Dynamic Negation, The One and Only

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Abstract

We consider the variety of Dynamic Relation Algebras $\mathbf{V}(\mathsf{DRA})$. We show that the monoid of an algebra in this variety determines dynamic negation uniquely.

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1 Introduction

A Dynamic Relation Algebra or DRA is the algebra of all relations on a given domain D with operations id, the identity or diagonal relation, $\perp$, the empty relation, ;, composition (in the order of application), and dynamic negation $\sim$. Here:

- $a(\sim R)b :\Leftrightarrow a = b$ and, for no c, aRc.

Subsets of the given domain D can be represented as relations via the standard embedding diag, where $\mathsf{diag}(X) = \{\langle x, x\rangle \mid x \in X\}$. The elements of the range of diag are subrelations of the diagonal id. These subrelations are called *tests* or *conditions*. If we write $\mathsf{dom}(R)$ for the domain of R and $(.)^c$ for complementation in D, we have: $\sim R = \mathsf{diag}((\mathsf{dom}(R))^c)$.

The study of DRA's is motivated by *Dynamic Predicate Logic* (DPL, see [1]), a variant of classical predicate logic tailored to deal with such dynamic aspects of natural language semantics as anaphora. In DPL, formulas are not interpreted by a set of assignments (i.e., those that make them true), but by a binary relation between such assignments. DRA focusses on the propositional connectives of DPL, and generalizes the setting from relations on assignments to arbitrary binary relations.

DRA's where studied by the first author in his paper [2] and by the second author in his paper [6]. Hollenberg gives a finite axiomatization of the equations

A0:	$\sim\bot = \mathsf{id}$	(identity definition)
A1:	$\sim x; x = \bot$	(negation elimination)
A2:	$x; \bot = \bot$	(falsum right)
A3:	$\mathsf{id}; x = x$	(identity left)
A4:	$x; (y; z) = (x; y); z$	(associativity)
A5:	$\sim x; \sim y = \sim y; \sim x$	(test permutation)
A6:	$x = (\sim\sim x); x$	(domain test)
A7:	$\sim\sim(\sim x; \sim y) = \sim x; \sim y$	(test composition)
A8:	$\sim(x; y); x = (\sim(x; y); x); \sim y$	(modus ponens)
A9:	$\sim(x; \sim(\sim y; \sim z)) = \sim\sim(\sim(x; y); \sim(x; z))$	(distribution)

Table 1: AX.

valid in all DRA's. Visser shows that the equations valid in all DRA's are precisely the schematic equations valid in all DPL-models, thus firmly establishing that 'DRA is the propositional logic of DPL'.

In table 1 we give the finite axiomatization, AX, of equational validity in DRA's. We write $\vdash t_1 = t_2$ if this equation is derivable from the equations in AX and the rules of equational logic. So completeness means: $\models t_1 = t_2$ iff $\vdash t_1 = t_2$.

We repeat a few useful consequences of **AX** that are proved in [2].

- Identity right: $x; \mathsf{id} = x$.
- Triple negation law: $\sim\sim\sim x = \sim x$.
- Test idempotency: $\sim x; \sim x = \sim x$.
- Range test: $\sim(x; y) = \sim(x; \sim\sim y)$.
- Double negation law for falsum: $\sim\sim\bot = \bot$.
- Falsum left: $\bot; x = \bot$.

The following equations are *invalid*: $x; x = x$ and $x; \sim x = \bot$.

In [2] it is shown that a product of DRA's is (isomorphic to) a subalgebra of a DRA. However Hollenberg produces an example to show that there are algebras in $\mathbf{V}(\mathsf{DRA})$, i.e. algebras satisfying the equations of AX, that are still not sub-DRA's. We will call in this paper an algebra that satisfies AX a *Dynamic Negation Monoid* or DNM.

An *annihilator monoid* is a monoid with an element $\bot$ or 0 such that $\bot; x = x; \bot = \bot$. The reducts of DNM's obtained by omitting $\sim$ are annihilator monoids. We show in section 2 that not every annihilator monoid can be extended to a DNM. In section 3 we show that all extensions of a given monoid to a DNM are the same. Our result is analogous to the well known result described in the example below.

Example 1.1 This example, which we learned from [3], inspired the questions we answer in this paper.

Consider a domain D and suppose on D we have a monoid with operator ; and an upper semi-lattice with operator $\vee$. Residuations over this structure are binary operations $\rightarrow$ and $\leftarrow$ satisfying:

$$✠ \quad x;y \leq z \;\Leftrightarrow\; x \leq (z \leftarrow y) \;\Leftrightarrow\; y \leq (x \rightarrow z).$$

One can show that the 'pseudo identities' used in formulating ✠ can be replaced by identities.

The presence of residuations implies distributivity of ; over $\vee$. Hence, not every combination of monoid and upper semilattice can be extended with residuations. On the other hand, it is immediate from ✠ that *if* one can define residuations, *then* these are uniquely determined. This example subsumes similar results in Boolean Algebra and Heyting Algebra. ❑

There is a philosophical reason to be interested in our result. In dynamic semantics we would like to view the merge, the operation of 'adding up' items of information, as fundamental. In the relational semantics the merge takes the form of relation composition ;. What is the relationship of other operations to the merge? It is clear that it cannot be expected that these operations can be explicitly defined in terms of the merge. One strategy to answer this question is simply to eliminate all other operations in favour of the merge. See [4] and [5] for a first attempt in this direction. Another route would be to show that other operations are *implicitly definable* in terms of the merge. We think no such result is to be expected for generalized quantifiers and the like. However, our present result shows that at least dynamic negation is implicitly equationally definable.

2 A Non Extendable Annihilator Monoid

We show that not every annihilator monoid can be extended to a DNM. Consider the annihilator monoid $\mathcal{M}$ with domain $\{0, 1, 2\}$ with, as binary operation, multiplication modulo 4. It is easy to verify that we have indeed specified an annihilator monoid with 1 in the role of id, 0 in the role of $\perp$ and $\cdot$ in the role of ;.

Suppose $\mathcal{M}$ were extendable to a DNM. By A1, $\sim 1 \cdot 1 = 0$, so $\sim 1 = 0$. By A6, $\sim 0 = \sim 0 \cdot 1 = \sim\sim 1 \cdot 1 = 1$. By A8 we have:

$$2 = \sim 0 \cdot 2 = \sim(2 \cdot 2) \cdot 2 = (\sim(2 \cdot 2) \cdot 2) \cdot \sim 2 = \sim 0 \cdot 2 \cdot \sim 2 = 2 \cdot \sim 2$$

So ~ 2 must be 1. This would make $\sim\sim 2 = 0$, but then $2 = \sim\sim 2; 2 = 0$. Quod non.

3 Dynamic Negation Is Unique

We give the argument that negation in a DNM is fixed by the underlying monoid.[1] Suppose we have a non-empty set M with associative operation ;. Suppose further that we have elements id_1, id_2, $\bot_1$, $\bot_2$ and unary operations $\sim_1$ and $\sim_2$ such that both ;, id_1, $\bot_1$, $\sim_1$ and ;, id_2, $\bot_2$, $\sim_2$ satisfy AX.

It is easy to see that $\mathsf{id}_1 = \mathsf{id}_2$ and $\bot_1 = \bot_2$. We will write: $\mathsf{id} := \mathsf{id}_1 = \mathsf{id}_2$ and $\bot := \bot_1 = \bot_2$. To increase readability we will write $\sim := \sim_1$ and $\neg := \sim_2$. We prove that these negations must be equal. If we want to stress that an axiom is used for the operations of kind i, we add a subscript i to the axiom name.

step 1 We show that $\sim x; \neg x = \sim x$. By symmetry we also have: $\neg x; \sim x = \neg x$. In other words: the first negation prevails. Here is the argument:

$$
\begin{array}{rcll}
\sim x & = & \neg\bot; \sim x & \mathsf{A3}_2, \mathsf{A0}_2 \\
 & = & \neg(\sim x; x); \sim x & \mathsf{A1}_1 \\
 & = & \neg(\sim x; x); \sim x; \neg x & \mathsf{A8}_2 \\
 & = & \neg\bot; \sim x; \neg x & \mathsf{A1}_1 \\
 & = & \sim x; \neg x & \mathsf{A3}_2, \mathsf{A0}_2
\end{array}
$$

step 2 We show that $\sim\sim x; \neg x = \bot$. We have:

$$
\begin{array}{rcll}
\sim\sim x; \neg x & = & \sim\sim x; \neg\sim x; \neg x; \sim x & \text{step 1, twice} \\
 & = & \sim\sim x; \neg x; \neg\sim x; \sim x & \mathsf{A5}_2 \\
 & = & \bot & \mathsf{A1}_2, \mathsf{A2}
\end{array}
$$

step 3 We show that: $\sim\neg x = \sim\sim x$ (another case where the first negation prevails). Define $(y \vee z) := \sim(\sim y; \sim z)$. We will use the following obvious facts:

1. $\sim y \vee y = \mathsf{id}$,
2. $\bot \vee y = \sim\sim y$.

Here is the proof:

$$
\begin{array}{rcll}
\sim\neg x & = & \sim(\mathsf{id}; \neg x) & \mathsf{A3} \\
 & = & \sim((\sim\sim x \vee \sim x); \neg x) & \text{obvious fact 1} \\
 & = & \sim((\sim\sim x \vee \sim x); \sim\sim\neg x) & \text{(range test)}_1 \\
 & = & \sim(\sim\sim\neg x; (\sim\sim x \vee \sim x)) & \mathsf{A5}_1 \\
 & = & \sim((\sim\sim\neg x; \sim\sim x) \vee (\sim\sim\neg x; \sim x)) & \mathsf{A9}_1 \\
 & = & \sim((\sim\sim x; \sim\sim\neg x) \vee (\sim x; \sim\sim\neg x)) & \mathsf{A5}_1 \\
 & = & \sim((\sim\sim x; \neg x) \vee (\sim x; \neg x)) & \text{(range test)}_1 \\
 & = & \sim(\bot \vee \sim x) & \text{steps 2,1} \\
 & = & \sim\sim\sim\sim x & \text{obvious fact 2} \\
 & = & \sim\sim x & \text{(triple negation)}_1
\end{array}
$$

[1] The proof for DRA's was discovered by Albert Visser. It was generalized to cover all DNM's by Marco Hollenberg.

step 4 We have:

$$\begin{array}{rcll} \neg x & = & {\sim}{\sim}\neg x;\neg x & \mathsf{A6}_1 \\ & = & {\sim}{\sim}{\sim}x;\neg x & \text{step 3} \\ & = & {\sim}x;\neg x & (\text{triple negation})_1 \\ & = & {\sim}x & \text{step 1} \end{array}$$

This ends our proof: negation, if it exists, is unique.

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