Modal Logic of Projective Geometries of Finite Dimension

Vera Stebletsova
Department of Philosophy
Utrecht University
Heidelberglaan 8
3584 CS Utrecht
the Netherlands
Vera.Stebletsova@phil.ruu.nl *

1. Introduction

Extending an original idea of B. Jonsson, R. Lyndon showed in [Lyndon 1961] how to construct relation algebras — boolean algebras with the binary operator ; (composition), the unary operator $\smile$ (converse) and the constant 1' (identity) which satisfy the so-called Tarski axioms (see [Henkin et alii 1997], [Jónsson & Tarski 1952]) — from projective geometries, thus providing "a method for deriving consequences in the algebraic theory of binary relations from certain familiar facts of projective geometry" [Lyndon 1961]. By a projective geometry he meant the following (the definition below is valid throughout the present paper):

1.1. Definition. *A* projective geometry (or projective space) *is a structure* $\mathfrak{G} = (P, L, In)$, *where* P *is a non-empty set of* points, L *is a family of subsets of* P *called lines, and* $In \subseteq P \times L$ *is a binary relation of incidence between points and lines.* $\mathfrak{G}$ *satisfies the following axioms:*

(A) every line has at least four points;

(B) each pair of distinct points (x, y) *lies on a unique line,* $\overline{xy}$*;*

(C) if x, y *and* z *are distinct points, and a line meets lines* $\overline{xy}$ *and* $\overline{xz}$ *in distinct points, then it meets* $\overline{yz}$.

Given a projective geometry $\mathfrak{G}$, one can define a relation algebra $\mathfrak{A}(\mathfrak{G}) = (A, +, -, ;, \smile, 1')$ associated with $\mathfrak{G}$. The atoms of $\mathfrak{A}(\mathfrak{G})$ are all points of $\mathfrak{G}$ plus e, a neutral element not in $\mathfrak{G}$. The domain consists of all subsets of $\mathfrak{G} \cup \{e\}$. The boolean operations are interpreted in the usual set-theoretical sense, 1' is $\{e\}$, the converse $\smile$ of a set is that set itself. Composition ; (which is commutative and associative) is defined on atoms as

$$\{x\};\{y\} = \begin{cases} \{e, x\}, \text{ if } x = y \\ \overline{xy} \setminus \{x, y\}, \text{ if } x \neq y \text{ and } e \notin \{x, y\} \\ \{x\}, \text{ if } y = e \end{cases}$$

This definition extends to arbitrary subsets of $\mathfrak{G} \cup \{e\}$ by complete additivity.

*The research of the author is sponsored by the Netherlands Organization for the Advancement of Research (NWO).

Given a projective geometry $\mathfrak{G}$, one can also define a relation structure $\mathfrak{B}(\mathfrak{G})$ associated with $\mathfrak{G}$. Recall that a relation structure is a quadruple $\mathfrak{B} = (B, C, R, I)$ such that B is non-empty set, $C \subseteq B^3, R \subseteq B^2, I \subseteq B$. The definition below is very close to the one the reader can find in [Givant 1997]. The only difference between these two definitions is that Givant recovers a geometric relation structure from a geometry re-defined as a *one sorted* structure (P, Co), where Co is a ternary relation on P (his definition of a geometry is equivalent to the standard one). We use the standard definition of geometries as two-sorted structures to define geometric relation structures. It is easy to observe from the definition below that Lyndon's algebras are complex algebras of the relation structures associated with projective geometries, and these relation structures are, in fact, atomic structures of Lyndon's algebras in the sense of R. Maddux [Maddux 1982]:

1.2. Definition. *Let $\mathfrak{G} = (P, L, In)$ be a projective geometry. We say $\mathfrak{B}(\mathfrak{G}) = (B, C, R, I)$ is a* relation structure associated with geometry $\mathfrak{G}$ *if:*

- $B = P \bigcup \{e\}$, *where e is a* special element *not in P*
- Ix *iff* $x = e$
- Rxy *iff* $x = y$
- $Cxyz$ *iff either*
    1. *$e \notin \{x, y, z\}$, and x, y, z are all distinct and collinear, or*
    2. *$x = y = z$, or*
    3. *one of x, y, z is e, and two others are equal.*

Note that according to the definition, $Cxxz$ doesn't hold unless $z \in \{e, x\}$.

Let $\mathsf{B} = \{\mathfrak{B} : \mathfrak{B} \simeq \mathfrak{B}(\mathfrak{G})$ for some geometry $\mathfrak{G}\}$, that is B is the class of all relational structures isomorphic to those associated with projective geometries. This class is characterized by the conditions which are formulated in the lemma below:

1.3. Lemma. *([Givant 1997], Lemma 3.6) A relation structure $\mathfrak{B}$ is in B iff $\mathfrak{B}$ satisfies the following conditions:*
(i) there is a unique point in I and at least two points not in I;
(ii) Rxy iff $x = y$;
(iii) if $Cxyz$ then $Cyxz$ and $Cxzy$;
(iv) if $Cxyz$ and $Czuv$ then there is a w with $Cxwv$ and $Cyuw$;
(v) if $x \in I$ then $Cxyz$ iff $y = z$;
(vi) if $x \notin I$ then $Cxxz$ iff $z = x$ or $z \in I$.

Every (projective) geometry is characterized by a unique dimension d (finite or infinite) which is defined via a number of related geometrical notions. In the definition below we follow [Garner 1981]. Note that $\overline{xy}$ is identified with the set of points on the line through x and y.

1.4. Definition. *Let $\mathfrak{G}$ be a projective geometry.*

(i) $S \subseteq \mathfrak{G}$ [1] is a subspace *of $\mathfrak{G}$ if for all $x, y \in S$, $x \neq y$, we have $\overline{xy} \subseteq S$.*

(ii) For any $X \subseteq \mathfrak{G}$ define

$$\langle X \rangle = \bigcap \{S : S \text{ is a subspace of } \mathfrak{G} \text{ and } S \supseteq X\}$$

to be the smallest subspace containing X, called the span *of X.*

If $\langle X \rangle = \mathfrak{G}$ then X is called a spanning set of $\mathfrak{G}$.

[1] In many cases, for the sake of simplicity, we make no distinction between $\mathfrak{G}$ and P, the set of points of $\mathfrak{G}$.

(iii) A subset S of $\mathfrak{G}$ is independent *if for all $p \in S$ we have $\langle S \setminus \{p\} \rangle \neq \langle S \rangle$. We call elements of S independent points.*

(iv) An independent spanning set of $\mathfrak{G}$ is called a basis *of $\mathfrak{G}$.*

(v) The number of points in a basis of $\mathfrak{G}$ is called the rank *r of $\mathfrak{G}$. The* dimension *d of $\mathfrak{G}$ is defined as $d = r - 1$.*

From the theory of geometry we know that any two bases of the same projective space have the same rank, and hence, the same dimension.

EXAMPLES. The dimension of a space which is $\emptyset$ is -1, the dimension of a space which is a point is 0, the dimension of a space which is a single line is 1, the dimension of a projective plane is 2.

In the present paper we restrict the class B to the class $\mathsf{B}_d = \{\mathfrak{B} : \mathfrak{B} \simeq \mathfrak{B}(\mathfrak{G})$ for some geometry $\mathfrak{G}$ of dimension $d\}$ of all relational structures isomorphic to those associated with projective geometries of an arbitrary but fixed finite dimension d, and develop a modal formalism which language contains one constant 1', one unary operator $\otimes$, and one binary operator $\circ$. The motivation for introducing this formalism is twofold. First, it axiomatizes the class B_d thus axiomatizing the equational theory of the class of relation algebras of projective geometries of a fixed finite dimension (the completeness of the algebraic counterpart of a logic is straightforward from the completeness of that logic, see for details [Venema 1993]). Second, it might be interesting on its own as a *modal logic of projective geometries of finite dimension d, MPG_d*. Involving the dimension of projective geometries into logical considerations we provide ourselves with a possibility to reason about sequences of subspaces called in geometries *flags* (point — line — plane — subspaces of higher dimension). We also get an opportunity to speak about classes of projective spaces like Pappian spaces and Desarguesian spaces thus obtaining a possibility to use known geometric results in modal theory.

Our logical derivation system can be classified as non-orthodox in the sense of [Venema 1993] for it contains an irreflexivity rule of inference. In many cases non-orthodox rules are associated with the presence of the difference operator D in a language (see [Venema 1993], p.1016, or the proof of Fact 2.8 below for its definition) which allows to define special formulas. Each of such formulas holds precisely at one point in a model, and therefore could be considered as a name of that point. The latter provides us with useful tools for constructing our canonical model in a shape we need, and so play an important role in proving completeness of a system w.r.t. intended semantics. Although in our language the D-operator is undefinable (see Fact 2.8 below), we can model — due to the fact that each projective space has a uniquely determined dimension — the same effect obtained in the presence of the D-operator.

The results of the paper include the following. In the next section we construct the modal logic of projective spaces with semantics in the class B_d of all relation structures isomorphic to those associated with projective geometries of finite dimension, and give its axiomatization. The proof of the Completeness Theorem appears in Section 3. We establish that our logic lacks the finite model property using Pappus' theorem in Section 4.

Acknowledgements. I am indebted to Yde Venema for iterated comprehensive reading of the manuscript and detailed comments which have led to many improvements, and to Albert Visser for helpful discussions.

2. Modal logic of projective spaces

In this section we introduce a modal logic (MPG_d) of projective geometries with semantics in the class B_d of all relational structures isomorphic to those derived from geometries of dimension

d, $1 \leq d < \omega$.

2.1. Definition. *The modal language MLG of our logic is built up from a modal similarity type having one constant 1', one unary operator $\otimes$, one binary operator $\circ$, and a set of propositional variables. The set of formulas is given by*

$$\phi ::= p \mid \bot \mid 1' \mid \neg\phi \mid \phi_1 \vee \phi_2 \mid \otimes\phi \mid \phi_1 \circ \phi_2.$$

MLG is of the same similarity type as Lyndon's algebras: $\otimes$ corresponds to $\smile$, $\circ$ to ;.

MLG-formulas are interpreted over elements of the class B_d. A model is $\mathfrak{M} = (\mathfrak{B}, V)$, where $\mathfrak{B} \in \mathsf{B}_d$ and $V(p) \subseteq \mathfrak{B}$ for any propositional variable p:

- $\mathfrak{M}, x \models 1'$ *iff* $x = e$
- $\mathfrak{M}, x \models \otimes\psi$ *iff* $\mathfrak{M}, x \models \psi$
- $\mathfrak{M}, x \models \phi \circ \psi$ *iff there exist* y, z *such that* $Cxyz$ *and* $\mathfrak{M}, y \models \phi$ *and* $\mathfrak{M}, z \models \psi$.

Taking into account the definition of C from 1.2, the truth condition for $\circ$ can be rewritten in an equivalent form as follows:

If $x \in \mathfrak{G}$ then $\mathfrak{M}, x \models \phi \circ \psi$ iff either

- there are y, z such that x, y, z are distinct and collinear, and $\mathfrak{M}, y \models \phi$ and $\mathfrak{M}, z \models \psi$, or
- $\mathfrak{M}, x \models \phi \wedge \psi$, or
- $\mathfrak{M}, x \models \phi$ and $\mathfrak{M}, e \models \psi$, or
- $\mathfrak{M}, x \models \psi$ and $\mathfrak{M}, e \models \phi$.

If $x = e$ then $\mathfrak{M}, x \models \phi \circ \psi$ iff there exists y such that $\mathfrak{M}, y \models \phi \wedge \psi$.

We will also use two modalities $\Box$ and $\Diamond$. The universal modality $\Box$ is defined semantically as $\mathfrak{M}, x \models \Box\phi$ iff $\mathfrak{M}, y \models \phi$ for all $y \in \mathfrak{M}$, and modality $\Diamond$ as $\mathfrak{M}, x \models \Box\phi$ iff $\mathfrak{M}, y \models \phi$ for some $y \in \mathfrak{M}$. $\Diamond$ is definable via operator $\circ$ as $\Diamond\phi \stackrel{\text{def}}{=} \top \circ \phi$, and $\Box$ is a dual of $\Diamond$, that is $\Box\phi \stackrel{\text{def}}{=} \neg\Diamond\neg\phi$.

Let us take a closer look at formulas which will play roles of names in our further steps. Consider the following formula:

$$\textbf{subspace}(\phi) \stackrel{def}{\equiv} \Box(\phi \circ \phi \to \phi \vee 1') \wedge \Box(1' \to \neg\phi).$$

This formula says that ϕ (or more precise, sets of points which satisfy formula ϕ) is a *subspace* for we have the following lemma:

2.2. Lemma. $\mathfrak{M}, x \models \textbf{subspace}(\phi)$ *iff* $\{x \in \mathfrak{G} \cup \{e\} : \mathfrak{M}, x \models \phi\}$ *is a subspace of* $\mathfrak{G}$ *of dimension* d.

Proof of Lemma 2.2. ($\Rightarrow$). Recall that domain of $\mathfrak{M}$ is $B = \mathfrak{G} \bigcup \{e\}$. Suppose $e \notin \{z, w\}$, $z \neq w$, and $z, w \in \{x \in \mathfrak{G} \cup \{e\} : \mathfrak{M}, x \models \phi\}$. Then, by the semantic definition of $\circ$ we have $\mathfrak{M}, s \models \phi \circ \phi$ for any $s \in \overline{zw}$ such that $s \notin \{z, w, e\}$. At the same time, since $\mathfrak{M}, x \models \textbf{subspace}(\phi)$, we have $\mathfrak{M}, y \models \phi \circ \phi \to \phi \vee 1'$ for any $y \in \mathfrak{G}$. Hence $\mathfrak{M}, s \models \phi$, and so $\overline{zw} \subseteq \{x \in \mathfrak{G} \cup \{e\} : \mathfrak{M}, x \models \phi\}$. Thus, $\{x \in \mathfrak{G} \cup \{e\} : \mathfrak{M}, x \models \phi\}$ is a subspace, according to Definition 1.4(i).

$\Leftarrow$: Obvious. QED

Now let $d = 2$ and

$$O_{\psi_1\psi_2}\phi \stackrel{def}{\equiv} \begin{array}{l} \textbf{subspace}(\phi) \wedge \textbf{subspace}(\psi_1) \wedge \textbf{subspace}(\psi_2) \wedge \phi \wedge \\ \Box(\phi \to \psi_1) \wedge \Box(\psi_1 \to \psi_2) \wedge \Diamond(\neg\phi \wedge \psi_1) \wedge \Diamond(\neg\psi_1 \wedge \psi_2) \end{array}$$

The formula says that there is a sequence of subspaces ϕ, ψ_1, ψ_2 of a projective plane $\mathfrak{G}$ such that ϕ is a proper subspace of ψ_1, and ψ_1 is a proper subspace of ψ_2, and we can prove (see Lemma 2.4 below for a general case of dimension d) that ϕ is a *point*, and ψ_1, ψ_2 are a line and a plane, correspondingly. Generalization for the case of dimension $d < \omega$ is obvious:

$$O_{\psi_1\dots\psi_d}\phi \stackrel{\text{def}}{\equiv} \begin{array}{l} \textbf{subspace}(\phi) \wedge \bigwedge_{i=d}^{1} \textbf{subspace}(\psi_i) \wedge \phi \wedge \\ \Box(\phi \to \psi_1) \wedge \bigwedge_{i=d-1}^{1} \Box(\psi_i \to \psi_{i+1}) \wedge \Diamond(\neg\phi \wedge \psi_1) \wedge \\ \bigwedge_{i=d}^{1} \Diamond(\neg\psi_i \wedge \psi_{i+1}), \end{array}$$

Formulas $O_{\psi_1\dots\psi_d}\phi$ play an important role in our considerations: we use them as name-formulas for we know — due to Lemma 2.3 below — that each such a formula holds only in one point and nowhere else. The latter is possible because the structure of such a formula reflects geometric properties connected with finite dimensions of subspaces. Throughout the text we refer to these formulas as O-formulas.

2.3. Lemma. *Let $\mathfrak{M} = (\mathfrak{B}, V)$ be a model such that $\mathfrak{B} \in \mathsf{B}_d$.*
If $\mathfrak{M}, x \models O_{\psi_1\dots\psi_d}\phi$ then $\mathfrak{M}, x \models \phi$ and x is the only *point where ϕ is satisfied.*
We prove Lemma 2.3 by means of two following lemmas: Lemma 2.2 and

2.4. Lemma. *Let $\mathfrak{G}$ be a projective geometry of dimension d, and $S_i, i = -1, 0, \dots, d$, be subspaces of $\mathfrak{G}$ such that $S_i \subsetneqq S_{i+1}$. Then S_i is of dimension i. In particular S_{-1} is $\emptyset$, and S_0 is a point.*

PROOF OF LEMMA 2.4. Here we use the geometric fact that if subspaces S and S' are such that $S \subsetneqq S'$ then the rank of S is strictly less than the rank of S'. From the simple calculations it follows that S_0 is of dimension 0, so S_0 is a point. QED

PROOF OF LEMMA 2.3. Assume $\mathfrak{M}, x \models O_{\psi_1\dots\psi_d}\phi$. Then, by Lemma 2.2, $S(\phi) = \{x \in \mathfrak{G} \cup \{e\} : \mathfrak{M}, x \models \phi\}$ and $S(\psi_i) = \{x \in \mathfrak{G} \cup \{e\} : \mathfrak{M}, x \models \psi_i\}$ are subspaces of $\mathfrak{G}$ of dimension d. The formula $O_{\psi_1\dots\psi_d}\phi$ informs us that all subspaces $S(\phi)$ and $S(\psi_i)$ are not empty, $S(\phi)$ is properly included in $S(\psi_1)$, each $S(\psi_j)$ is properly included in $S(\psi_{j+1})$, $1 \leq j < d-1$. From these observations, by Lemma 2.4, we conclude that $S(\phi)$ has dimesion 0, so it is a point. QED

Below we define our logical derivation system.

2.5. Definition. By the modal logic of projective geometries of the dimension d, MPG_d, we mean the modal logic of MLG-similarity type axiomatized by the following axiom schemas and rules of inference.

A1. All tautologies of the propositional logic

A2. $((\phi \vee \phi') \circ \psi) \leftrightarrow (\phi \circ \psi \vee \phi' \circ \psi)$

A3. $\Diamond(1' \wedge \phi) \wedge \Diamond(1' \wedge \psi) \to \Diamond(1' \wedge \phi \wedge \psi)$

A4. $\otimes\phi \leftrightarrow \phi$

A5. $\phi \circ \psi \leftrightarrow \psi \circ \phi$

A6. $\phi \circ \neg(\phi \circ \psi) \wedge \psi \to \bot$

A7. $\phi \to \phi \circ \phi$

A8. $\phi \leftrightarrow 1' \circ \phi$

A9. $(\phi \circ \psi) \circ \chi \leftrightarrow \phi \circ (\psi \circ \chi)$

A10. $O_{\psi_1 \ldots \psi_d}\phi \wedge \neg\xi \wedge \Diamond(\xi \wedge \neg 1') \rightarrow \neg(\phi \circ (\xi \wedge \neg 1'))$

A11. $\phi \circ (\psi \wedge O_{\chi_1 \ldots \chi_d}\xi) \wedge \phi' \circ (\psi' \wedge O_{\chi_1 \ldots \chi_d}\xi) \rightarrow \phi \circ (\psi \wedge \psi' \wedge O_{\chi_1 \ldots \chi_d}\xi) \wedge \phi' \circ (\psi \wedge \psi' \wedge O_{\chi_1 \ldots \chi_d}\xi)$

A12. $\neg O_{\psi_1 \ldots \psi_{d+1}}\phi$.

Rules of inference:

R1. Modus Ponens

R2. $\vdash \phi \Rightarrow \vdash \neg(\neg\phi \circ \neg\psi)$

R3. $\vdash (1' \vee O_{q_1 \ldots q_d}p) \rightarrow \phi \Rightarrow \vdash \phi$ if $p, q_1, \ldots, q_d \notin \phi$.

COMMENTS. Some of axioms, such as A2, A6, A8, A9, are well-known from relation algebras/arrow logics (cf. [Jónsson & Tarski 1952], [Venema 1996]), and need no comments. A3 says that e is unique in a relation structure from the class B_d, A4 states that the converse of a set is that set itself. Commutativity of the composition is expressed by A5. Axiom A7 reflects the fact that for any point x in a relation structure from the class B_d, the ternary relation C holds between x, x, x. Axiom A10 mirrors the fact that in a relation structure built up from geometries of finite dimension there is no ternary relation between x, x, z unless $z \in \{e, x\}$ (cf. Definition 1.2). Axiom A11 says that formulas $O_{q_1 \ldots q_d}p$ play roles of names. Axiom A12, which is the crucial one in formalizing dimensionality of geometries, states that for geometries of dimension d, there is no sequences of subspaces longer than $d+1$, and it can be viewed as an axiom restricting the dimension from above.

2.6. Lemma. *Let $\mathfrak{G}$ be a geometry. Then $\mathfrak{B}(\mathfrak{G}) \models$ A12 iff the dimension of $\mathfrak{G}$ is less or equal d.*

PROOF. $\Leftarrow$. Let $\mathfrak{G}$ is of dimension $\leq d$. Assume, for a contradiction, that there is a model $\mathfrak{M}$ and $x \in \mathfrak{B}$ such that $\mathfrak{M}, x \models O_{\psi_1 \ldots \psi_{d+1}}\phi$. Then $\mathfrak{M}, x \models \textbf{subspace}(\psi_{d+1})$. By Lemma 2.2 this means that $S(\psi_{d+1}) = \{x \in \mathfrak{G} \cup \{e\} : \mathfrak{M}, x \models \psi_{d+1}\}$ is a subspace of $\mathfrak{G}$. On the other hand, the formula $O_{\psi_1 \ldots \psi_{d+1}}\phi$ informs us that there is a sequence of subspaces $S(\phi) \subsetneqq S(\psi_1) \subsetneqq \ldots \subsetneqq S(\psi_{d+1})$. Since the dimension of $S(\phi)$ can't be less than 0, and the rank of each predecessor subspace is strictly less than the rank of its successor, the dimension of $S(\psi_{d+1})$ is $\geq d+1$. So it can't be a subspace of $\mathfrak{G}$ of dimension is $\leq d$.

$\Rightarrow$. Let $\mathfrak{B}(\mathfrak{G}) \models \neg O_{\psi_1 \ldots \psi_{d+1}}\phi$. Assume towards a contradiction that dimension of $\mathfrak{G}$ bigger than d. Then there is a sequence of subspaces $S(\phi) \subsetneqq S(\psi_1) \subsetneqq \ldots \subsetneqq S(\psi_{d+1})$, and we can construct some model $\mathfrak{M}$ such that $\mathfrak{M}, x \models O_{\psi_1 \ldots \psi_{d+1}}\phi$. ∎

To prove the Completeness theorem in the next section we need some arithmetical facts about our derivation system

2.7. Lemma. *The following facts hold in MPG_d:*

(a) $\vdash (\phi \wedge \psi) \circ \chi \rightarrow \phi \circ \chi \wedge \psi \circ \chi$

(b) $\vdash ((\phi \rightarrow \psi) \wedge (\phi' \rightarrow \psi')) \rightarrow (\phi \circ \phi' \rightarrow \psi \circ \psi')$

(c) $\vdash \bigwedge_{i=1}^{n}((\psi_i \wedge O_{q_1 \ldots q_d}p) \circ O_{t_1 \ldots t_d}s) \rightarrow (\bigwedge_{i=1}^{n} \psi_i \wedge O_{q_1 \ldots q_d}p) \circ O_{t_1 \ldots t_d}s$

(d) $\vdash (\phi \wedge O_{q_1 \ldots q_d}p) \circ O_{t_1 \ldots t_d}s \wedge O_{q_1 \ldots q_d}p \circ (\psi \wedge O_{t_1 \ldots t_d}s) \rightarrow (\phi \wedge O_{q_1 \ldots q_d}p) \circ (\psi \wedge O_{t_1 \ldots t_d}s)$

(e) $\vdash \neg(\phi \wedge \psi \circ \chi) \Leftrightarrow \vdash \neg(\chi \wedge \psi \circ \phi)$

PROOF. (a), (b) is due to A2; (c) and (d) — by A11. ∎

2.8. Fact. *The difference operator D is not expressible in MLG interpreted in B_d, $d > 1$.*

PROOF. The D-operator is a modal operator which holds at a point x in a model M iff there is a point y such that $x \neq y$ and p holds at y. Consider the unary operator O with the following semantics: Op holds at a point x in a model M iff x is the only point where p holds. Observe that if Op holds at a point in a model then it can't hold at any other point of that model. Op is definable via the D-operator as $p \wedge \neg Dp$. (For more details on operators D and O see [Venema 1993].) So, if D is expressible in a language then O is too. In the rest of the proof we show that, in fact, O is not expressible in MLG interpreted in B_d.

Suppose Op is expressible by a MPG_d-formula $A(p)$. Since Op depends on only one variable p, we can assume without loss of generality that $A(p)$ contains no variables other than p.

Consider two projective geometries $\mathfrak{G}$ and $\mathfrak{G}'$ such that both of dimension > 1. Let x be a point in $\mathfrak{G}$, and let l be a line in $\mathfrak{G}'$. Notice that $\mathfrak{G} \setminus \{x\} \neq \emptyset$ and $\mathfrak{G}' \setminus l \neq \emptyset$.

Define two models $\mathfrak{M} = (\mathfrak{B}(\mathfrak{G}), V)$ and $\mathfrak{M}' = (\mathfrak{B}(\mathfrak{G}'), V')$ such that $\mathfrak{B}(\mathfrak{G})$, $\mathfrak{B}(\mathfrak{G}')$ are relation structures associated with geometries $\mathfrak{G}$ and $\mathfrak{G}'$, respectively, and $V(p) = \{x\}, V'(p) = l$.

For points $a \in \mathfrak{B}(\mathfrak{G})$ and $b \in \mathfrak{B}(\mathfrak{G}')$ define a binary relation I:

$$aIb \equiv \begin{cases} a = x & \text{iff} \quad b \in l \\ a \in \mathfrak{G} \setminus \{x\} & \text{iff} \quad b \in \mathfrak{G}' \setminus l \\ a = e & \text{iff} \quad b = e \end{cases}$$

Let ϕ be any MPG_d-formula containing no variables other than p. By induction on ϕ we can verify

CLAIM. For any $a \in \mathfrak{B}(\mathfrak{G})$ and $b \in \mathfrak{B}(\mathfrak{G}')$, if aIb then $\mathfrak{M}, a \models \phi$ iff $\mathfrak{M}', b \models \phi$.

Now assume $A(p)$, which we suppose to define Op, holds in x from $\mathfrak{M}$. So, by the definition of I and the above Claim, $A(p)$ holds at all points b of line l from $\mathfrak{M}'$, and therefore can't express Op because, as we have noticed at the beginning of the proof, Op can hold only at one point in a model, due to the truth conditions of Op. ∎

3. Main theorem

In this section we focus ourself on the proof for the Completeness Theorem, in which we used ideas and methods from [Venema 1993].

3.1. Theorem. (Completeness Theorem) *MPG_d is strongly complete with respect to B_d.*

We begin the way towards the proof of the Completeness Theorem constructing the canonical model which consists of maximally consistent sets (MCS's) of a special form. We call such MCS's good MCS's. A good MSC, an MCS extended with O-formulas (name-formulas), containes, due to its form, the information about names of all other good MCS's connected with it. The latter plays an important role in establishing that the ternary relation C^* of our canonical structure is according to Definition 1.2, and that our canonical structure belongs to the class of relation structure B_d.

NOTATION. Throughout the section $\overline{O}_{q_1 \ldots q_d} p$ denotes $(1' \vee O_{q_1 \ldots q_d} p)$.

3.2. Definition. *Let η and ϕ be formulas in which $p, q_1, \ldots, q_d$ do not occur. If η is a subformula of ϕ we don't identify different occurences of η in ϕ. Define $\phi[{}^{\overline{O}_{q_1 \ldots q_d} p \wedge \eta}/_{\eta}]$:*

$$\begin{aligned}
\phi[{}^{\overline{O}_{q_1\cdots q_d}p\wedge\eta}/_{\eta}] &= \phi \textit{ if } \eta \textit{ is not a subformula of } \phi,\\
\phi[{}^{\overline{O}_{q_1\cdots q_d}p\wedge\eta}/_{\eta}] &= \overline{O}_{q_1\ldots q_d}p \wedge \phi \textit{ if } \eta \textit{ is } \phi,\\
\neg\phi[{}^{\overline{O}_{q_1\cdots q_d}p\wedge\eta}/_{\eta}] &= \neg\phi \textit{ if } \eta \textit{ is a subformula of } \phi,
\end{aligned}$$

$$(\phi \wedge \psi)[{}^{\overline{O}_{q_1\cdots q_d}p\wedge\eta}/_{\eta}] = \begin{cases} \phi[{}^{\overline{O}_{q_1\cdots q_d}p\wedge\eta}/_{\eta}] \wedge \psi & \textit{if } \eta \textit{ is a subformula of } \phi,\\ \phi \wedge \psi[{}^{\overline{O}_{q_1\cdots q_d}p\wedge\eta}/_{\eta}] & \textit{if } \eta \textit{ is a subformula of } \psi,\end{cases}$$

$$(\phi \circ \psi)[{}^{\overline{O}_{q_1\cdots q_d}p\wedge\eta}/_{\eta}] = \begin{cases} \phi[{}^{\overline{O}_{q_1\cdots q_d}p\wedge\eta}/_{\eta}] \circ \psi & \textit{if } \eta \textit{ is a subformula of } \phi,\\ \phi \circ \psi[{}^{\overline{O}_{q_1\cdots q_d}p\wedge\eta}/_{\eta}] & \textit{if } \eta \textit{ is a subformula of } \psi.\end{cases}$$

A maximal consistent set Δ *is* good *(gMCS) if for every* $\phi \in \Delta$ *and every subformula* η *of* ϕ *there are propositional variables* $p, q_1, \ldots, q_d$ *with* $\phi[{}^{\overline{O}_{q_1\cdots q_d}p\wedge\eta}/_{\eta}] \in \Delta$.

Our next step is to construct such a good maximal consistent set.

3.3. Definition. *Let* Γ *be a consistent set. We call a pair* (ϕ, η)*, where* $\phi \in \Gamma$ *and* η *is a subformula of* ϕ*, a* defect *of* Γ *if* $\phi[{}^{\overline{O}_{q_1\cdots q_d}p\wedge\eta}/_{\eta}] \notin \Gamma$ *for some propositional variables* $p, q_1, \ldots, q_d$.

Let's fix a consistent set Δ of formulas. We construct a sequence of consistent sets of formulas such that $\Delta_0 \subseteq \Delta_1 \subseteq \ldots$

NOTATION. $Var(\Delta_i)$ denotes the set of all propositional variables of a consistent set Δ_i.

basic step $\Delta_0 = \Delta$.

odd steps

$$\Delta_{2n+1} = \begin{cases} \Delta_{2n} \cup \{\chi\} & \text{if } \Delta_{2n+1} \cup \{\chi\} \text{ is consistent}\\ \Delta_{2n} \cup \{\neg\chi\} & \text{otherwise.}\end{cases}$$

even steps

$$\Delta_{2n} = \begin{cases} \Delta_{2n-1} \cup \{\phi[{}^{\overline{O}_{q_1\cdots q_d}p\wedge\eta}/_{\eta}]\}, p, q_1, \ldots, q_d \notin Var(\Delta_{2n-1}), \\ \text{ if } \Delta_{2n-1} \text{ has a defect } (\phi, \eta), \\ \\ \Delta_{2n-1}, \text{ otherwise.}\end{cases}$$

Define $\overline{\Delta} = \bigcup_{n<\omega} \Delta_n$.

In a standard way we can define enumerations of all MPG_d-formulas and all defects, and fix a procedure guaranteeing that for each MPG_d-formula there is an odd step in our construction on which this formula is considered, and for each defect there is an even step in our construction on which this defect is repeared (an appropriate $\phi[{}^{\overline{O}_{q_1\cdots q_d}p\wedge\eta}/_{\eta}]$ is added).

Now we check whether the set constructed in the above way is a good MCS.

3.4. Lemma. *(i) On each step* i *of the construction* Δ_i *is consistent.*

(ii) $\overline{\Delta} = \bigcup_{n<\omega} \Delta_n$ *is a good* MCS.

PROOF. (i). Δ_0 is obviously consistent. Each odd step brings a consistent set in a standard way. Each even step yields a consistent set by the following Claim:

CLAIM. Let Δ be a consistent set of MPG_d-formulas such that $p, q_1, \ldots, q_d \notin Var(\Delta)$, and (ϕ, η) be a defect of Δ. Then $\Delta \cup \{\phi[{}^{\overline{O}_{q_1\cdots q_d}p\wedge\eta}/_{\eta}]\}$ is a consistent set for all subformulas η of ϕ.

Proof of Claim. Suppose $\Delta \cup \{\phi[{}^{\overline{O}_{q_1\ldots q_d} p \wedge \eta}/_{\eta}]\}$ is inconsistent. Then

$$\vdash (\phi[{}^{\overline{O}_{q_1\ldots q_d} p \wedge \eta}/_{\eta}] \wedge \bigwedge_{i=n}^{1} \xi_i) \to \bot,$$

for some $\xi_i \in \Delta$. For each clause of Definition 3.2 we can show, by applying rule R3, Lemma 2.7(e), that

$$\vdash (\phi[{}^{\overline{O}_{q_1\ldots q_d} p \wedge \eta}/_{\eta}] \wedge \bigwedge_{i=n}^{1} \xi_i) \to \bot \Rightarrow \vdash (\phi \wedge \bigwedge_{i=n}^{1} \xi_i) \to \bot.$$

The latter contradicts that Δ is consistent. ◀

(ii). Clearly, the union of consistent sets Δ_i such that for each $0 \leq i < \omega$ we have $\Delta_i \subseteq \Delta_{i+1}$, is a consistent set. The step-by-step way we construct $\overline{\Delta}$ provides maximality and goodness obviously. ■

We define relations C^*, R^* and I^* between $gMCS$'s exactly as it would be for standard canonical relations:

3.5. Definition. *Let Σ, Γ, Δ be $gMCS$'s.*

$C^*\Sigma\Gamma\Delta$	*iff*	*for all* $\phi \in \Gamma$, *for all* $\psi \in \Delta, \phi \circ \psi \in \Sigma$
$I^*\Gamma$	*iff*	1' $\in \Gamma$
$R^*\Gamma\Delta$	*iff*	$\Gamma = \Delta$.

To guarantee that our canonical structure is of a proper shape, namely, that there is only one world E such that 1' $\in E$, as well as any formula $O_{q_1\ldots q_d} p$ belongs only to one world in our structure, we need some special condition, namely, connectedness of worlds in the structure.

3.6. Definition. *Two $gMCS$'s Γ and Δ are* connected, *$\Gamma \approx \Delta$, if there is a $gMCS$ Σ such that $C^*\Gamma\Delta\Sigma$.*

3.7. Lemma. *$\approx$ is an equivalence relation.*

Proof. (1) For reflexivity we show that $C^*\Gamma\Gamma\Gamma$ for any $gMCS$ Γ, i.e. for all $\phi, \psi \in \Gamma$ we have $\phi \circ \psi \in \Gamma$. Clearly, starting with $\phi \wedge \psi \in \Gamma$ and applying axiom A7 we obtain $(\phi \wedge \psi) \circ (\phi \wedge \psi) \in \Gamma$, and hence $\phi \circ \psi \in \Gamma$, by Lemma 2.7(a).

(2) For symmetry ($\Gamma \approx \Delta \Leftrightarrow \Delta \approx \Gamma$) we show that there is an $gMCS$ Σ such that $C^*\Gamma\Delta\Sigma$ iff there is an $gMCS$ Σ such that $C^*\Delta\Gamma\Sigma$. Let us check 'only if' direction. There is an $gMCS$ Σ such that for all $\phi \in \Delta, \psi \in \Sigma$, we have $\phi \circ \psi \in \Gamma$, hence $\phi \circ \psi \wedge \chi \in \Gamma$, for all $\chi \in \Gamma, \phi \in \Delta, \psi \in \Sigma$. Assume, towards contradiction, that for all Σ there are some $\chi \in \Gamma, \psi \in \Sigma$ such that $\neg(\chi \circ \psi) \in \Delta$. Then $\neg(\chi \circ \psi) \circ \psi \wedge \chi \in \Gamma$, whence $\bot \in \Gamma$, by axioms A5 and A6. So we conclude that $\Delta \approx \Gamma$. The other direction is similar.

(3) Transitivity: $\Gamma \approx \Delta, \Delta \approx \Sigma \Rightarrow \Gamma \approx \Sigma$. By Definition 3.6 and the definition of C^* there is Θ such that for all $\phi \in \Delta, \theta \in \Theta$ we have $\phi \circ \theta \in \Gamma$, and there is Υ such that for all $\xi \in \Sigma, \gamma \in \Upsilon$ we have $\xi \circ \gamma \in \Delta$. So $(\xi \circ \gamma) \circ \theta \in \Gamma$, and hence $\xi \circ (\gamma \circ \theta) \in \Gamma$, by A9. Since Γ is a $gMCS$ we have $(\xi \wedge \overline{O}_{q_{1\Sigma}\ldots q_{d\Sigma}} p_\Sigma) \circ (\gamma \circ \theta \wedge \overline{O}_{q_{1\Psi}\ldots q_{d\Psi}} p_\Psi) \in \Gamma$ for some propositional letters $p_\Sigma, q_{1\Sigma}, \ldots, q_{d\Sigma}, p_\Psi, q_{1\Psi}, \ldots, q_{d\Psi}$. Define

$$\Psi = \{\psi : \overline{O}_{q_{1\Sigma}\ldots q_{d\Sigma}} p_\Sigma \circ (\psi \wedge \overline{O}_{q_{1\Psi}\ldots q_{d\Psi}} p_\Psi) \in \Gamma\}.$$

We have to show that:

(a) Ψ is consistent;
(b) Ψ is maximal;
(c) Ψ is good;
(d) $C^*\Gamma\Sigma\Psi$.

Proof of (a). Assume Ψ is inconsistent. Then

$$\vdash (\bigwedge_{i=1}^{n} \psi_i) \to \bot,$$

for some $\psi_1, \ldots, \psi_n \in \Psi$ such that $\bigwedge_{i=1}^{n}(\overline{O}_{q_{1\Sigma}\ldots q_{d\Sigma}} p_\Sigma \circ (\psi_i \wedge \overline{O}_{q_{1\Psi}\ldots q_{d\Psi}} p_\Psi)) \in \Gamma$. Using Lemma 2.7(c) and (a) we obtain $\overline{O}_{q_{1\Delta}\ldots q_{d\Delta}} p_\Delta \circ \bot \in \Gamma$ facing a contradiction.

Proof of (b). Since $\overline{O}_{q_{1\Sigma}\ldots q_{d\Sigma}} p_\Sigma \circ \overline{O}_{q_{1\Psi}\ldots q_{d\Psi}} p_\Psi \in \Gamma$, for each ψ either $\overline{O}_{q_{1\Sigma}\ldots q_{d\Sigma}} p_\Sigma \circ (\psi \wedge \overline{O}_{q_{1\Psi}\ldots q_{d\Psi}} p_\Psi) \in \Gamma$ or $\overline{O}_{q_{1\Sigma}\ldots q_{d\Sigma}} p_\Sigma \circ (\neg\psi \wedge \overline{O}_{q_{1\Psi}\ldots q_{d\Psi}} p_\Psi) \in \Gamma$, so Ψ is maximal.

Proof of (c). Let $\psi \in \Psi$ and η is a subformula of ψ. We show that there are propositional variables $s, t_1, \ldots, t_d$ with $\psi[^{\overline{O}_{t_1\ldots t_d} s \wedge \eta}/_\eta] \in \Psi$. Γ is a $gMCS$, so

$$(\overline{O}_{q_{1\Sigma}\ldots q_{d\Sigma}} p_\Sigma \circ (\psi \wedge \overline{O}_{q_{1\Psi}\ldots q_{d\Psi}} p_\Psi))[^{O_{t_1\ldots t_d} s \wedge \eta}/_\eta] \in \Gamma,$$

for some fresh $s, t_1, \ldots, t_d$, whence

$$\overline{O}_{q_{1\Sigma}\ldots q_{d\Sigma}} p_\Sigma \circ (\overline{O}_{q_{1\Psi}\ldots q_{d\Psi}} p_\Psi \wedge \psi[^{\overline{O}_{t_1\ldots t_d} s \wedge \eta}/_\eta]) \in \Gamma,$$

by Definition 3.2. Therefore $\psi[^{\overline{O}_{t_1\ldots t_d} s \wedge \eta}/_\eta] \in \Psi$, by our definition of Ψ.

Proof of (d). Observe two facts:

$$(\xi \wedge \overline{O}_{q_{1\Sigma}\ldots q_{d\Sigma}} p_\Sigma) \circ (\gamma \circ \theta \wedge \overline{O}_{q_{1\Psi}\ldots q_{d\Psi}} p_\Psi) \in \Gamma \text{ for all } \xi \in \Sigma,$$

and

$$\overline{O}_{q_{1\Sigma}\ldots q_{d\Sigma}} p_\Sigma \circ (\psi \wedge \overline{O}_{q_{1\Psi}\ldots q_{d\Psi}} p_\Psi) \in \Gamma \text{ for all } \psi \in \Psi$$

($\gamma \circ \theta \in \Psi$, indeed). By axioms A11 and A5, they bring us

$$(\xi \wedge \overline{O}_{q_{1\Sigma}\ldots q_{d\Sigma}} p_\Sigma) \circ (\psi \wedge \overline{O}_{q_{1\Psi}\ldots q_{d\Psi}} p_\Psi) \in \Gamma$$

for all $\xi \in \Sigma, \psi \in \Psi$. Hence $\xi \circ \psi \in \Gamma$, by Lemma 2.7(a), and so $C^*\Gamma\Sigma\Psi$, by Definition 3.5 of C^*. ∎

Finally, we are ready to define our canonical structure:

3.8. Definition. *A* good canonical frame *is*

$$\mathfrak{F} = (B^*, C^*, R^*, I^*),$$

where B^* is $\approx$-equivalence class of $gMCS$'s, and C^*, R^*, I^* as in Definition 3.5.

A good canonical model *is $\mathfrak{M}^* = (\mathfrak{F}, V^*)$ where $V^*(p) = \{\Gamma \in B^* : p \in \Gamma\}$.*

The following lemma is a technical result important for the proof of the Truth Lemma:

3.9. Lemma. *Let Σ be a $gMCS$ and $\phi \circ \psi \in \Sigma$. Then there are $gMCS$'s Γ and Δ such that $\phi \in \Gamma, \psi \in \Delta$ and $C^*\Sigma\Gamma\Delta$.*

PROOF. Since Σ is a $gMCS$ and $\phi \circ \psi \in \Sigma$ we have $(\phi \wedge \overline{O}_{q_{1\Gamma}\dots q_{d\Gamma}} p_\Gamma) \circ (\psi \wedge \overline{O}_{q_{1\Delta}\dots q_{d\Delta}} p_\Delta) \in \Sigma$ for some propositional letters $p_\Gamma, q_{1\Gamma}, \dots, q_{d\Gamma}, p_\Delta, q_{1\Delta}, \dots, q_{d\Delta}$.

Define:

$$\Gamma = \{\psi : (\psi \wedge \overline{O}_{q_{1\Gamma}\dots q_{d\Gamma}} p_\Gamma) \circ \overline{O}_{q_{1\Delta}\dots q_{d\Delta}} p_\Delta \in \Sigma\},$$
$$\Delta = \{\vartheta : \overline{O}_{q_{1\Gamma}\dots q_{d\Gamma}} p_\Gamma \circ (\vartheta \wedge \overline{O}_{q_{1\Delta}\dots q_{d\Delta}} p_\Delta) \in \Sigma\}.$$

The proof that Γ, Δ are $gMCS$'s goes similarly as the proof of clause 3(a)—(c) in Lemma 3.7. To show that $C^*\Sigma\Gamma\Delta$ we use definitions of Γ, Δ, Lemma 2.7(d),(a). ∎

3.10. Lemma. *(Truth Lemma) For all good models $\mathfrak{M}^*$ and all $\Gamma \in \mathfrak{M}^*$:*

$$\mathfrak{M}^*, \Gamma \models \psi \text{ iff } \psi \in \Gamma.$$

PROOF. By induction on ψ.
$\psi = \phi \circ \psi$. Assume $\mathfrak{M}^*, \Gamma \models \phi \circ \psi$. Then there are Δ, Σ such that $\mathfrak{M}^*, \Delta \models \phi$ and $\mathfrak{M}^*, \Sigma \models \psi$, and $C^*\Gamma\Delta\Sigma$. By inductive hypothesis, $\phi \in \Delta, \psi \in \Sigma$. Therefore $\phi \circ \psi \in \Gamma$, by the definition of C^*.

Assume $\phi \circ \psi \in \Gamma$. Then, by Lemma 3.9, there are Δ, Σ ($gMCS$'s) such that $\phi \in \Delta, \psi \in \Sigma$, and $C^*\Gamma\Delta\Sigma$. The latter, by Definition 3.6, means $\Gamma \approx \Delta$, and hence $\Delta \in \mathfrak{M}^*$. We obtain $C^*\Gamma\Sigma\Delta$ from $C^*\Gamma\Delta\Sigma$ as follows. We have $\phi \circ \psi \in \Gamma$ for all $\phi \in \Delta, \psi \in \Sigma$, therefore, by A5, $\psi \circ \phi \in \Gamma$. So $C^*\Gamma\Sigma\Delta$ and — by Definition 3.6 — $\Sigma \in \mathfrak{M}^*$.
By inductive hypothesis, $\mathfrak{M}^*, \Delta \models \phi$ and $\mathfrak{M}^*, \Sigma \models \psi$. So $\mathfrak{M}^*, \Gamma \models \phi \circ \psi$. ∎

Due to the connectedness, our canonical structure has a proper shape: there is only one world E such that 1' $\in E$, as well as any formula $O_{q_1\dots q_d} p$ belongs only to one world in our structure. The following two lemmas make it obvious:

3.11. Lemma. *Let A be $\approx$-equivalence class of $gMCS$'s. Then there is a unique $gMCS$ E in A such that 1' $\in E$.*

PROOF. The existence of a $gMCS$ E such that 1' $\in$ E is provided by A8 and Lemma 3.9. For uniqueness of such a $gMCS$ assume there are two distinct $gMCS$'s $\mathrm{E}, \mathrm{E}' \in A$ with 1' $\in$ E and 1' $\in \mathrm{E}'$. Since they are distinct there is a formula ϕ such that $\phi \in \mathrm{E}$ and $\neg\phi \in \mathrm{E}'$. E and E' are connected, so there is some Γ such that $C^*\mathrm{E}\mathrm{E}'\Gamma$. By the definition of C^* we have $(1' \wedge \neg\phi) \circ \top \in \mathrm{E}$. Hence $\Diamond(1' \wedge \neg\phi) \in \mathrm{E}$, by axiom A5 and the definition of $\Diamond$. We have also $\Diamond(1' \wedge \phi) \in \mathrm{E}$, by axiom A8, Lemma 2.7(b), and the definition of $\Diamond$. Using axiom A3 we obtain $\Diamond(1' \wedge \neg\phi \wedge \phi) \in \mathrm{E}$, i.e. there exists $gMCS$ containing $\perp$, which is a contradiction. ∎

3.12. Lemma. *If $\Gamma \in \mathfrak{M}^*$, and $O_{\psi_1\dots\psi_d}\phi \in \Gamma$ then Γ is the only $gMCS$ in $\mathfrak{M}^*$ with $O_{\psi_1\dots\psi_d}\phi \in \Gamma$.*

PROOF. Suppose there are two different $gMCS$'s Γ and Δ in $\mathfrak{M}^*$, both containing $O_{\psi_1\dots\psi_d}\phi$. So there is a formula χ such that $\chi \in \Gamma$ and $\neg\chi \in \Delta$. Since all $gMCS$'s are connected, there is Σ such that $C^*\Gamma\Delta\Sigma$, and so $C^*\Sigma\Gamma\Delta$. The latter means that $\top \circ (\chi \wedge O_{q_1\dots q_d} p) \in \Sigma$ and $\top \circ (\neg\chi \wedge O_{q_1\dots q_d} p) \in \Sigma$. Applying A11 we get $\top \circ (\neg\chi \wedge \chi \wedge O_{\psi_1\dots\psi_d}\phi) \in \Sigma$, hence there is a $gMCS$ Ψ s.t. $\perp \in \Psi$. ∎

We have reached the final step in our way towards the proof of the Completeness Theorem: we show that our canonical structure $\mathfrak{F}$ is a relation structure from the class B_d. First we establish that $\mathfrak{F}$ is in the class B of all relation structures isomorphic to those associated with projective geometries.

3.13. Lemma. *Let $\mathfrak{F}$ be a good canonical frame. Then $\mathfrak{F}$ is in* B.

PROOF. We show that $\mathfrak{F}$ satisfies conditions (i) — (vi) from Lemma 1.3.

(i) Lemma 3.11 guarantees the existence of a unique $gMCS$ E such that 1' $\in$ E. Since E is a good MCS it contains $O_{q_1 \ldots q_d} p$ for some propositional variables $p, q_1, \ldots, q_d$. By Lemma 3.12, E is the only $gMCS$ in our good canonical frame with $O_{q_1 \ldots q_d} p$, and so with p. Hence, $\Diamond(\neg p \wedge \neg 1') \in$ E. By Lemma 3.9, there is an gMCS Δ such that $(\neg p \wedge \neg 1') \in \Delta$, and $C^* \mathrm{EE}\Delta$. Thus we have one $gMCS$ Δ besides E. Δ also contains some $O_{r_1 \ldots r_d} s$ for some propositional variables $s, r_1, \ldots, r_d$. So, by Lemma 3.12, Δ is the only $gMCS$ in $\mathfrak{F}$ with $O_{r_1 \ldots r_d} s$, and so with s. Hence, $\Diamond(\neg s \wedge \neg 1') \in \Delta$, i.e. $(\top; (\neg s \wedge \neg 1')) \in \Delta$. Again by Lemma 3.9, there are good MCS's Σ and Ψ such that $\top \in \Sigma$, and $\neg s \wedge \neg 1' \in \Psi$, and $C^* \Delta\Sigma\Psi$. So Ψ is different from Δ and from E. Thus we have at least two $gMCS$'s, Δ and Ψ, besides E.

(ii) is provided due to A4.
(iii) We have $C^* \Gamma\Delta\Sigma$, i.e. for all $\delta \in \Delta, \xi \in \Sigma$, $\delta \circ \xi \in \Gamma$, hence $\delta \circ \xi \wedge \phi \in \Gamma$ for all $\phi \in \Gamma$. Assume, towards contradiction, that there are some $\phi \in \Gamma, \xi \in \Sigma$ with $\neg(\phi \circ \xi) \in \Delta$. Then $\neg(\phi \circ \xi) \circ \xi \wedge \phi \in \Gamma$, whence $\bot \in \Gamma$, by axioms A5 and A6. So $C^* \Delta\Gamma\Sigma$. To obtain $C^* \Gamma\Sigma\Delta$ from $C^* \Gamma\Delta\Sigma$ we use A5.

(iv) Assume $C^* \Gamma\Delta\Sigma$ and $C^* \Sigma\Theta\Phi$, i.e. for all $\delta \in \Delta, \theta \in \Theta, \phi \in \Phi$ we have $\delta \circ (\theta \circ \phi) \in \Gamma$. Then, by A9 (associativity of $\circ$), we get $(\delta \circ \theta) \circ \phi \in \Gamma$. Since Γ is a $gMCS$ we have $(\delta \circ \theta \wedge \overline{O}_{q_{1\Psi} \ldots q_{d\Psi}} p_\Psi) \circ (\phi \wedge \overline{O}_{q_{1\Phi} \ldots q_{d\Phi}} p_\Phi) \in \Gamma$ for some propositional letters $p_\Psi, q_{1\Psi}, \ldots, q_{d\Psi}, p_\Phi, q_{1\Phi}, \ldots, q_{d\Phi}$. Define

$$\Psi = \{\psi : (\psi \wedge \overline{O}_{q_{1\Psi} \ldots q_{d\Psi}} p_\Psi) \circ \overline{O}_{q_{1\Phi} \ldots q_{d\Phi}} p_\Phi \in \Gamma\}.$$

The proof that Ψ is an $gMCS$ goes similarly as the proof of clause (3) in Claim 3.7.

$C^* \Gamma\Psi\Phi$ holds due to two facts — $(\delta \circ \theta \wedge \overline{O}_{q_{1\Psi} \ldots q_{d\Psi}} p_\Psi) \circ (\phi \wedge \overline{O}_{q_{1\Phi} \ldots q_{d\Phi}} p_\Phi) \in \Gamma$ for all $\phi \in \Phi$, and $(\psi \wedge \overline{O}_{q_{1\Psi} \ldots q_{d\Psi}} p_\Psi) \circ \overline{O}_{q_{1\Phi} \ldots q_{d\Phi}} p_\Phi \in \Gamma$ for all $\psi \in \Psi$ ($\delta \circ \theta \in \Psi$, indeed) — which yield, by axioms A11 and A5, $(\psi \wedge \overline{O}_{q_{1\Psi} \ldots q_{d\Psi}} p_\Psi) \circ (\phi \wedge \overline{O}_{q_{1\Phi} \ldots q_{d\Phi}} p_\Phi) \in \Gamma$, whence $\psi \circ \phi \in \Gamma$ for all $\phi \in \Phi$, $\psi \in \Psi$, by Lemma 2.7(a).

Since $(\delta \circ \theta) \in \Psi$ for all $\delta \in \Delta, \theta \in \Theta$ (by the definition of Ψ) we also have that $C^* \Psi\Delta\Theta$, and therefore $C^* \Delta\Theta\Psi$ due to clause (iii) of the present lemma.

(v) use A8 and Lemma 3.12.
(vi) Let Γ be a $gMCS$ such that 1' $\notin \Gamma$ and $C^* \Gamma\Gamma\Delta$. $\Rightarrow$: Assume for contradiction that $\Delta \neq \Gamma$ and 1' $\notin \Delta$. Since Γ and Δ are different there is a MPG_d-formula ξ which belongs to Δ and does not belong to Γ. Γ is a $gMCS$, so $O_{q_1 \ldots q_d} p \in \Gamma$ for some $p, q_1, \ldots, q_d$. Now we have $(O_{q_1 \ldots q_d} p \wedge \neg \xi \wedge \Diamond(\xi \wedge \neg 1')) \in \Gamma$. By A10, $\neg(p \circ (\xi \wedge \neg 1')) \in \Gamma$. On the other hand, $p \in \Gamma$, $(\xi \wedge \neg 1') \in \Delta$ and $C^* \Gamma\Gamma\Delta$ together with the assumption that $\Delta \neq \Gamma \neq$ E yield $p \circ (\xi \wedge \neg 1')) \in \Gamma$, according to the definition of C^*.

$\Leftarrow$: use A7 if $\Delta = \Gamma$, or A8 if Δ is E. ∎

To prove that our good canonical frame is a relation structure belonging to the class B_d, we show that the dimension of the geometry built from our good canonical frame is d; this guarantees that our good canonical frame is in B_d.

As in [Givant 1997], we construct geometry $\mathfrak{G}(\mathfrak{F}) = (P, L, In)$ from a good canonical frame $\mathfrak{F} = (B^*, C^*, R^*, I^*)$. Obviously, $P = B^* \setminus \{E\}$. To define L we need the notion of collinearity in terms of our frame. Three points Γ, Δ, Σ are collinear iff $e \notin \{\Gamma, \Delta, \Sigma\}$, and either $\Gamma = \Delta$, or $\Gamma \neq \Delta$ and one of $C^* \Gamma\Delta\Sigma$, or $\Sigma = \Gamma$, or $\Sigma = \Delta$ holds. For each pair of distinct $gMCS$'s Γ, Δ, a *line* $\overline{\Gamma\Delta}$ is the set of $gMCS$'s that are collinear with Γ and Δ. L is the set of such lines. In is the relation between a point Γ and a line $\overline{\Delta\Sigma}$ when $C^* \Gamma\Delta\Sigma$ and Γ, Δ, Σ are collinear and distinct. To check whether the resulting structure satisfies the axioms (A) — (C) one should use the clauses from Lemma 1.3. The proof goes very close to the proof of Lemma 3.4 in [Givant 1997], so we omit the details.

3.14. Lemma. *Let $\mathfrak{F} = (B^*, C^*, R^*, I^*)$ be a good canonical frame. Then the dimension of the geometry $\mathfrak{G}$ built from $\mathfrak{F}$ is d.*

Proof.

Suppose $\mathfrak{F} = (B^*, C^*, R^*, I^*)$ is a good canonical frame and $\mathfrak{G}(\mathfrak{F})$ is the geometry built from $\mathfrak{F}$. We have $\mathfrak{F} \models \neg O_{\psi_1 \ldots \psi_{d+1}} \phi$. Assume, for a contradiction, that the dimension of $\mathfrak{G}(\mathfrak{F})$ is greater than d. Then there is a sequences of subspaces $S_1 \subsetneq \ldots \subsetneq S_{d+1}$. Then we can construct a model $\mathfrak{M}'$ based on the relation structure $\mathfrak{B}(\mathfrak{G}(\mathfrak{F}))$ such that $\mathfrak{M}', x \models O_{\psi_1 \ldots \psi_{d+1}} \phi$, for some $x \in \mathfrak{G}(\mathfrak{F})$. It is easy to verify, using Definition 1.2, that the relation structure $\mathfrak{B}(\mathfrak{G}(\mathfrak{F}))$ associated with $\mathfrak{G}(\mathfrak{F})$ is isomorphic to $\mathfrak{F}$, so $\mathfrak{M}', x \models \neg O_{\psi_1 \ldots \psi_{d+1}} \phi$. A contradiction.

Now we have to show that the dimension of $\mathfrak{G}(\mathfrak{F}) \geq d$. Each $gMCS$ Γ_i contains name formulas $O_{q_1 \ldots q_d} p$. So $\mathfrak{M}^*, \Gamma_i \models O_{q_1 \ldots q_d} p$, by Truth Lemma. $O_{q_1 \ldots q_d} p$ describes a sequence of subspaces $S(q_i), 1 \leq i < d$, such that each $S(q_i)$ properly contains in $S(q_{i+1})$, and $S(p)$ is a point, by Lemma 2.4. This means, by the geometric fact, that $S(q_d) \subseteq \mathfrak{G}(\mathfrak{F})$ is of dimension greater or equal to d. ∎

3.15. Corollary. *Let $\mathfrak{F}$ be a good canonical frame. Then $\mathfrak{F} \in \mathsf{B}_d$.*

PROOF. From $\mathfrak{F}$ we obtain the geometry $\mathfrak{G}(\mathfrak{F})$ which dimension is d, by Lemma 3.14. Since the relation structure $\mathfrak{B}(\mathfrak{G}(\mathfrak{F}))$ associated with $\mathfrak{G}(\mathfrak{F})$ is isomorphic to $\mathfrak{F}$, $\mathfrak{F} \in \mathsf{B}_d$. ∎

PROOF OF THEOREM 3.1. Suppose $\Delta \not\vdash \phi$. So $\Delta \cup \{\neg\phi\}$ is consistent set, and therefore is extendable to a $gMCS$ Δ', as we saw at the beginning of this section and in Lemma 3.4. Δ' together with all $gMCS$'s $\approx$-equivalent to Δ' constitute the domain of our good canonical frame $\mathfrak{F}$ and a corresponding good canonical model $\mathfrak{M}^*$. By Lemma 3.10, $\mathfrak{M}^*, \Delta' \models \chi$ for all $\chi \in \Delta \cup \{\neg\phi\}$, hence $\Delta \not\models \phi$. QED

4. Pappus' theorem and finite model property

In this section we use Pappus' theorem, a well-known concept in projective geometry, in order to establish that for $d \geq 3$ our logic MPG_d lacks the finite model property. On the one hand, Pappus' theorem holds in any *finite* projective geometry $\mathfrak{G}$ of dimension ≥ 3 (see [Garner 1981], p.104, Theorem 12, and section 6.2). On the other hand, one can construct an infinite geometry in which Pappus' theorem fails (for details see [Ellis 1992], Chapter 6). In order to show that MPG_d does not have the finite model property we construct in our language a formula (F_d), and prove that it is equivalent to Pappus' theorem (Lemma 4.2 below). Therefore in any finite model associated with a finite projective geometry of dimension ≥ 3 formula F_d holds, meanwhile F_d fails in an infinite model associated with the infinite geometry in which Pappus' theorem doesn't hold. So we obtain the following result

4.1. Theorem. *MPG_d, $d \geq 3$, does not have the finite model property.*

In the above argument the only step we are left to prove is that formula F_d is equivalent to Pappus' theorem. So in the rest of the section we will go through details of Lemma 4.2. First recall

Pappus' Theorem.
Let l and l' be two distinct lines incident with a point x. Let x_1, x_2, x_3 be three distinct points incident with line l, and y_1, y_2, y_3 three distinct points incident with line l' such that none of x_i or y_i, $i = 1, 2, 3$, are on both l and l'. Let z_1 be the point incident with the lines $\overline{x_2y_3}$ and $\overline{x_3y_2}$. Let z_2 be the point incident with the lines $\overline{x_1y_3}$ and $\overline{x_3y_1}$. Let z_3 be the point incident with the lines $\overline{x_1y_2}$ and $\overline{x_2y_1}$.
Then z_1, z_2, z_3 are collinear.

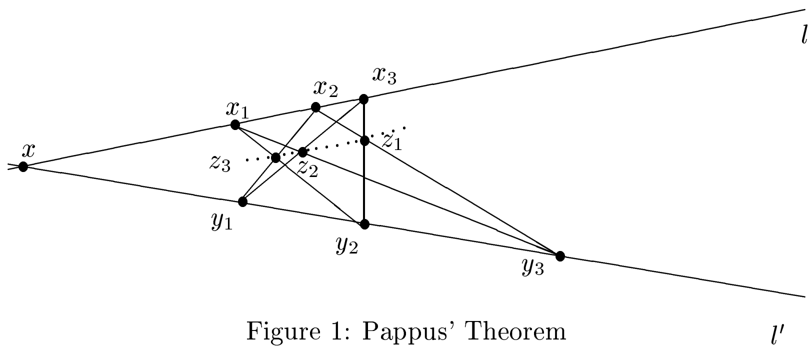

Figure 1: Pappus' Theorem

Now we construct formula F_d which turns out to be equivalent to Pappus' theorem.

Let $\mathbf{Inc}((\phi_1 \dots \phi_n), \psi, \overline{\chi})$ denotes the conjunction $\Diamond O_{\psi\chi_2\dots\chi_d}\phi_1 \wedge \dots \wedge \Diamond O_{\psi\chi_2\dots\chi_d}\phi_n$ (the points $\phi_1, \dots, \phi_n$ are incident with the line ψ).

$$\begin{aligned}
\text{Let } \mathbf{A} ::= \;& \mathbf{Inc}((p_1p_2p_3), q_1, \overline{r}) \wedge \Box(p_1 \to \neg p_2 \wedge \neg p_3) \wedge \Box(p_2 \to \neg p_3) \wedge \\
& \mathbf{Inc}((t_1t_2t_3), q_2, \overline{r}) \wedge \Box(t_1 \to \neg t_2 \wedge \neg t_3) \wedge \Box(t_2 \to \neg t_3) \wedge \\
& \Box(p_i \to \neg q_2) \wedge \Box(t_i \to \neg q_1) \wedge \Diamond(q_1 \wedge q_2) \wedge \\
& \mathbf{Inc}((p_2t_3s_1), q_3, \overline{r}) \wedge \mathbf{Inc}((p_3t_2s_1), q_4, \overline{r}) \wedge \mathbf{Inc}((p_1t_3s_2), q_5, \overline{r}) \wedge \\
& \mathbf{Inc}((p_3t_1s_2), q_6, \overline{r}) \wedge \mathbf{Inc}((p_1t_2s_3), q_7, \overline{r}) \wedge \mathbf{Inc}((p_2t_1s_3), q_8, \overline{r}),
\end{aligned}$$

for $i = 1, 2, 3$.

Let $\mathbf{Col}(s_1s_2s_3)$ denote $\Box(s_1 \leftrightarrow s_2) \vee \Box(s_2 \leftrightarrow s_3) \vee \Box(s_1 \leftrightarrow s_3) \vee \Box(s_1 \to s_2 \circ s_3)$ (points s_1, s_2, s_3 are collinear).

Finally, let $F_d = \mathbf{A} \to \mathbf{Col}(s_1s_2s_3)$.

4.2. Lemma. *For any finite projective geometry $\mathfrak{G}$ of dimension $2 \leq d < \omega$, Pappus' theorem holds in $\mathfrak{G}$ iff $\mathfrak{B}(\mathfrak{G}) \models F_d$.*

PROOF. $\Leftarrow$: Assume Pappus' theorem fails in $\mathfrak{G}$, i.e. there are distinct points $x, x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3$, and lines l, l', $\overline{x_2y_3}$, $\overline{x_3y_2}$, $\overline{x_1y_3}$, $\overline{x_3y_1}$, $\overline{x_1y_2}$ and $\overline{x_2y_1}$ satisfying the assumption of Pappus' theorem, but z_1, z_2, z_3 are not collinear. Consider the relation structure $\mathfrak{B}(\mathfrak{G}) = (\mathfrak{G} \cup \{e\}, C, R, I)$ associated with $\mathfrak{G}$. We construct a valuation V that falsifies F_d. Let $V(p_i) = \{x_i\}, V(t_i) = \{y_i\}$, $V(s_i) = \{z_i\}, V(q_1) = l$, $V(q_2) = l', V(q_3) = \overline{x_2y_3}$, $V(q_4) = \overline{x_3y_2}, V(q_5) = \overline{x_1y_3}, V(q_6) = \overline{x_3y_1}$, $V(q_7) = \overline{x_1y_2}, V(q_8) = \overline{x_2y_1}$; $V(r_2) = \langle l, l' \rangle$, i.e. the smallest plane containing the lines l and l'; and the $V(r_j)$'s, $3 \leq j < d$, are subspaces such that $V(r_2) \subsetneqq \dots \subsetneqq V(r_d)$ (this flag exists since the dimension of geometry $\mathfrak{G}$ is d).

Under such a valuation the antecedent of F_d holds while $\mathbf{Col}(s_1s_2s_3)$ fails. We leave to check details to the reader.

$\Rightarrow$. Assume there are V and x such that $(\mathfrak{B}(\mathfrak{G}), V), x \not\models F_d$, i.e. $(\mathfrak{B}(\mathfrak{G}), V), x \models \mathbf{A}$ and $(\mathfrak{B}(\mathfrak{G}), V), x \models \neg\mathbf{Col}(s_1s_2s_3)$.

Due to Lemma 2.3, $\mathbf{Inc}((p_1p_2p_3), q_1, \overline{r})$ says that there are points, say x_1, x_2, x_3, incident with line l, and $\Box(p_1 \to \neg p_2 \wedge \neg p_3) \wedge \Box(p_2 \to \neg p_3)$ says these points are distinct. $\mathbf{Inc}((t_1t_2t_3), q_2, \overline{r}) \wedge \Box(t_1 \to \neg t_2 \wedge \neg t_3) \wedge \Box(t_2 \to \neg t_3)$says that there are points, say y_1, y_2, y_3, incident with line l', and these points are distinct. $\Box(p_i \to \neg q_2) \wedge \Box(t_i \to \neg q_1) \wedge \Diamond(q_1 \wedge q_2)$ says that none of x_i or y_i, $i = 1, 2, 3$, are on both l and l', and there is a point, say x, in which lines l and l' intersect. $\mathbf{Inc}((p_2t_3s_1), q_3, \overline{r}) \wedge \mathbf{Inc}((p_3t_2s_1), q_4, \overline{r})$ states that point z_1 is incident with the line going through the points x_2 and y_3, and the line going through the points x_3 and y_2. $\mathbf{Inc}((p_1t_3s_2), q_5, \overline{r}) \wedge$

$\mathbf{Inc}((p_3 t_1 s_2), q_6, \overline{r})$, says the same about point z_2 and lines $\overline{x_1 y_3}$ and $\overline{x_3 y_1}$. $\mathbf{Inc}((x_1 s_2 r_3), q_7, \overline{r}) \wedge \mathbf{Inc}((x_2 s_1 r_3), q_8, \overline{r})$ says the same about point z_3 and lines $\overline{x_1 y_2}$ and $\overline{y_2 x_1}$. So, in fact, we have the antecedent of Pappus' theorem. Meanwhile formula $\neg\mathbf{Col}(s_1 s_2 s_3)$ describes points z_1, z_2, z_3 as being not collinear. Hence, Pappus' theorem doesn't hold in $\mathfrak{G}$. ∎

References

[Ellis 1992] G. Ellis. *Rings and Fields.* Clarendon press, Oxford, 1992.

[Garner 1981] L. Garner. *An Outline of Projective Geometry.* Elsevier North Holland, 1981.

[Givant 1997] S. Givant. Universal classes of simple relation algebras. *JSL* (forthcoming).

[Henkin et alii 1997] L. Henkin, J.D. Monk, A. Tarski, *Cylindric Algebras Part 2*, North-Holland, Amsterdam, 1985.

[Jónsson & Tarski 1952] B. Jónsson and A. Tarski, Boolean algebras with operators, Part 2". *American Journal of Mathematics*, **74** (1952), pp. 127-162.

[Lyndon 1961] R. Lyndon. Relation Algebras and projective geometries. *Michigan Mathematics Journal*, **8**, (1961), pp. 21-28.

[Maddux 1982] R. Maddux, Some varieties containing relation algebras. *Transactions of the American Mathematical Society*, 272(2), (1982), pp. 501-526.

[Venema 1996] Y. Venema. A crash course in arrow logic. In: Arrow Logic and Multi-Modal Logic, ed. M.Marx et alii, CSLI, (1996), pp. 3-34.

[Venema 1993] Y. Venema. Cylindric modal logic. *JSL*, 60(2), (1995), pp. 591-623.

[Venema 1993] Y. Venema. Derivation rules as anti-axioms in modal logic. *JSL*, **58**(3), (1993), pp. 1003-1034.