer. Figure 2 shows a plot of $E(\tilde{\rho}_n \| \tilde{\rho}_1)$ for $X^2 = S^2$.

2. The torus

Here, just as for the sphere, we have at our disposal many explicit solutions of the Liouville equation with distributional sources.^{17,18}

We want to emulate our discussion for the sphere, but immediately we are faced with the problem that on the torus with $\Lambda > 0$ there does not exist a zero particle solution (this follows from the Gauss-Bonnet theorem for generalized Riemannian surfaces, compare, Ref. 18). The least number of particles allowed is equal to one. The solution with precisely one particle of mass $m = -2\pi$ located at the point $(0, 0)$ on the torus was discovered by Olesen^{19,20} and is given by

$$\rho_0 = \frac{4 |g'|^2 |e_1|^2}{\Lambda (|e_1|^2 + |g|^2)^2}$$

Here,

$$g(z) \equiv g_{2,2i}(z)$$

is the Weierstrass p-function associated with the lattice $2\mathbb{Z} + 2i\mathbb{Z}$, and $e_1 = g(1)$.

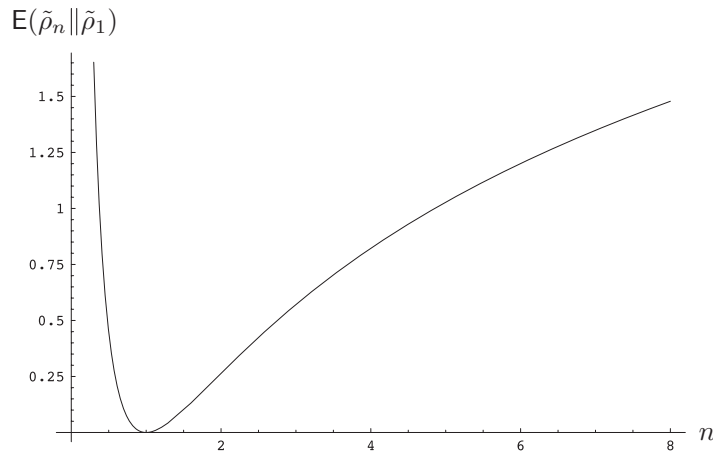


FIG. 2. $E(\tilde{\rho}_n \| \tilde{\rho}_1)$ for $X^2 = S^2$.

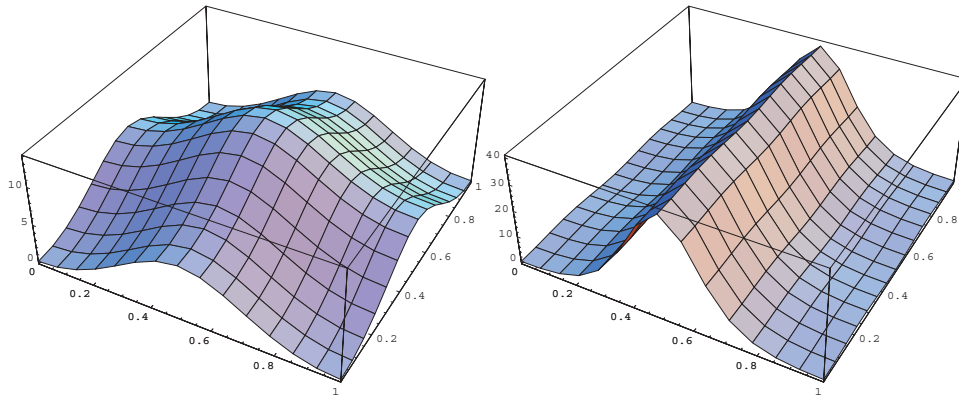


FIG. 3. (Color online) Two solutions of the Liouville equation for $X^2 = T^2$. Left is Olesen’s periodic solution of the Liouville equation $\rho_1 = \rho_{\mathcal{O}}$, corresponding to a single particle at $(0, 0)$; right is ρ_2 , a solution of the Liouville equation corresponding to one particle at $(0, 0)$ and another one at $(0, 1/2)$.

By considering analogous solutions on the lattices

$$2\mathbb{Z} + \frac{1}{n}2i\mathbb{Z} \quad (n = 1, 2, 3, \dots),$$

it is easy to construct n particle solutions of the Liouville equation on the torus (cf. Ref. 17, Eqs. (25), (65), and (66)) corresponding to one particle of mass $m = -2\pi$ sitting at each of the points

$$(0, 0), (0, 1/n), (0, 2/n), \dots$$

Call these solutions ρ_n (so that $\rho_1 = \rho_{\mathcal{O}}$). Examples are shown in Figure 3.

Calculating the relative entropy

$$E(\tilde{\rho}_n \| \tilde{\rho}_{\mathcal{O}})$$

by hand seems like a formidable task. However, numerical computation is simple enough and the outcome is shown in Figure 4. Remarkably, the result as a function of n is the same as that for the sphere (within the bounds of numerical accuracy)! This result is the basis for our claim in the Introduction that the relative entropy appears to be quite insensitive to global topology.

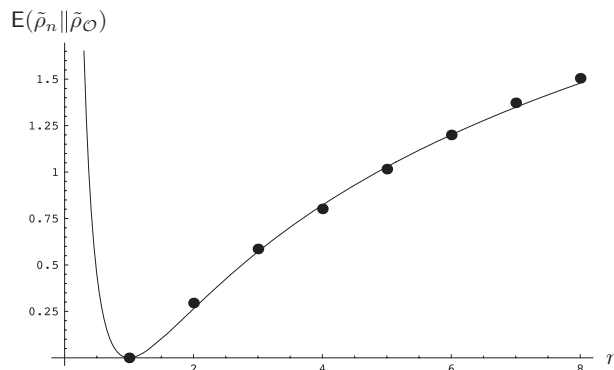


FIG. 4. $E(\tilde{\rho}_n \| \tilde{\rho}_{\mathcal{O}})$ for $X^2 = T^2$ (dots) superimposed on the analogous plot for the sphere.

C. Gravitational waves

Our final example is the relative volume entropy of an exact gravitational wave on a torus with respect to Minkowski space. (To the best of our knowledge the class of periodic gravitational waves presented below is new.)

We first exhibit a class of gravitational waves in toroidal space. Let

$$M = \mathbb{R}_t \times [0, L]^3 \quad (L \cong 0).$$

For the metric, we make the plane wave ansatz

$$ds^2 = -dt^2 + f(z-t)^2 dx^2 + g(z-t)^2 dy^2 + dz^2,$$

where $f(u)$ and $g(u)$ are L -periodic functions. The vacuum Einstein equations now imply

$$\frac{f''}{f} + \frac{g''}{g} = 0. \quad (8)$$

We can write down L -periodic solutions to this equation in terms of Mathieu functions (cf. Whittaker and Watson^{21,22}) as follows.

Let $q_0 > 0$ be a solution to the equation

$$a_n(q_0) = -b_m(-q_0) \quad (m, n \in \mathbb{N}^*),$$

where the a s and b s are the characteristic values of even and odd Mathieu functions, respectively.

Then, using the differential equation satisfied by Mathieu functions, it is easy to see that the following is a doubly infinite system of real L -periodic solutions to (8):

$$g(u) = C \left(a_m(q_0), q_0, \frac{2\pi u}{L} \right) \quad (\text{Mathieu cosine function}),$$

and

$$f(u) = S \left(-b_n(-q_0), -q_0, \frac{2\pi u}{L} \right) \quad (\text{Mathieu sine function}).$$

Thus, our solutions depend on two parameters $m, n \in \mathbb{N}^*$.

We have studied the relative volume entropy of a sample of these solutions with respect to Minkowski space as functions of the period-length L . By a change of variables, the relative volume entropy scales like L^3 , but we have not attempted to find a general formula for the coefficient. Figure 5 is one example of this behavior.

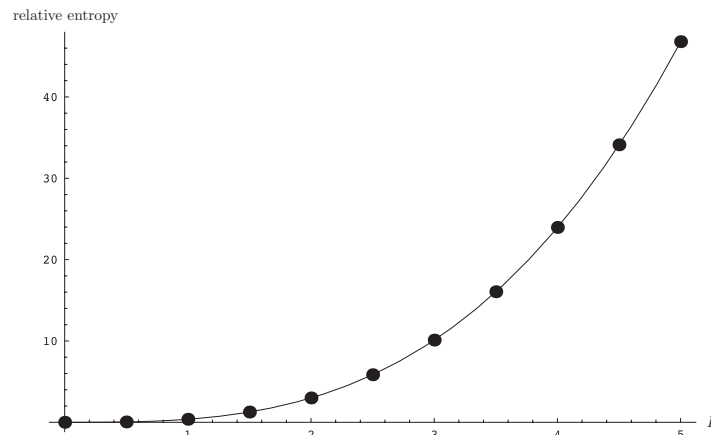


FIG. 5. Relative volume entropy with respect to Minkowski space of a gravitational wave with $m = 1$ and $n = 2$ as a function of the period-length L (dots), superimposed on a plot of the function $x \mapsto 0.374x^3$.

III. DISCUSSION

In this paper, we have introduced the relative entropy of volume, an information theoretical measure in classical gravity, measuring part of the information content of one solution of the Einstein equation relative to another, on a fixed manifold. We have studied in some examples how it detects and quantifies the deviation from homogeneity as caused by gravitational radiation and matter. The moral for now is that if we implement an entropy that measures the amount of information needed to describe (local) volume fluctuation from one solution to the other, we find this to measure matter/energy inhomogeneity. Philosophically, this of course concurs with Einstein's equations as an identification between geometric and energy content of a model. In fact, the form of the energy-momentum-based entropy of Buchert *et al.*^{11–13} can be *deduced* from the non-commutation of space averaging and time evolution, thus confirming that relative entropy arises naturally from physical principles (cf. the remark on the relation between the two entropies in the second-to-last paragraph of Introduction)

In this paper, we have studied some very basic examples of relative entropy (based on $2 + 1$ -dimensional toy models à la Deser–Jackiw–'t Hooft and periodic gravitational waves), which make very clear the dependence on inhomogeneity in the solutions, and the relative independence of the entropy on cosmic topology. Thus, we have neglected issues related to the presence of a non-trivial Weyl tensor, and time dependence, which we hope to return to in the future. Indeed, it would be interesting to study the relative volume entropy in the presence of a non-trivial Weyl tensor, for example, in the light of Penrose's proposal of non-activation of gravitational degrees of freedom (encoded in the Weyl tensor) at the Big Bang.²³ It would also be interesting to study an example of a cosmology in which the relative volume entropy of two solutions becomes time dependent, and investigate whether an analogue of the second law of thermodynamics holds for this entropy, namely: (eventual) monotonicity in time. This is also one of the main conjectural issues in Refs. 11–13. Cornelissen and Wylleman have computed the relative entropy for the spacetime of an inhomogeneous perfect fluid (with respect to Minkowski space). In this case, the relative entropy is monotonous in time, but we feel this example is too restrictive to draw any conclusions or propose conjectures at this point.

In addition, we believe that relative volume entropy could play a role in the theory of cosmic structures (compare Ref. 24), as a computational tool in choosing models. Minimizing the relative entropy of a putative analytical model of the resulting spacetime with respect to numerical data could provide a way of fixing parameters in the analytical model (applying the “principle of mean discrimination of information” due to Kullback). In Newtonian structure formation, one may minimize the relative entropy of the Lagrangian volume elements (i.e., Jacobians of the transformation from Eulerian to Lagrangian coordinates, as described in Ref. 25). This method is compatible with the fact that, in general, backreaction cannot be neglected (cf. Sec. 7.1 in Ref. 13).

A speculative issue is whether relative volume entropy concurs with actual theories of gravitational or cosmological entropy. The natural approach to entropy in classical and quantum gravity often is the von Neumann entropy (see e.g., Refs. 9 and 26), and it would certainly be very interesting to study its relation to the present relative entropy. As an example of this, in loop quantum gravity, volume is an (trace class) operator \mathbf{V} (compare, Refs. 27–29), so it makes sense to consider its von Neumann entropy $-\text{Tr}(\mathbf{V} \log \mathbf{V})$ (as a regularized value). Observe that in this case, there is no need for a *relative* version of entropy (or one may consider it as a relative entropy over empty spacetime). Does this “loop volume” entropy have a relation to our relative volume entropy (over a suitable background)?

Finally, in his proof of the geometrization conjecture, Perelman³⁰ uses the monotonicity of a certain “entropy” functional, a variation of which was put into the context of Fokker-Planck diffusion by Carfora.³¹ It would also be worthwhile to study the variation of our entropy in such a dynamical context, where spacetime evolves according to some flow equation.

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