

# Brouwer and Weyl: The phenomenology and mathematics of the intuitive continuum\*

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L.E.J. Brouwer and Hermann Weyl both recognized that there is something called ‘the intuitive continuum’. It is the phenomenon of the intuitive continuum that motivates their developments of constructive real analysis on the basis of choice sequences. Brouwer already mentions the intuitive continuum and describes a few of its features in his doctoral thesis (1907). Weyl, evidently unaware of Brouwer’s early work, discusses the intuitive continuum in Chapter 2, section 6, of *Das Kontinuum* (1918), which is entitled ‘The Intuitive and the Mathematical Continuum’. The view of the intuitive continuum in *Das Kontinuum* is based largely on Husserlian phenomenological descriptions of the consciousness of internal time, although Weyl also mentions here and elsewhere some other historical views related to the idea of the intuitive continuum.<sup>1</sup> In this section of *Das Kontinuum* Weyl takes the experience of the flow of internal time as the model of the intuitive continuum.

Brouwer and Weyl both distinguish ‘internal’, intuitive time from ‘external’ time. Brouwer (1907, p.61) calls the latter ‘scientific’ or measurable time. Brouwer’s notion of the primordial intuition of mathematics is concerned only with internal, intuitive time. Weyl (1921, p.88), following Husserl, distinguishes ‘phenomenal’ time from ‘objective’ time and says that he will, in effect, ‘suspend’ the latter and focus only on the former. It is the phenomenon of the intuitive continuum that he feels he has not captured with the mathematical (or ‘arithmetic’) theory of the continuum developed in *Das Kontinuum*. It is this phenomenon, in Weyl’s eyes, to which Brouwer’s development of intuitionistic real analysis does far more justice, and it is for just this reason that Weyl

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<sup>1</sup>Weyl mentions in *Das Kontinuum* and later writings (1921, 1925), e.g., ideas of Anaxagoras, Aristotle, Bergson. There is a long tradition of similar views on the intuitive continuum. One finds such a notion, for example, in Kant, Peirce, and others. See, e.g., the citations in the Introduction to *A Primer of Infinitesimal Analysis*, by John Bell (1998). For some early phenomenological discussions of the intuitive continuum and choice sequences see Becker’s monograph on geometry (1923) and Kaufmann (1930). For a critique of these two discussions, see sections 3.3.1 and 4.2.2 of (van Atten 1999).

declares his allegiance to Brouwer in his famous paper on the ‘Grundlagenkrise’ (1921).

We will describe below some of the features of the phenomenon on which Brouwer and Weyl are focused and then consider their mathematical treatments of it. A mathematics of the intuitive continuum should be founded on the formal or structural features of the intuitive continuum and, for Brouwer and Weyl, these will be structural features of the stream of consciousness that are abstracted from its qualitative aspects. It is useful to think of the views of Weyl and Brouwer in connection with the phenomenology of the consciousness of internal time because Husserl probably invested more effort in describing the phenomenon in question than any other philosopher (Husserl 1966). What should be of interest to those who know phenomenology is that there is a mathematics associated with the phenomenon of internal time. What should be of interest to students of Brouwerian intuitionism is that there is a phenomenology of the intuitive continuum. Focusing on internal time as a model of the intuitive continuum should also help to give logicians, philosophers and mathematicians a firmer grasp of what it is that is being analyzed by people like Brouwer and Weyl, and it should help to allay some of the skepticism that has been expressed about whether Brouwer’s description of the move of time has any real content (e.g., in (Tait 1983)).<sup>2</sup>

We open the paper with some informal descriptions of the intuitive continuum. The mathematics of the intuitive continuum depends on formalizing aspects of the informal descriptions. In their subsequent mathematical and logical analyses of the intuitive continuum Brouwer and Weyl do not agree on all details. Both Brouwer and Weyl use choice sequences to characterize real numbers and they describe modifications of classical logic that are required for this purpose, but there are some important differences between their views. One central difference is that Weyl evidently thinks it is possible to take full advantage of the *concept* of choice sequence without being ontologically committed to sequences other than those that are lawlike. In his account, existential quantification over non-lawlike sequences is not permitted. Since this view is quite different from Brouwer’s, and since it may have been motivated in part by his reading of Husserl, we will consider whether this is a workable alternative to Brouwer’s development of intuitionistic real analysis.

We should clear up a matter of terminology right from the beginning. For Brouwer, ‘choice sequence’ is the all-inclusive concept, with lawless sequences and lawlike sequences as the two limiting cases. Even as, according to Brouwer, all sequences are generated by the subject, the subject may limit its own freedom in generating a particular sequence. In particular, it may freely decide to let a rule or law prescribe its next choice. (In a Kantian vein, following a rule may be a perfect expression of freedom.) In Weyl’s usage, on the other hand, ‘choice sequence’ or ‘free choice sequence’ never refers to a lawlike sequence. We will adhere to Brouwer’s terminology.

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<sup>2</sup>For an argument in favour of intuitionistic analysis based on considerations in physics, see (Dummett 2000).

# 1 Internal Time and the Intuitive Continuum

Brouwer and Weyl both distinguish ‘internal’, intuitive time from ‘external’ time. They follow the idealist tradition in holding that time is the basic form of the stream of consciousness. Brouwer makes a number of comments about the intuitive continuum that can be readily understood in connection with Husserl’s phenomenological descriptions of internal time. First, Brouwer describes the primordial intuition of mathematics as

the substratum, divested of all quality, of any perception of change, a unity of continuity and discreteness, a possibility of thinking together several entities, connected by a ‘between’, which is never exhausted by the insertion of new entities (Brouwer 1907, p.17)

The reason for divesting this ‘substratum’ of all quality is simple. In order to do mathematics we need to focus on the form or structure of the phenomenon. Thus, acts of abstraction or idealization are already at work in the primordial intuition of mathematics. In this basic intuition, Brouwer says,

continuity and discreteness occur as inseparable complements, both having equal rights and being equally clear, it is impossible to avoid one of them as a primitive entity, trying to construe it from the other one, the latter being put forward as self-sufficient (Brouwer 1907, p.17)

Brouwer says that the intuition of continuity, of ‘fluidity’, is as primitive as that of intuiting several things together as a unity.

What Brouwer in this early work calls the discrete aspect of the primordial intuition of mathematics is expressed in his later work in the so-called ‘first act of intuitionism’ (e.g., (1952)). Brouwer says that this first act separates mathematics from mathematical language and recognizes that intuitionist mathematics is a languageless activity of the mind having its origin in the perception of a move of time, i.e., of the falling apart of a life moment into two distinct things, one of which gives way to the other, but is retained by memory. This ‘two-ity’, divested of all quality, is the empty form of the common substratum of all two-ities. In this common substratum, this empty form of all two-ities, lies the basis of the discrete aspect of the primordial intuition of mathematics. It successively generates each natural number and arbitrary finite sequences. It is natural to suppose that it lies at the basis of the finite combinatorial objects that could be generated from the natural numbers.

Brouwer gives precedence to ‘two-ity’ in his account of the natural numbers. It is not good enough to merely start with a unit to obtain the natural numbers. Once we become aware of a sensation passing into another sensation, for example, we have the foundation for the abstract two-ity out of which the natural numbers are generated. Suppose we were to mark this awareness as  $( | ) |$  to indicate the retention in memory of what was sensed earlier. This ‘two-ity’ can then be an element of a new two-ity:  $(( | ) | ) |$ , and so on. From the perspective of Husserl’s phenomenology, one could call what occurs at the next stage of this

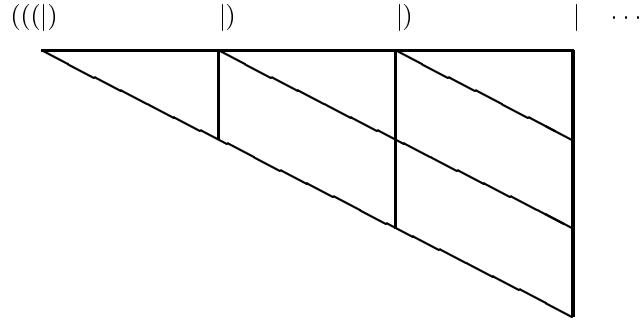


Figure 1.

process after  $(|)$ , by adding another stroke, a successor act of intuition. In phenomenology one could make the following observations about this successive aspect of intuition. At later (successor) stages of intuition the earlier stages sink back in time but are retained in an appropriately modified manner, indicating the importance of the role of memory in our constructions. To be more specific, as the construction begins and continues earlier parts of it sink into the past and out of our immediate awareness even though they are retained and remain active in processing present parts of the construction. Indeed, this must be the case or it would not be possible to take in the construction as a unified whole. We can picture this process of retention as in Figure 1 (cf. (Husserl 1966, §10)).

The horizontal axis indicates the ‘flowing present’ of successive ‘now’ moments or stages. The diagonal lines below the horizontal axis indicate ‘retentions’ that are part of the awareness at the chosen stage. The retentions are continuously modified as they sink back into the less immediate part of our present experience. Along the horizontal axis we have a multiplicity of successors but the vertical axis at each stage indicates how they are all held together or unified in one consciousness at that stage. Along the horizontal axis we have the ‘many’, while along the vertical axes we have the ‘one’ in which the many are unified or synthesized (see also (Tieszen 1989, ch.5)). Brouwer emphasizes how in this successive, sequential structure with its ordering of ‘before’ and ‘after’ the ‘before-after’ or ‘first-second’ are held together in consciousness so that we have unity in multitude.

While the construction is in process there will also be some more or less determinate protentions (which are similar to expectations—but only roughly, see below) at any stage about how it will unfold and about its completion. In the case at hand this is completely determinate. It is clear that this will be a lawlike or rule-governed becoming. The future course of experience is fixed: it is simply the iteration of successor. In Husserl’s language, we could say that the ‘horizon’ of possibilities of experience here is fixed. (The situation is of course quite different in the case of non-lawlike choice sequences.) At any stage in this form we obtain a definite or determinate object, e.g., the object expressed

by  $(( | ) | ) |$  is clearly distinct from the object expressed by  $( | ) |$ . We can also indicate protentions in our diagram. Protentions could be indicated by including diagonal lines above the horizontal axis that point to the future.

One could hold that the natural numbers are ‘complete’ objects in the sense that each natural number can be obtained in a finite construction process. This is not something that we could expect in the case of real numbers. One of the features involved in obtaining the natural numbers on the basis of abstraction is that they are themselves durationless. They are arrived at by abstraction from a temporal process in which we are aware, for example, of a sensation passing into a different sensation. The concept of duration simply does not apply to natural numbers themselves. One cannot suppose that the same is necessarily true of real numbers if we are to have a constructive account of the real numbers. Indeed, real numbers in intuitionism are typically viewed as ‘incomplete’ or ‘unfinished’ objects for just these kinds of reasons. On the intuitionistic view, an act of abstraction that would give us a real number as a durationless point is not something of which we would be capable.

If we now put together our remarks on retentions and protentions we see that the particular stages are not cut off from one another as though there were isolated, atomic points. Rather, there will always be connections between earlier and later phases by way of retentions and protentions, along with secondary memory. There is always an overlap of phases. The stream of consciousness is a continuous fabric. While we can construct a successor and then another successor, and so on, we are always doing this against the background of the flow of inner time. Constructing natural numbers through time is of course not the same thing as actually experiencing a durationless now point in the stream of consciousness. In the former case we are simply registering or willing successor intuitions as the flow of time proceeds. We are, as it were, constructing an idealized grid over the continuum. In our experience of the flow of time itself we are not conscious of a durationless now point. Rather, what we experience is a ‘specious’ or ‘extended’ present comprised, in our description, of a retention-primal impression-protention structure. Using metaphorical language, Husserl describes this specious present as a ‘halo’ or a ‘comet’s tail’. Its extension is indeterminate and shades off continuously.

According to phenomenology, retentions and protentions are associated with all acts of consciousness. They concern the awareness of the immediate past and future. Retention or ‘primary memory’ is not itself an act we undertake. Similarly, protention is not an act that we affect. Retention and protention occur passively or automatically and they are to be distinguished from acts of remembering or acts of anticipating or planning. Acts of remembering are *recollections* or a type of *representation*. Acts of remembering themselves have a retention-protention structure. This type of ‘secondary memory’ is of course very important in retaining what is past. It is a fundamental feature of the constitution of any given phase of our awareness. In actuality it is of course not perfect. This leads Brouwer sometimes to speak of the ‘idealized mathematician’ for whom (secondary) memory is perfect. It also leads him to assign a role to the use of language as an aid to memory.

The role of memory is related to another feature of the intuitive continuum. There is in experience an asymmetry between past and future. What is past, as retained in secondary memory, is determinate and fixed. We do not have the freedom to change it. The future, however, is open and more or less indeterminate. What this suggests, as Husserl and Brouwer intimate, is that we can distinguish a future that is lawlike from one that may involve choices. In the case of a non-lawlike future there may be degrees of freedom/indeterminacy. We discuss this in more detail below.

In all cases the subject's intuition is finite. The subject (or transcendental ego) carries out only finitely many acts of intuition. There are no completed infinite sequences of intuitions or, if you like, no completed infinite sets of acts of intuition. Infinite sequences of intuitions do not, as it were, have being. The notion of infinity for our intuitions can be thought of only as potential, in the sense of becoming.

As Brouwer remarks in the quotation at the beginning of this section, there is always something 'between' the discrete units we obtain on the basis of the first act of intuitionism. Brouwer says in effect that the discrete and the continuous are complementary because a requirement for discreteness or apartness of moments in time is the existence of a 'between', while awareness of change or the move of time is only possible by recognition of a before and now, or a past and present moment. The distinction of moments in time is only possible because of the 'between' but the 'between' in turn presupposes the existence of endpoints. It is this 'between' that, mathematically speaking, founds the infinite potential for insertion of new points. This, however, cannot be understood along the lines of the Cantorian continuum. The later is conceived as a pre-given set of static points. The aspect of continuity that is part of the primordial intuition of mathematics is provided for by Brouwer's 'second act of intuitionism' (e.g., (1952)). It is this act that provides for real analysis by taking us beyond lawlike sequences (i.e., beyond what can be done in arithmetic, as the constructivists before Brouwer and indeed the early Brouwer himself did).

The second act recognizes the possibility of generating more or less free choice sequences as well as mathematical species. If we have natural or rational numbers at our disposal and we use them in a sequence then it seems that there is nothing about our description of the open-endedness of the future of the subject that prohibits the subject from choosing freely among these numbers to extend the sequence. Indeed, one could say that this is just what it means in a mathematical setting to have an open-ended future. This allows for the possibility of modeling the intuitive continuum mathematically as the species of the more or less freely proceeding convergent infinite sequences of rational intervals.

We have already pointed out how we do not in fact experience a durationless now point in our awareness of the flow of internal time. We could try to approach a durationless point in this intuitive continuum by an infinite sequence of nested rational intervals the lengths of which converge to 0 (Figure 2). Indeed, such a durationless point would just be a real number according to the classical characterization of the real numbers. In *Das Kontinuum*, Weyl says that a point

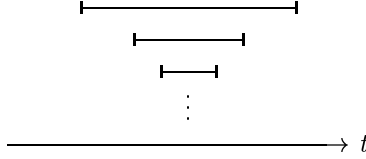


Figure 2.

or real number in this sense is something that is only thought or conceived. It is an ‘idealization’. It is not something that we intuit or experience. Brouwer would certainly agree with this latter claim, as he sees that the discrete and the continuous cannot be reduced to one another. For Weyl, it is something we can talk about ‘in theory’ but then we see that there is a significant gap between theory and intuition. Weyl follows the Kantian tradition in distinguishing concepts from intuitions and in arguing that both concepts and intuitions are required for knowledge. This is similar to Husserl’s distinction between an empty intention and the intuitive fulfillment of the intention. Weyl argues that classical real analysis does not provide an intuitive foundation for the concept or intention ‘ $x$  is a real number’ since it posits the existence of objects (durationless points) that we do not and cannot intuit. It would follow that classical real analysis could not count as knowledge. What Brouwer tries to do is precisely to provide an intuitive foundation for the category ‘ $x$  is a real number’. Already in (1907) Brouwer says that ‘The continuum as a whole was intuitively given to us by intuition; a construction of the continuum, an act which would create by means of the mathematical intuition “all its points” is inconceivable and impossible’ (p.45).

Since Weyl distinguishes between intuition and concepts, and since he and Brouwer believe that the intuitive basis of the concept ‘real number’ is to be found in choice sequences, it evidently follows that in Cantorian set theory we have an empty, purely conceptual (and possibly inconsistent) view of the continuum. It is an interesting question whether there is only one thing that could possibly answer to the notion of ‘continuum’, that being the intuitive continuum. If so, then what should we say about the ‘set-theoretic continuum’? Brouwer, for one, seems to hold that the so-called ‘set-theoretic continuum’ is pure illusion. On Weyl’s later views, the matter is not as clear. It seems that Weyl, given his distinction between the intuitive and the symbolic (see, e.g., his (1925, 1949); also (Tieszen 2000)), would be committed to providing some kind of explanation of the set-theoretic view of the continuum. The set-theoretic continuum would have to be understood as purely ‘symbolic’, conceptual or theoretical. It would require some work to specify exactly what, if anything, this means.

It is worth noting that, just as we do not experience durationless points in time, we do not experience extensionless points in space. Brouwer and Weyl,

however, both focus on inner time. Brouwer (1907) in fact says that ‘time is the only a priori of mathematics’. Inner time arguably provides a better model than space for a number of reasons. Inner time yields dynamical aspects of the continuum such as order and progression. It also yields the open-endedness of the future. It provides the possibility of a more or less determinate horizon that can be structured through free choices. The model provided by space, independent of time, would be static. (Of course, the introduction of mathematical points as sequences of nested intervals on the line chosen by the mathematician brings time into play again. The choices are made in time, and the horizon of the process of choosing is generally open-ended. But what we want to stress now is that this link between spatial continua and time is not a direct one, but is mediated by the process of choosing intervals.)

Robert Tragesser, in correspondence with the authors, has made some additional observations about why inner time is important as a foundation.

First, if we thought of spatial continua as fundamental and we considered them to be potential then we immediately face a kind of messiness that we avoid by taking inner time as fundamental. For example, if we could have only finite but ever extendible line segments then it becomes complicated to say what it means for lines to intersect. Inner time has the feature of potentiality but we avoid such messiness.

Second, since inner time is one-dimensional we avoid certain complications involved in working out its real properties that we would have to face in the case of space. We would need to ask whether the properties that a line has by virtue of being embedded (or allowing to be embedded) in a higher dimensional space are intrinsic properties.

Third, the two-ity that emerges by abstraction from the phenomenon of the present sinking into the past gives a natural, uncontrived purchase on the real line by discrete things. It is not at all clear how one would obtain the discrete from geometric continua without already having something discrete in hand (e.g., a ruler, ideal rod).

Finally, temporal continua give us a foundation for continuous change, whereas if we started with geometric continua we would have to import temporal continua if we were interested in continuous changes.

As Brouwer and Weyl see the matter, the intuitive continuum cannot be understood as a set of durationless or extensionless points. Classical analysis represents real numbers this way by identifying them with the points that are obtained in infinite sequences of nested intervals. On this view, the real line would consist of the set of such points. This is an atomistic, static view of the continuum. The problem is precisely that it makes the intuitive continuum disappear. We simply have a set of durationless points. Brouwer and Weyl hold that in order to capture the fluid continuum we should replace the element/set relation with a part/whole relation. There are in fact many kinds of part/whole relations and it happens that Husserl distinguished many of these and discussed them in the third of his *Logische Untersuchungen*. The intuitive continuum would be what Weyl (1949, p.52), following Husserl (1984, p.276), calls an ‘extensive whole’ (‘extensives Ganzes’). (Curiously, Husserl himself would not



have accepted choice sequences as mathematical objects. See section 3, below.)

An extensive whole is a whole whose parts are of the same lowest genus as the undivided whole itself. If a particular temporal interval is an extensive whole then its parts must themselves be temporal intervals. This conforms to our experience since we have no intuition of durationless points. (Similar observations can be made about extensionless points in space.) On the basis of this part/whole analysis we would not be able to obtain real numbers as idealized, durationless or extensionless points, as purely conceptual objects that are not given in intuition. Brouwer, in a similar manner, says that the ‘synthetic construction’ of sets (which consists of combining discrete elements into a new and different kind of mathematical entity, a set) is wholly inappropriate for generating the continuum. The characteristic construction of the continuum is ‘analysis’, i.e., decomposition into homogeneous parts. ‘Analysis’ splits the continuum or interval into two parts, subintervals, homogeneous to each other and to the whole interval. Each subinterval is an interval in its own right—it is of the same nature as its parent interval. The act of insertion and the natural order of the continuum generate a relation of ‘inclusion’ and an order relation between subintervals. So the relation of ‘inclusion’ or ‘subinterval’ (i.e., a whole-part relation) is the fundamental relation of the continuum, not the set-element relation. The order relation between disjoint subintervals is the natural order of the continuum abstracted from the progression of time. One of Brouwer’s characterizations of the real numbers mirrors several of our comments here. Let  $\lambda$ -intervals be intervals of the form

$$\left[ \frac{a}{2^{\nu-1}}, \frac{a+1}{2^{\nu-1}} \right]$$

Then Brouwer (1992, p.69) defines real numbers as follows:

We [ . . . ] consider an indefinitely proceedable sequence of *nested*  $\lambda$ -intervals  $\lambda_{\nu_1}, \lambda_{\nu_2}, \lambda_{\nu_3}, \dots$  which have the property that every  $\lambda_{\nu_{i+1}}$  lies *strictly* inside its predecessor  $\lambda_{\nu_i}$  ( $i = 1, 2, \dots$ ). Then [according to the definition of  $\lambda$ -intervals] the length of the interval  $\lambda_{\nu_{i+1}}$  at most equals half the length of  $\lambda_{\nu_i}$ , and therefore the lengths of the intervals converge to 0 [ . . . ] *We call such an indefinitely proceedable sequence of nested  $\lambda$ -intervals a point P or a real number P.* We must stress that for us the point P is the sequence

$$(1) \lambda_{\nu_1}, \lambda_{\nu_2}, \lambda_{\nu_3}, \dots$$

*itself*; not something like ‘the limiting point to which according to classical conception the  $\lambda$ -intervals converge and which could according to this conception be defined as the unique accumulation point of midpoints of these intervals’.

Every one of the  $\lambda$ -intervals (1) is therefore *part of* the point P. [original emphases]

Note how Brouwer says that the point P is the sequence itself and that the  $\lambda$ -intervals are parts of the point P.

## 2 The Intuitive Continuum and Free Choice

An intuitive foundation for real analysis is provided by choice sequences, such as a sequence of  $\lambda$ -intervals just mentioned. These sequences may but need not be lawlike. One may put various restrictions on one's own choices, the limiting cases being a law (algorithm) or, on the other hand, no restriction on one's choices at all (lawless sequences). Moreover, there is the possibility to pose restrictions on restrictions, and so on, to any finite order. Brouwer also allows that these choices of numbers and restrictions are made dependent on possible future mathematical experiences of the creating subject.

We can think of a non-lawlike choice sequence as a kind of intuition even though it is quite different in some respects from a lawlike intuition. It is still a sequence carried out in time by a subject (or transcendental ego), only part of which is actually completed. We would actually complete only a finite initial segment of it, it will be associated with filling out the horizon of an intention directed toward a real number, and we should think of it as a 'medium of free becoming'. It should be noted that we are speaking here of a choice sequence as a process, as a sequence of acts developing in time. Instead of simply engaging in the process of constructing the sequence one can also reflect on choice sequences. In this case the choice sequence itself becomes an object. The choice sequence as an object is constituted on the basis of the choice sequence as a process (see (van Atten 1999, ch.5)).

What distinguishes a lawlike sequence from other choice sequences, is that it is determinate and gives us the same definite object at each stage. Its horizon is fixed. A non-lawlike sequence is more or less indeterminate or indefinite. Its horizon is not fixed in advance but is quite open. Apart from some very general specifications, there is nothing in particular that should occur in it. There is a mere requirement to posit something further. This is quite formal and abstract compared with a lawlike sequence. With non-lawlike sequences we are considering intentions that can be fulfilled by any arbitrary postulation within the specifications, if any, that have been indicated.

Relative to a non-lawlike sequence, nothing essentially new occurs in carrying out a lawlike sequence. We can overview or survey lawlike sequences in a way that is not possible with non-lawlike sequences. With a lawlike sequence we are directed toward an endless lawlike succession. The infinite can in this sense be grasped by a finite mind (transcendental ego) by way of grasping the law that generates the sequence. The law can be finitely specified. With a lawlike sequence we need not continue carrying out the sequence. Since nothing essentially new occurs we can break off further acts and simply appeal to the law. With a non-lawlike sequence we have no grounds to break off further acts. In this sense, the idea of non-lawlike sequences forces the idea of 'becoming in time', of infinity as potential, much more impressively than the idea of lawlike sequences. With lawlike sequences we can downplay the 'becoming' and incompleteability by appealing to the law.

Even so, it is clear that for a correct understanding of a choice sequence *as* representing a point on the intuitive continuum, the sequence should be

considered as a sequence in progress, whether it is lawlike or not. In the case of non-lawlike sequences this may be easiest to grasp, but the same holds for lawlike sequences. For if, on the contrary, a lawlike sequence is conceived of as a finished object, we may be seduced into thinking of the point in the classical, atomistic way again. But then the continuum would be disrupted. In other words, the condition that the point never ‘is’ but always ‘becomes’ preserves the continuum. This is Brouwer’s rationale for writing that

For us a point and therefore also the points in a set are always something becoming and often something remaining undetermined, contrary to the classical conception, where a point is considered determined as well as finished. (Brouwer 1992, p.71)

This unfinished character of choice sequences has repercussions for logic. It means that a sequence can not, at any stage, have (or lack) a certain property if that could not be demonstrated from the information available at that stage. It follows that bivalence, and hence the Principle of the Excluded Middle, does not hold generally for statements about choice sequences. For example, consider a lawless sequence  $\alpha$  of which so far the initial segment  $\langle 1, 2, 3 \rangle$  has been generated, and the statement  $P =$  ‘The number 4 occurs in  $\alpha$ ’. Then we cannot say that  $P \vee \neg P$  holds. For the same reason, extensional identity of choice sequences (i.e. having the same values at the same places) is not generally decidable. (In the case of two lawlike sequences, one may be able to show extensional identity by proving equivalence of the laws governing them.) Note how this argument against the validity of the Principle of the Excluded Middle depends on both the freedom of generation and the potential infinity of the sequences. Acceptance of choice sequences as mathematical objects forces a revision of logic. As Placek (1999) has argued, of all arguments in favour of intuitionistic logic, this one is probably the most cogent. (The theme of the dependency of logic on one’s ontology has been elaborated by Tragesser (1977), taking his cue from Husserl (1974).)

The asymmetry between past, which is fully determined, and future, which is largely undetermined, figures in Brouwer’s work in an important way. A choice sequence will be split by the present moment of choice into a completed and determinate initial segment and an extension that has not yet been constructed. But how, then, can a choice sequence as a whole ever be the input to a function or predicate? Brouwer found the right principle. He saw that a choice sequence  $\alpha$  can be taken as an argument of a total function because in that case the function assignment must be constructable from just a suitable initial segment of  $\alpha$ :

$$\forall \alpha \exists x (F(\alpha) = x) \Rightarrow \forall \alpha \exists m \exists x \forall \beta [\bar{\beta}m = \bar{\alpha}m \rightarrow F(\beta) = x]$$

where  $\alpha$  and  $\beta$  range over choice sequences of natural numbers,  $m$  and  $x$  over natural numbers, and  $\bar{\alpha}m$  stands for  $\langle \alpha(0), \alpha(1), \dots, \alpha(m-1) \rangle$ , the initial segment of  $\alpha$  of length  $m$ . This is nowadays known as ‘weak continuity for numbers’ (weak, as it only deals with each  $\alpha$  individually). It is the basis for Brouwer’s

development of real analysis and function theory and from it one can prove Brouwer's well-known theorem that all fully defined real functions are continuous (Veldman 1982). We will come back to this principle below.

It seems that for Husserl and Weyl the completed initial finite segment of a choice sequence stands out in a special way: it can claim full mathematical existence (in *their* senses of 'existence'). This is a point at which there seems to be a divergence of views. According to Brouwer's second act of intuitionism, non-lawlike sequences as such can also claim full mathematical existence. Thus, Brouwer held that mathematical objects may be unfinished or incomplete. We discuss this in more detail below.

In reflecting on the continuity of internal time one would expect a uniform continuity theorem for all real-valued functions. Although it was Brouwer who introduced the choice sequences, Weyl (1921) preceded him in asserting the unsplitability of the continuum and the continuity of real functions. As van Dalen (1999, p.325) has noted, however, Weyl did not *prove* the continuity of all real functions. Weyl's statement is a corollary of his definition of real function since on his account an approximation of the value of a function was by definition determined by the approximation of the argument. Continuity was in effect part of the definition of 'function'. This is different from Brouwer. Brouwer was concerned with functions from choice sequences to choice sequences, a more complicated type than Weyl used. A few years later Brouwer proved the assertions of continuity and unsplitability by developing ideas that are not to be found in Weyl's work (the fan and bar theorems).

Once the notion of choice sequence emerges from our reflections on the intuitive continuum we can distinguish and explore different types and features of choice sequences. Reflecting on choice sequences in this manner is not the same thing as reflecting on the intuitive continuum although there may to some extent be a dialectical interplay between the two types of reflection in which each can inform the other. Thus, one might also engage in a phenomenology of choice sequences (van Atten 1999).

### 3 Choice Sequences: Some Differences Between Brouwer and Weyl

Weyl, along with Oskar Becker, was one of the first to interpret intuitionistic logic and mathematics from the standpoint of Husserl's phenomenology. The position he arrived at differed in some respects from Brouwer's.<sup>3</sup> It is not always clear to what extent these differences arose because of Weyl's own phenomenological research, or because of elements that he simply took from Husserl's writings. In particular, according to Husserl, mathematical objects are characterized

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<sup>3</sup>Grattan-Guinness (2000) mentions that, in opposition to Hilbert, Brouwer and Weyl each had alternative versions of the continuum, and then adds: 'Perhaps sensing weakness in the camp, (Fraenkel 1925) noted differences between the two in a long piece for the *Jahrbuch* [über die Fortschritte der Mathematik]'. But note that Fraenkel does not bring out any difference not mentioned by Weyl himself.

as invariant with respect to time. At first, Husserl expressed this by saying that such objects are ‘atemporal’ (‘unzeitlich’: outside of time altogether). By the time that Weyl began his philosophical investigations, Husserl had changed this into ‘omnitemporal’ (‘allzeitlich’: existing at all times, without changing).<sup>4</sup> In Husserlian terminology, omnitemporal objects are ‘ideal’ objects. Objects that have temporal extension or duration are called ‘real’ objects. Included among real entities are mental acts or processes. Some real entities also have spatial extension. These are ordinary physical objects. If one accepts this from Husserl, then Brouwer’s position faces the following problem. A lawlike sequence may be conceived of as an unchanging or omnitemporal object. One can come back to precisely the same object at any given time. Brouwer himself would of course not look at the matter this way; as we explained above, doing so leads one to forget that a point is always becoming. Still, the abstract possibility remains. The non-lawlike sequences, however, change through time in a way that is not wholly predetermined. That means that from Husserl’s point of view, non-lawlike sequences cannot be genuine mathematical objects. They would presumably be viewed as ‘real’, not ‘ideal’. It is very natural to read Weyl as having found a possible way out of this situation, where it seems one could take full advantage of the concept of choice sequence without being ontologically committed to sequences other than the lawlike. We will now consider whether this is a workable alternative to Brouwer’s analysis.

In fact, Weyl seems to recognize only two concepts of sequences, the lawless and the lawlike. (The term ‘lawless’ was introduced in print only later, by Kreisel (1968), following a suggestion by Gödel). This has become the standard term. Weyl (1921) speaks of ‘the freely developing choice sequence’ (p.100), which is ‘unrestricted by any law in the freedom of its development’ (p.94). (This also makes clear that by ‘freely developing sequence’ does not mean ‘sequence developing in an arbitrary way’, but the stronger ‘lawless’.) The lawless sequences, according to him, are not individual objects, whereas the lawlike are: ‘a single particular (and up to infinity determined) sequence can only be defined by a law’; and ‘the individual real number is represented by a law’ (1921, p.94). (As we saw above, Brouwer thinks of a point as ‘always something becoming and often remaining undetermined’, so for Brouwer non-lawlike sequences are individual objects just as well.) However, Weyl also holds that there is a use in mathematics for the *concept* of lawless sequence, in that it allows one to conceptualize the continuum in the right way. ‘It is one of the fundamental insights of Brouwer that number sequences, developing through free acts of choice, are possible objects of mathematical concept formation’ (1921, p.94).

This seems to be a way to do justice to both Husserl’s and Brouwer’s intuitions while avoiding actual conflict, and therefore it must be treated at some length. Brouwer received a copy of Weyl’s manuscript for the ‘Grundlagenkrise’ paper, and commented on it. We will quote from Brouwer’s reaction later on (see (van Dalen 1995) for the historical details).

Weyl (1921, p.96) claims that the range of the existential quantifier over

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<sup>4</sup>For a discussion of this change, see (Lohmar 1993) and (van Atten 1999, section 4.2.1).

sequences is only the lawlike sequences: ‘The expression “There is” commits us to Being and law, while “every” releases us into Becoming and Freedom.’

Weyl’s explanation seems to be that it is essential to pure mathematics that its objects can be coded in natural numbers:

Every application of mathematics must set out from certain objects that are to be subjected to mathematical treatment, and that can be distinguished from one another by means of a number character. The characters are natural numbers. The connection to pure mathematics and its constructions is achieved by the symbolic method, which replaces these objects by their characters. The point geometry on the straight line is, in this way, based on the system of the above-mentioned dual intervals, which we are able to identify by means of two whole number characters. (Weyl 1921, pp.100–101)

When reading this quote one has to be aware that for Weyl ‘application of mathematics’ does not always mean what it normally would, and does not necessarily contrast with what is usually understood as pure mathematics. In Weyl’s usage, the point geometry on a line already is an application of mathematics. The coded basic objects are rational segments. This example also illustrates that the substitution of numbers for objects in the process of symbolization cannot be a mere labeling of those objects. Some information needs to be preserved, depending on what we want to use the mathematics for. When that is done, Weyl continues, we are back in the realm of ‘pure mathematics and its constructions’. From this we conclude that according to Weyl, all individual objects of pure mathematics can be coded into natural numbers.

Laws, being finitely specifiable, can be coded in natural numbers and therefore Weyl, by the criterion above, accepts lawlike sequences as genuine (individual) objects of mathematics. Lawless sequences cannot be thus coded, hence are not to be regarded as individual objects. This gives Weyl’s game away: although he recognizes a role for lawless sequences in conceptualizing the continuum, in the end mathematics is only about numerically codifiable objects.

Weyl says that from conceptual truths about lawless sequences, one arrives at genuine mathematical statements by substituting lawlike sequences for lawless sequence variables:

The proper judgments that can be gained from these universal judgments come into being [ . . . ] in the case of the freely developing choice sequence, by substituting for it a law  $\phi$  that determines an individual number sequence in infinitum.’ (Weyl 1921, p.100)

The concept of lawless sequence is meaningful, but there are no individual mathematical objects falling under it. They are beyond the reach of Weyl’s methods of construction. Only objects that can be coded into natural numbers can be constructed in that strong sense. Weyl, however, has to accept lawless sequences as objects of some, albeit not mathematical, sort. Perhaps they could be accepted as empirical objects, as Gödel seems to suggest (Kreisel 1980, p.202).

In Husserl's terms, Weyl and Gödel could think of such sequences as 'real', whereas the concepts or essences of which they are instances are 'ideal'.

An incoherency in Weyl's conception seems to be that he on the one hand only talks about lawlike and lawless sequences, but on the other hand also gives the following example of forming a sequence  $n$  by operating on a lawless sequence  $m$  (Weyl 1921, p.94):

$$n(h) = m(1) + m(2) + \dots + m(h)$$

The trouble here is that  $n$ , because of its dependence on  $m$ , is neither lawlike nor lawless. It isn't lawlike, because it functionally depends on a lawless sequence; it isn't lawless, because it depends on another sequence at all. (So lawless sequences are not closed under even the most innocent operations, except for the identity operation.) Brouwer was aware of this need for types of choice sequence in between lawless and lawlike, we turn to that below.

Leaving this matter aside for the moment, let us review the argument that Weyl develops to defend the thesis that the concept of lawless sequence has mathematical applicability:

1. The concept of lawless sequence is an 'object for mathematical concept (Begriff) formation'. (premise, (Weyl 1921, p.94))
2. Lawless sequences may be so chosen as to follow a lawlike sequence. (premise, implicit)
3. All individual (i.e. lawlike) sequences can be embedded in the continuum of lawless sequences. (from 1 and 2; see (Weyl 1921, p.100))
4. It may be part of the meaning of 'lawless sequence' that it does or does not have a given property E. (premise, (Weyl 1921, p.96))
5. Hence there will be general judgement directions for lawlike sequences. (from 3 and 4; see (Weyl 1921, p.100))

Premise 2 is implicit in (Weyl 1921, p.96):

Then it can happen that it is part of the essence of a developing sequence, that is, of a sequence where each individual choice step is completely free to possess the property not-E. This is not the place to discuss how such insights into the essence of a developing sequence are to be gained [Here Weyl thinks of Husserl's 'Wesenschau'.] Yet only this sort of insight provides a justification for the fact that, when given a law  $\phi$  by someone, we can, without examination, reply: The sequence determined in infinitum by this law does not possess the property E.

This premise 2 is already problematic. One can see the idea behind it—can't a lawless sequence be chosen such that it happens to coincide extensionally with a lawlike sequence? But the answer is 'no'. One cannot specify a lawless

sequence  $\alpha$  by saying that it follows some lawlike sequence  $a$ ; for that would contradict the lawless character of  $\alpha$ . As non-lawlike sequences are necessarily unfinished objects, one cannot say that a law and a sequence of free choices may simply be alternative ways to describe the same completed infinite sequence. (One might be able to establish that an already chosen initial segment of a lawless sequence coincides with an initial segment of a lawlike sequence.)

Weyl goes on to argue that there is no corresponding general theory of functions. He recognizes three kinds of functions (all constructive, i.e., laws):

**functiones discretæ**  $\mathbb{N} \rightarrow \mathbb{N}$ , from numbers to numbers (i.e. lawlike sequences)

**functiones mixtæ**  $(\mathbb{N} \rightarrow \mathbb{N}) \rightarrow \mathbb{N}$ , assigning numbers to choice sequences, or  $\mathbb{N} \rightarrow (\mathbb{N} \rightarrow \mathbb{N})$ , from numbers to lawlike functions

**functiones continuæ**  $(\mathbb{N} \rightarrow \mathbb{N}) \rightarrow (\mathbb{N} \rightarrow \mathbb{N})$ , from choice sequences to choice sequences

The argument is based on the ‘fact’ (that is, if one accepts premise 2) that functiones discretæ, and only those, can be embedded into a conceptually acceptable continuum. That embedding is based on the concept of lawless sequence; but there is, according to Weyl, nothing analogous to that in the case of functiones mixtæ and continuæ. There is no continuum then that the mixtæ and continuæ can be embedded in, hence, there is no interpretation of general statements about them, and there is no general theory of functions.

This point Weyl stresses again and again:

But let me emphasize again that individual, determined functions of this sort occur from case to case in the theorems of mathematics, yet one never makes general theorems about them. The general formulation of these concepts is therefore only required if one gives an account of the meaning and the method of mathematics; for mathematics itself, and for the content of its theorems, it does not come into consideration at all [ . . . ] For these theorems, as far as they are self-sufficient and not purely individual judgments, are general statements about numbers or choice sequences, but not about ‘functions’.  
(Weyl 1921, p.106)

Sets of functions and sets of sets, however, shall be wholly banished from our minds. There is therefore no place in our analysis for a general set theory, as little as there is room for general statements about functions [ . . . ] As far as I understand, I no longer completely concur with Brouwer in the radical conclusions drawn here. After all, he immediately begins with a general theory of functions (the name ‘set’ is used by him to refer to what I call here *functio continua*), he looks at properties of functions, properties of properties, etc., and applies the identity principle to them. (I am unable to find a sense for many of his statements). (Weyl 1921, p.109)



[A]rithmetic and analysis merely contain general statements about numbers and freely developing sequences; there is no general theory of functions or sets of independent content! (Weyl 1921, p.110)

Now we turn to premise 5. In modern language, Weyl's semantics of  $\forall\alpha A(\alpha)$  is that  $A(a)$  holds for every lawlike  $a$ . The universal quantifier ranges over lawless sequences, but statements about individual sequences must be about lawlike ones. Let us repeat this quotation:

The proper judgments that can be gained from these universal judgments come into being [ . . . ] in the case of the freely developing choice sequence, by substituting for it a law  $\phi$  that determines an individual number sequence in infinitum. (Weyl 1921, p.100)

This semantics for universal quantification is the clearest sign that Weyl does not accept non-lawlike sequences as individuals.

However, given Weyl's stress on the concept of lawless sequence, and given his mentioning of constructing new sequences through operations on other sequences, one may conjecture that he had in mind something like the following. A plausible attempt to obtain a satisfying universe of choice sequences is to start by accepting the lawless sequences and then get further sequences by applying lawlike continuous operations to them. One might hope to get from this all the sequences one needs, without opening the gates to many different primitive types of sequence. But Troelstra (1977, ch.4) has shown that this attempt must fail, in so far as it allows counterexamples to basic principles such as weak continuity (see above).

So Weyl's effort to avail himself of the use of the concept of lawless sequence without allowing them into his mathematical ontology suffers from an incoherency (implicitly admitting sequences in between lawless and lawlike), a faulty premise (after all, lawless sequences cannot be stipulated to follow lawlike sequences), and two unwelcome consequences. The first is that it requires an unnatural (asymmetrical) interpretation of the quantifiers, according to which the universal quantifier ranges over lawless, but the existential over lawlike sequences. The second is that it forbids a general theory of functions. (We will see how Brouwer reacts to especially this second consequence.)

Moreover, Weyl's approach is subject to two phenomenological reservations.

First, that it should be an essential property of mathematical objects that they can be coded into natural numbers might be true, but in phenomenology one needs a constitution analysis to justify such a claim. Unless that is done, it is only a presupposition. In (van Atten 1999), an attempt is made to show that this presupposition is false, by bringing out evidence that non-lawlike sequences indeed are mathematical objects (and hence that Husserl's 'omnitemporality' condition is a false presupposition too).

Secondly, there is Husserl's anti-reductionism, applied to choice sequences: the discussion should be carried on in terms of the (counter)evidence we can get for these alleged objects. This application of anti-reductionism, an essential part of Husserl's transcendental idealism, to choice sequences can be connected

to Troelstra's (1983, pp.201–203) distinction of three approaches to the study of choice sequences:

**analytic** One starts with (a conceptual analysis of) the idea of an individual choice sequence of a certain type (say  $\tau$ ) and attempts to derive from the way such a choice sequence is supposed to be given to us (i.e. from the type of data available at any given moment of its generation) the principles which should hold for choice sequences of type  $\tau$ .

**holistic** One accepts (in one way or another) the universe of all intuitionistic (choice) sequences (or alternatively, 'quantification over choice sequences') as a single primitive intuition.

**figure of speech** This approach might be presented as an extreme variant of the holistic approach and regards all talk about choice sequences as a figure of speech, which expresses that 'quantifying over all sequences' means somehow more than just 'quantifying over all determinate sequences'

Troelstra states a preference for the analytic approach and proposes to take recourse to the other two only when it has demonstrably failed. He has this preference because the analytic approach enables one to study the role of intensional aspects in intuitionism. There are many different types of choice sequences, depending on the information we have about them, and this leads to the question what the effect is of varying types of intensional information. The analytic approach is the one that is sensitive to this variety.

Husserl's principled anti-reductionism may serve as a theoretical underpinning of this preference. But Weyl does not conduct his investigation according to Husserl's methodology, for although there is in Weyl's work a phenomenological analysis of the continuum, there is no corresponding analysis of choice sequence as objects.

Husserl, to whom Weyl sent an offprint of his 'Grundlagenkrise' paper, did not react explicitly to Weyl's theory. Brouwer did. It is important to see how Brouwer's view on choice sequences differs from Weyl's: for Brouwer they are individual objects, individuated by their moment of beginning, hence not omnitemporal. They cannot be given to us (extensionally) via a linguistic representation, but that doesn't bother him as he holds that mathematics is an activity independent from language to begin with. For Weyl, on the other hand, the impossibility of a finite numerical coding (and such coding is equivalent to the existence of a linguistic representation) rules out their status as mathematical objects. Brouwer writes, in an undated draft of a letter to Weyl: 'In your restrictions of the object of mathematics you are in fact much more radical than I am' (van Dalen 1995, p.148).

In that same draft, Brouwer comments:

It seems to me that the whole purpose of your paper is endangered by the end of the second paragraph of page 34 [(Weyl 1921, p.106)]. After you have roused the sleeper, he will say to himself: 'So the author admits that the real mathematical theorems are not affected

by his considerations? Then he should no longer disturb me!’ and turns away and sleeps on. Thereby you do our cause injustice, for with the existence theorem of the accumulation point of an infinite point set, many a classical existence theorem of a minimal function, as well as the existence theorems for geodesics without the second differentiability condition, loses its justification! (van Dalen 1995, p.149)

Brouwer does accept non-lawlike sequences as mathematical individuals. They are free constructions of the subject (the ‘second act of intuitionism’, mentioned above). So when Weyl writes ‘The concept of a sequence alternates, according to the logical connection in which it occurs, between “law” and “choice”, that is, between “Being” and “Becoming” ’ (Weyl 1921, p.109) Brouwer comments in the margin

for me ‘emerging sequence’ is neither one; one considers the sequences from the standpoint of a helpless spectator, who does not know at all in how far the completion has been free. (van Dalen 1995, p.160)

In other words, whether a sequence should be considered as an individual object does not depend on our knowledge of its being lawlike or not. Brouwer agrees with Weyl that the concept of law is extensionally indefinite, but he does not accept Weyl’s further premise that sequences can only be individuated by a law; they are always individuated by their moment of beginning. This amounts to allowing dynamical objects as individuals. Hence it makes sense to build a theory on basis of these sequences. They can be elements of a spread (Brouwer’s term is ‘Menge’):

A spread is a law on the basis of which, if again and again an arbitrary complex of digits [a natural number] of the sequence  $\zeta$  [the natural number sequence] is chosen, each of these choices either generates a definite symbol, or nothing, or brings about the inhibition of the process together with the definitive annihilation of its result; for every  $n$ , after every uninhibited sequence of  $n - 1$  choices, at least one complex of digits can be specified that, if chosen as  $n$ -th complex of digits, does not bring about the inhibition of the process. Every sequence of symbols generated from the spread in this manner (which therefore is generally not representable in finished form) is called an element of the spread. We shall also speak of the common mode of formation of the elements of a spread  $M$  as, for short, the spread  $M$ . (Brouwer 1918, pp.2–3)

In 1947, Brouwer adds the following clarification to this definition:

Because mathematics is independent of language, the word ‘symbol’ (‘Zeichen’) and in particular the words ‘complex of digits’ (‘Ziffernkomples’) must be understood in this definition in the sense of

‘mental symbols’, consisting in previously obtained mathematical concepts. (Brouwer 1947, original emphases)

And in the next sentence after the definition of a spread, he refers to these elements as ‘choice sequences’: ‘Wenn verschiedene Wahlfolgen [ . . . ]’. Later, Brouwer directed his attention from choice sequences as elements of a spread to the way individual choice sequences are given to us (Troelstra 1982).

Brouwer notes in his draft letter to Weyl that spreads can accommodate the three kinds of functions that Weyl describes. For the details we refer to (van Dalen 1995).

The second case of the functiones mixtae requires a law that assigns numbers to choice sequences. Here Weyl appeals to a continuity principle (for lawless sequences, now known as the ‘principle of open data’):

A law that, from a developing number sequence, generates a number  $n$  that is dependent on the outcome of the choices is necessarily such that the number  $n$  is fixed as soon as a certain finished segment of the choice sequence is present, and  $n$  remains the same however the choice sequence may further develop. (Weyl 1921, p.95)

And later he adds that for every mode of construction of functiones mixtae (‘Erzeugungsgesetz’), whatever it may be, the following ‘essential insight’ holds:

According to this law, there will always be a moment in the developing sequence where the sequence, regardless of how it develops, creates a number. This feature is all that is essential to the f.[unctiones] m.[ixtae]. (Weyl 1921, p.105)

Brouwer, in contrast to Weyl, realized that one needs other non-lawlike sequences than just the lawless, and formulated a continuity principle on the universal spread, of which all sequences, lawless, lawlike and all in between, are elements:

In 1918 Brouwer (p.13) formulates the following continuity principle for C:

A law that assigns to each element  $g$  of  $C$  [the universal spread] an element  $h$  of  $A$  [the natural numbers], must have determined the element  $h$  completely after a certain initial segment  $\alpha$  of the sequence of numbers of  $g$  has become known. But then to every element of  $C$  that has  $\alpha$  as an initial segment, the same element  $h$  of  $A$  will be assigned.

Brouwer neither then nor later gives a justification that this principle indeed holds for the universe of all choice sequences (see (van Atten and van Dalen forthcoming) for a justification) but he never questioned it, and it shows that Brouwer was aware that choice sequences come in a wide variety.

## 4 Conclusion

Brouwer and Weyl had closely related views on the existence and nature of the intuitive or ‘fluid’ continuum. There is a phenomenological datum here on which

they agreed. As we have indicated, they differed in some important respects in their mathematical and logical analyses of this phenomenon. In particular, we have to conclude that Weyl's method does not succeed in both saving the conceptual advantages of non-lawlike sequences and escaping ontological commitment to them. It would be rather unfair, however, to ask for today's insights in Weyl's pioneering paper. The surprising thing is not that Weyl's conception of choice sequence was awkward, but that Brouwer hit on the right notion right away.

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