Submodels of Kripke Models

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October 11, 1998

Abstract

A Kripke model $\mathcal{K}$ is a *submodel* of another Kripke model $\mathcal{M}$ if $\mathcal{K}$ is obtained by restricting the set of nodes of $\mathcal{M}$. In this paper we show that the class of formulas of Intuitionistic Predicate Logic that is preserved under taking submodels of Kripke models is precisely the class of semipositive formulas. This result is an analogue of the Łoś-Tarski theorem for the Classical Predicate Calculus.

In appendix A we prove that for theories with decidable identity we can take as the embeddings between domains in Kripke models of the theory, the identical embeddings. This is a well known fact, but we know of no correct proof in the literature. In appendix B we answer, negatively, a question posed by Sam Buss: whether there is a classical theory T, such that $\mathcal{H}T$ is HA. Here $\mathcal{H}T$ is the theory of all Kripke models $\mathcal{M}$ such that the structures assigned to the nodes of $\mathcal{M}$ all satisfy T in the sense of classical model theory.

Key words: Kripke models, Intuitionistic Logic, Heyting Arithmetic

MSC codes: Primary: 03B20, 03C40, 03F50, Secondary: 03C90

Contents

1 Introduction

What is a submodel of a given Kripke model for Intuitionistic Predicate Logic (IQC)? Several answers suggest themselves. First, we might consider models on the same frame, where the structures assigned to the nodes of the submodels are substructures, in the classical sense, of the structures assigned to the corresponding nodes of the original model. Secondly, we might take a submodel to be the result of simply restricting the frame of the given model and keeping the structures assigned to the remaining nodes the same. Thirdly, we might combine the strategies by both restricting the frame and considering substructures in the nodes. Moreover, these three possibilities allow all kinds of possible further variations. E.g. we might the replace *subframe* by frame that has a forward embedding into the given frame. (See [10], for an explanation of this notion for the case of propositional logic.)

In the present paper, we will consider only the second idea of a submodel: frame restriction. There are no strong reasons for this choice. Here are some weak ones. (i) The present notion is familiar to me from the literature. (ii) The other notions are as far as I know untested. (iii) The results obtained by considering the present notion look fairly natural.

From this point on submodel *will mean submodel obtained by erasing nodes.*

Now that we have settled upon a notion of submodel, the following question suggests itself. Which formulas are preserved under taking submodels of Kripke models in Intuitionistic Predicate Logic (IQC)? Thus, we ask for an analogue of the well known Łoś-Tarski theorem, which states that the formulas of Classical Predicate Logic (CQC) preserved under submodels (of classical models) are precicely the purely universal formulas (modulo CQC-provable equivalence). See [7] and [5] for the originale result. See [3], theorem 3.2.2 for the treatment in a well known textbook. An analogue for Intuitionistic *Propositional* Logic (IPC) was independently formulated and proved by Johan van Benthem and the author. See [10] and [9].

In the present paper we will show that, modulo IQC-provable equivalence, the formulas preserved under submodels are precisely *the semipositive formulas*. The semipositive formulas are the formulas one obtains by restricting, in the definition of the set of formulas, the formation rule for implication by the demand that the antecedent always be an atomic formula.

In a sense, both the result of the paper and the method of proof are unsurprising. Everything is precisely as one —after some experimentation— would expect. Here are some reasons why one would expect things to be as they, in fact, turn out to be. First, if one tries to provide the formulas preserved under submodels *following the inductive construction of formulas*,[1] then one arrives

[1] Admittedly, not all classes defined via a model theoretic property admit a characterization that follows the syntactic construction.

automatically at the semipositive formulas.[2] E.g. the proof of lemma 2.38 of [11] follows precisely this path. Secondly, we already have a characterization of the formulas of Intuitionistic Propositional Logic preserved under submodels. See [10] and [9]. The propositional formulas preserved under submodels are the NNIL-formulas, i.e. the formulas obtained by restricting the formation rule for implication by allowing only propositional variables in the antecedents. Clearly, the NNIL-formulas are a propositional analogue of the semipositive formulas. In fact they *are*, modulo provable equivalence, the semipositive formulas of the predicate logical language with only 0-ary predicates. Finally, the method of proof used in the present paper extends, in a straightforward way, the method of proof used in [10].

The interest of the present result lies primarily in the fact that it is an obvious first step in a closer investigation of Kripke models for IQC. A second source of interest is provided by the fact that the semipositive formulas play a role in some nice results on constructive theories. A first example is as follows. A self-completion of an arithmetical theory T is a theory T^*, such that, T-provably, $T^* = T + (\phi \to \Box_{T^*}\phi)$. In [8] it is shown that for many constructive arithmetical theories, such as Heyting's Arithmetic, T^* is conservative over T w.r.t. the semipositive formulas. A second example is Buss' characterization of the theories of so-called T-normal Kripke models, i.e. Kripke models whose nodes, viewed as classical models, satisfy the classical theory T. The semipositive formulas play an prominent role in this characterization. See [2]. A third example is Wolfgang Burr's study of analogues of the arithmetical hierarchy. See [1]. The second of these analogues is the Θ-hierarchy. If we formulate things in an appropriate way we can see that Θ_2 is equal to semipositive. Burr shows a.o. that $i\Theta_n$ (= intuitionistic arithmetic with Θ_n-induction) is Π_2-conservative over $I\Pi_n$ (= classical arithmetic with Π_n-induction).

In some cases we can work with a simpler kind of Kripke model. The proof of such a theorem involves transforming a given Kripke model into one of the simpler kind in such a way that what is forced is preserved. In appendix A we provide some results in this style. Specifically, we give a proof of the fact that, for theories with decidable identity we may restrict ourselves to considering Kripke models where the embeddings between the domains associated with the nodes are the identical embeddings of the subset relation. All the proofs in the literature of this elementary fact seem to be wrong.

In appendix B we show that *no* class of Kripke models can be both sound and complete for Heyting's Arithmetic HA and at the same time be closed under submodels. An immediate corrolary of this result is that, for no classical theory T, HA is the theory, $\mathcal{H}T$, of all T-normal[3] models.

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[2]Modulo some variations that disappear modulo provable equivalence, like allowing conjunctions, disjunctions and existential quantifiers in the antecedents of implications.

[3]A Kripke model is T-normal if the all the classical structures associated with its nodes are classical models of T.

2 Basic Notions and Elementary Facts

Consider a signature $\mathcal{L}$ for predicate logic. A *Kripke* model $\mathcal{K}$ of signature $\mathcal{L}$ is a pair $\langle \mathcal{P}, \Phi \rangle$, where $\mathcal{P} = \langle K, \preceq \rangle$ is a partial order. Φ is a functor from $\mathcal{P}$, considered as an order category[4], to the category of structures of signature $\mathcal{L}$ (in the sense of ordinary classical predicate logic) with as morphisms weakly structure preserving functions.[5] Let $k, k' \in K$ and suppose $k \preceq k'$. Let P be an n-ary predicate symbol of $\mathcal{L}$ and let F be an n-ary function symbol of $\mathcal{L}$. We will write D_k for $D_{\Phi(k)}$, the domain of the structure $\Phi(k)$; we write P_k for $I_{\Phi(k)}(P)$, the interpretation of P in $\Phi(k)$; we write F_k for $I_{\Phi(k)}(F)$, the interpretation of F in $\Phi(k)$; finally we write $\mathsf{e}_{k,k'}$ for $\Phi(\langle k, k' \rangle)$. The following persistence conditions are implied by the functoriality of Φ. Suppose $d_1, \ldots d_n \in D_k$, then:

- $P_k(d_1, \cdots, d_n) \Rightarrow P_{k'}(\mathsf{e}_{k,k'}(d_1), \cdots, \mathsf{e}_{k,k'}(d_n))$;
- $\mathsf{e}_{k,k'}(F_k(d_1, \cdots, d_n)) = F_{k'}(\mathsf{e}_{k,k'}(d_1), \cdots, \mathsf{e}_{k,k'}(d_n))$.

We fix some set of variables Var. We associate an ordinary predicate logical language to $\mathcal{L}$ and Var. Par abus de langage, we call this language $\mathcal{L}$. Our repertoire of connectives will be $\bot$, $\top$, $\wedge$, $\vee$, $\to$, $\forall$, $\exists$. As usual $\neg\phi$ is considered as an abbreviation of $(\phi \to \bot)$.

Consider a node k. An *assignment* f for k is a function from Var to D_k. The interpretation $[\![t]\!]_{f,k}$ of a term t in a node k is defined by the usual recursive clauses. We have, by an easy induction, $\mathsf{e}_{k,k'}([\![t]\!]_{f,k}) = [\![t]\!]_{\mathsf{e}_{k,k'} \circ f, k'}$. We define the forcing relation as follows.

- $k, f \Vdash P(t_1, \cdots, t_n) :\Leftrightarrow P_k([\![t_1]\!]_{f,k}, \cdots, [\![t_n]\!]_{f,k})$,
- $k, f \nVdash \bot$,
- $k, f \Vdash \top$,
- $k, f \Vdash \phi \wedge \psi :\Leftrightarrow k, f \Vdash \phi$ and $k, f \Vdash \psi$,
- $k, f \Vdash \phi \vee \psi :\Leftrightarrow k, f \Vdash \phi$ or $k, f \Vdash \psi$,
- $k, f \Vdash \phi \to \psi :\Leftrightarrow \forall k' \succeq k\ (k', \mathsf{e}_{k,k'} \circ f \Vdash \phi \Rightarrow k', \mathsf{e}_{k,k'} \circ f \Vdash \psi)$,
- $k, f \Vdash \forall v\, \phi :\Leftrightarrow \forall k' \succeq k\, \forall d' \in D_{k'}\ k', (\mathsf{e}_{k,k'} \circ f)[v := d'] \Vdash \phi$
(here '$f'[v := d']$' means: the result of resetting f' at v to d'),
- $k, f \Vdash \exists v\, \phi :\Leftrightarrow \exists d \in D_k\ \ k, f[v := d] \Vdash \phi$.

By an easy induction one proves persistence: for $k \preceq k'$, we have $k, f \Vdash \phi \Rightarrow k', \mathsf{e}_{k,k'} \circ f \Vdash \phi$. Whether a sentence is forced is independent of the chosen assigment. Thus we will often omitt the assignment when stating that a sentence is forced.

[4]Everything would work if we took $\mathcal{P}$ to be an arbitrary category. However there seems to be no point to the added generality.

[5]Weak preservation means that e.g. $\langle a, b \rangle \in I_{\mathcal{M}}(P) \Rightarrow \langle f(a), f(b) \rangle \in I_{\mathcal{N}}(P)$, but that the converse need not necessarily hold.

A Kripke model $\mathcal{K}$ is a *submodel* of a Kripke model $\mathcal{M}$ or if $\mathcal{M} = \langle\langle M, \preceq\rangle, \Phi\rangle$, $\mathcal{K} = \langle\langle K, \preceq'\rangle, \Phi'\rangle$, $K \subseteq M$ and $\preceq' = \preceq\restriction K$, $\Phi' = \Phi \restriction (K\cup \preceq')$. In other words, a submodel is the restriction of the given model *qua functor* to a full subcategory. We will write $\mathcal{K} = \mathcal{M} \restriction K$. We define $\mathcal{M}[k]$ by $\mathcal{M} \restriction (\uparrow k)$, where $\uparrow k := \{m \in M \mid k \preceq m\}$.

We take theories to be sets of sentences closed under IQC-derivable consequence. Let T be an $\mathsf{IQC}_{\mathcal{L}}$-theory. A model is a T-model if it forces the axioms of T at all nodes. We say that a formula ϕ of $\mathcal{L}$ is preserved under T-submodels if, for any Kripke T-models $\mathcal{K}$ and $\mathcal{M}$ such that $\mathcal{K}$ is a submodel of $\mathcal{M}$, and for any node k of $\mathcal{K}$ and for any assignment f for k, we have: $k, f \Vdash_{\mathcal{M}} \phi \Rightarrow k, f \Vdash_{\mathcal{K}} \phi$. We define preservation of a theory U under T-submodels in a similar way.

The version of Kripke models that we defined above is, we think, the correct notion. It is, however, as in ordinary model theory, often more convenient to work with a slightly different format. In the present format the interpretation of the identity sign at a node, i.o.w. $I_{\Phi(k)}(=)$, is true identity. This blocks, generally, the possibility to take the $\Phi(\langle k, k'\rangle)$ to be identical embeddings associated with the subset relation. The reason is that domain elements that differ at a given node may be identified at a higher node. In the other format we allow identity to be interpreted as an equivalence relation (which is a congruence w.r.t. the other operations and relations). We will call models employing equivalence relations and identical embeddings: *eq-models*.

We can always transform, preserving what is forced in a given node, an eq-model to a model in our earlier format by dividing out the equivalence relations locally. Conversely, we can transform a model of the original format to an eq-model. The proof is given given in appendix A. In case identity is decidable in a model we can even do better. We can transform the given model into a model with true identity in the nodes *and* with standard embeddings between the nodes. This transformation preserves what is forced in a precise sense explained in appendix A. The construction is slightly more involved: we are forced, in general, to change the structure of the nodes. Since all the proofs in the literature that I am aware of are fallacious, I give the correct —I hope— proof in appendix A.

We briefly describe the Henkin construction. Let C be any set of constants which are not part of the signature $\mathcal{L}$. $\mathcal{L}(C)$ is the signature we obtain by extending $\mathcal{L}$ with C. We say that a set Γ of $\mathcal{L}(C)$-sentences is C-saturated if:

- For any finite, possibly empty, set of $\mathcal{L}(C)$-sentences X, if $\Gamma \vdash \bigvee X$, then, for some $\phi \in X$, we have $\phi \in \Gamma$;
  (we take $\bigvee \emptyset := \bot$, so saturated sets are consistent);
- If $\Gamma \vdash \exists x\, \phi$, then, for some $c \in C$, $\phi[x := c] \in \Gamma$.

The Henkin model $\mathcal{H}$ is specified as an eq-model as follows.

- The nodes are the sets that are pairs $\langle C, \Gamma\rangle$, where C is countable and where Γ is C-saturated;
- The ordering is the component-wise subset ordering;

- $D_{\langle C,\Gamma\rangle} := C$;
- $P_{\langle C,\Gamma\rangle}(c_1, \cdots, c_n) :\Leftrightarrow P(c_1, \cdots, c_n) \in \Gamma$;
- $F_{\langle C,\Gamma\rangle}(c_1, \cdots, c_n) = c :\Leftrightarrow (F(c_1, \cdots, c_n) = c) \in \Gamma$.

One can show $\langle C, \Gamma\rangle, f \Vdash \phi \Leftrightarrow \sigma_f(\phi) \in \Gamma$. Here σ_f is the substitution that sends v to $f(v)$. The Henkin construction as described here delivers a model of class size. However, with some care, e.g. chosing the C's to be co-infinite subsets of a fixed countable set C^*, we can get it down to the power of the continuum. With some more care we can even get both the set of nodes and the theories in the nodes countable.

The set of atomic formulas of $\mathcal{L}$ is denoted by '$\mathcal{A}$'. The atomic formulas of $\mathcal{L}(C)$ are denoted by '$\mathcal{A}(C)$'. The set $\mathcal{S}$ of semipositive formulas of signature $\mathcal{L}$ is the smallest set given by the usual inductive clauses for the language except that in the formation rule for implication we restrict the antecedents to atomic formulas. The semipositive formulas of signature $\mathcal{L}(C)$ are denoted by '$\mathcal{S}(C)$'.

3 The Main Theorem

Fix a signature $\mathcal{L}$. Let T and U be theories of signature $\mathcal{L}$ and suppose that $T \subseteq U$. Suppose that U is preserved under taking T-submodels.

Theorem 3.1 *Under the conditions described above, we have: U can be axiomatized by $\mathcal{S}$-sentences over T.*

Before we give the proof of the main theorem, we draw an immediate consequence.

Corollary 3.2 Suppose ϕ is any formula of $\mathcal{L}$ and suppose that ϕ is preserved under taking T-submodels. Then, ϕ is T-provably equivalent to an $\mathcal{S}$-formula. ❑

Here is the proof of corollary 3.2. We will write $[W]_C$ for the IQC-derivable closure of the set of sentences W in $\mathcal{L}(C)$.

Proof

Let σ be a substitution that maps the free variables of ϕ injectively to fresh constants from among the finite set $\vec{c}$. Let $U := [T + \sigma(\phi)]_{\vec{c}}$. It is not difficult to see that the conditions of the theorem are fulfilled by $[T]_{\vec{c}}$ and U. By the theorem in combination with compactness, we find that $\sigma(\phi)$ is T-provably equivalent to an $\mathcal{S}(\vec{c})$-formula. Since $\vec{c}$ does not occur in T we find that ϕ is T-provably equivalent to a $\mathcal{S}$-formula. ❑

We proceed with the proof of the theorem. To obtain a contradiction, we assume that U cannot be semipositively axiomatized over T. Our proof strategy is direct. We construct two models, $\mathcal{H}_2$ with forcing relation $\Vdash$ and $\mathcal{H}_2^{\mathsf{f}}$ with forcing relation $\Vdash^{\mathsf{f}}$. Here $\mathcal{H}_2^{\mathsf{f}}$ is a submodel of $\mathcal{H}_2$. We will produce a node $\mathfrak{r}$ of $\mathcal{H}_2^{\mathsf{f}}$ such that $\mathfrak{r} \Vdash^{\mathsf{f}} T$, $\mathfrak{r} \nVdash^{\mathsf{f}} U$, $\mathfrak{r} \Vdash U$. It follows that $\mathcal{H}_2[\mathfrak{r}]$ and $\mathcal{H}_2^{\mathsf{f}}[\mathfrak{r}]$ are T-models, that $\mathcal{H}_2^{\mathsf{f}}[\mathfrak{r}]$ is a submodel of $\mathcal{H}_2[\mathfrak{r}]$, and that $\mathfrak{r}$ forces U in $\mathcal{H}_2[\mathfrak{r}]$ but not in $\mathcal{H}_2^{\mathsf{f}}[\mathfrak{r}]$. Thus we obtain the desired contradiction. We first describe $\mathcal{H}_2$.

Definition 3.3 Consider a triple $\mathfrak{x} := \langle X, C, Y\rangle$. We say that $\mathfrak{x}$ is *acceptable* if C is a countable set of constants and X and Y are $\mathsf{IQC}(\mathcal{L}(C))$-deductively closed sets of sentences of $\mathcal{L}(C)$. We put:

$$\langle X, C, Y\rangle \leq \langle X', C', Y'\rangle :\Leftrightarrow X \subseteq X',\ C \subseteq C' \text{ and } Y \subseteq Y'.$$

An acceptable triple $\mathfrak{k} := \langle \Gamma, C, \Delta\rangle$ is an *h2-node* if Γ and Δ are C-saturated. ❑

We specify $\mathcal{H}_2$ as an eq-model.

- The nodes of $\mathcal{H}_2$ are the h2-nodes.
- The ordering relation is $\leq$.
- The domain $D_{\mathfrak{x}}$ associated to $\mathfrak{x} = \langle \Gamma, C, \Delta\rangle$ is C.
- For $\mathfrak{x} = \langle \Gamma, C, \Delta\rangle$, we put $P_{\mathfrak{x}}(c_1, \cdots, c_n) :\Leftrightarrow P(c_1, \cdots, c_n) \in \Delta$.
- For $\mathfrak{x} = \langle \Gamma, C, \Delta\rangle$, we put $F_{\mathfrak{x}}(c_1, \cdots, c_n) = c :\Leftrightarrow (F(c_1, \cdots, c_n) = c) \in \Delta$.

It is easy to see that $\mathcal{H}_2$ is indeed an eq-model.

Theorem 3.4 *Let $\mathfrak{b} = \langle \Gamma, C, \Delta\rangle$ and let f be an $\mathfrak{b}$-assignment. For $\psi \in \mathcal{L}$, we have: $\mathfrak{b}, f \Vdash \psi \Leftrightarrow \sigma_f(\psi) \in \Delta$. (Remember that σ_f is the substitution corresponding to f.)*

Proof

Our model is just a blown-up version of the standard Henkin model $\mathcal{H}$, since the Γ's really do no work. They are just there to make room for $\mathcal{H}_2^{\mathsf{f}}$ as a submodel. So the proof is just a trivial variation of the usual proof of the Henkin Theorem. ❑

We proceed to construct $\mathcal{H}_2^{\mathsf{f}}$. We write $\mathsf{Th}_W(Z)$ for the set of consequences of Z that are in W.

Definition 3.5 We say that $\mathfrak{x}$ is *f-acceptable*[6] in case it is acceptable and (f1) $\mathsf{Th}_{\mathcal{A}(C)}(X) \subseteq Y$ and (f2) $\mathsf{Th}_{\mathcal{S}(C)}(Y) \subseteq X$. ❑

[6]The 'f' refers to the connection of this definition with the *forward property*. See [10].

Note that f-acceptability implies that $\mathsf{Th}_{\mathcal{A}(C)}(X) = \mathsf{Th}_{\mathcal{A}(C)}(Y)$.

$\mathcal{H}_2^{\mathsf{f}}$ is obtained by restricting $\mathcal{H}_2$ to the f-acceptable h2-nodes. We show that the forcing in $\mathcal{H}_2^{\mathsf{f}}$ 'listens' to the first components of our triples.

Theorem 3.6 *Let* $\mathfrak{b} = \langle \Gamma, C, \Delta \rangle$ *and let* f *be an* $\mathfrak{b}$*-assignment. For* $\psi \in \mathcal{L}$, $\mathfrak{b}, f \Vdash^{\mathsf{f}} \psi \Leftrightarrow \sigma_f(\psi) \in \Gamma$.

To prove our theorem we need the following two lemmas.

Lemma 3.7 Let $\mathfrak{u} := \langle U, F, V \rangle$ be an acceptable triple. Suppose $\mathfrak{u}$ satisfies clause (f2). Let $V' := [V \cup \mathsf{Th}_{\mathcal{A}(F)}(U)]_F$. Then, $\mathfrak{u}' := \langle U, F, V' \rangle$ is f-acceptable. ❑

Lemma 3.8 [Saturation Lemma] Suppose $\mathfrak{x} := \langle X, C, Y \rangle$ is f-acceptable and $X \nvdash \psi$. Then there is an f-acceptable h2-node $\mathfrak{g} := \langle \Gamma, D, \Delta \rangle$ such that $\mathfrak{x} \leq \mathfrak{g}$ and $\Gamma \nvdash \psi$. ❑

Here is the proof of lemma 3.7.

Proof

(Under the assumptions of the lemma.) $\mathfrak{u}'$ is clearly acceptable. Moreover it is immediate that it satisfies clause (f1). We verify clause (f2). Suppose $\chi \in V' \cap \mathcal{S}(F)$. Then, for some finite conjunction β of α's in $\mathcal{A}(F)$ such that $U \vdash_F \alpha$, we have $V \vdash_F \beta \to \chi$. Since, modulo $\mathsf{IQC}(F)$-provable equivalence, $(\beta \to \chi)$ is again in $\mathcal{S}(F)$, we have, applying (f2) for $\mathfrak{u}$, $U \vdash_F \beta \to \chi$. Moreover, $U \vdash_F \beta$, so $U \vdash_F \chi$. ❑

We postpone the laborious proof of the saturation lemma 3.8 to the next section. Here is the proof of theorem 3.6.

Proof

We have to show that, for any f-acceptable h2-node $\mathfrak{b} = \langle \Gamma, C, \Delta \rangle$ and $\mathfrak{b}$-assignment f, we have $\mathfrak{b}, f \Vdash^{\mathsf{f}} \psi \Leftrightarrow \sigma_f(\psi) \in \Gamma$. We prove this by induction on ψ in $\mathcal{L}$. Here are the only two interesting cases.

Suppose that $\psi = (\nu \to \rho)$ and $\sigma_f(\psi) \notin \Gamma$. Let $\nu' := \sigma_f(\nu)$ and $\rho' := \sigma_f(\rho)$. We have $\Gamma, \nu' \nvdash_C \rho'$. Clearly, $\langle [\Gamma \cup \{\nu'\}]_C, C, \Delta \rangle$ satisfies property f2. Hence by lemma 3.7,

$$\mathfrak{c} := \langle [\Gamma \cup \{\nu'\}]_C, C, [\Delta \cup \mathsf{Th}_{\mathcal{A}(C)}(\Gamma \cup \{\nu'\})]_C \rangle$$

is f-acceptable. By lemma 3.8, we can extend $\mathfrak{c}$ to an f-acceptable h2-node $\mathfrak{d} := \langle \Gamma', C', \Delta' \rangle$ such that $\Gamma' \nvdash_{C'} \rho'$. By the induction hypothesis we will have $\mathfrak{d}, f \Vdash^{\mathsf{f}} \nu$ and $\mathfrak{d}, f \nVdash^{\mathsf{f}} \rho$. Ergo $\mathfrak{b}, f \nVdash \psi$.

Suppose that $\psi = \forall x\, \rho$ and $\sigma_f(\psi) \notin \Gamma$. Let d be a fresh constant. $C' := C \cup \{d\}$. Let ρ' be the result of substituting d for x in ρ and subsequently applying

σ_f. (In other words: $\rho' := \sigma_{f[x:=d]}(\rho)$.) We have $\Gamma \nvdash_{C'} \rho'$. We verify that $\langle[\Gamma]_{C'}, C', [\Delta]_{C'}\rangle$ satisfies property f2. Suppose $\nu(\vec{c}, d) \in \mathcal{S}(C')$ and $\Delta \vdash_{C'} \nu(\vec{c}, d)$. Then, $\Delta \vdash_C \forall z\, \nu(\vec{c}, z)$. Here z is some variable not occurring in $\nu(\vec{c}, d)$. Since $(\forall z\, \nu(\vec{c}, z)) \in \mathcal{S}(C)$ and since $\mathfrak{b}$ is f-acceptable, we find: $\Gamma \vdash_C \forall z\, \nu(\vec{c}, z)$. Ergo: $\Gamma \vdash_{C'} \nu(\vec{c}, d)$.

By lemma 3.7, $\mathfrak{c} := \langle[\Gamma]_{C'}, C', [\Delta \cup \mathsf{Th}_{\mathcal{A}(C')}(\Gamma)]_{C'}\rangle$ is f-acceptable. By lemma 3.8, We can extend $\mathfrak{c}$ to an f-acceptable h2-node $\mathfrak{d} := \langle\Gamma'', C'', \Delta''\rangle$ such that $\Gamma'' \nvdash_{C''} \rho'$. By the induction hypothesis we will have $\mathfrak{d}, f[x := d] \nVdash^{\mathsf{f}} \rho$. Ergo $\mathfrak{b}, f \nVdash \psi$. ❑

Finally we construct an f-acceptable h2-node $\mathfrak{r} := \langle\Gamma_0, C_0, \Delta_0\rangle$ such that $T \subseteq \Gamma_0$, $U \nsubseteq \Gamma_0$, $U \subseteq \Delta_0$.

Consider the triple $\mathfrak{a} := \langle[T \cup \mathsf{Th}_{\mathcal{S}}(U)], \emptyset, U\rangle$. It is clear that our triple is f-acceptable. Moreover, since U is supposed to be not $\mathcal{S}$-axiomatizable over T, there is a ψ such that $[T \cup \mathsf{Th}_{\mathcal{S}}(U)] \nvdash \psi$ and $U \vdash \psi$. By lemma 3.8, we can find an f-acceptable h2-node $\mathfrak{r} := \langle\Gamma_0, C_0, \Delta_0\rangle$ extending $\mathfrak{a}$, such that $\Gamma_0 \nvdash \psi$. We obviously have: $T \subseteq \Gamma_0$, $U \nsubseteq \Gamma_0$, $U \subseteq \Delta_0$.

Thus we obtain our desired contradiction!

4 A Saturation Lemma

In this section we will prove the basic saturation result. Let $\mathcal{L}$, T, U be as in the preceding section. Suppose $\mathfrak{x} := \langle X, C, Y\rangle$ is f-acceptable and $X \nvdash \psi$. We will produce is an f-acceptable h2-node $\mathfrak{g} := \langle\Gamma, D, \Delta\rangle$ such that $\mathfrak{x} \leq \mathfrak{g}$ and $\Gamma \nvdash \psi$.

We construct a $\leq$-ascending sequence of f-acceptable triples such that $\mathfrak{x}_0 = \mathfrak{x}$ and $\mathfrak{x}_n := \langle X_n, C_n, Y_n\rangle$ with $X_n \nvdash \psi$. We define $\Gamma := \bigcup_n X_n$, $D := \bigcup_n C_n$ and $\Delta := \bigcup_n Y_n$. Let $E := \{e_0, e_1, \ldots\}$ be a set of fresh constants. C_n will be of the form $C \cup \{e_0, \ldots, e_{k_n - 1}\}$. Here k_n is defined simultaneously with $\mathfrak{x}_n$.

We take $D := C \cup E$. Let $\theta_0, \theta_1, \ldots$ be an enumeration with infinite repetitions of the sentences of $\mathcal{L}(D)$. The construction of the $\mathfrak{x}_n$ is as follows.

step 0
$\mathfrak{x}_0 := \langle X, C, Y\rangle$, $k_0 := 0$.

step 2n+1
We assume that $\mathfrak{x}_{2n}$ has the desired properties.

In case θ_n is not in $\mathcal{L}(C_{2n})$ or in case θ_n is not a disjunction or an existentially quantified formula or in case $X_{2n} \nvdash_{C_{2n}} \theta_n$, we take: $\mathfrak{x}_{2n+1} := \mathfrak{x}_{2n}$, $k_{2n+1} := k_{2n}$.

Suppose θ_n is in $\mathcal{L}(C_{2n})$, $\theta_n = (\chi_0 \vee \chi_1)$ and $X_{2n} \vdash_{C_{2n}} \theta_n$. Define:

- $X^{(i)}_{2n} := [X_{2n} \cup \{\chi_i\}]_{C_{2n}}$,
- $Y^{(i)}_{2n} := [Y_{2n} \cup \mathsf{Th}_{\mathcal{A}(C_{2n})}(X^{(i)}_{2n})]_{C_{2n}}$,

- $\mathfrak{x}_{2n}^{(i)} := \langle X_{2n}^{(i)}, C_{2n}, Y_{2n}^{(i)} \rangle$.

Then, by lemma 4.1 below, for some $i \in \{0,1\}$, $\mathfrak{x}_{2n}^{(i)}$ is f-acceptable and $X_{2n}^{(i)} \nvdash_{C_{2n}} \psi$. We take $\mathfrak{x}_{2n+1} := \mathfrak{x}_{2n}^{(i)}$, for the smallest such i, and we set $k_{2n+1} := k_{2n}$.

Suppose that θ_n is in $\mathcal{L}(C_{2n})$, $\theta_n = \exists x\, \chi$ and $X_{2n} \vdash_{C_{2n}} \theta_n$. We define: $e := e_{k_{2n}}$. Clearly, e is a fresh variable. We define:

- $C_{2n+1} := C_{2n} \cup \{e\}$,
- $X_{2n+1} := [X_{2n} \cup \{\chi(e)\}]_{C_{2n+1}}$,
- $Y_{2n+1} := [Y_{2n} \cup \mathsf{Th}_{\mathcal{A}(C_{2n+1})}(X_{2n+1})]_{C_{2n+1}}$,
- $\mathfrak{x}_{2n+1} := \langle X_{2n+1}, C_{2n+1}, Y_{2n+1} \rangle$.

By lemma 4.3 below, $\mathfrak{x}_{2n+1}$ is f-acceptable and $X_{2n+1} \nvdash_{C_{2n+1}} \psi$. We take $k_{2n+1} := k_{2n} + 1$.

Step 2n+2
We assume that $\mathfrak{x}_{2n+1}$ has the desired properties.

In case θ_n is not in $\mathcal{L}(C_{2n+1})$ or in case θ_n is not a disjunction or an existentially quantified formula or in case $Y_{2n+1} \nvdash_{C_{2n+1}} \theta_n$, we take: $\mathfrak{x}_{2n+2} := \mathfrak{x}_{2n+1}$, $k_{2n+2} := k_{2n+1}$.

Suppose that θ_n is in $\mathcal{L}(C_{2n+1})$, $\theta_n = (\chi_0 \vee \chi_1)$ and $Y_{2n+1} \vdash_{C_{2n+1}} \theta_n$. We define, for $i \in \{0,1\}$:

- $X_{2n+1}^{(i)} := [X_{2n+1} \cup \mathsf{Th}_{\mathcal{S}(C_{2n+1})}(Y_{2n+1} \cup \{\chi_i\})]_{C_{2n+1}}$,
- $Y_{2n+1}^{(i)} := [Y_{2n+1} \cup \{\chi_i\} \cup \mathsf{Th}_{\mathcal{A}(C_{2n+1})}(X_{2n+1}^{(i)})]_{C_{2n+1}}$,
- $\mathfrak{x}_{2n+1}^{(i)} := \langle X_{2n+1}^{(i)}, C_{2n+1}, Y_{2n+1}^{(i)} \rangle$.

By lemma 4.2, for some $i \in \{0,1\}$, $\mathfrak{x}_{2n+1}^{(i)}$ is f-acceptable and $X_{2n+1}^{(i)} \nvdash_{C_{2n+1}} \psi$. We take $\mathfrak{x}_{2n+2} := \mathfrak{x}_{2n+1}^{(i)}$, for the smallest such i and we set $k_{2n+1} := k_{2n}$.

Suppose that θ_n is in $\mathcal{L}(C_{2n+1})$, $\theta_n = \exists x\, \chi$ and $Y_{2n+1} \vdash \theta_n$. We define: $e := e_{k_{2n+1}}$. Clearly, e is a fresh variable. We define:

- $C_{2n+2} := C_{2n+1} \cup \{e\}$,
- $X_{2n+2} := [X_{2n+1} \cup \mathsf{Th}_{\mathcal{S}(C_{2n+2})}(Y_{2n+1} \cup \{\chi(e)\})]_{C_{2n+2}}$,
- $Y_{2n+2} := [Y \cup \{\chi(e)\} \cup \mathsf{Th}_{\mathcal{A}(C_{2n+2})}(X_{2n+2})]_{C'}$,
- $\mathfrak{x}_{2n+2} := \langle X_{2n+2}, C_{2n+2}, Y_{2n+2} \rangle$.

Then, by lemma 4.4, $\mathfrak{x}_{2n+2}$ is f-acceptable and $X_{2n+2} \nvdash_{C_{2n+2}} \psi$. Finally, we put $k_{2n+2} := k_{2n+1} + 1$.

Let $\Gamma := \bigcup_n X_n$ and $\Delta := \bigcup_n Y_n$. It is easy to see that $\langle \Gamma, D, \Delta \rangle$ has the desired properties.

In the rest of this section we supply the lemmas necessary for verifying the details of the construction described above.

Lemma 4.1 Suppose that $\langle U, F, V \rangle$ is f-acceptable and $U \nvdash \psi$. Suppose further that $U \vdash_F \chi_0 \vee \chi_1$. Let

- $U^{(i)} := [U \cup \{\chi_i\}]_F$,
- $V^{(i)} := [V \cup \mathsf{Th}_{\mathcal{A}(F)}(U^{(i)})]_F$.

Then, for some $i \in \{0, 1\}$, $\langle U^{(i)}, F, V^{(i)} \rangle$ is f-acceptable and $U^{(i)} \nvdash_F \psi$. ❑

Proof

(Under the assumptions of the lemma.) By the usual argument, we can find an i such that $U, \chi_i \nvdash_F \psi$. Clearly property (f2) is not lost by adding χ_i to U. So $\langle U^{(i)}, F, V \rangle$ satisfies (f2). By lemma 3.7 addition of the atoms implied by $U^{(i)}$ to V preserves (f2) and guarantees (f1). Thus, $\langle U^{(i)}, F, V^{(i)} \rangle$ is f-acceptable. ❑

Lemma 4.2 Suppose that $\langle U, F, V \rangle$ is f-acceptable and $U \nvdash \psi$. Suppose further that $V \vdash_F \chi_0 \vee \chi_1$. Let,

- $U^{(i)} := [U \cup \mathsf{Th}_{\mathcal{S}(F)}(V \cup \{\chi_i\})]_F$,
- $V^{(i)} := [V \cup \{\chi_i\} \cup \mathsf{Th}_{\mathcal{A}(F)}(U^{(i)})]_F$

Then, for some $i \in \{0, 1\}$, $\langle U^{(i)}, F, V^{(i)} \rangle$ is f-acceptable and $U^{(i)} \nvdash_F \psi$. ❑

Proof

(Under the assumptions of the lemma.) Suppose $U^{(i)} \vdash_F \psi$ for $i = 0, 1$. Then, there are $\tau_0, \tau_1 \in \mathcal{S}(F)$ such that $V, \chi_i \vdash_F \tau_i$ and $U, \tau_i \vdash_F \psi$ for $i = 0, 1$. It follows that $V \vdash_F \tau_0 \vee \tau_1$. Since $(\tau_0 \vee \tau_1) \in \mathcal{S}(F)$, we find that $U \vdash_F \tau_0 \vee \tau_1$. But then $U \vdash_F \psi$. Quod non.

Pick i such that $U^{(i)} \nvdash_F \psi$. It is clear that $\langle U^{(i)}, F, [V \cup \{\chi_i\}]_F \rangle$ satisfies (f2). So, by lemma 3.7, $\langle U^{(i)}, F, V^{(i)} \rangle$ is f-acceptable. ❑

Lemma 4.3 Suppose that $\langle U, F, V\rangle$ is f-acceptable and $U \nvdash \psi$. Suppose further that $U \vdash_F \exists x\, \chi(x)$. Let e be a new constant occurring neither in $\mathcal{L}$ nor in F. Define:

- $F' := F \cup \{e\}$,
- $U' := [U \cup \{\chi(e)\}]_{F'}$,
- $V' := [V \cup \mathsf{Th}_{\mathcal{A}(F')}(U')]_{F'}$.

Then, $\langle U', F', V'\rangle$ is f-acceptable and $U' \nvdash_{F'} \psi$. ❑

Proof

(Under the assumptions of the lemma.) By the usual argument, $U, \chi(e) \nvdash_{F'} \psi$. Suppose $V \vdash_{F'} \rho(\vec{f}, e)$, for $\rho \in \mathcal{S}(F')$. Since e does not occur in V, it follows that $V \vdash_F \forall z\, \rho(\vec{f}, z)$, (for suitably chosen z). Now $\forall z\, \rho(\vec{f}, z)$ is in $\mathcal{S}(F)$, so $U \vdash_F \forall z\, \rho(\vec{f}, z)$. Ergo, $U \vdash_{F'} \rho(\vec{f}, e)$. It follows that $\langle U', F', [V]_{F'}\rangle$ satisfies (f2). By lemma 3.7 addition of the atoms implied by U' to $[V]_{F'}$ preserves (f2) and guarantees (f1). Thus, $\langle U', F', V'\rangle$ is f-acceptable. ❑

Lemma 4.4 Suppose that $\langle U, F, V\rangle$ is f-acceptable and $U \nvdash \psi$. Suppose further that $V \vdash_F \exists x\, \chi(x)$. Let e be a new constant occurring neither in $\mathcal{L}$ nor in F. Define:

- $F' := F \cup \{e\}$,
- $U' := [U \cup \mathsf{Th}_{\mathcal{S}(F')}(V \cup \{\chi(e)\})]_{F'}$,
- $V' := [V \cup \{\chi(e)\} \cup \mathsf{Th}_{\mathcal{A}(F')}(U')]_{F'}$

Then, $\langle U', F', V'\rangle$ is f-acceptable and $U' \nvdash_{F'} \psi$. ❑

Proof

(Under the assumptions of the lemma.) Suppose that $U' \vdash_{F'} \psi$. Then, there is a $\tau(\vec{f}, e) \in \mathcal{S}(F')$ such that $V, \chi(e) \vdash_{F'} \tau(\vec{f}, e)$ and $U, \tau(\vec{f}, e) \vdash_{F'} \psi$. It follows that $V \vdash_F \exists z\, \tau(\vec{f}, z)$. Since $(\exists z\, \tau(\vec{f}, z)) \in \mathcal{S}(F)$, it follows that $U \vdash_F \exists z\, \tau(\vec{f}, z)$. But then $U \vdash_F \psi$. Quod non.

It is clear that $\langle U', F', [V \cup \{\chi(e)\}]_{F'}\rangle$ satisfies (f2). So, by lemma 3.7, $\langle U', F', V'\rangle$ is f-acceptable. ❑

5 Some Consequences

5.1 Heyting's Arithmetic

We show that HA is not preserved under taking submodels. If it were so preserved, then, by theorem 3.1, it would be $\mathcal{S}$-axiomatizable. We show that this cannot be the case. Suppose that HA is $\mathcal{S}$-axiomatizable. We write ϕ^{r} for $\exists x\, x\,\mathbf{r}\,\phi$, where $\mathbf{r}$ stands for Kleene realizability. $i\Sigma_1$ is intuitionistic arithmetic with Σ_1-induction. Remember that $i\Sigma_1$ is finitely axiomatizable. Consider the theory $T := i\Sigma_1 + A^{*,\mathsf{r}}$, where A^* is the statement $\forall \phi{\in}\Sigma_3\,(\Box_{\mathsf{HA}}\phi \to \mathsf{True}_{\Sigma_3}\phi)$. We will show that T is both consistent and inconsistent, thus arriving at a contradiction.

First note that A^* follows from $\mathsf{HA}+\mathsf{RFN}(\mathsf{HA})$. Here $\mathsf{RFN}(\mathsf{HA})$ is the uniform reflection principle for HA. Since $\mathsf{HA}+\mathsf{RFN}(\mathsf{HA})$ proves its own realizability, it follows that $\mathsf{HA}+\mathsf{RFN}(\mathsf{HA}) \vdash A^{*,\mathsf{r}}$. So, a fortiori, T is consistent. We sketch the proof of the inconsistency of T. We refer the reader to [4] for the basic facts on realizability employed in the argument.

a) For any $\phi \in \mathcal{S}$, there is a ψ in classical prenex normal form, such that ψ is $i\Sigma_1$-provably equivalent to ϕ. The proof is by induction of ϕ. Here is a sample of a reduction. $(\chi \vee \forall x\,\rho)$, with x not free in χ, is provably equivalent to $\exists y\,\forall x\,(\,(y=0\to\chi)\wedge(y\neq 0\to\rho)\,)$. (Note that the usual reduction in classical predicate logic is constructively invalid.)

b) For any ψ in classical prenex normal form, there is a τ in Σ_3 such that $i\Sigma_1 + \mathsf{ECT}_0 \vdash \psi \leftrightarrow \tau$. This may be proved by reducing $\forall\exists$ using the existence of a recursive choice function.

c) HA is conservative over $\mathsf{HA}+\mathsf{ECT}_0$ w.r.t. formulas in prenex normal form. Here is the argument. One may prove, by a simple induction that, for ψ in prenex normal form, $i\Sigma_1 \vdash \psi^{\mathsf{r}} \to \psi$. Suppose that, for ψ in prenex normal form, $\mathsf{HA}+\mathsf{ECT}_0 \vdash \psi$. Then $\mathsf{HA} \vdash \psi^{\mathsf{r}}$, and, hence, $\mathsf{HA} \vdash \psi$.

d) Since $i\Sigma_1 + \mathsf{ECT}_0$ proves the equivalence of every ϕ with ϕ^{r}, we have: $T + \mathsf{ECT}_0 \vdash A^*$.

e) Consider any HA-axiom α. By our assumption that HA is $\mathcal{S}$-axiomatizable, it follows that there is a $\sigma \in \mathcal{S}$, such that $\mathsf{HA} \vdash \sigma$ and $\sigma \vdash_{\mathsf{IQC}} \alpha$ (†). By (a) and (b), there is a $\tau \in \Sigma_3$, such that $i\Sigma_1 + \mathsf{ECT}_0 \vdash \sigma \leftrightarrow \tau$ (‡). It follows that $\mathsf{HA}+\mathsf{ECT}_0 \vdash \tau$, and, hence, by (c), that $\mathsf{HA} \vdash \tau$. We may conclude that $T + \mathsf{ECT}_0 \vdash \Box_{\mathsf{HA}}\tau$. Thus, by (d), $T + \mathsf{ECT}_0 \vdash \tau$. It follows, by (†) and (‡), that $T + \mathsf{ECT}_0 \vdash \alpha$.

f) By (e), $T+\mathsf{ECT}_0$ extends HA. Since T is finitely axiomatized and since HA is *essentially reflexive*, it follows that $T+\mathsf{ECT}_0$ proves its own consistency, and, hence, by the Second Incompleteness Theorem, is inconsistent.

g) Since $i\Sigma_1$ proves its own realizability and since formulas of the form ϕ^{r} are self-realizing, T proves its own realizability. So all theorems of $T + \mathsf{ECT}_0$ are realizable in T. We may conclude that T is inconsistent.

The present proof that HA is not $\mathcal{S}$-axiomatizable uses lots of specific properties of $\mathcal{S}$. This makes one wonder whether its conclusion can be generalized to wider classes. This suggests the following question.

Open Question 5.1 Can HA be axiomatized in one of Wolfgang Burr's classes Θ_n? See [1]. I conjecture: *no.* ▯

In appendix B, we give a direct proof of the fact that HA is not closed under taking submodels. This result immediately yields an alternative proof of the fact that HA cannot be $\mathcal{S}$-axiomatizable.

5.2 Further Consequences

It is easy to see that e.g. $i\Delta_0$ and PA (considered as a theory over IQC) are preserved under taking submodels. It follows that these theories are $\mathcal{S}$-axiomatizable over IQC. Of course, these modest facts can also be directly verified.

Acknowledgements

I thank Domenico Zambella and Freek Wiedijk for listening patiently to presentations of the main proof.

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A Comparing Classes of Models

Since we are going to compare different classes of models, it is pleasant to work in a bigger class of models extending both. First consider classical structures of signature $\mathcal{L}$ where identity is allowed to be an equivalence relation, which is a congruence w.r.t. the operations and relations of the structure. As before we morphisms will preserve structure in the weak sense. We will count two structural morphisms $f, g : \mathfrak{k} \to \mathfrak{m}$ the same if, for all $d \in D_{\mathfrak{k}}$, $f(d) =_{\mathfrak{m}} g(d)$. Thus an isomorphism between $\mathfrak{k}$ and $\mathfrak{m}$ can be given by two morphisms $i : \mathfrak{k} \to \mathfrak{m}$ and $j : \mathfrak{m} \to \mathfrak{k}$ such that $(j \circ i)(d) =_{\mathfrak{k}} d$ and $(i \circ j)(e) =_{\mathfrak{m}} e$.

Let's say that an eq$^+$-model is a Kripke model as originally defined where we allow identity to be locally an equivalence relation which respects structure. So a Kripke model is an eq$^+$-model where identity is, locally, true identity. Moreover, an eq-model is an eq$^+$-model where all the embeddings are the standard embeddings associated with the subset relation.

An *isomorphism* between eq$^+$-models $\mathcal{K} = \langle\langle P, \preceq\rangle, \Phi\rangle$ and $\mathcal{M} = \langle\langle Q, \leq\rangle, \Psi\rangle$ is a pair of functions $\langle F, G\rangle$, where F is an isomorphism between $\langle P, \preceq\rangle$ and $\langle Q, \leq\rangle$, and where $G(k)$ is an isomorphism $\langle i, j\rangle$ between k and $F(k)$. We demand that

- $i(k)(\mathsf{e}_{k',k}(d')) =_{F(k)} \mathsf{e}_{F(k'),F(k)}(i(k')(d'))$,
- $j(m)(\mathsf{e}_{m',m}(e')) =_{F^{-1}(m)} \mathsf{e}_{F^{-1}(m'),F^{-1}(m)}(j(m')(e'))$.

It is easy to see that if $\langle F, G\rangle$ is an isomorphism and $G(k) = \langle i, j\rangle$ then:

1. $k, f \Vdash \phi \Leftrightarrow F(k), i \circ f \Vdash \phi$.
2. $k, j \circ g \Vdash \phi \Leftrightarrow F(k), g \Vdash \phi$.

Theorem A.1 *Every eq$^+$-model is isomorphic to a Kripke model.*

The proof is simply by dividing out the local equivalence relations in the obvious way.

Theorem A.2 *Every eq$^+$-model $\mathcal{K}$ is isomorphic to an eq-model $\mathcal{K}^\circ$.*

Proof

Suppose that $\mathcal{K}$ is $\langle\langle P, \preceq\rangle, \Phi\rangle$. $\mathcal{K}^\circ$ will be $\langle\langle P, \preceq\rangle, \Phi^\circ\rangle$. We will write P_k° for $\Phi^\circ(k)(P)$, etcetera. Define:

- $D_k^\circ := \{\langle k', d'\rangle \mid k' \preceq k \text{ and } d' \in D_{k'}\}$
- $P_k^\circ(\langle k_1', d_1'\rangle, \cdots, \langle k_n', d_n'\rangle) :\Leftrightarrow P_k(\mathsf{e}_{k_1',k}(d_1'), \cdots, \mathsf{e}_{k_n',k}(d_n'))$
  (This includes the definition of the new '='. Of course, one has to verify that this is an equivalence relation which is a congruence w.r.t the functions and relations of the new structure.)
- $F_k^\circ(\langle k_1', d_1'\rangle, \cdots, \langle k_n', d_n'\rangle) = F_k((\mathsf{e}_{k_1',k}(d_1'), \cdots, \mathsf{e}_{k_n',k}(d_n'))$
- $\mathsf{e}_{k',k}^\circ(\langle k'', d''\rangle) := \langle k'', d''\rangle$.
  Note that since $k'' \preceq k' \preceq k$, we have $k'' \preceq k$. So $\langle k'', d''\rangle$ is indeed in D_k°.

The isomorphism between $\mathcal{K}$ and $\mathcal{K}^\circ$ has as first component F the identity morphism on $\langle P, \preceq\rangle$. We take $G(k) := \langle i, j\rangle$, where $i(d) := \langle k, d\rangle$ and, for $k' \preceq k$ and $d' \in D_{k'}$, $j(\langle k', d'\rangle) := \mathsf{e}_{k',k}(d')$.

The verification that all this works is long and boring. We a few samples.

We verify a property of the new model. Suppose that $P_k^\circ(\cdots, \langle k', d'\rangle, \cdots)$ and $\langle k', d'\rangle =_k^\circ \langle k'', d''\rangle$. We show that $P_k^\circ(\cdots, \langle k'', d''\rangle, \cdots)$. From our assumptions we have: $P_k(\cdots, \mathsf{e}_{k',k}(d'), \cdots)$ and $\mathsf{e}_{k',k}(d') =_k \mathsf{e}_{k'',k}(d'')$. Ergo, since $=_k$ is a congruence, $P_k(\cdots, \mathsf{e}_{k'',k}(d''), \cdots)$. Hence, $P_k^\circ(\cdots, \langle k'', d''\rangle, \cdots)$.

We verify that the new model is persistent for predicates. Suppose $k' \preceq k$ and $P_{k'}^\circ(\langle k_1'', d_1''\rangle, \cdots, \langle k_n'', d_n''\rangle)$. Here the k_i'' are equal to or below k'. We want to show: $P_k^\circ(\langle k_1'', d_1''\rangle, \cdots, \langle k_n'', d_n''\rangle)$. From our assumption, we have

$$P_{k'}(\mathsf{e}_{k_1'',k'}(d_1''), \cdots, \mathsf{e}_{k_n'',k'}(d_n'')).$$

Hence, by persistence in $\mathcal{K}$,

$$P_k(\mathsf{e}_{k',k}(\mathsf{e}_{k_1'',k'}(d_1'')), \cdots, \mathsf{e}_{k',k}(\mathsf{e}_{k_n'',k'}(d_n''))).$$

And, thus, $P_k(\mathsf{e}_{k_1'',k}(d_1''), \cdots, \mathsf{e}_{k_n'',k}(d_n''))$. I.o.w., $P_k^\circ(\langle k_1'', d_1''\rangle, \cdots, \langle k_n'', d_n''\rangle)$.

We verify a property of the isomorphism: $(i \circ j)(\langle k', d'\rangle) =_k^0 \langle k', d'\rangle$. The following are equivalent:

$$(i \circ j)(\langle k', d'\rangle) =_k^\circ \langle k', d'\rangle \quad (1)$$
$$i(\mathsf{e}_{k',k}(d')) =_k^\circ \langle k', d'\rangle \quad (2)$$
$$\langle k, \mathsf{e}_{k',k}(d')\rangle =_k^\circ \langle k', d'\rangle \quad (3)$$
$$\mathsf{e}_{k,k}(\mathsf{e}_{k',k}(d')) =_k \mathsf{e}_{k',k}(d') \quad (4)$$
$$\mathsf{e}_{k',k}(d') =_k \mathsf{e}_{k',k}(d') \quad (5)$$

The other verifications are more of the same. ❑

Let's say that an eq$^+$-model is an emb-model if it is both a Kripke model (identity is, locally, true identity) and an eq-model (embeddings of domains are subset embeddings). In case identity in $\mathcal{K}$ is decidable, we can produce a emb-model $\mathcal{K}^\circ$ that is, in a sense, the same as the original model. An attempted proof can be found in Smoryński's paper [6]: see the proof of his theorem 5.1.23. However, Smoryński's proof contains a mistake. The mistake was pointed out by Kai Wehmeier in his [12]. Unfortunately, Wehmeier's proof is also mistaken: see the proof of his lemma 1.2. It is easy to see that these proofs *cannot* be right: their new model is isomorphic (in our sense) to the original model. However, the following example gives a Kripke model with decidable identity that cannot be isomorphic to an emb-model.

Example A.3 Consider the model $\mathcal{A}$ that is 'generated' by the following specifications.

- The nodes are k, m, n, p;
- we have $k < n$, $k < p$, $m < n$, $m < p$;
- $D_k = D_n = D_p = \{a, b\}$, $D_m = \{c\}$;
- $\mathsf{e}_{k,n}(a) = \mathsf{e}_{k,p}(a) = a$, $\mathsf{e}_{k,n}(b) = \mathsf{e}_{k,p}(b) = b$, $\mathsf{e}_{m,n}(c) = a$, $\mathsf{e}_{m,p}(c) = b$.

Suppose there was such a model, say $\mathcal{A}^\circ$. Without loss of generality we can assume:

- The new nodes are k, m, n, p;
- $k <^\circ n$, $k <^\circ p$, $m <^\circ n$, $m <^\circ p$;
- $D^\circ_k = D^\circ_n = D^\circ_p = \{a, b\}$;
- $\mathsf{e}^\circ_{k,n}(a) = \mathsf{e}^\circ_{k,p}(a) = a$, $\mathsf{e}^\circ_{k,n}(b) = \mathsf{e}^\circ_{k,p}(b) = b$.

But now to what can c correspond? It must correspond to both a and b. Quod impossibile. ❑

The lack of a correct statement and proof is especially embarassing, since (i) completeness for theories with decidable identity in emb-models is always used in the literature and (ii) the Henkin construction does *not* automatically deliver an emb-model.

The correct diagnosis is of our problem that isomorphism is a too restricted notion of sameness. The right notion of sameness is *bisimulation*.

Consider two models $\mathcal{K} = \langle\langle P, \preceq\rangle, \Phi\rangle$ and $\mathcal{M} = \langle\langle Q, \leq\rangle, \Psi\rangle$. We will write P°_m for $I_{\Psi(m)}(P)$, etc. A bisimulation $\mathcal{B}$ between $\mathcal{K}$ amd $\mathcal{M}$ is a set of quadruples $\langle k, i, j, m\rangle$, where $k \in P$, $m \in Q$ and where $\langle i, j\rangle$ witnesses an isomorphism between $\Phi(k)$ and $\Psi(m)$. We write $k\,\mathcal{B}_{i,j}\,m$ for $\langle k, i, j, m\rangle \in \mathcal{B}$. We demand:

zig Suppose $k\,\mathcal{B}_{i,j}\,m$ and $k \preceq k'$. Then there are i', j', k' such that $k'\,\mathcal{B}_{i',j'}\,m'$ and $m \leq m'$. Moreover, $\mathsf{e}^{\circ}_{m,m'} \circ i = i' \circ \mathsf{e}_{k,k'}$.

zag Suppose $k\,\mathcal{B}_{i,j}\,m$ and $m \leq m'$. Then there are i', j', k' such that $k'\,\mathcal{B}_{i',j'}\,m'$ and $k \preceq k'$. Moreover, $\mathsf{e}_{k,k'} \circ j = j' \circ \mathsf{e}^{\circ}_{m,m'}$.

Lemma A.4 Suppose $k, f \Vdash \phi$ and $k\,\mathcal{B}_{i,j}\,m$. Then $m, i \circ f \Vdash \phi$. Conversely, suppose $m, g \Vdash \phi$ and $k\,\mathcal{B}_{i,j}\,m$. Then $k, j \circ g \Vdash \phi$. ❑

Proof

We start with an observation. Suppose $k\,\mathcal{B}_{i,j}\,m$. Then $k, j \circ i \circ f \Vdash \chi \Leftrightarrow k, f \Vdash \chi$ and $m, i \circ j \circ g \Vdash \chi \Leftrightarrow m, g \Vdash \chi$. This follows by a simple induction on χ using the fact that $d =_k (j \circ i)(d)$ and $e =_m (i \circ j)(e)$.

We prove the lemma by induction on ϕ. We treat the case of the universal quantifier in one of the directions. Suppose $k, f \Vdash \forall x\, \psi$ and $k\,\mathcal{B}_{i,j}\,m$. We want to show $m, i \circ f \Vdash \forall x\, \psi$. Consider $m' \geq m$ and suppose $e' \in D^{\circ}_{m'}$. We have to show that $m', (\mathsf{e}^{\circ}_{m,m'} \circ i \circ f)[x := e'] \Vdash \psi$. By the zag-property, there is a $k' \succeq k$ with $k'\,\mathcal{B}_{i',j'}\,m'$ and $\mathsf{e}_{k,k'} \circ j = j' \circ \mathsf{e}^{\circ}_{m,m'}$. So, by the induction hypothesis in combination with our opening observation, it is sufficient to show that $k', j' \circ ((\mathsf{e}^{\circ}_{m,m'} \circ i \circ f)[x := e']) \Vdash \psi$ (‡). We have:

$$\begin{aligned} j' \circ ((\mathsf{e}^{\circ}_{m,m'} \circ i \circ f)[x := e']) &= (j' \circ \mathsf{e}^{\circ}_{m,m'} \circ i \circ f)[x := j'(e')] \\ &= (\mathsf{e}^{\circ}_{k,k'} \circ j \circ i \circ f)[x := j'(e')] \end{aligned}$$

Since, by assumption, $k, f \Vdash \forall x\, \psi$, it follows that $k, j \circ i \circ f \Vdash \forall x\, \psi$. We find $k', (\mathsf{e}^{\circ}_{k,k'} \circ j \circ i \circ f)[x := j'(e')] \Vdash \psi$. We may conclude ‡. ❑

We say that a bisimulation is *full* if every $k \in P$ is $\mathcal{B}$-connected to an m in Q, and vice versa. $\mathcal{K}$ and $\mathcal{M}$ are *bisimilar* if there is a full bisimulation between them. It is easy to see that isomorphism can be subsumed under bisimulation: an isomorphism can be redescribed as a specific full bisimulation. Thus isomorphic models are certainly bisimilar. It is easy to see that bisimulations can be composed. It follows that bisimilarity is an equivalence relation between models. Here is a standard lemma concerning bisimulation. A model is *tree-like* if it satisfies: if $k' \preceq k$ and $k'' \preceq k$, then $k'' \preceq k'$ or $k' \preceq k''$.

Lemma A.5 Every eq$^+$-model $\mathcal{K}$ is bisimilar to a *tree-like* eq$^+$-model $\mathcal{K}^{\circ}$. The same holds for eq-models and for ordinary Kripke models. ❑

Proof

The proof employs te standard unraveling construction. Here is the specification of the new model.

- The nodes are non-empty sequences $\langle k_1, \cdots, k_n \rangle$, where $k_i \preceq k_{i+1}$.

- $\preceq^\circ$ is the weak extension order on sequences. I.o.w. $\langle k_1, \cdots, k_n\rangle \preceq^\circ \langle k'_1, \cdots, k'_m\rangle$ iff $n \leq m$ and, for all $i \leq n$, $k_i = k'_i$.
- $\Phi^\circ(\langle k_1, \cdots, k_n\rangle) := \Phi(k_n)$.
- $\mathsf{e}^\circ_{\langle \cdots, k'\rangle, \langle \cdots, k\rangle}(d') = \mathsf{e}_{k',k}(d')$.

It is easy to see that we have constructed a tree-like model that is eq$^+$, eq or ordinary depending on whether the original model is eq$^+$, eq or ordinary. The promised bisimulation is given by:

- $\langle k_1, \cdots, k_n\rangle \, \mathcal{B}_{i,j} \, k :\Leftrightarrow k = k_n, \ i = j = \mathsf{id}_{\Phi(k)}$.

We omitt the trivial verification that this specifies a full bisimulation. ❑

Here is the promised theorem.

Theorem A.6 *Suppose $\mathcal{K}$ is an eq$^+$-model with decidable identity —i.e. $k, f \Vdash x = y \vee \neg x = y$, for all nodes k of $\mathcal{K}$ and all k-assignments f. Then there is a emb-model $\mathcal{K}^\circ$ bisimilar to $\mathcal{K}$.*

Proof

It is most convenient to do the construction is steps.

Step 1 It is sufficient to prove our theorem for a real Kripke model, since by theorem A.1 and the preceding remarks, there a Kripke model bisimilar to an eq$^+$-model. Since identity is still decidable, our embeddings $\mathsf{e}_{k',k}$ will be injections.

Step 2 It is sufficient to prove our theorem for a tree-like Kripke model by lemma A.5. Clearly, the decidability of identity is preserved and, hence, our embeddings are still injections.

Step 3 By the above we may assume that $\mathcal{K}$ is a tree-like Kripke model. Say $\mathcal{K} = \langle\langle P, \preceq\rangle, \Phi\rangle$. Define $A := \{\langle k, d\rangle \mid k \in P, \ d \in D_k\}$. We define the following binary relations on A:

- $\langle q, f\rangle \preceq \langle k, d\rangle :\Leftrightarrow q \preceq k$ and $d = \mathsf{e}_{q,k}(f)$.
- $\alpha \, \mathsf{E} \, \beta :\Leftrightarrow \exists \gamma \in A \ \ \gamma \preceq \alpha$ and $\gamma \preceq \beta$

It is easy to see that $\preceq$ is a partial ordering on A. We show that it is tree-like. Suppose $\langle q, e\rangle \preceq \langle k, d\rangle$ and $\langle q', e'\rangle \preceq \langle k, d\rangle$. Since $\preceq$ is treelike on P, we have $q' \preceq q$ or $q \preceq q'$. Suppose e.g. $q' \preceq q$. It follows that $\mathsf{e}_{q,k}(\mathsf{e}_{q',q}(e')) = d$. Moreover, $\mathsf{e}_{q,k}(e) = d$, so, by injectivity, $e = \mathsf{e}_{q',q}(e')$. Ergo: $\langle q', e'\rangle \preceq \langle q, e\rangle$.

We show that E is an equivalence relation on A. Reflexivity and symmetry are trivial. We verify transitivity. Supppose (i) $\alpha \, \mathsf{E} \, \beta$ and (ii) $\beta \, \mathsf{E} \, \gamma\rangle$. Suppose δ witnesses (i) and that ϵ witnesses (ii). Evidently, $\delta \preceq \beta$ and $\epsilon \preceq \beta$. Hence $\epsilon \preceq \delta$

or $\delta \preceq \epsilon$. Let η be the minimum of δ, ϵ. It is now clear that $\eta \preceq \alpha$ and $\eta \preceq \gamma$. Ergo $\alpha \,\mathsf{E}\, \gamma$. We will write $[\alpha]$ for the E-equivalence class of α. We write $[k, d]$, for $[\langle k, d\rangle]$.

A next point is that if $\langle k, d\rangle \,\mathsf{E}\, \langle k, d'\rangle$, then $d = d'$. This is immediate from the injectivity of the embeddings. Thus the function $d \mapsto [k, d]$ is an injection from D_k in A_{E}.

We are finally ready to specify $\mathcal{K}^\circ$.

- The new nodes are the old nodes and the new ordering is the old ordering.
- $D_k^\circ := \{[k, d] \mid d \in D_k\}$.
- $P_k^\circ([k, d_1], \cdots, [k, d_n]) :\Leftrightarrow P_k(d_1, \cdots, d_n)$.
  The definition is meaningful, since d_i is fixed by k and $[k, d_i]$.
- $F_k^\circ([k, d_1], \cdots, [k, d_n]) = F_k(d_1, \cdots, d_n)$.
- $\mathsf{e}_{k',k}([k', d']) = [k', d']$.
  $[k', d']$ is in D_k°, since $\langle k', d'\rangle \,\mathsf{E}\, \langle k, \mathsf{e}_{k',k}(d)\rangle$.

It is now easy to see that $\mathcal{K}^\circ$ is isomorphic to $\mathcal{K}$. In fact we only changed the elements of the domains. ❑

B Solution of a Problem of Buss

Sam Buss (see [2], p173) asks whether there is a theory T such that HA is $\mathcal{H}T$, i.e. the theory of all T-normal Kripke models. A Kripke model is T-normal if the structures associated to its nodes all satisfy T in the sense of classical model theory. Buss' question can be a bit generalized. Let $\mathcal{X}$ be *any* class of classical models (of the signature of arithmetic). A Kripke model is $\mathcal{X}$-normal when the structure associated to any node is an element of $\mathcal{X}$. So we might ask: is there any $\mathcal{X}$ so that HA is the theory of all $\mathcal{X}$-normal models.

The answer is *no*. We prove a more general statement. Consider any class of Kripke models (of the signature of arithmetic) $\mathfrak{K}$. Assume that $\mathfrak{K}$ is sound and complete for HA, i.e. every $\mathcal{K}$ in $\mathfrak{K}$ satisfies HA and, whenever $\mathsf{HA} \nvdash A$, there is a $\mathcal{K} \in \mathfrak{K}$ such that $\mathcal{K} \nVdash A$. Then $\mathfrak{K}$ cannot be closed under submodels.

Since the class of $\mathcal{X}$-normal models is certainly closed under submodels, HA cannot be $\mathcal{X}$-normal.

Consider a $\mathfrak{K}$ that is sound and complete for HA. We show that $\mathfrak{K}$ is not closed under submodels.

We write $\Box$ for provability in HA and $\Diamond$ for $\neg\Box\neg$. We write $\Box_x$ for provability from $i\mathsf{Q}$ plus all induction axioms with formulas of complexity below x. We assume that the complexity classes exhaust all HA-formulas and that inside $\Box_{x+1}$ we can prove, verifiably in $i\Sigma_1$, $\Diamond_x\top$, the consistency statement for $\Box_x$. It is clear that we can always find a suitable complexity measure that does this. We also assume that $\Box_0$ verifiably includes $i\Sigma_1$. It is well known that such complexity classes can be found.

Lemma B.1 $i\Sigma_1 \vdash \Box_x \Box_x \bot \to \Box_{x+1} \bot$ ❑

Proof

Reason in $i\Sigma_1$. Suppose $\Box_x \Box_x \bot$. Then $\Box_{x+1} \Box_x \bot$, because $\Box_{x+1}$ extends $\Box_x$. Since $\Box_{x+1} \Diamond_x \top$, we may conclude that $\Box_{x+1} \bot$. ❑

Lemma B.2 Let π be the smallest HA-proof of $\Box\bot$, i.e. the smallest witness of $\Box\Box\bot$. Clearly 'π *exists*' is equivalent to $\Box\Box\bot$. We have:

$$\mathsf{HA} \nvdash \pi \text{ exists } \to \exists x > \pi\, (\Box_{x+1}\bot \to \Box_x \bot).$$

❑

Proof

Suppose that HA did prove the above implication.

The de Jongh translation $(.)^n$ for n, commutes with atoms and all connectives except $\to$ and $\forall$. Further:

- $(A \to B)^n := (A^n \to B^n) \wedge \Box_n (A^n \to B^n)$.
  We can also make a 'q-variant' with:
  $(A \to B)^n := (A^n \to B^n) \wedge \Box_n (A \to B)$.
  For our application both are equally good.

- $(\forall x\, A)^n := \forall x\, A^n \wedge \Box_n \forall x\, A^n$.
  The q-variant would be:
  $(\forall x\, A)^n := \forall x\, A^n \wedge \Box_n \forall x\, A$.

One can show that, whenever $\mathsf{HA} \vdash A$, then there is an n such that $\mathsf{HA} \vdash A^n$. Moreover S^n is provably equivalent with S for any Σ^0_1-formula S. Using this result we find that, for some standard n,

$$\mathsf{HA} \vdash \pi \text{ exists } \to \exists x > \pi\, ((\Box_{x+1}\bot \to \Box_x\bot) \wedge \Box_n (\Box_{x+1}\bot \to \Box_x \bot)).$$

Reason in HA. Suppose π exists. Then we have, for some $x > \pi$, $(\Box_{x+1}\bot \to \Box_x\bot)$ and $\Box_n(\Box_{x+1}\bot \to \Box_x\bot)$. Thus we find $\Box_n(\Box_x\Box_x\bot \to \Box_x\bot)$. Hence, since $\pi > n$, $\Box_x(\Box_x\Box_x\bot \to \Box_x\bot)$ and so $\Box_x\Box_x\bot$. It follows that $\Box_{x+1}\bot$. We may conclude, using $(\Box_{x+1}\bot \to \Box_x\bot)$, that $\Box_x\bot$ and, a fortiori, $\Box\bot$.

So $\mathsf{HA} \vdash \Box\Box\bot \to \Box\bot$. Hence, by Löb's Theorem, $\mathsf{HA} \vdash \Box\bot$. Quod non. ❑

By our assumption there is a $\mathcal{K} \in \mathfrak{K}$ and a node k in $\mathcal{K}$ such that $k \Vdash \pi$ *exists* and $k \nVdash \exists x > \pi\ (\Box_{x+1}\bot \to \Box_x\bot)$. Without loss of generality we may assume that, for every $j \succeq k$, the embedding $\mathsf{e}_{k,j}$ is a standard identical embedding associated with $\subseteq$. (This is possible since we only make the assumption *for the single node k*.) We find that, for every $a > \pi$ in the domain of k, we can find a $j_a \succeq k$, such that $j_a \Vdash \Box_{a+1}\bot$ and $j_a \nVdash \Box_a\bot$.

$\mathcal{K}_0$ is the submodel of $\mathcal{K}$ we obtain by restricting the nodes to k plus the elements j_a for $a > \pi + \omega$. Since in k, $2 \cdot \pi > \pi + \omega$, there are such a.

We show that the new model refutes HA. We use the well-known fact that Σ_1-formulas are preserved under taking submodels of models of HA and under extending Kripke models to models of HA. (This fact is easily proved using the provable decidability of Δ_0-formulas.) We prove that the set of all b in the domain of k such that $k \Vdash_0 \Diamond_b\top$, is precisely the set of $b < \pi + \omega$. First suppose $k \Vdash_0 \Diamond_b\top$. If b were above $\pi + \omega$ we would have $j_{b-1} \Vdash \Box_b\bot$ and, hence, $j_{b-1} \Vdash_0 \Box_b\bot$. This contradicts $k \Vdash_0 \Diamond_b\top$. So $b < \pi + \omega$. Conversely, suppose $b < \pi + \omega$ and that, for some i, we have $i \Vdash_0 \Box_b\bot$. It follows that, for some $a > \pi + \omega$, $j_a \Vdash_0 \Box_b\bot$. But then $b < a$ and $j_a \Vdash \Box_b\bot$ and, hence, $j_a \Vdash \Box_a\bot$. Quod non.

Clearly each of the j_a in the new model forces $\Box\bot$, but k does not. We find that:

$$k \Vdash_0 \forall x\, (\,(\Box\bot \vee \Diamond_x\top) \to (\Box\bot \vee \Diamond_{x+1}\top)\,).$$

But $k \nVdash_0 \forall x\, (\Box\bot \vee \Diamond_x\top)$. ❑