

# On the limit existence principles in elementary arithmetic and related topics

Lev D. Beklemishev & Albert Visser

February 24, 2004

## Abstract

We study the arithmetical schema asserting that every eventually decreasing primitive recursive function has a limit. Some other related principles are also formulated. We establish their relationship with restricted parameter-free induction schemata.

Using these results we show that **ILM** is the logic of  $\Pi_1$ -conservativity of any reasonable extension of parameter-free  $\Pi_1$ -induction schema. This result, however, cannot be much improved: by adapting a theorem of D. Zambella and G. Mints we show that the logic of  $\Pi_1$ -conservativity of primitive recursive arithmetic properly extends **ILM**.

In the third part of the paper we give an ordinal classification of  $\Sigma_n$ -consequences of the standard fragments of Peano arithmetic in terms of reflection principles. This is interesting in view of the general program of ordinal analysis of theories, which in the most standard cases classifies  $\Pi$ -classes of sentences (usually  $\Pi_1^1$  or  $\Pi_2^0$ ).

## 1 Introduction

This paper was motivated by a particular question in the area of interpretability logic. It has been open for some time if a proof of the arithmetical completeness theorem for the logic of  $\Pi_1$ -conservativity **ILM** can be carried through in a theory weaker than  $I\Sigma_1$ . The essential principle used in this proof is the statement that a certain elementary function  $h$  (Solovay function), for which one can prove that it is eventually descending, has a limit. We consider the schema

$$\exists m \forall n \geq m \ h(n+1) \leq h(n) \rightarrow \exists m \forall n \geq m \ h(n) = h(m), \quad (Lim)$$

for each elementary function  $h$ . The schema  $(Lim)$  obviously follows from  $I\Sigma_1$  (see Lemma 1 below), but is properly weaker, because its arithmetical

complexity is  $\Sigma_2 \rightarrow \Sigma_2$ . ( $I\Sigma_1$  does not follow from any set of true  $\Sigma_3$ -sentences.)

We establish that over elementary arithmetic EA this schema is equivalent to the parameter-free  $\Sigma_1$ -induction schema  $I\Sigma_1^-$ . Also,  $(Lim)$  turns out to be equivalent to some other natural limit existence principles in elementary arithmetic. On the other hand, in the proof of completeness theorem for **ILM**, e.g., in the one due to G. Japaridze [9], the principle  $(Lim)$  is actually only applied once in the form of a rule

$$\frac{\exists m \forall n \geq m \ h(n+1) \geq h(n)}{\exists m \forall n \geq m \ h(n) = h(m)}, \quad (LimR)$$

for a certain elementary function  $h$ . We show that the closure of EA under unnested applications of  $(LimR)$  is equivalent to the fragment  $III_1^-$  of PA axiomatized over EA by the parameter-free induction schema for  $\Pi_1$ -formulas.  $III_1^-$  is known to be much weaker than  $I\Sigma_1^-$ , it has the same provably total computable functions as EA (elementary functions).

In the second part of this paper we briefly formulate the corollaries of this result for conservativity logics. In particular, **ILM** is the conservativity logic for any sound extension of  $III_1^-$ . We also give an example showing that this cannot be much improved. Based on a result of D. Zambella and G. Mints we demonstrate that the conservativity logic of primitive recursive arithmetic properly extends **ILM**.

The limit existence rule and  $III_1^-$  provide concrete examples of mathematically meaningful theorems of PA of complexity  $\Sigma_2$ . In the third part we consider the general question of characterizing  $\Sigma_n$ -consequences of arithmetical theories. For standard fragments of PA we give an ordinal classification of such consequences based on iterated reflection principles of a special kind. This result presents an independent interest in view of the general program of ordinal analysis and is largely based on the ideas in [3]. The third part of the paper can be read independently from the first two.

## 2 Limit existence principles

We relate some natural limit existence principles to the standard hierarchies of fragments of PA. As our basic fragment we take elementary arithmetic EA (or  $I\Delta_0 + \text{exp}$ ) which we formulate in the language containing terms for all Kalmar elementary functions. EA can be axiomatized over basic defining equations for all such functions by the schema of induction for bounded ( $\Delta_0$ ) formulas in its language. PA is defined by the schema of

induction for arbitrary formulas in the language.  $I\Sigma_n$  is axiomatized over EA by the induction schema for  $\Sigma_n$ -formulas, and  $I\Sigma_n^-$  is axiomatized by the parameter-free induction schema for  $\Sigma_n$ -formulas. See [8, 10] for more details.

First we study the schema  $(Lim)$ . The most natural proof of  $(Lim)$  can be carried out in  $I\Sigma_1$ .

**Lemma 1**  $I\Sigma_1$  proves  $(Lim)$ .

**Proof.** We reason inside  $L\Sigma_1$ , the least element principle for  $\Sigma_1$ -formulas, which is equivalent to  $I\Sigma_1$ .

Let  $m_0$  be such that

$$\forall n \geq m_0 \ h(n) \geq h(n+1),$$

where  $h$  is the given function. Using  $L\Sigma_1$  pick an  $x$  such that

$$\exists n \geq m_0 \ h(n) = x \wedge \forall y < x \ \neg \exists n \geq m_0 \ h(n) = y.$$

We have  $\forall n \geq m_0 \ h(n) \geq x$ . By an auxiliary  $\Delta_0$ -induction we also obtain

$$\forall n \geq m_0 \ h(m_0) \geq h(n).$$

Now if  $m_1$  is such that  $h(m_1) = x$ , then  $\forall n \geq \max(m_0, m_1) \ h(n) = x$ . Hence, the limit of  $h$  exists.  $\boxtimes$

As a corollary of a general proof-theoretic result from [10] we obtain the following improvement.

**Corollary 2** The parameter-free  $\Sigma_1$ -induction schema  $I\Sigma_1^-$  proves  $(Lim)$ .

**Proof.** By [10],  $I\Sigma_1$  is conservative over  $I\Sigma_1^-$  for  $\Sigma_3$ -sentences, which includes  $(Lim)$ .  $\boxtimes$

The converse to this corollary is also true, which follows from the next theorem.

**Theorem 1** The following schemata are equivalent over EA:

- (i)  $I\Sigma_1^-$
- (ii)  $\exists x \forall m \ f m \leq x \rightarrow \exists m \forall n \ f n \leq f m$
- (iii)  $\forall n \ f n \leq f(n+1) \wedge \exists x \forall m \ f m \leq x \rightarrow \exists m \forall n > m \ f n = f m$

$$(iv) \exists m \forall n \geq m f(n+1) \leq f(n) \rightarrow \exists m \forall n \geq m f(n) = f(m).$$

Here  $f$  ranges over all elementary terms in one variable.

**Proof.** Notice that (iv) is (*Lim*). By Corollary 2 it is sufficient to prove (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i). We reason in EA.

(iii) $\Rightarrow$ (ii): We argue by contraposition. Assume  $\forall m \exists n f n > f m$ . Consider the elementary function

$$f'(n) := \max_{i \leq n} f(i).$$

The graph of  $f'$  provably in EA satisfies

$$f'(n) = y \leftrightarrow (\exists m \leq n f(m) = y \wedge \forall m \leq n f(m) \leq y).$$

Hence, EA proves  $\forall n f'(n) \leq f'(n+1)$ .

We claim:  $\forall m \exists n > m f'(n) > f'(m)$ .

Indeed, if  $f'(m) = a$ , then  $\exists k \leq m f(k) = a$ . The assumption about  $f$  implies  $\exists n f n > f k = a$ . Hence, for this  $n$ ,  $f'(n) \geq f(n) > a = f'(m)$ , and by the provable monotonicity of  $f'$  we also have  $n > m$  (for otherwise  $f'(n) \leq f'(m)$ ). This proves the claim.

Applying now (iii) to  $f'$  we conclude that  $\forall x \exists m f'(m) > x$ , which implies  $\forall x \exists m f(m) > x$ .

(ii) $\Rightarrow$ (i): We have to show, for each  $\Delta_0$ -formula  $\varphi(x, u)$ , with only free variables  $x$  and  $u$ ,

$$\exists u \varphi(0, u) \wedge \forall x (\exists u \varphi(x, u) \rightarrow \exists u \varphi(x+1, u)) \rightarrow \forall x \exists u \varphi(x, u).$$

Let  $\text{IH}_\varphi$  denote the premise of the above implication. We introduce an elementary function

$$f(u) = \begin{cases} 0, & \text{if } u \text{ does not code a sequence} \\ \max_{x \leq \text{lh}(u)} \forall i < x \varphi(i, (u)_i), & \text{otherwise.} \end{cases}$$

Here  $\text{lh}(u)$  denotes the length of the sequence  $u$ .

We claim:  $\text{IH}_\varphi \rightarrow \forall u \exists v f(v) > f(u)$ .

Indeed, assume  $f(u) = a$ . If  $a = 0$ , consider a  $y$  such that  $\varphi(0, y)$  and let  $v = \langle y \rangle$ . Obviously,  $f(v) = 1 > 0$ .

If  $a > 0$  we have  $\forall i < a \varphi(i, (u)_i)$ . Let  $u'$  denote the initial segment of  $u$  of length  $a$  and let  $j = a - 1$ . We have  $\varphi(j, (u)_j)$ , hence by  $\text{IH}_\varphi$  there is a  $z$

such that  $\varphi(a, z)$ . Let  $v$  be the concatenation of  $u'$  and  $\langle z \rangle$ . We obviously have  $\forall i \leq a \varphi(i, (v)_i)$ , hence  $f(v) = a + 1 > a$ . This proves the claim.

From this using (ii) we obtain  $\forall x \exists u f(u) > x$ . However, if  $f(u) > x$ , then  $\varphi(x, (u)_x)$  by the definition of  $f$ . Hence,  $\forall x \exists y \varphi(x, y)$ .

(iv) $\Rightarrow$ (iii) Suppose  $\forall x (f(x) \leq f(x + 1))$  and  $f$  is bounded by  $a$ . We define elementary functions  $h$  and  $j$  as follows.

- $j0 = 0, h0 = 0.$
- $$j(z + 1) = \begin{cases} jz, & \text{if } fz \leq jz \\ z, & \text{otherwise,} \end{cases}$$
- $$h(z + 1) = \begin{cases} jz - fz, & \text{if } fz \leq jz \\ 0, & \text{otherwise.} \end{cases}$$

We show that  $h$  is eventually descending. In case  $fy \leq ja$  for all  $y$ , we find that, for all  $y > a$ ,  $jy = ja$  and hence  $hy = ja - fy$ . So clearly  $h$  will be descending from  $a + 1$  on.

Suppose that, for some  $y$ ,  $fy > ja$ . Since  $f$  is weakly ascending, we may assume that  $y \geq a$ . Let  $y^*$  be the smallest  $y \geq a$  such that  $fy > ja$  (it exists by the  $\Delta_0$ -least element principle). It follows that  $j(y^* + 1) = y^* \geq a$ . Hence, for all  $z$ ,  $fz \leq j(y^* + 1)$ . Thus, from  $y^* + 2$  on,  $h$  will be weakly descending.

By (iv) we find that  $h$  is eventually constant. If the limit of  $h$  is 0, then, for almost all  $z$ ,  $fz \geq jz$ . Suppose for infinitely many  $z$ ,  $fz > jz$ . Then, for any such  $z$ ,  $j(z + 1) = z$  and hence  $f(z + 1) \geq j(z + 1) = z$ . This contradicts the boundedness of  $f$ .

Otherwise,  $fz = jz$ , for almost all  $z$ . Then, by the definition of  $j$ ,  $j(z + 1) = jz$ , for almost all  $z$ . Hence, the functions  $j$  and  $f$  are eventually constant.

If the limit of  $h$  is  $c \neq 0$ , then, similarly to the previous case, for almost all  $z$ ,  $jz = fz + c$ . It follows that  $j(z + 1) = jz$ , for almost all  $z$ , and the functions  $j$  and  $f$  are eventually constant.  $\square$

If one wants to get a more direct proof of  $(Lim)$  in  $I\Sigma_1^-$  than the one using Corollary 2, one can prove (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv).

(i) $\Rightarrow$ (ii): We argue by contraposition. Assume  $\forall m \exists n fn > fm$  and prove  $\forall x \exists m f(m) > x$  by induction on  $x$ . BASIS of induction is easy. For the induction STEP assume that, for some  $m$ ,  $f(m) > x$ . Then  $\exists n f(n) > f(m) > x$  by our assumption. Hence,  $f(n) > x + 1$ , q.e.d.

(ii) $\Rightarrow$ (iii): From  $\forall n \, fn \leq f(n+1)$  we obtain

$$\forall m \forall n (m < n \rightarrow fm \leq fn)$$

by  $\Delta_0$ -induction. Hence,  $\forall m \exists n > m \, fm \neq fn$  implies  $\forall m \exists n > m \, fn > fm$ . Axiom (ii) then yields  $\forall x \exists m \, fm > x$ .

Now we derive (iv), i.e., (*Lim*) from (iii) using an idea of J. Joosten.

Let  $h$  be the given elementary function such that  $\forall n \geq m \, h(n) \geq h(n+1)$  for some  $m$ . Consider the function

$$g(x) = \sum_{i \leq x} |h(i) - h(i+1)|.$$

Obviously,  $g$  is provably nondecreasing. We show that  $g$  is bounded from above.

We claim:  $\forall x \, g(x) \leq g(m) + h(m+1)$ .

Indeed, if  $x \leq m$ , then  $g(x) \leq g(m)$  by monotonicity. If  $x > m$  we have

$$g(x) = g(m) + \sum_{i=m+1}^x |h(i) - h(i+1)|.$$

Now by an auxiliary  $\Delta_0$ -induction on  $x$  with parameter  $m$  we show

$$\forall x > m \sum_{i=m+1}^x |h(i) - h(i+1)| = h(m+1) - h(x+1).$$

We use the fact that for  $x > m$  the function  $h$  is weakly descending. If  $x = m+1$ , this is obviously true, because  $h(x) \geq h(x+1)$ . Similarly, for the induction step we have

$$\sum_{i=m+1}^{x+1} |h(i) - h(i+1)| = h(m+1) - h(x+1) + (h(x+1) - h(x+2)) = h(m+1) - h(x+2),$$

as required.

Applying (iii) we may conclude that  $g$  has a limit. This implies that  $\exists m \forall n \geq m \, g(n) = g(n+1)$ , which yields  $|h(n) - h(n+1)| = 0$ . Hence,  $h$  must have a limit, as well.

Let us now consider the limit existence rule

$$\frac{\exists m \forall n \geq m \, h(n+1) \geq h(n)}{\exists m \forall n \geq m \, h(n) = h(m)}, \quad (\textit{LimR})$$

for each elementary function  $h$ . The following result shows that the closure of EA under one application of (*LimR*) is contained in  $\text{III}_1^-$ .

**Theorem 2** *If*  $\text{EA} \vdash \exists m \forall n \geq m \ h(n) \geq h(n+1)$ ,  
*then*  $\text{III}_1^- \vdash \exists m \forall n \geq m \ h(n) = h(m)$ .

**Proof.** We use a version of the Herbrand Theorem for  $\exists\forall$ -formulas. In our treatment EA is formulated as a purely universal theory with terms for all Kalmar elementary functions. So, the assumption of the theorem yields a sequence of terms  $m_0, m_1(x_0), m_2(x_0, x_1), \dots, m_k(x_0, \dots, x_{k-1})$  such that the following disjunction is provable in EA:

$$(x_0 \geq m_0 \rightarrow h(x_0) \geq h(x_0 + 1)) \vee \quad (1)$$

$$(x_1 \geq m_1(x_0) \rightarrow h(x_1) \geq h(x_1 + 1)) \vee \quad (2)$$

$$\dots \quad (3)$$

$$(x_k \geq m_k(x_0, \dots, x_{k-1}) \rightarrow h(x_k) \geq h(x_k + 1)). \quad (4)$$

Notice that the variables  $x_0, \dots, x_k$  are free.

We prove  $\exists m \forall n \geq m \ h(n) = h(m)$  using the parameter-free least element principle for  $\Sigma_1$ -formulas  $L\Sigma_1^-$ . The standard argument shows that  $L\Sigma_1^-$  is equivalent to  $\text{III}_1^-$ .

Consider the least  $y_0$  such that  $\exists x_0 \geq m_0 \ h(x_0) = y_0$  (such an  $y_0$  exists by  $\text{III}_1^-$ ). If  $\forall x_0 \geq m_0 \ h(x_0) \geq h(x_0 + 1)$ ,  $x \geq m_0$  and  $h(x) = y_0$ , then by  $\Delta_0$ -induction

$$\forall n \geq x \ y_0 \leq h(n) \leq y_0,$$

hence  $y_0$  is the limit of  $h$ .

Otherwise,  $\exists x_0 \geq m_0 \ h(x_0) < h(x_0 + 1)$ , so we can take the least such  $x_0 \geq m_0$ . For this  $x_0$  the first line (1) of the disjunction is false.

Consider now the least  $y_1$  such that  $\forall x_1 \geq m_1(x_0) \ h(x_1) \geq h(x_1 + 1)$ . Notice that  $\text{III}_1^-$  proves the existence of  $y_1$  because  $x_0$  is  $\Delta_0$ -definable:

$$\begin{aligned} y_1 = \mu y. \exists x_0, x_1 (x_0 \geq m_0 \wedge h(x_0) < h(x_0 + 1) \wedge \\ \forall x < x_0 \neg(x \geq m_0 \wedge h(x) < h(x + 1)) \wedge x_1 \geq m_1(x_0) \wedge h(x_1) = y). \end{aligned} \quad (5)$$

Therefore we may proceed with the second line of the disjunction in the same way as with the first one: either  $\forall x_1 \geq m_1(x_0) \ h(x_1) \geq h(x_1 + 1)$  and then  $y_1$  is the limit of  $h$ , or (2) is false and using  $\Delta_0$ -induction we take the least  $x_1 \geq m_1(x_0)$  such that  $h(x_1) < h(x_1 + 1)$ . The point  $x_1$  is  $\Delta_0$ -definable, so we may proceed in a similar manner and successively obtain the elements  $y_2, \dots, y_k$  until all the remaining lines of the disjunction are falsified.  $\square$

The following theorem shows that unnested applications of  $(\text{Lim}R)$  are, in fact, sufficient to derive  $\text{III}_1^-$ .

**Theorem 3** Any instance of  $I\Pi_1^-$  is provable by one application of  $(LimR)$  over EA.

**Proof.** We prove any instance of  $L\Sigma_1^-$  (and thereby of  $I\Pi_1^-$ ) by an application of  $(LimR)$ . Assume  $\varphi xu$  is a  $\Delta_0$ -formula for which we want to prove

$$\exists x \exists u \varphi xu \rightarrow \exists x (\exists u \varphi xu \wedge \forall y < x \forall v \neg \varphi yv).$$

We construct the following elementary functions using the standard coding of pairs.

$$g_1(n) = \begin{cases} n, & \text{if } \forall v \leq n. \neg \varphi((v)_0, (v)_1) \\ (\mu v \leq n. \varphi((v)_0, (v)_1))_0, & \text{otherwise.} \end{cases}$$

In other words,  $g_1(n) = n$  for all  $n$  smaller than the code of the first pair  $\langle x, u \rangle$  such that  $\varphi xu$ . Thereafter, the value of  $g_1$  does not change and is equal to that particular  $x$ .

$$g(n) = \begin{cases} (n)_0, & \text{if } \exists u \leq (n)_1. \varphi((n)_0, u) \\ g_1(n), & \text{otherwise.} \end{cases}$$

The function  $g(n)$  enumerates the set  $\{x : \exists u \varphi xu\} \cup X$ , where  $X$  is a finite set coming from the values of  $g_1$ . (More precisely,  $X$  will be finite, if  $\exists x \exists u \varphi xu$ .) Notice that the points from  $X \setminus \{x : \exists u \varphi xu\}$  are enumerated only once, whereas the points from  $\{x : \exists u \varphi xu\}$  are enumerated infinitely often. Thus, it would be sufficient for us to find the minimum of  $\{x : \exists^{>1} n g(n) = x\}$ .

We define the function  $h$  by setting  $h(0) = g(0)$ .  $h(n+1)$  is defined as follows. If  $\forall k, m \leq n (k \neq m \rightarrow g(k) \neq g(m))$ , then  $h(n+1) = g(n+1)$ . If  $\exists m \leq n g(n+1) = g(m)$  and  $g(n+1) < h(n)$ , then  $h(n+1) = g(n+1)$ . Otherwise,  $h(n+1) = h(n)$ .

$h(n)$  coincides with  $g(n)$  until some value of  $g$  is repeated. Thereafter,  $h$  can only decrease (or not change) its value. So,  $\exists x \exists u \varphi xu$  implies

$$\exists m \forall n \geq m h(n) \geq h(n+1).$$

Define another function  $h'$  as follows:

$$h'(x) = \begin{cases} h(x), & \text{if } \exists n \leq x \varphi((n)_0, (n)_1) \\ 0, & \text{otherwise.} \end{cases} \quad (6)$$



Thus,  $h'$  eventually coincides with  $h$ , if  $\exists x \exists u \varphi x u$ , and is identically 0, otherwise. In any case, it is provable in EA that  $h'$  is eventually descending. So, applying  $(LimR)$  once, we conclude that  $h'$  has a limit  $y$ . If  $\exists x \exists u \varphi x u$ , then  $y$  is also the limit of  $h$ .

By the definition of  $h$ ,  $y$  will also be the minimum of  $\{x : \exists^{>1} n g(n) = x\}$ , because  $h$  decreases every time a repeated value of  $g$  smaller than the current value of  $h$  appears. Hence,  $y$  is also the minimum of  $\{x : \exists u \varphi x u\}$ .  $\square$

Remaining question: Do nested applications of  $(LimR)$  over EA yield new theorems (beyond  $\Pi_1^-$ )?

### 3 An application to conservativity logic

Interpretability logic studies the binary modality  $\varphi \triangleright \psi$  which is translated in the language of arithmetic as the formalized statement

$$“T + \varphi \text{ interprets } T + \psi”.$$

Here  $T$  is any fixed theory containing enough arithmetic and interpretability is understood as the relative interpretability in the usual sense of Tarski. *The interpretability logic*  $\mathbf{IL}(T)$  of a theory  $T$  is the set of all propositional modal formulas in the language with  $\triangleright$  whose arithmetical translations are always provable in  $T$ . This notion attracted the attention of researchers from the late 80's, particularly, because it represented a significant strengthening of the more usual *provability logic*. The modality  $\Box\varphi$  meaning “ $\varphi$  is provable in  $T$ ” can easily be expressed in terms of  $\triangleright$  by  $\Box\varphi \leftrightarrow (\neg\varphi) \triangleright \perp$ .

An important result, obtained by A. Berarducci [4] and V. Shavrukov [12], states that if  $T$  is an essentially reflexive theory, in particular, if  $T$  is a (sound) extension of Peano Arithmetic PA in the language of PA, then the interpretability logic of  $T$  is decidable and axiomatized by a certain axiom system  $\mathbf{ILM}$ .

By the well-known Orey-Hájek characterization (see [8]), for essentially reflexive theories, the notion of interpretability coincides with that of  $\Pi_1$ -conservativity. In general, however, the two notions diverge, most notably for finitely axiomatizable theories.

This suggests an alternative translation of the modality  $\varphi \triangleright \psi$  as the arithmetized statement

$$\forall \pi \in \Pi_1 (T + \psi \vdash \pi \Rightarrow T + \varphi \vdash \pi)$$

expressing the  $\Pi_1$ -conservativity of  $T + \psi$  over  $T + \varphi$ . Accordingly, the *logic of  $\Pi_1$ -conservativity*  $\mathbf{CL}(T)$  of  $T$  is defined as the set of all modal formulas that are always provable in  $T$  under the  $\Pi_1$ -conservativity translation of  $\triangleright$ .

P. Hájek and F. Montagna [7] showed that  $\mathbf{CL}(T) = \mathbf{ILM}$  not only for extensions of PA, but also for any sound theory  $T$  extending  $I\Sigma_1$ . G. Japaridze [9], among others, found a simplified proof of their result. However, it was not clear from any of these proofs if the bound  $I\Sigma_1$  is optimal.

Using the result of the previous section the Berarducci–Shavrukov–Hájek–Montagna theorem can now be improved to the following statement.

**Theorem 4** *If  $T$  is a sound extension of  $III_1^-$ , then  $\mathbf{CL}(T) = \mathbf{ILM}$ .*

**Proof.** We analyze the proof given in [5]. Without going into the details, we briefly indicate where the use of  $III_1^-$  is essential. We assume the knowledge of Theorem 14.2 in [5] and the terminology used there.

The proof rests on a diagonal definition of a certain primitive recursive function  $g$ , which is associated with an arbitrary finite  $\mathbf{ILM}$ -model. Firstly, it is easy to see that  $g$  is Kalmár elementary, so it can be introduced within EA. The essential point in the proof is Lemma 14.3 (a), that is, the statement that  $g$  has a limit. The rest of the reasoning is formalizable in EA.

We prove Lemma 14.3 (a) using  $(LimR)$ . With the function  $g$  we associate another elementary function  $h$  defined by  $h(0) = 0$  and

$$h(n + 1) = \begin{cases} 0, & \text{if } n \text{ is an } R'\text{-transfer of } g, \\ \text{rank}(n), & \text{if } n \text{ is an } S'\text{-transfer of } g, \\ h(n), & \text{otherwise.} \end{cases}$$

The notions of  $S'$ - and  $R'$ -transfer are taken from [5], and  $\text{rank}(n)$  indicates the rank of the  $S'$ -transfer at  $n$ .

The fact that the  $\mathbf{ILM}$ -model at hand is finite and the relation  $R'$  is irreflexive implies that only finitely many  $R'$ -transfers of  $g$  are possible. Moreover, this fact is verifiable purely within EA. One can establish it by external induction on the  $R'$ -depth of the model.

So, EA proves that, for almost all  $n$ ,  $g$  only makes  $S'$ -transfers or remains unchanged. As in [5], page 526, we see directly from the definition of  $g$  that the ranks of consecutive  $S'$ -transfers must decrease (strictly, if the last transfer is a move to a different node). This implies within EA that  $h$  is eventually descending.

Applying Theorem 2, by  $(LimR)$  we conclude that  $h$  has a limit. This can only happen in the case that eventually no  $R'$ - and  $S'$ -moves are made. Hence,  $g$  has a limit, too.  $\square$

Now we turn to some examples showing that the above bound  $III_1^-$  cannot be much improved. The following theorem has been discovered by D. Zambella [14]. G. Mints (private correspondence) gave a finitary proof of this statement based on the Herbrand Theorem. Below, we reproduce a modified version of that proof.

We write  $T \equiv_{\Pi_1} U$  if  $T$  and  $U$  are mutually  $\Pi_1$ -conservative.

**Theorem 5 (Zambella)** *Let  $T, S \subseteq \Pi_2$  and  $T \equiv_{\Pi_1} S$ . Then  $T + S \equiv_{\Pi_1} T$ .*

**Proof.** Assume  $S + T \vdash M(\vec{a})$ , where  $M$  is a quantifier-free formula with the parameters  $\vec{a}$ . Then the following disjunction is logically provable:

$$M(\vec{a}) \vee \exists x_1 \forall y P(x_1, y) \vee \exists x_2 \forall z Q(x_2, z),$$

where the second and the third formula are refutable in  $T$  and  $S$ , respectively. The Herbrand Theorem for  $\exists\forall$ -formulas delivers a sequence of terms  $t_0(\vec{a}), s_0(\vec{a}), t_1(\vec{a}, y_0, z_0), s_1(\vec{a}, y_0, z_0), \dots, s_n(\vec{a}, y_0, z_0, \dots, y_{n-1}, z_{n-1})$  such that

$$\vdash K_n \vee M(\vec{a}),$$

where  $K_n$  denotes

$$\bigvee_{i=0}^n (P(t_i, y_i) \vee Q(s_i, z_i)).$$

We prove by induction on  $j$  that for all  $j = 0, \dots, n$ ,

$$S \vdash M \vee K_{n-j}.$$

The statement is obvious for  $j = 0$ . For the induction step assume

$$S \vdash M \vee K_{l+1},$$

where  $l = n - j - 1$ . We have

$$S \vdash M \vee K_l \vee P(t_{l+1}, y_{l+1}) \vee Q(s_{l+1}, z_{l+1}).$$

Hence,

$$S \vdash M \vee K_l \vee P(t_{l+1}, y_{l+1}) \vee \forall z_{l+1} Q(s_{l+1}, z_{l+1}),$$

because the variable  $z_{l+1}$  only occurs in  $Q(s_{l+1}, z_{l+1})$ . Since

$$S \vdash \neg \exists x_2 \forall z_{l+1} Q(x_2, z_{l+1}),$$

this yields

$$S \vdash M \vee K_l \vee P(t_{l+1}, y_{l+1}).$$

This formula is quantifier-free, so by the  $\Pi_1$ -conservativity of  $S$  over  $T$  we obtain

$$T \vdash M \vee K_l \vee P(t_{l+1}, y_{l+1}).$$

Now we can get rid of  $P(t_{l+1}, y_{l+1})$  in the same way using the fact that  $T \vdash \forall x_1 \exists y_{l+1} \neg P(x_1, y_{l+1})$ . So, we obtain  $T \vdash M \vee K_l$ , whence  $S \vdash M \vee K_l$  by the  $\Pi_1$ -conservativity of  $T$  over  $S$ . This proves the induction step and the theorem.  $\square$

Let  $\text{EA}^+$  be axiomatized over  $\text{EA}$  by the axiom stating that the superexponentiation function is total (see [8]). The Herbrand Theorem is formalizable in  $\text{EA}^+$ , hence so is the proof of Zambella's theorem.

Let us consider the conservativity logic  $\text{CL}(T)$  for a  $\Pi_2$ -axiomatizable theory  $T$  containing  $\text{EA}^+$ . Examples of such  $T$  are  $\text{EA}^+$  itself or the *primitive recursive arithmetic* PRA.

Notice that any arithmetical translation of a modal formula of the form  $\varphi \triangleright \psi$  is  $\Pi_2$ , and the translations of the formulas of the form  $\diamond\varphi$  and  $\Box\varphi$  are  $\Pi_1$  and  $\Sigma_1$ , respectively.

Call a modal formula  $\Pi_2$ , if it is obtained from formulas of the forms  $\varphi \triangleright \psi$ ,  $\Box\varphi$  and  $\diamond\varphi$  by positive boolean connectives  $\vee$  and  $\wedge$ . By the above remark any arithmetical translation of such a formula belongs to the class  $\Pi_2$  of the arithmetical hierarchy.

**Corollary 3** *If  $T \subseteq \Pi_2$  and contains  $\text{EA}^+$ , then the following schema belongs to  $\text{CL}(T)$ :*

$$(\varphi \triangleright \psi \wedge \psi \triangleright \varphi) \rightarrow \varphi \triangleright (\varphi \wedge \psi), \quad (\text{Zam})$$

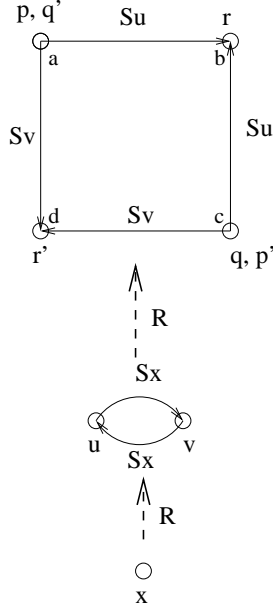
where  $\varphi$  and  $\psi$  are  $\Pi_2$ -formulas.

**Proof.** This is Zambella's theorem applied to the theories  $T + \varphi$  and  $T + \psi$ . Mints' proof shows that this statement is actually provable in  $\text{EA}^+$  and hence in  $T$ .  $\square$

Now we prove that the principle (*Zam*) is independent from **ILM**.

**Theorem 6** *The instance of (*Zam*) with  $\varphi = (\diamond p \wedge (q \triangleright r))$  and  $\psi = (\diamond p' \wedge (q' \triangleright r'))$  is not provable in **ILM**.*

**Proof.** We exhibit an **ILM**-model that falsifies the given instance of (*Zam*). For the notion of **ILM**-model see [5].



Here, each of the nodes  $a, b, c, d$  is  $R$ -accessible from any of the nodes  $u, v$  and both  $u$  and  $v$  are  $R$ -accessible from  $x$ . The  $S$ -arrows are all indicated except for the obvious reflexivity arrows for each of the nodes.

At every node the shown variables are true; all the other variables are assumed to be false. It is not difficult to see that this model falsifies the given instance of  $(Zam)$  at the node  $x$ .  $\square$

By the previous theorem, the principle  $(Zam)$  is independent from **ILM**. Thus, the  $\Pi_1$ -conservativity logics of  $\Pi_2$ -axiomatizable theories such as PRA properly extend **ILM**. We know from [2] that the schema  $III_1^-$  is equivalent to the local  $\Sigma_2$ -reflection principle over **EA** w.r.t. cut-free provability. This implies that  $III_1^-$  is not contained in any consistent r.e. extension of **EA** of quantifier complexity  $\Pi_2$ , which agrees with Theorem 4 above.

To find a complete axiomatization of **CL(PRA)** and **CL(EA)** remains an intriguing open question.

## 4 On $\Sigma_n$ -consequences of $I\Sigma_m$

Results on proof-theoretic (ordinal) analysis of theories  $T$  can be viewed as characterizations of consequences of  $T$  of specific logical complexity, e.g., of complexity  $\Pi_1^1$  or  $\Pi_2^0$ . Typically, given a logical complexity class  $\Gamma$  one considers a mapping  $P : \alpha \mapsto P_\alpha$  of ordinals to schemata in the language of  $T$  satisfying the following demands:

1. For each  $\alpha$ ,  $P_\alpha \in \Gamma$ .
2.  $\{\varphi \in \Gamma : T \vdash \varphi\}$  can be axiomatized by  $\{P_\beta : \beta < \alpha\}$ , where

$$\alpha = |T|_P := \min\{\beta : T \not\vdash P_\beta\}.$$

The ordinal  $\alpha$  is then called the *proof-theoretic ordinal of  $T$  w.r.t. the mapping  $P$* . Somewhat loosely, one also calls it the  $\Gamma$ -*ordinal of  $T$* , if there is one distinguished hierarchy  $P$  associated with  $\Gamma$  (see [3] for a more detailed discussion).

Thus,  $P$  serves as a measure of proof-theoretic strength of theories. Assuming fixed some natural system of ordinal notation one can define the following typical hierarchies considered in the literature:

- (a)  $P_\alpha := \text{WF}(\alpha) \in \Pi_1^1$ , the transfinite induction axiom up to  $\alpha$ ;
- (b)  $Q_\alpha := (\forall x \exists y F_\alpha(x) = y) \in \Pi_2^0$ , where  $F_\alpha$  is the  $\alpha$ -th function of the fast growing hierarchy;
- (c)  $R_\alpha := (\forall x \exists y G_\alpha(x) = y) \in \Pi_2^0$ , where  $G_\alpha$  is the  $\alpha$ -th function of the slow growing hierarchy.

For example, it follows from Gentzen's results that the ordinal of Peano arithmetic (conservatively extended by second order variables) w.r.t. the above measure  $P$  is  $\epsilon_0$ . This was improved later by H. Schwichtenberg and S. Wainer to the statement that the ordinal of PA w.r.t.  $Q$  is  $\epsilon_0$ . On the other hand, the ordinal of PA w.r.t.  $R$  is much higher and was calculated by J.-Y. Girard.

In [3], developing the ideas in [13, 6] and [11], the hierarchies of uniform reflection principles were used to classify  $\Pi_n^0$ -consequences of theories, for any  $n$ . Below we shall work in the language of EA. Recall that the  *$n$ -consistency assertion*  $n\text{-Con}(T)$  for a theory  $T$  formalizes the fact that  $T$  together with all true  $\Pi_n$ -sentences is consistent. This can be naturally expressed using a truth-definition for  $\Pi_n$ -formulas in EA.

*Iterated consistency assertions*, assuming fixed an elementary system of ordinal notation, are defined by the following fixed point equation:

$$\text{EA} \vdash n\text{-Con}^\alpha(T) \leftrightarrow n\text{-Con}(T \cup \{n\text{-Con}^\beta(T) : \beta < \alpha\}).$$

It is proved in [1, 3] that the solution of this equation is unique up to provable equivalence in EA. We also denote the theory  $T \cup \{n\text{-Con}^\beta(T) : \beta < \alpha\}$  by  $(T)_\alpha^n$ . Notice that, provably in EA,  $n\text{-Con}^0(T) \leftrightarrow n\text{-Con}(T)$  and  $n\text{-Con}^\alpha(T) \leftrightarrow n\text{-Con}((T)_\alpha^n)$ . See [3] for more information on such hierarchies.

As shown in [3], the statements  $P_\alpha^n := n\text{-Con}^\alpha(\text{EA})$  can be used to classify  $\Pi_n$ -consequences of theories, for any  $n$ . For  $n = 2$  the ordinals obtained in this way agree with the ordinals w.r.t. the traditional  $\Pi_2^0$ -measure  $Q$  defined by the fast growing hierarchy. However, to the best of our knowledge, there were no results on ordinal classifications of  $\Gamma$ -consequences of theories, for  $\Sigma$ -type classes of sentences  $\Gamma$ .

In the realm of fragments of arithmetic some suggestive relationships are known. Thus, by a result of [10],  $I\Sigma_m$  is  $\Sigma_{m+2}$ -conservative over  $I\Sigma_m^-$ . Hence,  $\Sigma_{m+2}$ -consequences of  $I\Sigma_m$  are axiomatizable by  $I\Sigma_m^-$  which has the smaller complexity  $\Sigma_{m+1} \rightarrow \Sigma_{m+1}$ . However, already the question about  $\Sigma_2$ -consequences of  $I\Sigma_1$  was open. Among the  $\Sigma_2$ -consequences of  $I\Sigma_1$  we find the system  $I\Pi_1^-$ , but it does not exhaust all its  $\Sigma_2$ -consequences.

Below we give a complete classification of  $\Sigma_n$ -consequences of the fragments  $I\Sigma_m$  of PA, for any  $n$  and  $m$ . We notice some trivial relationships. Firstly,  $\Sigma_1$ -consequences of any sound theory containing EA coincide with the set of true  $\Sigma_1$ -sentences. Secondly, if  $n > m + 2$ , then  $I\Sigma_m$  itself provides a  $\Sigma_n$ -axiomatization of its  $\Sigma_n$ -consequences. The other cases, except for  $n = m + 2$ , are covered by the following theorem.

Define:  $\omega_0 = 1$ ,  $\omega_{k+1} = \omega^{\omega^k}$ . We consider the schema

$$S_\alpha^n := \{\pi \rightarrow n\text{-Con}^\alpha(\text{EA} + \pi) : \pi \in \Pi_{n+2}\}.$$

**Theorem 7** *Let  $0 \leq n < m$ . Then  $\Sigma_{n+2}$ -consequences of  $I\Sigma_m$  are axiomatized over EA by the schemata*

$$\{S_\alpha^n : \alpha < \omega_{m-n+1}\}. \quad (*)$$

**Proof.** The proof of this theorem is based on the results of [3]. We note that the arithmetical complexity of the schema  $S_\alpha^n$  is, indeed,  $\Sigma_{n+2}$ .

First, we derive (\*) from  $I\Sigma_m$ . Consider a  $\Pi_{n+2}$ -sentence  $\pi$ . We know that  $I\Sigma_m$  is equivalent to  $(m+1)\text{-Con}(\text{EA})$ . Since  $m > n$ , we have that  $I\Sigma_m + \pi$  is equivalent to  $(m+1)\text{-Con}(\text{EA} + \pi)$ . Applying now Theorem 4 from [3] we obtain that

$$(\text{EA} + \pi)_1^{m+1} \equiv_{\Pi_{n+1}} (\text{EA} + \pi)_{\omega_{m+1-n}}^n. \quad (**)$$

Hence,

$$I\Sigma_m + \pi \vdash n\text{-Con}^\alpha(\text{EA} + \pi),$$

for all  $\alpha < \omega_{m+1-n}$ . Deduction theorem implies

$$I\Sigma_m \vdash \pi \rightarrow n\text{-Con}^\alpha(\text{EA} + \pi),$$

for any such  $\alpha$ .

For the converse direction, assume  $I\Sigma_m \vdash \neg\pi$ , where  $\pi \in \Pi_{n+2}$ . Then  $I\Sigma_m + \pi \vdash \perp$ . By (\*\*) we have  $(\mathbf{EA} + \pi)_{\omega_{m+1-n}}^n \vdash \perp$ . Hence, there is  $\alpha < \omega_{m+1-n}$  such that  $\mathbf{EA} + \pi + n\text{-Con}^\alpha(\mathbf{EA} + \pi) \vdash \perp$ . Therefore,

$$\mathbf{EA} + \pi + (\pi \rightarrow n\text{-Con}^\alpha(\mathbf{EA} + \pi)) \vdash \perp,$$

and by the Deduction Theorem

$$\mathbf{EA} + (\pi \rightarrow n\text{-Con}^\alpha(\mathbf{EA} + \pi)) \vdash \neg\pi.$$

This proves the theorem.  $\square$

**Remark 4**  $\Sigma_{m+2}$ -consequences of  $I\Sigma_m$  axiomatized by  $I\Sigma_m^-$  have a characterization in terms of reflection principles which is similar to the one in Theorem 7. By the results of [2],  $I\Sigma_m^-$  is equivalent to

$$\mathbf{EA} + \{\pi \rightarrow m\text{-Con}(\mathbf{EA} + \pi) : \pi \in \Pi_{m+1}\}.$$

Therefore, Theorem 7 also holds with  $m = n$ , but with the complexity of  $\pi$  decreased by 1.

Now we mention some corollaries of Theorem 7.

**Corollary 5**  $\Sigma_2$ -consequences of  $I\Sigma_m$  are axiomatized by

$$\{\pi \rightarrow \text{Con}^\alpha(\mathbf{EA} + \pi) : \pi \in \Pi_2, \alpha < \omega_{m+1}\}.$$

In particular, for  $I\Sigma_1$  one obtains all  $S_\alpha^0$  with ordinals  $\alpha < \omega^\omega$ .

**Corollary 6**  $\Sigma_{n+2}$ -consequences of PA are axiomatized by  $S_\alpha^n$  with  $\alpha < \epsilon_0$ .

Thus, the hierarchy of schemata  $S_\alpha^n$  allows to define proof-theoretic ordinals of theories for logical complexity  $\Sigma_{n+2}$ . According to this measure the  $\Sigma_2$ -ordinals of  $I\Sigma_1$  and PA are  $\omega^\omega$  and  $\epsilon_0$ , respectively.

We show that  $S_\alpha^n$  can also be perceived as iterated local reflection schemata. For any theory  $T$ , let  $S^n(T)$  denote the relativized local reflection schema

$$\{\pi \rightarrow n\text{-Con}(T + \pi) : \pi \in \Pi_{n+2}\}.$$

Iterated version thereof is a progression of theories  $S_\alpha^n(T)$  obtained, similarly to the iterated consistency assertions, by formalizing the fixed point equation

$$S_\alpha^n(T) \equiv T + \{S^n(S_\beta^n(T)) : \beta < \alpha\}.$$

See [3] for more details. We have the following relationship.



**Theorem 8** *Verifiably in EA, for any  $\alpha$ ,*

$$S_\alpha^n(\text{EA}) \equiv \text{EA} + \{S_\beta^n : \beta < \omega^\alpha\}.$$

For the proof, first we need the following lemma.

**Lemma 7** *For any  $\alpha$  and  $\pi \in \Pi_{n+2}$ , provably in EA,*

$$S_\alpha^n(\text{EA}) + \pi \equiv S_\alpha^n(\text{EA} + \pi).$$

**Proof.** The proof goes by the so-called reflexive induction on  $\alpha$  in EA (see [3]). We give an informal argument.

Assume the statement holds provably for all  $\beta < \alpha$ . Consider the axioms of  $S_\alpha^n(\text{EA} + \pi)$  that have the form

$$\pi_1 \rightarrow n\text{-Con}(S_\beta^n(\text{EA} + \pi) + \pi_1),$$

for some  $\beta < \alpha$  and  $\pi_1 \in \Pi_{n+2}$ . By the reflexive induction hypothesis the latter is equivalent to

$$\pi_1 \rightarrow n\text{-Con}(S_\beta^n(\text{EA}) + \pi + \pi_1). \quad (7)$$

Notice that  $S_\alpha^n(\text{EA})$  proves

$$\pi \wedge \pi_1 \rightarrow n\text{-Con}(S_\beta^n(\text{EA}) + \pi + \pi_1),$$

for both  $\pi$  and  $\pi_1$  are  $\Pi_{n+2}$ . It follows that  $S_\alpha^n(\text{EA}) + \pi$  proves (7).

The proof in the opposite direction is easy.  $\square$

**Proof of Theorem 8.** Again, we use reflexive induction. We also rely on the results of [1, 3]. Recall that  $\alpha$ -times iterated local reflection principle contains (and is  $\Pi_1$ -conservative over)  $\omega^\alpha$ -times iterated consistency assertion, and similar relationship holds for the relativized versions of these schemata.

To prove ( $\subseteq$ ), consider any  $\beta < \omega^\alpha$  and any  $\pi \in \Pi_{n+2}$ . We have:

$$\begin{aligned} S_\alpha^n(\text{EA}) + \pi &\vdash S_\alpha^n(\text{EA} + \pi) \\ &\vdash (\text{EA} + \pi)_{\omega^\alpha}^n, \quad \text{by Proposition 6.2 of [3]} \\ &\vdash n\text{-Con}^\beta(\text{EA} + \pi), \quad \text{for } \beta < \omega^\alpha. \end{aligned}$$

Hence, by the Deduction Theorem

$$S_\alpha^n(\text{EA}) \vdash \pi \rightarrow n\text{-Con}^\beta(\text{EA} + \pi),$$

q.e.d.

To prove  $(\supseteq)$  consider any axiom of  $S_\alpha^n(\text{EA})$  of the form

$$\pi \rightarrow n\text{-Con}(S_\beta^n(\text{EA}) + \pi),$$

with  $\beta < \alpha$  and  $\pi \in \Pi_{n+2}$ . By Proposition 6.2 of [3] we have, provably in EA,

$$S_\beta^n(\text{EA} + \pi) \equiv_{\Pi_{n+1}} (\text{EA} + \pi)_{\omega^\beta}^n.$$

Hence,  $n\text{-Con}(S_\beta^n(\text{EA} + \pi))$  is equivalent to  $n\text{-Con}^{\omega^\beta}(\text{EA} + \pi)$ . Since  $\omega^\beta < \omega^\alpha$  we obtain

$$\begin{aligned} \text{EA} + \{S_\gamma^n : \gamma < \omega^\alpha\} &\vdash \pi \rightarrow n\text{-Con}^{\omega^\beta}(\text{EA} + \pi) \\ &\vdash \pi \rightarrow n\text{-Con}(S_\beta^n(\text{EA} + \pi)) \\ &\vdash \pi \rightarrow n\text{-Con}(S_\beta^n(\text{EA}) + \pi). \end{aligned}$$

Hence,  $\text{EA} + \{S_\gamma^n : \gamma < \omega^\alpha\}$  contains  $S_\alpha^n(\text{EA})$ .  $\square$

## References

- [1] L.D. Beklemishev. Iterated local reflection versus iterated consistency. *Annals of Pure and Applied Logic*, 75:25–48, 1995.
- [2] L.D. Beklemishev. Parameter free induction and provably total computable functions. *Theoretical Computer Science*, 224(1–2):13–33, 1999.
- [3] L.D. Beklemishev. Proof-theoretic analysis by iterated reflection. *Archive for Mathematical Logic*, 42:515–552, 2003.
- [4] A. Berarducci. The interpretability logic of Peano Arithmetic. *The Journal of Symbolic Logic*, 55:1059–1089, 1990.
- [5] D. de Jongh and G. Japaridze. The Logic of Provability. In S.R. Buss, editor, *Handbook of Proof Theory*. Studies in Logic and the Foundations of Mathematics, Vol.137., pages 475–546. Elsevier, Amsterdam, 1998.
- [6] S. Feferman. Transfinite recursive progressions of axiomatic theories. *The Journal of Symbolic Logic*, 27:259–316, 1962.
- [7] P. Hájek and F. Montagna. **ILM** is the logic of  $\Pi_1$ -constervativity. *Archive for Mathematical Logic*, 30:113–123, 1990.

- [8] P. Hájek and P. Pudlák. *Metamathematics of First Order Arithmetic*. Springer-Verlag, Berlin, Heidelberg, New York, 1993.
- [9] G. Japaridze. A simple proof of arithmetical completeness for  $\Pi_1$ -conservativity logic. *Notre Dame Journal of Formal Logic*, 35:346–354, 1994.
- [10] R. Kaye, J. Paris, and C. Dimitracopoulos. On parameter free induction schemas. *The Journal of Symbolic Logic*, 53(4):1082–1097, 1988.
- [11] U.R. Schmerl. A fine structure generated by reflection formulas over Primitive Recursive Arithmetic. In M. Boffa, D. van Dalen, and K. McAloon, editors, *Logic Colloquium'78*, pages 335–350. North Holland, Amsterdam, 1979.
- [12] V.Yu. Shavrukov. The logic of relative interpretability over Peano arithmetic. Preprint, Steklov Mathematical Institute, Moscow, 1988. In Russian.
- [13] A.M. Turing. System of logics based on ordinals. *Proc. London Math. Soc.*, ser. 2, 45:161–228, 1939.
- [14] D. Zambella. *Chapters on bounded arithmetic and on provability logic*. PhD thesis, University of Amsterdam, 1994.