Q-algebras

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1 Introduction

The paper [Jónsson 1991] contains a discussion and comparison of a number of possible operations on binary relations. Among these are the so-called Q-operations that are defined as follows.

Definition 1.1 *Let $\underline{R}$ be a matrix of binary relations R_{ij}, $0 \leq i,j < n$, and let k,l be two numbers smaller than n, to be called reference points*[1]. *Then $\mathsf{Q}_n^{kl}(\underline{R})$ is the binary relation defined by*

$$s\mathsf{Q}_n^{kl}(\underline{R})t \text{ iff there are } u_0,\ldots,u_{n-1} \text{ such that } s=u_k,\ t=u_l \text{ and } u_iR_{ij}u_j \text{ for all } i,j\in n.$$

As an example of such an operation, consider the following figure. It depicts the definition of

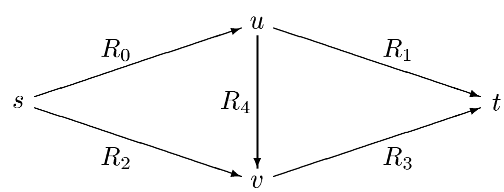

Figure 1: A quinary operation

a quinary operation $Q(R_0,R_1,R_2,R_3,R_4)$ that holds between s and t iff there are points u and v such that the binary relations R_i hold between these four points as in the picture. In the notation of the previous definition, the resulting relation could be expressed as

$$\mathsf{Q}_4^{01}\begin{pmatrix}1 & 1 & R_0 & R_2\\ 1 & 1 & 1 & 1\\ 1 & R_1 & 1 & R_4\\ 1 & R_2 & 1 & 1\end{pmatrix}$$

where 1 denotes the universal relation on the base set.

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[1] For the sake of a more transparent presentation of our axiom schemas, we deviate slightly from Jónsson's original approach where the reference points are *fixed* as $k=0$, $l=1$. It is not difficult to show that in fact, the two approaches are term-definitionally equivalent.

These Q-operations have a geometrical origin, but in [Jónsson 1991] Jónsson discusses them only in a context where he compares the expressive power of various clones of operations on binary relations. For instance, the operation defined in the picture above does not belong to the Tarski clone; that is, Q cannot be defined by using the boolean operations together with the identity constant and the converse and composition operations. One of the main problems posed by Jónsson is to find a simple set of operations that taken together will provide all the expressive power of first order logic. A natural candidate for this seemed to be the Q-clone, and indeed, if one only considers algebras of relations over a finite base set then we obtain a positive result, as was shown in [Németi & Andréka 1991]. On the other hand, these authors prove that if one also allows relations over infinite sets, then there are first order definable operations on binary relations that do not belong to Jónsson's Q-clone. The latter result was also proven independently in [Venema 1991].

Nevertheless, we believe that Q-operations provide interesting algebras. One of our reasons to believe so is that the representable Q-algebras (to be formally defined below) allow an axiomatization that is quite simple and transparent — at least, given the fact that we have a relatively complex similarity type. This representation result, Theorem 2 below, is the main technical result of our paper. Our axiomatization and the proof of our representation theorem are inspired by the paper [Nishimura 1980]. In that article a completeness result is proven for a temporal logic of intervals of which the modal operators bear a close resemblance to the Q-operations discussed in our paper. A second reason is formed by the interesting connections that seem to exist between our Q-calculus and some approaches in the area of term graph rewriting, cf. [Kahl 1996].

Definition 1.2 *Let U be some set; with $Re(U)$ we denote the set $\mathcal{P}(U \times U)$ of all binary relations over U. The* full relation set Q-algebra over U, *notation: $\mathfrak{Q}(U)$, is defined as the structure*

$$\mathfrak{Q}(U) = (Re(U), \cap, \cup, (\cdot)^c, \emptyset, Id_U, \mathsf{Q}_n^{kl})_{n\in\omega,k,l<n},$$

where $Id_U = \{(x,x) \mid x \in U\}$ is the diagonal or identity relation on U, and the operations Q_n^{kl} are as in Definition 1.1 above. The class of these algebras is denoted by FQ.

*Algebras $(A, \cdot, +, -, 0, 1', Q_n^{kl})_{n\in\omega,k,l<n}$ of this similarity type are simply called Q-*type algebras.

Let $\mathfrak{A}$ be a Q-type algebra. A representation *of $\mathfrak{A}$ is an embedding of $\mathfrak{A}$ into a product of full relation set Q-algebras. $\mathfrak{A}$ is* representable *if it has a representation. The class of representable Q-type algebras is denoted by RQ.*

We also mention some of our notational conventions. We write $i \in n$ with the understanding that $n = \{0, \ldots, n-1\}$; underlined symbols are used to denote matrices. The notation $x^{\smile}$ abbreviates $Q_2^{01}\begin{pmatrix} 1 & 1 \\ x & 1 \end{pmatrix}$, and $x \ ; \ y$ is short for $Q_3^{01}\begin{pmatrix} 1 & 1 & x \\ 1 & 1 & 1 \\ 1 & y & 1 \end{pmatrix}$. Note that in this way $R^{\smile}$ denotes the converse relation of R on full relation set Q-algebras, and $R \ ; \ S$ the composition or relative product of R and S.

The first result concerning RQ that we want to mention is the following.

Theorem 1 RQ *is a canonical discriminator variety of Boolean algebras with operators.*

Since RQ is defined as $\mathsf{S\,P\,RQ}$, saying that it is a variety is equivalent to stating that it is closed under taking homomorphic images. A variety of boolean algebras with operators is canonical if it is closed under taking canonical embedding algebras, as defined in [Jónsson & Tarski 1951]. [Jipsen 1993] showed that a variety of Boolean algebras with operators is a discriminator variety iff it is generated by a class K of algebras having a so-called unary discriminator term. This is a term $c(x)$ such that $\mathsf{K} \models x \neq 0 \to c(x) = 1$, while $c(0) = 0$. In the present case, it is easy to see that the term

$$\mathsf{Q}_4^{01}\begin{pmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&x&1\end{pmatrix}$$

is a unary discriminator term on FQ.

The main problem that we concentrate on in this paper is how to axiomatize the equational theory of RQ. In Definition 2.1 of the next section we will give a set of axioms defining a class Q of algebras that we will call Q-algebras. We can now formulate the main theorem of this paper as follows.

Theorem 2 *A Q-type algebra is representable iff it is a Q-algebra. In brief:*

$$\mathsf{Q} = \mathsf{RQ}.$$

Proofs for each of these theorems will be supplied in section 3, while in section 4, we discuss the third result of the paper, establishing a link between Q-algebras and the so-called relation algebras of finite dimension (formerly called matrix algebras of finite degree), cf. [Maddux 1983, Maddux 1992]. ¿From the discussion following Definition 1.2 it is easily seen that every Q-type algebra contains a Tarski (generalized) reduct. In section 4 we will define, for each $n \geq 3$, a finitely based equational class Q_n; the main result of the section will then state the following.

Theorem 3 *Let $3 \leq n < \omega$. An algebra $\mathfrak{A}$ is a relation algebra of dimension n iff it can be embedded in the Tarski reduct of a Q_n-algebra. In brief:*

$$\mathsf{MA}_n = \mathsf{S}\,\mathsf{Rd}_T\,\mathsf{Q}_n.$$

This theorem is in contrast to recent results obtained in [Hirsch & Hodkinson 1997] stating that for every $n \geq 4$, the variety MA_{n+1} is not finitely based over MA_n.

2 Q-algebras

It is the aim of this section to give the equations axiomatizing the class of representable Q-algebras. To be more precise, we define the equational class of Q-algebras which we will show in the next section to coincide with the class of representable Q-algebras. We also gather some basic facts concerning these Q-algebras, in Lemma 2.2.

In the next definition, the Q-axioms are presented. While reading and trying to understand these axioms, the reader is strongly advised to have a simultaneous glimpse at the examples that are provided right after the axiom schemas. In the presentation of the axioms we frequently use meta-variables $\underline{s}$ and $\underline{t}$ that stand for matrices of terms. In general, '$\underline{x}$' stands for a matrix of variables; we denote syntactical identity of the terms s and t by '$s \equiv t$'. For example, instances of the axiom schema Q2 involve precisely those matrices of terms in which all terms are variables except for one which must be the constant 0.

Definition 2.1 *By a Q-algebra we mean a Q-type algebra $\mathfrak{A} = (\mathfrak{B}, 1', \{Q_n^{kl}\})_{1<n<\omega,k,l\in n}$ such that $\mathfrak{B}$ is a boolean algebra and $\mathfrak{A}$ satisfies all well-typed instances[2] of the following axiom schemas Q1–Q10. The class of Q-algebras is denoted by Q.*

Q1. $Q_2^{01}\begin{pmatrix}1&1\\1&1\end{pmatrix} = 1.$

Q2. $Q_n^{kl}(\underline{t}) = 0,$

[2] That is to say, every instance of the schema is an axiom if it is a well-formed equation in the language of Q-algebras.

where, for some fixed $i, j \in n$, *the matrix* $\underline{t}$ *of terms is given by*

$$t_{pq} \equiv \begin{cases} 0 & \text{if } (p,q) = (i,j), \\ x_{pq} & \text{otherwise.} \end{cases}$$

Q3. $Q_n^{kl}(\underline{x}) \cdot x' = Q_n^{kl}(\underline{t})$,

where the matrix $\underline{t}$ *of terms is given by*

$$t_{pq} \equiv \begin{cases} x_{kl} \cdot x' & \text{if } (p,q) = (k,l) \\ x_{pq} & \text{otherwise.} \end{cases}$$

Q4. $Q_n^{kl}(\underline{x}) \leq Q_n^{kl}(\underline{t})$,

where, for some fixed $i \in n$, *the matrix* $\underline{t}$ *of terms is given by*

$$t_{pq} \equiv \begin{cases} x_{ii} \cdot 1\text{'} & \text{if } (p,q) = (i,i) \\ x_{pq} & \text{otherwise.} \end{cases}$$

Q5. $Q_{n+1}^{kl}(\underline{s}) \leq Q_n^{f(k)f(l)}(\underline{t})$,

for any surjective map $f : n+1 \to n$ *such that, for some fixed* $i, j \in n$, $f(i) = f(j)$. *Here the matrices* $\underline{s}$ *and* $\underline{t}$ *of terms are given by*

$$\begin{aligned} s_{pq} &\equiv \begin{cases} x_{ij} \cdot 1\text{'} & \text{if } (p,q) = (i,j) \\ x_{pq} & \text{otherwise,} \end{cases} \\ t_{pq} &\equiv \prod_{(f(p'),f(q'))=(p,q)} x_{p'q'} \end{aligned}$$

Q6. $Q_n^{kl}(\underline{x}) = Q_n^{kl}(\underline{t})$,

where, for some fixed $i, j \in n$, *the matrix* $\underline{t}$ *of terms is given by*

$$t_{pq} \equiv \begin{cases} x_{ji}^{\smile} \cdot x_{ij} & \text{if } (p,q) = (i,j) \\ x_{pq} & \text{otherwise.} \end{cases}$$

Q7. $x^{\smile\smile} = x$.

Q8. $Q_m^{f(k)f(l)}(\underline{x}) \leq Q_n^{kl}(\underline{t})$,

for any map $f : n \to m$. *Here the matrix* $\underline{t}$ *of terms is given by*

$$t_{pq} \equiv x_{f(p)f(q)}.$$

Q9. $Q_m^{kl}(\underline{s}) \leq Q_{m+n}^{kl}(\underline{t})$, *where for some fixed* $i, j \in m$, $k', l' \in n$, *the matrices* $\underline{s}$ *and* $\underline{t}$ *of terms are given by*

$$s_{pq} \equiv \begin{cases} x_{ij} \cdot Q_n^{k'l'} y & \text{if } (p,q) = (i,j) \\ x_{pq} & \text{otherwise} \end{cases}$$

$$t_{pq} \equiv \begin{cases} x_{pq} & \text{if } p, q < m \\ y_{p-m,q-m} & \text{if } p, q \geq m \\ 1\text{'} & \text{if } \{p,q\} = \{i, k'+m\} \text{ or } \{p,q\} = \{j, l'+m\} \\ 1 & \text{otherwise.} \end{cases}$$

Q10. $Q_n^{kl}(\underline{x}) \leq Q_n^{kl}(\underline{t})$,

where, for some fixed $i, j \in n$, *the matrix* $\underline{t}$ *of terms is given by*

$$t_{pq} \equiv \begin{cases} Q_n^{ij}(x) & \text{if } (p,q) = (i,j) \\ x_{pq} & \text{otherwise.} \end{cases}$$

The meaning of these axioms will be easier to understand by an inspection of the following examples.

Example of Q3.

$$Q_3^{20} \begin{pmatrix} x_{00} & x_{01} & x_{02} \\ x_{10} & x_{11} & x_{12} \\ x_{20} & x_{21} & x_{22} \end{pmatrix} \cdot x' = Q_3^{20} \begin{pmatrix} x_{00} & x_{01} & x_{02} \\ x_{10} & x_{11} & x_{12} \\ x_{20} \cdot x' & x_{21} & x_{22} \end{pmatrix}.$$

Example of Q4.

$$Q_3^{02} \begin{pmatrix} x_{00} & x_{01} & x_{02} \\ x_{10} & x_{11} & x_{12} \\ x_{20} & x_{21} & x_{22} \end{pmatrix} \leq Q_3^{02} \begin{pmatrix} x_{00} & x_{01} & x_{02} \\ x_{10} & x_{11} \cdot 1\text{'} & x_{12} \\ x_{20} & x_{21} & x_{22} \end{pmatrix}$$

Example of Q5.
$f : 0 \mapsto 0, 1 \mapsto 1, 2 \mapsto 1, 3 \mapsto 2.$

$$Q_4^{30} \begin{pmatrix} x_{00} & x_{01} & x_{02} & x_{03} \\ x_{10} & x_{11} & x_{12} & x_{13} \\ x_{20} & x_{21} \cdot 1\text{'} & x_{22} & x_{23} \\ x_{30} & x_{31} & x_{32} & x_{33} \end{pmatrix} \leq Q_3^{20} \begin{pmatrix} x_{00} & x_{01} \cdot x_{02} & x_{03} \\ x_{10} \cdot x_{20} & x_{11} \cdot x_{12} \cdot x_{21} \cdot x_{22} & x_{13} \cdot x_{23} \\ x_{30} & x_{31} \cdot x_{32} & x_{33} \end{pmatrix}$$

Example of Q6.

$$Q_3^{21} \begin{pmatrix} x_{00} & x_{01} & x_{02} \\ x_{10} & x_{11} & x_{12} \\ x_{20} & x_{21} & x_{22} \end{pmatrix} = Q_3^{21} \begin{pmatrix} x_{00} & x_{01} \cdot \breve{x_{10}} & x_{02} \\ x_{10} & x_{11} & x_{12} \\ x_{20} & x_{21} & a_{22} \end{pmatrix}.$$

Examples of Q8.
$f : 0 \mapsto 0, 1 \mapsto 1, 2 \mapsto 1.$

$$Q_2^{01} \begin{pmatrix} x_{00} & x_{01} \\ x_{10} & x_{11} \end{pmatrix} \leq Q_3^{01} \begin{pmatrix} x_{00} & x_{01} & x_{01} \\ x_{10} & x_{11} & x_{11} \\ x_{10} & x_{11} & x_{11} \end{pmatrix}$$

$f : 0 \mapsto 2, 1 \mapsto 1.$

$$Q_3^{11}\begin{pmatrix} x_{00} & x_{01} & x_{02} \\ x_{10} & x_{11} & x_{12} \\ x_{20} & x_{21} & x_{22} \end{pmatrix} \leq Q_2^{11}\begin{pmatrix} x_{22} & x_{21} \\ x_{12} & x_{11} \end{pmatrix}$$

EXAMPLE OF Q9.

$$Q_2^{10}\begin{pmatrix} x_{00} & x_{01} \\ x_{10} \cdot Q_3^{21}(\underline{y}) & x_{11} \end{pmatrix} \leq Q_5^{10}\begin{pmatrix} x_{00} & x_{01} & 1 & 1 & 1' \\ x_{10} \cdot Q_3^{21}(\underline{y}) & x_{11} & 1 & 1' & 1 \\ 1 & 1 & y_{00} & y_{01} & y_{02} \\ 1 & 1' & y_{10} & y_{11} & y_{12} \\ 1' & 1 & y_{20} & y_{21} & y_{22} \end{pmatrix}.$$

EXAMPLE OF Q10.

$$Q_3^{02}\begin{pmatrix} x_{00} & x_{01} & x_{02} \\ x_{10} & x_{11} & x_{12} \\ x_{20} & x_{21} & x_{22} \end{pmatrix} \leq Q_3^{02}\begin{pmatrix} x_{00} & x_{01} & x_{02} \\ x_{10} & x_{11} & x_{12} \\ x_{20} & Q_3^{21}\begin{pmatrix} x_{00} & x_{01} & x_{02} \\ x_{10} & x_{11} & x_{12} \\ x_{20} & x_{21} & x_{22} \end{pmatrix} & x_{22} \end{pmatrix}.$$

Lemma 2.2 below contains some arithmetical facts concerning Q-algebras which are useful later. One of the most important examples concerns conjugacy. Recall that two unary functions f and g on a Boolean algebra are conjugated if we have that $f(x) \cdot y = 0$ iff $x \cdot g(y) = 0$. For operations of higher arity, we use the following definition which goes back to Jónsson and Tarski [Jónsson & Tarski 1951]. Let $f : A^n \to A$ be an n-ary operation on the Boolean algebra $\mathfrak{A}$. Fix a number $k < n$ and a sequence $a_0, \ldots, a_{k-1}, a_{k+1}, \ldots, a_{n-1}$ of elements of $\mathfrak{A}$. Consider the unary map $\bar{f} : A \to A$ given by

$$a \mapsto f(a_0, \ldots, a_{k-1}, a, a_{k+1}, a_{n-1}).$$

Such maps are called *sections* of f. Now an n-ary operation f is conjugated if each of its sections is conjugated in the sense meant before for unary operations.

Conjugated operations behave fairly nice; for instance, they are completely additive, which means that they distribute over arbitrary sums in each of their arguments. A fortiori, conjugated operations distribute over finite sums; in our particular case, this means that all Q-operations are additive.

Lemma 2.2 *The following hold in* Q.

(1) $Q_2^{01}\begin{pmatrix} 1 & x \\ 1 & 1 \end{pmatrix} = x.$

(2) Each Q-operation is conjugated; we have

$$Q_n^{kl}(\underline{x}) \cdot z = 0 \;\leftrightarrow\; Q_n^{ij}(\underline{t}) \cdot x_{ij} = 0,$$

where the matrix $\underline{t}$ *of terms is given by*

$$t_{pq} \equiv \begin{cases} x_{kl} \cdot z & \text{if } (p,q) = (k,l) \\ 1 & \text{if } (p,q) = (i,j) \\ x_{pq} & \text{otherwise} \end{cases}$$

if $(k,l) \neq (i,j)$*, or by*

$$t_{pq} \equiv \begin{cases} z & \textit{if } (p,q) = (k,l) \\ x_{pq} & \textit{otherwise} \end{cases}$$

if $(k,l) = (i,j)$.

(3) Each Q-operation is completely additive (in each of its arguments). In particular, for each i,j, *we have*

$$Q_n^{kl}(\underline{s}) + Q_n^{kl}(\underline{s'}) = Q_n^{kl}(\underline{t}),$$

where $\underline{s}$, $\underline{s'}$ *and* $\underline{t}$ *are matrices of terms such that* $s_{ij} = x_{ij}$, $s'_{ij} = x'_{ij}$ *and* $t_{ij} = x_{ij} + x'_{ij}$, *while* $s_{pq} = s'_{pq} = t_{pq} = x_{pq}$ *for* $(p,q) \neq (i,j)$.

(4) Each Q-operation is monotone; that is, we have the following quasi-equation:

$$(\bigwedge_{i,j\in n} x_{ij} \leq y_{ij}) \ \rightarrow \ Q_n^{kl}(\underline{x}) \leq Q_n^{kl}(\underline{y}).$$

(5) $Q_n^{kl}(\underline{x}) = Q_n^{kl}(\underline{s}) + Q_n^{kl}(\underline{s'})$, *where for some fixed* $i,j \in n$, $\underline{s}$ *and* $\underline{s'}$ *are the term matrices given by* $s_{ij} = x_{ij} \cdot x'_{ij}$, $s'_{ij} = x_{ij} - x'_{ij}$, *while* $s_{pq} = s'_{pq} = x_{pq}$ *for* $(p,q) \neq (i,j)$.

(6) $Q_n^{kl}(\underline{x}) \leq x_{kl}$.

(7) $Q_n^{kl}(\underline{x}) = Q_n^{kl}(\underline{t})$, *where, for some fixed* i,j, *the matrix* $\underline{t}$ *of terms is given by*

$$t_{pq} \equiv \begin{cases} x_{ij}^{\smile} & \textit{if } (p,q) = (i,j) \\ x_{ji}^{\smile} & \textit{if } (p,q) = (j,i) \\ x_{pq} & \textit{otherwise.} \end{cases}$$

(8) The operation $^{\smile}$ *is self-conjugate; that is,* $x^{\smile} \cdot z = 0$ *iff* $z^{\smile} \cdot x = 0$.

(9) The operation $^{\smile}$ *is completely additive; in particular,* $(x+y)^{\smile} = x^{\smile} + y^{\smile}$.

(10) $1^{\smile} = 1$.

(11) $1'^{\smile} = 1'$.

(12) $Q_n^{kl}(\underline{x}) = 0$ *iff* $Q_n^{k'l'}(\underline{x}) = 0$ *for any* $(k,l),(k',l') \in n \times n$.

Example of Lemma 2.2(2). Assume that we consider the section of Q_3^{02} that is obtained by fixing elements $a_{pq} \in A$ for $(p,q) \neq (i,j)$. We have

$$Q_3^{02}\begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a & a_{22} \end{pmatrix} \cdot b = 0 \text{ iff } g(b) \cdot a = 0,$$

where g is the map given by

$$g(b) = Q_3^{21}\begin{pmatrix} a_{00} & a_{01} & a_{02}\cdot b \\ a_{10} & a_{11} & a_{12} \\ a_{20} & 1 & a_{22} \end{pmatrix}.$$

Example of Lemma 2.2(7).

$$Q_3^{11}\begin{pmatrix} x_{00} & x_{01} & x_{02} \\ x_{10} & x_{11} & x_{12} \\ x_{20} & x_{21} & x_{22} \end{pmatrix} = Q_3^{11}\begin{pmatrix} x_{00} & x^{\smile}_{10} & x_{02} \\ x^{\smile}_{01} & x_{11} & x_{12} \\ x_{20} & x_{21} & x_{22} \end{pmatrix}$$

Proof.

(1). Immediate by the axioms Q1 and Q3.

(2). For the direction from left to right, assume $Q_n^{kl}(\underline{x}) \cdot z = 0$ and $(k,l) \neq (i,j)$. Consider the following instance of the the axiom Q10:

$$Q_n^{ij}(\underline{u}) \leq Q_n^{ij}(\underline{s}) \tag{$*$}$$

where the matrix $\underline{u}$ of terms is given by

$$u_{pq} \equiv \begin{cases} x_{kl}\cdot z & \text{if } (p,q) = (k,l) \\ x_{pq} & \text{otherwise,} \end{cases}$$

and the matrix $\underline{s}$ of terms is given by

$$s_{pq} \equiv \begin{cases} Q_n^{kl}(\underline{u}) & \text{if } (p,q) = (k,l) \\ u_{pq} & \text{otherwise.} \end{cases}$$

Axiom Q3 yields $Q_n^{kl}(\underline{u}) = Q_n^{kl}(\underline{x}) \cdot z$. Since by the assumption $Q_n^{kl}(\underline{x}) \cdot z = 0$, we have $Q_n^{ij}(\underline{s}) = 0$, by axiom Q2. So we obtain from (*)

$$Q_n^{ij}(\underline{u}) = 0,$$

whence, by axiom Q3, we get $Q_n^{ij}(\underline{t}) \cdot x_{ij} = 0$, where the matrix $\underline{t}$ of terms is given by

$$t_{pq} \equiv \begin{cases} x_{kl}\cdot z & \text{if } (p,q) = (k,l) \\ 1 & \text{if } (p,q) = (i,j) \\ x_{pq} & \text{otherwise.} \end{cases}$$

The other direction is similar. For $(k,l) = (i,j)$ we apply axiom Q4 directly.

(3). This is immediate by the fact that each Q-operation is conjugated, cf. Theorem 1.14 of [Jónsson & Tarski 1951].

(4) is obtained by (3).

(5) is immediate from the boolean identity $x_{ij} = (x_{ij}\cdot z) + (x_{ij} - z)$ and (3).

(6). $Q_n^{kl}(\underline{x}) = Q_n^{kl}(\underline{x}) \cdot x_{kl} \leq x_{kl}$, by axiom Q3.

(7). $Q^{kl}_n(\underline{x}) =_1 Q^{kl}_n(\underline{s}) =_2 Q^{kl}_n(\underline{s'})) =_3 Q^{kl}_n(\underline{t})$,

where, for some fixed $i, j \in n$, the matrix $\underline{s}$ of terms is given by

$$s_{pq} \equiv \begin{cases} x_{ji}^{\smile} \cdot x_{ij} & \text{if } (p,q) = (i,j) \\ x_{ji} \cdot x_{ij}^{\smile} & \text{if } (p,q) = (j,i) \\ x_{pq} & \text{otherwise,} \end{cases}$$

the matrix $\underline{s'}$ of terms is given by

$$s'_{pq} \equiv \begin{cases} x_{ji}^{\smile} \cdot x_{ij}^{\smile\smile} & \text{if } (p,q) = (i,j) \\ x_{ji}^{\smile\smile} \cdot x_{ij}^{\smile} & \text{if } (p,q) = (j,i) \\ x_{pq} & \text{otherwise,} \end{cases}$$

and the matrix $\underline{t}$ of terms is given by

$$t_{pq} \equiv \begin{cases} x_{ji}^{\smile} & \text{if } (p,q) = (i,j) \\ x_{ij}^{\smile} & \text{if } (p,q) = (j,i) \\ x_{pq} & \text{otherwise.} \end{cases}$$

Here $=_1$ is by Q6, $=_2$ is by Q7, and $=_3$ is again by Q6.

(8). Suppose $Q^{01}_2 \begin{pmatrix} 1 & 1 \\ x & 1 \end{pmatrix} \cdot z = 0$. Then by Lemma 2.2(2), we obtain $Q^{10}_2 \begin{pmatrix} 1 & z \\ 1 & 1 \end{pmatrix} \cdot x = 0$. The opposite direction is analogues.

(9). This is immediate by the fact that $^{\smile}$ is conjugated, cf. Theorem 1.14 of [Jónsson & Tarski 1951].

(10). Use axiom Q7 and additivity of $^{\smile}$, by Lemma 2.2(3).

(11). $1'^{\smile} = Q^{01}_2 \begin{pmatrix} 1 & 1'^{\smile} \\ 1 & 1 \end{pmatrix} \leq Q^{01}_2 \begin{pmatrix} 1' & 1'^{\smile} \\ 1 & 1 \end{pmatrix}$, by Lemma 2.2(1) and axiom Q4.

$$Q^{01}_2 \begin{pmatrix} 1' & 1'^{\smile} \\ 1 & 1 \end{pmatrix} \leq Q^{01}_3 \begin{pmatrix} 1' & 1'^{\smile} & 1' \\ 1 & 1 & 1 \\ 1' & 1'^{\smile} & 1' \end{pmatrix} \leq Q^{01}_3 \begin{pmatrix} 1 & 1 & 1' \\ 1 & 1 & 1 \\ 1 & 1'^{\smile} & 1 \end{pmatrix} = Q^{01}_3 \begin{pmatrix} 1 & 1 & 1' \\ 1 & 1 & 1'^{\smile\smile} \\ 1 & 1^{\smile} & 1 \end{pmatrix},$$

by axiom Q8, Lemma 2.2(4) and (7).

$$Q^{01}_3 \begin{pmatrix} 1 & 1 & 1' \\ 1 & 1 & 1'^{\smile\smile} \\ 1 & 1^{\smile} & 1 \end{pmatrix} = Q^{01}_3 \begin{pmatrix} 1 & 1 & 1' \\ 1 & 1 & 1' \\ 1 & 1 & 1 \end{pmatrix} \leq Q^{01}_2 \begin{pmatrix} 1 & 1' \\ 1 & 1' \end{pmatrix} \leq Q^{01}_2 \begin{pmatrix} 1 & 1' \\ 1 & 1 \end{pmatrix} = 1',$$

by axiom Q7 and Lemma 2.2(10), axiom Q5, Lemma 2.2(4), and Lemma 2.2(1).

$1' \leq 1'^{\smile}$ is proved similarly using Lemma 2.2(1), the axioms Q4, Q8, Q5, and Lemma 2.2(4).

(12). We obtain (12) using first (2) with $z = 1$, and then axiom Q3.

QED

3 Construction

In this section we provide proofs for our first two theorems. We concentrate on the hard part of Theorem 2, which states that every Q-algebra is in fact representable. For this aim we first introduce a number of useful technical notions; in our presentation of the proof and our choice of terminology we follow [Hirsch & Hodkinson 199?].

Definition 3.1 *Given a Q-algebra $\mathfrak{A}$, a network of size n on $\mathfrak{A}$, or, shortly, n-network on $\mathfrak{A}$, is a function α: $n \times n \to \mathfrak{A}$. A n-network α is consistent if $Q_n^{kl}(\alpha) \neq 0$ for each $(k,l) \in n \times n$.*

Note that it follows by the axioms Q4 and Q2 that for any consistent network α, $\alpha_{ii} \cdot 1' \neq 0$ for all $i \in n$. Also, by Lemma 2.2(12) it follows that in order to check whether a given network α is consistent, it suffices to check whether $Q_n^{kl}(\alpha) \neq 0$ for *one* pair (k,l).

The key part of the proof of Theorem 2 consists of a step-by-step construction of a representation of a given countable Q-algebra $\mathfrak{A}$. This construction is aimed towards the creation of 'chains' of networks $\alpha^0 \subseteq \alpha^1 \subseteq \ldots$ from which we can 'read off' the representation of the algebra. Each network in this chain can be seen as an approximation of (part of) the final representation, the approximations getting better and better as we proceed in this chain. In fact, we need some sort of limit construction over such 'nice' chains of networks. Such limits are not ordinary networks themselves, but related entities that we will call *ultrafilter* networks.

Definition 3.2 *Let $\mathfrak{A}$ be a countable Q-algebra and a an arbitrary non-zero element of A. An ultrafilter network over $\mathfrak{A}$ for a is a function $\Phi : \omega \times \omega \to$ Ultrafilters$(\mathfrak{A})$ such that $a \in \Phi_{ij}$ for some i, j and*

(a) $1' \in \Phi_{ii}$ for all $i \in \omega$,

(b) $Q_n^{kl}(\beta) \in \Phi_{ij}$, $i,j \in \omega$ iff there are nodes $u_0, \ldots, u_k = i, u_l = j, \ldots, u_{n-1}$ in the domain of Φ such that $\beta_{pq} \in \Phi_{u_p u_q}$ for all $p, q \in n$.

The importance of ultrafilter networks will be made clear by the following two lemmas, that together imply that countable Q-algebras are representable. This suffices to show that *any* Q-algebra is representable, as we will show in the Proof of Theorem 2, at the end of the section.

Lemma 3.3 *Let $\mathfrak{A}$ be a countable Q-algebra and suppose that for each non-zero $a \in A$ there is an ultrafilter network over $\mathfrak{A}$ for a. Then $\mathfrak{A}$ is representable.*

Lemma 3.4 *Let $\mathfrak{A}$ be a countable Q-algebra and a an arbitrary non-zero element of A. Then there is an ultrafilter network over $\mathfrak{A}$ for a.*

Proof of Lemma 3.3. Let Φ be an ultrafilter network over the countable Q-algebra $\mathfrak{A}$. Define a relation $\simeq$ over ω as follows:

$$i \simeq j \text{ iff } 1' \in \Phi_{ij}.$$

Claim 1. *$\simeq$ is an equivalence relation.*

Proof of Claim. Reflexivity of $\simeq$ is immediate by Definition 3.2(a). For symmetry, suppose that $i \simeq j$; that is, $1' \in \Phi_{ij}$. By Lemma 2.2(1), $Q_2^{01}\begin{pmatrix} 1 & 1' \\ 1 & 1 \end{pmatrix} \in \Phi_{ij}$. Then, by Lemma 2.2(7),(10), $Q_2^{01}\begin{pmatrix} 1 & 1 \\ 1'^{\smile} & 1 \end{pmatrix} \in \Phi_{ij}$. So, by Definition 3.2(b), $1'^{\smile} \in \Phi_{ji}$, and therefore, by Lemma 2.2(11), we have $1' \in \Phi_{ji}$. The latter means $j \simeq i$, by definition of $\simeq$.

Finally, in order to prove transitivity of $\simeq$, assume that $i \simeq j, j \simeq k$; we have to show that $i \simeq k$. By the definition of $\simeq$, 1' $\in \Phi_{ij}$ and 1' $\in \Phi_{jk}$. Since $1 \in \Phi_{pq}$ for all $p, q \in \{i, j, k\}$, this gives by Definition 3.2(b) (with indices in the order i, j, k):

$$Q_3^{01}\begin{pmatrix} 1 & 1' & 1 \\ 1 & 1 & 1' \\ 1 & 1 & 1 \end{pmatrix} \in \Phi_{ik}.$$

By the axiom Q5 and by monotonicity (Lemma 2.2(4)) we obtain

$$Q_3^{01}\begin{pmatrix} 1 & 1' & 1 \\ 1 & 1 & 1' \\ 1 & 1 & 1 \end{pmatrix} \leq Q_2^{01}\begin{pmatrix} 1 & 1' \cdot 1 \\ 1 \cdot 1 & 1 \cdot 1' \cdot 1 \cdot 1 \end{pmatrix} = Q_2^{01}\begin{pmatrix} 1 & 1' \\ 1 & 1' \end{pmatrix} \leq Q_2^{01}\begin{pmatrix} 1 & 1' \\ 1 & 1 \end{pmatrix}.$$

Thus by upwards closedness of ultrafilters,

$$Q_2^{01}\begin{pmatrix} 1 & 1' \\ 1 & 1 \end{pmatrix} \in \Phi_{ik},$$

and finally, 1' $\in \Phi_{ik}$, by Lemma 2.2(1). That is, $i \simeq k$, by definition. ◀

In fact, $\simeq$ behaves like a sort of congruence relation, in a sense to be made precise in the claim below:

Claim 2. *$\Phi_{ij} = \Phi_{kl}$ whenever $i \simeq k$ and $j \simeq l$.*

Proof of Claim. Assume that $i \simeq k$ and $j \simeq l$. In order to show that $\Phi_{ij} = \Phi_{kl}$, it suffices to prove that $\Phi_{ij} \subseteq \Phi_{kl}$ (since Φ_{ij} and Φ_{kl} are ultrafilters). Hence, assume $b \in \Phi_{ij}$ for some arbitrary b in $\mathfrak{A}$.

By definition of $\simeq$, and the fact that $1 \in \Phi_{pq}$ for any p, q, we have by Definition 3.2(b) (read dimensions in the order i, j, k, l:)

$$Q_4^{23}\begin{pmatrix} 1 & b & 1' & 1 \\ 1 & 1 & 1 & 1' \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \in \Phi_{kl}.$$

Then by two application of basically the axiom Q5, we obtain

$$Q_4^{23}\begin{pmatrix} 1 & b & 1' & 1 \\ 1 & 1 & 1 & 1' \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \leq Q_3^{12}\begin{pmatrix} 1' & b & 1 \\ 1 & 1 & 1' \\ 1 & 1 & 1 \end{pmatrix} \leq Q_2^{01}\begin{pmatrix} 1' & b \\ 1 & 1' \end{pmatrix}.$$

It follows by monotonicity (Lemma 2.2(4)) and Lemma 2.2(1) that

$$Q_2^{01}\begin{pmatrix} 1' & b \\ 1 & 1' \end{pmatrix} \leq Q_2^{01}\begin{pmatrix} 1 & b \\ 1 & 1 \end{pmatrix} = b.$$

But then $b \in \Phi_{kl}$ since Φ_{kl} is an ultrafilter. ◀

We can now consider the set

$$U := \omega / \simeq$$

of $\simeq$-equivalence classes. This set U will be the base set of our representation; we introduce a function J from A to the full set of binary relations over U as follows: for any $a \in A$ we set

$$J(a) = \{([i],[j]) \in U \times U \mid a \in \Phi_{ij}\}.$$

It follows from the previous claim that this correctly defines a function. We will now show that in fact, it is a homomorphism.

CLAIM 3. *J is a homomorphism satisfying $J(a) \neq \emptyset$.*

PROOF OF CLAIM. We only treat the condition for sum and for the Q-operations. For sum:

$$\begin{aligned} J(a+a') &= \{([i],[j]) \mid a+a' \in \Phi_{ij}\} \\ &= \{([i],[j]) \mid a \in \Phi_{ij}\} \cup \{([i],[j]) \mid a' \in \Phi_{ij}\} \\ &= J(a) \cup J(a'). \end{aligned}$$

Here the crucial (second) identity follows from the fact that each Φ_{ij} is an ultrafilter and hence contains $a+a'$ iff it contains a or a'.

Now we turn to an arbitrary operation Q_n^{kl}. Consider the following equivalences:

$([i],[j]) \in J(Q_n^{kl}(\alpha))$
iff$_1$ $Q_n^{kl}(\alpha) \in \Phi_{ij}$
iff$_2$ there are nodes $h_0, \ldots, h_{n-1}$ such that $h_k = i$, $h_l = j$ and $\alpha_{pq} \in \Phi_{h_p,h_q}$ for all $p, q < n$
iff$_3$ there are nodes $h_0, \ldots, h_{n-1}$ such that $h_k = i$, $h_l = j$ and $([h_p],[h_q]) \in J(\alpha_{pq})$, for all $p, q < n$
iff$_4$ there are equivalence classes $H_0, \ldots, H_{n-1}$ such that $H_k = [i]$, $H_l = [j]$ and $(H_p, H_q) \in J(\alpha_{pq})$, for all $p, q < n$,
iff$_5$ $([i],[j]) \in \mathsf{Q}_n^{kl}(\underline{R})$, where $R_{pq} = J(\alpha_{pq})$, all $p, q < n$.

Here the equivalences 'iff$_1$' and 'iff$_3$' are by definition of J, while 'iff$_4$' is trivial. The second 'iff' is the second condition on ultrafilter networks, and the final equivalence is immediate by definition of the relation operation Q_n^{kl} (cf. Definition 1.1).

Finally, $J(a) \neq \emptyset$ since it was assumed that $a \in \Phi_{ij}$ for some $i, j \in \omega$. ◀

Now we are ready to prove Lemma 3.3. Assume that for each non-zero a in the algebra $\mathfrak{A}$, there is an ultrafilter network Φ^a for a. This implies that for each $a \neq 0$, there is a set U_a and a homomorphism $J_a : \mathfrak{A} \to \mathfrak{Q}(U_a)$ such that $J(a) \neq \emptyset$. But then it follows by a standard argument that the natural map $J : A \to \prod_{a \neq 0} Re(U_a)$ is an embedding of $\mathfrak{A}$ into the product algebra $\prod_{a \neq 0} \mathfrak{Q}(U_a)$. In other words, J is a representation.

This finishes the proof of Lemma 3.3. QED

We now turn to the proof of Lemma 3.4.

PROOF OF LEMMA 3.4. Let $\mathfrak{A}$ be a fixed countable Q-algebra and a an arbitrary non-zero element of A. We have to prove the existence of an ultrafilter network for a over $\mathfrak{A}$. We have already mentioned before that this ultrafilter network will be constructed as a sort of limit of a chain of ordinary networks. In order to make these notions more precise, we need some new concepts.

First, let α and β be two networks, of size m and n, respectively. A map $f : m \to n$ is an *embedding* of α into β if $\alpha_{f(i)f(j)} \leq \beta_{ij}$ for all i, j in m. If such a f exists, we say that β is an *extension* of α, notation: $\beta \subseteq \alpha$,

A *first-degree defect* of a consistent m-network α is a pair $\langle (i,j), a \rangle$ such that neither $\alpha_{ij} \leq a$ nor $\alpha_{ij} \leq -a$.

A *second-degree defect* of a consistent m-network α is a triple $\langle (i,j), (k,l), \mu \rangle$, where $i, j \in m$, μ is a network of size n and k, l are numbers in n such that $\alpha_{ij} \leq Q_n^{kl}(\mu)$, while there is no embedding $f : n \to m$ of μ into α such that $f(k) = i$ and $f(l) = j$.

Note that in a defect $\langle (i,j),(k,l),\mu\rangle$ of a network α, μ must be consistent. For, by the consistency of α it follows by the axiom Q2, that $\alpha_{ij} \neq 0$ for any $(i,j) \in m \times m$. So, $Q_n^{kl}(\mu) \neq 0$.

We first have to make sure that the sequence of networks we construct will indeed converge to a map sending pairs to ultrafilter. This is the meaning of the following claim.

CLAIM 1. *Let α be a consistent m-network, and d a first-degree defect of α. Then there is an extension β of α such that d is not a defect of β.*

PROOF OF CLAIM. Let α be a consistent m-network, $b \in \mathfrak{A}$, and $(i,j) \in m \times m$. Let α^+ and α^- be the networks obtained from α by changing the value of (i,j) into $\alpha_{ij} \cdot b$ and $\alpha_{ij} - b$, respectively. Then at least one of α^+ and α^- is consistent — this follows from Lemma 2.2(5). It is almost immediate that both α^+ and α^- are extensions of α. ◀

CLAIM 2. *Let α be a consistent m-network, and d a second-degree defect of α. Then there is an extension β of α such that d is not a defect of β.*

PROOF OF CLAIM. Let α be a consistent m-network, and assume that $d = \langle (i,j),(k',l'),\mu\rangle$ is a defect of α. That is, $(i,j) \in m \times m$, μ is an n-network, such that $\alpha_{ij} \leq Q_n^{kl}(\mu)$, while there is no map $f : n \to m$ such that $\alpha_{pq} \leq \mu_{pq}$ for all $p,q < n$ and $f(k) = i$ and $f(l) = j$. Without too much loss of generality we assume that $k \neq l$. Entirely without loss we take $k = n-2$, $l = n-1$; this latter assumption is justified by axiom Q8. Now define the $m+n$-network γ by

$$\gamma_{pq} := \begin{cases} \alpha_{pq} & \text{if } p,q < m, \\ \mu_{p-m,q-m} & \text{if } p,q \geq m, \\ 1\text{'} & \text{if } \{p,q\} = \{i, n+m-2\} \text{ or } \{p,q\} = \{j, n+m-1\}, \\ 1 & \text{otherwise.} \end{cases}$$

It follows easily from axiom Q9 that γ is consistent. Obviously, γ is an extension of both networks μ (by the function $g : p \mapsto m+p$) and α (by the identity map id of m into $m+n$). However, γ is not yet the network that we are looking for, since the embedding g of μ into γ does not satisfy $g(k) = i$ and $g(l) = j$. Fortunately, this can be easily fixed by collapsing γ to an $m+n-2$-network β, as follows. First, let $h : m+n \to m+n-2$ be the map given by

$$h(p) := \begin{cases} i & \text{if } p = m+n-2, \\ j & \text{if } q = m+n-1, \\ p & \text{otherwise.} \end{cases}$$

Now define the network β by

$$\beta_{pq} = \prod\{\gamma_{p'q'} \mid (f(p'), f(q') = (p,q)\}.$$

Clearly, the map $id : m \to m+n-2$ is an embedding of α into β, while the map $f : n \to m+n-2$ given by $f(p) = h(g(p))$ embeds μ into β. Since $f(k) = h(g(k)) = h(m+n-2) = i$ and $f(l) = h(g(l)) = h(m+n-1) = j$, the only thing that is left to show is that β is consistent. But this is immediate by the consistency of γ and two successive applications of axiom Q5 (with maps identifying $m+n-1$ with j and $m+n-2$ with i, respectively). We omit the rather cumbersome details. ◀

Now we are ready to define the main construction of the proof. We will build a sequence of consistent networks $\alpha^0 \subseteq \alpha^1 \subseteq \ldots \subseteq \alpha^k \subseteq \ldots$ as follows. To start with, since the algebra $\mathfrak{A}$ is countable, we can enumerate both kinds of possible defects. That is, we may assume that we have an enumeration $C_0, C_1, \ldots$ of the set $\omega \times \omega \times A$, and an enumeration $D_0, D_1, \ldots$ of all triples of the form $\langle (i,j),(k,l),\mu\rangle$, where i,j,k and l are natural numbers and μ is some matrix over $\mathfrak{A}$.

base step We start with the initial 2-network $\alpha^0 = \begin{pmatrix} 1 & a \\ 1 & 1 \end{pmatrix}$, where a is the element that we are constructing the ultrafilter network for. Since a is non-zero, α^0 is consistent by Lemma 2.2(1).

odd steps Assume that we have already defined the network α^{2r} for some natural number r. If α^{2r} does not have any defects of the first kind, then define $\alpha^{2r+1} := \alpha^{2r}$. Otherwise, let c be the *first* first-degree defect of α^{2r} (with respect to the enumeration C). By Claim 1, there is an extension β of α^{2r} such that c is not a defect of β. Then define $\alpha^{2r+1} := \beta$.

even steps Proceed as in the previous case, but for defects of the second kind. (Use Claim 2 here).

Obviously, this gives a chain $\alpha^0 \subseteq \alpha^1 \subseteq \alpha^2 \subseteq \dots$ of networks. We may and will assume that in fact $\alpha^l_{ij} \leq \alpha^k_{ij}$ whenever $l \leq k$. We now define α^ω as the limit of all networks α^k of the sequence, as follows. The domain of α^ω is $\omega \times \omega$ and for all (i,j) of α^ω,

$$\alpha^\omega_{ij} = \{ b \in A \mid \alpha^k_{ij} \leq b \text{ for some } k < \omega \}.$$

Claim 3. *α^ω is an ultrafilter network.*

Proof of Claim. We have to show that

(1) each (i,j) in α^ω is labelled by an ultrafilter of $\mathfrak{A}$, and

(2) For any n-network β, numbers $k,l < n$ and arbitrary numbers i,j: $Q^{kl}_n(\beta) \in \alpha^\omega_{ij}$ iff there are nodes $u_0, \dots, u_k = i, u_l = j, \dots, u_{n-1}$ in α^ω such that $\beta_{pq} \in \alpha^\omega_{u_p u_q}$ for all $p,q \in n$.

(3) 1' $\in \alpha^\omega_{ii}$ for all $i \in \omega$.

Proof of (1). Fix some natural numbers i and j. We will show that α^ω_{ij} is an ultrafilter. First, suppose b,c are elements of α^ω_{ij}. By the definition of α^ω_{ij}, there are $k,l < \omega$ such that $\alpha^k_{ij} \leq b$ and $\alpha^l_{ij} \leq c$. Without loss of generality we may assume that $k \leq l$. By construction, we have $\alpha^l_{ij} \leq \alpha^k_{ij}$, which implies $\alpha^l_{ij} \leq b \cdot c$. So, by the definition of α^ω_{ij}, $b \cdot c \in \alpha^\omega_{ij}$. Hence, α^ω_{ij} is closed under meets.

Second, $0 \notin \alpha^\omega_{ij}$. This follows from axiom Q2 and the consistency of all networks α^k. Third, upward closure of α^ω_{ij} is immediate by its definition. This shows that α^ω_{ij} is a filter.

For maximality, assume that neither b nor $-b$ is in α^ω_{ij}, for some element b of A. This implies that $\langle (i,j), b \rangle$ is a defect of *every* α^k. However, our construction guarantuees that every defect of a network will eventually be repaired. So we face a contradiction.

Proof of (2). Let β be an n-network over $\mathfrak{A}$ and let i and j be two natural numbers. For the 'only if'-direction, suppose that $Q^{kl}_n(\beta) \in \alpha^\omega_{ij}$. Then, by the definition of α^ω_{ij}, there is a number r such that $\alpha^r_{ij} \leq Q^{kl}_n(\beta)$. It follows by a standard argument on our construction that for some $s \geq r$, the triple $\langle (i,j), (k,l), \beta \rangle$ is *not* a defect of α^s. But this means by definition of second-degree defects that there is an embedding $f : n \to m$ (where n and m are the sizes of β and α^s, respectively) of β into α^s such that $f(k) = i$ and $f(l) = j$. By definition of an embedding, $\alpha^s_{f(p)f(q)} \leq \beta_{pq}$ for all $p,q < n$. By definition of α^ω, this means that for all $p,q < n$, $\beta_{pq} \in \alpha^\omega_{f(p)f(q)}$; since $f(k) = i$ and $f(l) = j$, we are finished.

For the other direction, suppose there are nodes $u_0, \dots, u_k = i, u_l = j, \dots, u_{n-1}$ (with $k,l \in n$), such that $\beta_{pq} \in \alpha^\omega_{u_p u_q}$ for all $p,q \in n$. Then, by construction of α^ω, there must be an m-network α^r, $r < \omega$, such that α^r is defined on all these edges and such that $\alpha^r_{u_p u_q} \leq \beta_{pq}$ for all $p,q \in n$. Since α^r is consistent, $Q^{kl}_m(\alpha^r) \neq 0$; it is not very difficult to prove that $Q^{kl}_m(\alpha^r) \in \alpha^\omega_{ij}$. Also, by axiom Q8, $Q^{ij}_m(\alpha^r) \leq Q^{kl}_n(\beta')$, where β' is defined by $\beta'_{pq} = \alpha^r_{u_p u_q}$. Then $\beta'_{pq} \leq \beta_{pq}$ for all $p,q \in n$, so by monotonicity, $Q^{kl}_n(\beta') \leq Q^{kl}_n(\beta)$. It follows that $Q^{kl}_m(\alpha^r) \leq Q^{kl}_n(\beta)$, and hence, that $Q^{kl}_n(\beta) \in \alpha^\omega_{ij}$.

Proof of (3). By construction of α^ω we have $Q^{kl}_n(\alpha^r) \neq 0$ for all $r < \omega, 1 < n < \omega$. Then we obtain from this $Q^{kl}_n(\ldots, \alpha^r_{ii} \cdot 0', \ldots) = 0$, by the axiom Q4. By construction (odd step) there is a step s such that $\alpha^s_{ii} \leq 0'$, which gives $1' \in \alpha^\omega_{ii}$ by definition of α^ω_{ii}. ◀

This finishes the proof of Lemma 3.4. QED

PROOF OF THEOREM 2. The lemmas 3.3, 3.4 imply that countable Q-algebras are representable. Now we complete the proof of Theorem 2 by showing that *any* Q-algebra is representable.

Suppose $\mathfrak{A}$ is a Q-algebra. Define $L_\mathfrak{A}$ to be the first-order language which contains a binary predicate symbol $P_a(v, w)$ for each element a of $\mathfrak{A}$. The theory $T_\mathfrak{A}$ (over $L_\mathfrak{A}$) associated with $\mathfrak{A}$ is determined by the following axioms:

(1) $\forall v\,(P_1(v, v))$; $\forall vw\,(P_1(v, w) \leftrightarrow P_1(w, v))$; $\forall vuw\,(P_1(v, u) \wedge P_1(u, w) \rightarrow P_1(v, w))$
(2) $\forall vw\,(P_{a \cdot b}(v, w) \leftrightarrow P_a(v, w) \wedge P_b(v, w))$
(3) $\forall vw\,(P_1(v, w) \rightarrow [P_{-a}(v, w) \leftrightarrow \neg P_a(v, w)]$
(4) $\forall vw\,(P_{1'}(v, w) \leftrightarrow v = w)$
(5) $\forall vw\,(P_{Q^{kl}_n(\underline{a})}(v, w) \leftrightarrow \exists u_0 \ldots u_{n-1}\,(v = u_k \wedge w = u_l \wedge \prod_{i,j \in n} P_{a_{ij}}(u_i, u_j)))$.

A representation of $\mathfrak{A}$ is (essentially) a model of $T_\mathfrak{A}$, and vice versa. Hence by the compactness theorem it suffices to show that each finite subset of $T_\mathfrak{A}$ has a model.

Let F be an arbitrary finite subset of $T_\mathfrak{A}$; let $\mathfrak{A}(F)$ denote the Q-algebra that is generated by those elements b of $\mathfrak{A}$ of which the predicate P_b occurs in F. Then by definition, $\mathfrak{A}(F)$ is a subalgebra of $\mathfrak{A}$, and since F is finite, $\mathfrak{A}(F)$ is countable. It follows that $\mathfrak{A}(F)$ is representable.

$T_{\mathfrak{A}(F)}$ denotes the theory associated with $\mathfrak{A}(F)$. Each sentence σ of F contains a finite number k of predicate symbols. Due to the way $\mathfrak{A}(F)$ and $T_{\mathfrak{A}(F)}$ were defined, $T_{\mathfrak{A}(F)}$ contains all possible sentences on these k predicate symbols, including σ. So $F \subseteq T_{\mathfrak{A}(F)}$. But since $\mathfrak{A}(F)$ is representable, $T_{\mathfrak{A}(F)}$ has a model, which is then also a model for F. QED

PROOF OF THEOREM 1. It follows immediately from Birkhoff's theorem that the equationally defined class Q is a variety. Now all the axioms Q1–Q10 defining Q are in so-called Sahlqvist form, cf. [de Rijke & Venema 1995]. It follows immediately that Q is canonical.

The corresponding statements for the class RQ are obtained by Theorem 2. We saw already in the Introduction that RQ is generated by a class FQ that has a discriminator term. QED

4 Relation algebras of finite dimension

In this final section we establish a link between Q-algebras and the so-called relation algebras of finite dimension. The latter, introduced in [Maddux 1983] under the name of matrix algebras of finite degree, come in varieties: there is a variety MA_n for each finite n (and also a MA_ω which will come into the picture only at the end of this section). In their turn these algebras correspond to a Gentzen-style sequent calculus for the set *Equ*RRA of the 'true' relational equations. Basically, this correspondence is such that the dimension of the variety MA_n reflects the *number* of variables used in a proof of this sequent calculus: an equation in the language of relation algebras belongs to the equational theory of MA_n iff it can be proved using only n variables.

In order to explain the link with Q-algebras, let us recall from the introduction that the Tarski operations of $\smile$ and ; (of converse and relative product, respectively) could easily be defined using the Q-operations. Thus Tarski-type algebras are *subreducts* of Q-type algebras; as we will see in this section, it turns out that the relation algebras of dimension n correspond precisely to the subreducts of Q-algebras satisfying a specified, finite part of the Q-axioms. For technical reasons however, we deviate from the Q-similarity type that we have worked in until now. In this section, our algebras contain operations Q^{kl}_n for one fixed n only, viz., the dimension of the corresponding variety of relation algebras.

Definition 4.1 *Let $3 \leq n < \omega$. A Q_n-*type algebra *is an algebra of the similarity type $(A, +, -, 1', \{Q_n^{kl}\}_{k,l<n})$, where $+$ and $-$ are binary operations on A, 1' is a constant, and Q_n^{kl} are n^2-ary operations on A.*

This switch does not imply any 'real' change: in a Q_n-type algebra, the operations Q_m^{kl} with m *smaller* than n are easily seen to be term-definable, using

$$Q_m^{kl}(x) = Q_n^{kl}(t)$$

where

$$t_{pq} \quad \equiv \quad \begin{cases} x_{pq} & \text{if } p, q \leq m \\ 1 & \text{otherwise.} \end{cases}$$

Operations Q_m^{kl} with m *bigger* than n are not term-definable, but then, they do not play any role in the correspondence with the Maddux relation algebras of dimension n. It is precisely in order to avoid cumbersome bookkeeping regarding the behaviour of such operations that we 'cut off' our Q-algebras in this section.

In order to make the promised connection precise, we now first define the concepts involved; we then proceed to prove Theorem 3. We first recall some definitions concerning Tarski-type algebras.

Definition 4.2 *Algebras of the form $\mathfrak{A} = (A, +, -, 1', \smile, ;)$ where A is a non-empty set, $+$ and ; are binary operations, $-$ and $\smile$ are unary operations, and 1' is a constant, are called* Tarski-type algebra.

A Tarski-type algebra $\mathfrak{A}$ is called a semi-associative relation algebra *if it satisfies the following axioms:*

A0. *$(A, +, -)$ is a Boolean algebra,*
A1. *$x = x;1' = 1';x$ (the identity law),*
A2. *$x;y \cdot z = 0$ iff $x^\smile;z \cdot y = 0$ (left Peircean law),*
A3. *$x;y \cdot z = 0$ iff $z;y^\smile \cdot x = 0$ (right Peircean law),*
A4. *$x;1;1 = x;1$ (semiassociative law).*

*Let $\mathfrak{A}$ be an atomic semi-associative relation algebra. $M \subseteq {}^{n\times n}At\mathfrak{A}$ is an n-*dimensional basis for *$\mathfrak{A}$ if the following are satisfied:*
(A) if $a \in M$, $i, j, p < n$ then $a_{ii} \leq 1', a_{ij}^\smile = a_{ji}, a_{ij} \leq a_{ip};a_{pj}$,
(B) if $a \in M$, $i, j, p < n$, $p \neq i, j$, and $x, y \in At\mathfrak{A}$, and $a_{ij} \leq x;y$ then there is some $b \in M$ such that $b_{ip} = x, b_{pj} = y$, and $b_{lm} = a_{lm}$ for $p \neq l, m < n$,
(C) for every *$x \in At\mathfrak{A}$ there is $a \in M$ such that $a_{01} = x$.*

A semi-associative algebra is called a relation algebra of dimension *n iff it is a subalgebra of some complete atomic semi-associative relation algebra with an n-dimensional basis; the class of such algebras is denoted with MA_n. The class of semi-associative algebras is denoted with SA.*

Definition 4.3 *Let $3 \leq n < \omega$. A Q_n-*type algebra *is an algebra of the similarity type $(A, +, -, 1', \{Q_n^{kl}\}_{k,l<n})$, where $+$ and $-$ are binary operations on A, 1' is a constant, and Q_n^{kl} are n^2-ary operations on A.*

By a Q_n-algebra we mean a Q_n-type algebra $\mathfrak{A} = (\mathfrak{B}, 1', \{Q_n^{kl}\}_{k,l<n})$ such that $\mathfrak{B}$ is a boolean algebra and $\mathfrak{A}$ satisfies the appropriate axiom schemas from Definition 2.1 — with the understanding that the axioms Q1, Q5, Q8 and Q9 are replaced, respectively, by Q1, Q5*, Q8*, and Q9* below.*

Q1*. $Q_n^{01}(s) = 1$

where the matrix s of terms is given by

$$s_{pq} \quad \equiv \quad 1$$

Q5*. $Q_n^{kl}(s) \leq Q_n^{f(k)f(l)}(t)$, *for any map $f : n \to n$. Here the matrices s and t of terms are given by*

$$
\begin{aligned}
s_{pq} &\equiv \begin{cases} x_{pq} \cdot 1' & \text{if } f(p) = f(q) \\ x_{pq} & \text{otherwise,} \end{cases} \\
t_{pq} &\equiv \prod_{(f(p'),f(q'))=(p,q)} x_{p'q'}
\end{aligned}
$$

Q8*. $Q_n^{f(k)f(l)}(x) \leq Q_n^{kl}(t)$,

for any map $f : n \to n$. Here the matrix t of terms is given by

$$t_{pq} \equiv x_{f(p)f(q)}.$$

Q9*. $Q_n^{kl}(s) \leq Q_n^{kl}(t)$,

where for some fixed $i, j, m \in n$, $m \neq i, j$, the matrices s and t of terms are given by

$$
\begin{aligned}
s_{pq} &\equiv \begin{cases} x_{ij} \cdot y \,;\, z & \text{if } (p, q) = (i, j) \\ x_{pq} & \text{otherwise} \end{cases} \\
t_{pq} &\equiv \begin{cases} x_{pq} & \text{if } p, q \neq m \\ y & \text{if } p = i, q = m \\ z & \text{if } p = m, q = j \\ 1 & \text{otherwise.} \end{cases}
\end{aligned}
$$

With Q_n we denote the class of Q_n-algebras.

To link up these two types of algebras, recall from the introduction that we could define $\smile$ (converse) and ; (relative product or composition) in Q-algebras. Note however, that we used the operations Q_2^{01} and Q_3^{01} in these definitions, whereas in a Q_n-type algebra we have only operations Q_n^{kl} in our disposal. Therefore from now on we use the following definitions:

Definition 4.4 *Let $\mathfrak{A} = (\mathfrak{B}, 1', \{Q_n^{kl}\}_{k,l<n})$ be a Q_n-type algebra. Define the following auxiliary operations:*

$$
\begin{aligned}
x^{\smile} &= Q_n^{01}(t), \quad \text{where} \quad t_{pq} \equiv \begin{cases} x & \text{if } p = 1, q = 0 \\ 1 & \text{otherwise.} \end{cases} \\
x;y &= Q_n^{01}(t), \quad \text{where} \quad t_{pq} \equiv \begin{cases} x & \text{if } p = 0, q = 2 \\ y & \text{if } p = 2, q = 1 \\ 1 & \text{otherwise.} \end{cases}
\end{aligned}
$$

The Tarski reduct of $\mathfrak{A}$ is defined as the Tarski-type algebra $\mathfrak{Rd}_T\,\mathfrak{A}$ given as:

$$\mathfrak{Rd}_T\,\mathfrak{A} = (\mathfrak{B}, 1', {}^{\smile}, ;),$$

where $\smile$ and ; are as defined as above. Given a class X of Q_n-type algebras, let $\mathsf{Rd}_T\,\mathsf{X}$ denote the class of associated Tarski reducts.

Now that we have defined this link between Q-type algebras and Tarski type algebras, we can start to prove the main result of this section, viz, Theorem 3. In order to do so, we apply some basic duality theory between boolean algebras with operators and relational structures or *frames* as we shall call them, cf. [Goldblatt 1989]. For readers unfamiliar with this theory, we just recall the definition of the (full) complex algebra of a frame.

Definition 4.5 *Let $\mathfrak{F} = \langle W, \{R_i\}_{i\in I}\rangle$ be a frame, where $R_i \subseteq {}^{n_i}W$. Its* complex algebra *is the algebra $\mathfrak{Cm}\,\mathfrak{F} = (\mathcal{P}(W), \cap, \cup, (\cdot)^c, \emptyset, \{\mathrm{m}_{R_i}\}_{i\in I})$, where m_{R_i} is the $n_i - 1$-ary operation defined by*

$$\mathrm{m}_{R_i}(X_1, \ldots, X_{n_i-1}) \;=\; \{y \in W \mid R_i y x_1 \ldots x_{n_i-1} \text{ for some } x_1 \in X_1, \ldots, x_{n_i-1} \in X_{n_i-1}\}.$$

This seems to be a natural move given the focus on complete and atomic algebras in the definition of relation algebras of finite dimension. The basic structure of the proof is to associate with each class of algebras (MA_n and Q_n) a class of frames (AF_n and QF_n, respectively); the precise nature of this association can be found in the Lemmas 4.8 and 4.10, respectively. Finally, the crux of the proof then lies in connecting these two frame classes; the precise link, which in fact is rather direct, is given in Lemma 4.11.

Definition 4.6 *Let W be a non-empty set, $C \subseteq {}^3W, F \subseteq {}^2W, I \subseteq W$. The structure $\mathfrak{M} = \langle W, C, F, I\rangle$ is called an arrow frame if*

(i) $\forall u, v, w\,(\exists z\,(Cuvz \wedge Iz) \leftrightarrow u = v)$,

(ii) $\forall u, v, w, z\,(Cuvz \wedge Fvw \rightarrow Czwu)$,

(iii) $\forall u, v, w, z\,(Cuvz \wedge Fzw \rightarrow Cvuw)$

(iv) $\forall w \exists! v(Fwv)$

(v) $\forall u, v, w, z, y(Cuvw \wedge Czuv \rightarrow \exists x(Czvx))$.

*Let $3 \leq n < \omega$; a set $B \subseteq {}^{n\times n}W$ is an n-*dimensional basis for an arrow frame $\mathfrak{M}$ *if the following are satisfied:*

(a) If $\underline{v} \in B$, $i, j, p < n$ then $Iv_{ii}, Fv_{ji}v_{ij}, Cv_{ij}v_{ip}v_{pj}$,

(b) If $\underline{v} \in B, i, j, p < n, p \neq i, j$, and $Cv_{ij}uz$ then there is some $\underline{w} \in B$ such that $w_{ip} = u$, $w_{pj} = z$ and whenever $p \neq l, m < n$, $w_{lm} = v_{lm}$,

(c) For every $w \in W$ there is some $\underline{v} \in B$ such that $v_{01} = w$.

Let $\underline{v}$ be an $n \times n$ matrix of elements of W. For any map $f : n \rightarrow n$ define $\underline{v}^f$ to be the matrix given by

$$v^f_{pq} \equiv v_{f(p)f(q)}.$$

Let B be a basis for $\mathfrak{M}$. Define $\tilde{B} = \{\underline{v}^f : \underline{v} \in B,\ f : n \rightarrow n \text{ is a bijection }\}$. B is called good *if $B = \tilde{B}$. Define $B_{\mathfrak{M}} = \bigcup\{B : B \text{ is a basis for } \mathfrak{M}\}$.*

Finally, we let AF denote the class of arrow frames, and AF_n the class of arrow frames with a good n-dimensional basis.

In the following lemma we prove that the set $B_{\mathfrak{M}}$ may serve as the canonical definition of an n-dimensional basis for $\mathfrak{M}$, whenever $\mathfrak{M}$ *has* an n-dimensional basis; furthermore, we may assume that $B_{\mathfrak{M}}$ is good.

Lemma 4.7 *Let $\mathfrak{M}$ be an arrow frame.*

1. *If B is a basis for $\mathfrak{M}$ then $\tilde{B}$ is also a basis for $\mathfrak{M}$.*
2. *If B is a basis for $\mathfrak{M}$ then $B_{\mathfrak{M}}$ is also a basis for $\mathfrak{M}$; furthermore, $B_{\mathfrak{M}}$ is good.*

Proof. For part 1, assume that B is a basis for $\mathfrak{M}$. Each $\underline{v}^f$ from $\tilde{B}$ satisfies 4.6(a) in an obvious way, and condition 4.6(c) holds because $B \subseteq \tilde{B}$.

To check 4.6(b), assume $\underline{v}^f \in \tilde{B}$, where $\underline{v} \in B$, $p \neq i, j$, and $Cv^f_{ij}xy$. So $Cv_{f(i)f(j)}xy$. Since $\underline{v} \in B$, and $Cv_{f(i)f(j)}xy$, and $f(p) \neq f(i), f(j)$ there is $\underline{w} \in B$ such that $w_{f(i)f(p)} = x$, $w_{f(p)f(j)} = y$, and $w_{f(l)f(m)} = v_{f(l)f(m)}$ whenever $f(l), f(m) \neq f(p)$. Then $\underline{w}^f \in \tilde{B}$ is such that $w^f_{ip} = w_{f(i)f(p)} = x$, $w^f_{pj} = w_{f(p)f(j)} = y$, and $w^f_{lm} = w_{f(l)f(m)} = v_{f(l)f(m)} = v^f_{lm}$ whenever $f(l), f(m) \neq f(p)$. Thus $\tilde{B}$ satisfies 4.6(b).

For part 2, it is straightforward to verify that $B_{\mathfrak{M}}$ is a basis. It is then rather easy to check (using part 1 and the definition of $B_{\mathfrak{M}}$) that $B_{\mathfrak{M}}$ is in fact good. QED

The following lemma links up some classes of Tarski-type algebras with some classes of arrow frames. We also recall the fact that every arrow frame has a 3-dimensional basis.

Lemma 4.8 *1. Any arrow frame has a 3-dimensional basis; that is:*

$$\mathsf{AF} = \mathsf{AF}_3.$$

2. A Tarski-type algebra is semi-associative iff it can be embedded in the complex algebra of an arrow frame, in brief:

$$\mathsf{SA} = \mathsf{S}\,\mathsf{Cm}\,\mathsf{AF}.$$

3. A Tarski-type algebra is a relation algebra of dimension n iff it can be embedded in the complex algebra of an arrow frame with a good n-dimensional basis. In brief:

$$\mathsf{MA}_n = \mathsf{S}\,\mathsf{Cm}\,\mathsf{AF}_n.$$

PROOF. The proof of part 1 and 2 is left to the reader (for closely related proofs, see [Maddux 1982] — the main difference is that in our set-up F is a functional relation, not a function).

For part 3, first let $\mathfrak{B}$ be a relation algebra of dimension n. By Definition 4.2 it can be embedded in some complete atomic semi-associative algebra $\mathfrak{A}$ with an n-dimensional basis M. Define the atom structure $\mathfrak{At}\,\mathfrak{A} = (At\mathfrak{A}, I, F, C)$ of $\mathfrak{A}$ as follows: $At\mathfrak{A}$ is the set of atoms of $\mathfrak{A}$, $I = \{x : x \leq 1'\}$, $F = \{\langle x,y\rangle : x \leq y^{\smile}\}$, and $C = \{\langle x,y,z\rangle : x \leq y;z\}$.

We leave it to the reader to verify that this structure satisfies condition (i) – (v) of Definition 4.6, and that $\mathfrak{A}$ is isomorphic to $\mathfrak{Cm}\,(\mathfrak{At}\,\mathfrak{A})$. It follows immediately from the definitions that the relation $M \subseteq {}^{n\times n}At\mathfrak{A}$ *itself* is a basis for the frame $\mathfrak{At}\,\mathfrak{A}$.

For the opposite direction of part 2, consider the complex algebra $\mathfrak{Cm}\,\mathfrak{F}$ of an arrow frame $\mathfrak{F} = \langle W, I, F, C\rangle$ with a good n-dimensional basis B. It follows from part 1 $\mathfrak{Cm}\,\mathfrak{F}$ is a semi-associative relation algebra. Note that the atoms of the algebra $\mathfrak{Cm}\,\mathfrak{F}$ are the singletons $\{x\}$, with $x \in W$. By this correspondence, the relation B naturally induces a relation on the atoms of $\mathfrak{Cm}\,\mathfrak{F}$; it is straightforward to verify that this relation is in fact a basis. QED

¿From this it is easy to deduce that $\mathsf{S}\,\mathsf{Cm}\,\mathsf{AF}_n \subseteq \mathsf{MA}_n$.

Now that we have established a link between frames and algebras for the similarity type of Tarski relation algebras, we do the same for the Q_n-similarity type. First of all we define Q_n-frames. Relations $R_n^{kl}, k,l < n$ of Q_n-frames are in correspondence to the algebraic operations Q_n^{kl}. We also introduce relations $D_{\mathfrak{F}}$ and $G_{\mathfrak{F}}$ which correspond to the defined operations ; and $^{\smile}$, respectively. Then in Lemma 4.10 we show a connection between Q_n-frames and Q_n-algebras.

Definition 4.9 *A structure $\mathfrak{F} = \langle W, I, R_n^{kl}\rangle_{k,l<n}$ with W a non-empty set, $I \subseteq W$ and $R^{kl} \subseteq W \times {}^{n^2}W$ is called a Q_n-*frame *if the conditions R1 – R9 below hold. We first define the following relations.*

D1. *$D_{\mathfrak{F}}$ consists of those triples $\langle x,y,z\rangle$ for which there is a n^2-matrix $\underline{v}$ of elements of W, and numbers $i,j,p,k,l < n$ such that $x = v_{ij}$, $y = v_{ip}$, $z = v_{pj}$ and $R_n^{kl}v_{kl}\underline{v}$.*

D2. *$G_{\mathfrak{F}}$ consists of those pairs $\langle x,y\rangle$ for which there is a n^2-matrix $\underline{v}$ of elements of W, and numbers $i,j,k,l < n$ such that $x = v_{ji}$, $y = v_{ij}$ and $R_n^{kl}v_{kl}\underline{v}$.*

In a similar way, we will use D and G as abbreviations in the first order language for describing Q_n-frames. With the help of these abbreviations, we can define the conditions as follows.

R1. $\forall w \exists \underline{v} \, (R_n^{01} w \underline{v})$,

R2. *Let* $i, j, m, p, q < n, m \neq i, j$.

$$\forall w \underline{v} x y \, (R_n^{kl} w \underline{v} \wedge D v_{ij} x y \rightarrow \exists \underline{z} \, (R^{kl} w \underline{z} \wedge z_{im} = x \wedge z_{mj} = y \wedge \bigwedge_{p,q \neq m} z_{p,q} = v_{p,q}))$$

R3. $\forall w \underline{v} \, (R_n^{kl} w \underline{v} \rightarrow w = v_{kl})$

R4. $\forall w \exists ! v \, (G w v)$

R5. $\forall w v \, (G u w \rightarrow G w u)$

R6. $\forall w \underline{v} \, (R_n^{kl} w \underline{v} \rightarrow I v_{ii})$

R7. *Let* $f : n \rightarrow n$.

$$\forall w, \underline{v} \, (R_n^{kl} w \underline{v} \wedge \bigwedge_{f(p)=f(q)} I v_{pq} \rightarrow \exists \underline{z} (R_n^{f(k)f(l)} v_{f(k)f(l)} \underline{z} \wedge \bigwedge z_{f(p)f(q)} = v_{pq}))$$

R8. *Let* $f : n \rightarrow n$.

$$\forall w, \underline{v} \, (R_n^{f(k)f(l)} w \underline{v} \rightarrow R_n^{kl} w \underline{v}^f)$$

R9. $\forall w, \underline{v} \, (R_n^{kl} w \underline{v} \rightarrow R_n^{ij} v_{ij} \underline{v})$.

The class of Q_n*-frames is denoted as* QF_n.

The next lemma links up frames and algebras on the Q-side.

Lemma 4.10 *Let* $3 \leq n < \omega$. *A* Q_n*-type algebra* $\mathfrak{A}$ *is a* Q_n*-algebra iff it can be embedded in the complex algebra of a* Q_n*-frame. In brief:*

$$\mathsf{Q}_n = \mathsf{S}\,\mathsf{Cm}\,\mathsf{QF}_n.$$

PROOF. (Sketch). We will show that for an arbitrary Q_n-type frame $\mathfrak{F}$, we have

$$\mathfrak{F} \text{ is in } \mathsf{QF}_n \text{ iff } \mathfrak{Cm}\,\mathfrak{F} \text{ is in } \mathsf{Q}_n. \tag{1}$$

¿From this the Lemma follows easily, since the variety Q_n is canonical — note that canonicity of Q_n follows from the simple (negation-free) syntactic shape of its defining axioms, cf. [Jónsson & Tarski 1951].

One can prove (1) easily using correspondence theory; that is, we will use the fact that with each Q-axiom there is a *corresponding* frame condition (effectively computable from the axiom) such that the axiom holds in $\mathfrak{Cm}\,\mathfrak{F}$ iff $\mathfrak{F}$ satisfies the frame condition; for details, see [de Rijke & Venema 1995].

We will confine ourselves to a few examples of the direction '$\Leftarrow$' of (1). Assume that $\mathfrak{Cm}\,\mathfrak{F}$ is a Q_n-frame. Then $\mathfrak{F} \models R1$ since $R1$ is the frame correspondent of axiom Q1; likewise, R2 corresponds to Q9*, and R3 to Q3. We leave the conditions R4 – R8 as exercises for the reader, and finish with showing that $\mathfrak{F} \models R9$. ¿From Lemma 2.2.(12) we may easily infer that $\mathfrak{F} \models \forall \underline{v} \, (\exists w \, R^{kl} w \underline{v} \leftrightarrow \exists u \, R^{ij} u \underline{v})$; taken together with R3 this immediately yields R9. QED

The final link that we need is between the two kinds of frames.

Definition 4.11 *Let $\mathfrak{M} = \langle W, C, F, I\rangle$ be an arrow frame in AF_n; recall the definition of the good n-dimensional basis $B_{\mathfrak{M}}$. Define, for $k, l < n$, the (n^2+1)-ary relation R^{kl} on W as follows:*

$$R^{kl}w\underline{v} \text{ iff } \underline{v} \in B_{\mathfrak{M}} \text{ and } w = v_{kl}.$$

With $\mathfrak{M}^Q$ we denote the associated Q_n-type frame $\langle W, I, R^{kl}\rangle_{k,l<n}$.

Conversely, given a Q_n-frame $\mathfrak{F} = \langle W, I, R_n^{kl}\rangle_{k,l<n}$, we define its associated arrow-type frame as $\mathfrak{F}^A = \langle W, D_{\mathfrak{F}}, G_{\mathfrak{F}}, I\rangle$ where $D_{\mathfrak{F}}$ and $G_{\mathfrak{F}}$ are as defined in 4.9.

In the main lemma of this section we show that on the frame level, there is an immediate correspondence between arrow-type frames and Q-type frames.

Lemma 4.12 *Let $\mathfrak{M} = \langle W, I, F, C\rangle$ be an arrow frame with a good n-dimensional basis B and let $\mathfrak{F} = \langle W, I, R_n^{kl}\rangle_k, l < n$ be a Q_n-frame. Then*

1. *$\mathfrak{M}^Q$ is a Q_n-frame;*
2. *$\mathfrak{F}^A$ belongs to AF_n.*
3. *$(\mathfrak{M}^Q)^A \simeq \mathfrak{M}$;*
4. *$\mathfrak{Cm}\,\mathfrak{F}^A = \mathfrak{Rd}_T\,\mathfrak{Cm}\,\mathfrak{F}$.*

Proof. We first show that

$$C = D_{\mathfrak{M}^Q} \text{ and } F = G_{\mathfrak{M}^Q}. \tag{2}$$

We only prove that $C = D_{\mathfrak{M}^Q}$; the other proof is similar. First assume $Cxyz$. Since $B_{\mathfrak{M}}$ is a basis, by Definition 4.6(c) and (b) there is some $\underline{w} \in B_{\mathfrak{M}}$ such that $w_{01} = x$, $w_{02} = y$, and $w_{21} = z$. Using the definition of R^{kl} on $\mathfrak{M}$ we obtain that $D_{\mathfrak{M}^Q}xyz$, by D1.

Assume $D_{\mathfrak{M}^Q}xyz$. By D1 this means that there is some $\underline{v}$ such that $x = v_{ij}, y = v_{ip}, z = v_{pj}$ and $R^{kl}v_{kl}\underline{v}$. So $\underline{v} \in B$, by definition of R^{kl}. Then $Cv_{ij}v_{ip}v_{pj}$, by 4.6(a).

For **part 1** of the Lemma, we show that $\mathfrak{M}^Q$ satisfies conditions R1 – R9 of Definition 4.9.

It is obvious that $\mathfrak{M}^Q$ satisfies R1 because of Definition 4.6(c). R2 follows from 4.6(b) and the fact that $C = D_{\mathfrak{M}^Q}$, R3 from the definition of R^{kl} on $\mathfrak{M}$. The fact that $F = G_{\mathfrak{M}^Q}$ makes R4 equal to 4.6(iv). For R5, first observe that any arrow frame satisfies the formula $(Fuw \wedge Fwy) \to u = y$; from this and (2) R5 follows immediately. R6 follows from 4.6(a).

For R7, consider a map $f : n \to n$, and a matrix $\underline{v} \in B$ such that Iv_{st} for all $s, t < n$ with $f(s) = f(t)$. We want to prove the existence of a matrix $\underline{z} \in B$ satisfying $z_{f(p)f(q)} = v_{pq}$ for all $p, q < n$.

We first prove

$$f(p) = f(q), f(r) = f(s) \text{ only if } v_{pr} = v_{qs}. \tag{3}$$

For a proof, observe that $f(p) = f(q)$ and $f(r) = f(s)$ imply that Iv_{pq} and Iv_{sr}. Since $\underline{v} \in B$, we have $Cv_{pr}v_{pq}v_{qr}$ and $Cv_{qr}v_{qs}v_{sr}$ by 4.6(a). We leave it to the reader to verify that in an arrow frame this implies $v_{pr} = v_{qr}$ and $v_{qr} = v_{qs}$. This proves (3).

Now let g be a bijection such that $f \circ g \circ f = f$. By (3) we have $(v^g)_{f(p)f(q)} = v_{g(f(p))g(f(q))} = v_{pq}$ for all p, q. Since $B_{\mathfrak{M}}$ is a good basis, $\underline{v}^g \in B_{\mathfrak{M}}$. Hence the matrix $\underline{v}^g$ satisfies our requirements.

For R8, it suffices to prove that for an arbitrary map $f : n \to n$ and an arbitrary matrix $\underline{v} \in B_{\mathfrak{M}}$, the matrix $\underline{v}^f$ is in $B_{\mathfrak{M}}$ as well.

Let us agree to call a function $h : n \to n$ simple if there is a p such that for all $k \neq p$ we have $h(k) = k$ while $h(p) \neq p$. Then every function is a composition of simple functions and permutations. Hence, since $B_{\mathfrak{M}}$ is good, it suffices to restrict ourselves to the case where f is a simple function f.

Fix $p \in n$ as the number such that for all $k \in n$, $f(k) = k$ if $k \neq p$, and let i, j be numbers distinct from p. By Definition 4.6(a), we have $Cv_{ij}v_{ii}v_{ij}$, so according to Definition 4.6(b), there is a matrix $\underline{w} \in B_{\mathfrak{M}}$ such that $w_{ip} = v_{ii}, w_{pj} = v_{ij}$, and $w_{st} = v_{st}$ whenever $s, t \neq p$. Using (a) and the properties of arrow frames it is not difficult to check that $\underline{w}$ is $\underline{v^f}$. But then indeed $\underline{v}^f$ is in $B_{\mathfrak{M}}$.

Finally, R9 follows from the definition of R^{kl} on $\mathfrak{M}$.

For **part 2**, let $\mathfrak{F} = (W, I, R^{kl}_n)_{k,l<n}$ be a Q_n-frame. We show that $\mathfrak{F}^A = (W, I, G_{\mathfrak{F}}, D_{\mathfrak{F}})$ is an arrow frame; of the conditions (i)–(v), we only check the first one; the other conditions are checked in the same manner. Assume $D_{\mathfrak{F}}uyz$ and Iz. By the definition of $D_{\mathfrak{F}}$ of 4.9 there is a n^2-matrix $\underline{v}$ of elements of W, and numbers $i, j, p, k, l < n$ such that $u = v_{ij}$, $y = v_{ip}$, $z = v_{pj}$ and $R^{kl}_n v_{kl}\underline{v}$. So Iz is Iv_{pj}. Let $f : n \to n$ be a map such that $f(p) = f(j)$; then by R7 we have

$$\exists \underline{x}(R_n^{f(k)f(l)} v_{f(k)f(l)}\underline{x} \wedge \bigwedge x_{f(s)f(t)} = v_{st}).$$

¿From this it follows that $u = v_{ij} = x_{f(i)f(j)} = x_{f(i)f(p)} = v_{ip} = y$.

Now we check that $B^{\mathfrak{F}}$ is an n-dimensional basis for it, where $B^{\mathfrak{F}}$ is defined as follows:

$$B^{\mathfrak{F}} = \{\underline{v} : R^{kl}_n(v_{kl}, \underline{v}), k, l < n\}.$$

For condition 4.6(a), assume $\underline{v} \in B^{\mathfrak{F}}$ and $i, j, p < n$. We obtain Iv_{ii} by R6, $G_{\mathfrak{F}}v_{ji}v_{ij}$ by using R4 and definition D2, and $D_{\mathfrak{F}}v_{ij}v_{ip}v_{pj}$ is immediate by D1. Then condition (b) follows from R2, and (c) is a direct consequence of R1.

Part 3 follows immediately from (2).

In order to prove **part 4**, it suffices to show that for arbitrary subsets $U, V \subseteq W$,

$$\begin{aligned} m_{D_{\mathfrak{F}}}(U,V) &= \{w \in W \mid \exists \underline{v} \in W\, (R^{01}_n w\underline{v}, v_{02} \in U, v_{21} \in V)\}, \\ m_{G_{\mathfrak{F}}}(U) &= \{w \in W \mid \exists \underline{v} \in W\, (R^{01}_n w\underline{v}, v_{10} \in U)\}. \end{aligned}$$

We only prove the first identity. The inclusion $\supseteq$ is immediate by the definitions. For the other direction, assume that $w \in m_{D_{\mathfrak{F}}}(U,V)$. By definition, there are $u \in U$ and $v \in V$ such that $D_{\mathfrak{F}}wuv$. This implies the existence of a matrix $\underline{w}$ of elements of W, and numbers k, l such that $R^{kl}w_{kl}\underline{w}$, $w = w_{ij}$, $u = w_{ip}$ and $v = w_{pj}$. Applying R9 to $R^{kl}w_{kl}\underline{w}$, we get $R^{ij}w\underline{w}$. Now consider a map $f : n \to n$ such that $f(0) = i$, $f(1) = j$ and $f(2) = p$; by R8, we get $R^{01}w\underline{w}^f$ where $w^f_{02} = w_{f(0)f(2)} = w_{ip} = u \in U$ and $w^f_{21} = w_{f(2)f(1)} = w_{pj} = v \in V$. But this is precisely what we required.

This finishes the proof of Lemma 4.12. QED

Note that in general, we need *not* have that $(\mathfrak{F}^A)^Q \simeq \mathfrak{F}$; when moving from $\mathfrak{F}$ to $(\mathfrak{F}^A)^Q$, we always choose the *maximal* basis of $\mathfrak{F}^A$.

Proof of Theorem 3. First assume that $\mathfrak{A}$ is a relation algebra of dimension n. By Lemma 4.8, there is some arrow frame $\mathfrak{M}$ in AF_n such that $\mathfrak{A}$ can be embedded in $\mathfrak{Cm}\,\mathfrak{M}$. By Lemma 4.12, (parts 1 and 3) there is a Q_n-frame $\mathfrak{F}$ such that $\mathfrak{M} \simeq \mathfrak{F}^A$. Lemma 4.12.4 then implies that $\mathfrak{Cm}\,\mathfrak{M}$ is isomorphic to $\mathfrak{Rd}_T\,\mathfrak{Cm}\,\mathfrak{F}$. By Lemma 4.10, $\mathfrak{Cm}\,\mathfrak{F}$ belongs to Q_n, so $\mathfrak{Cm}\,\mathfrak{M}$ belongs to $\mathsf{Rd}_T\,\mathsf{Q}_n$. But then clearly $\mathfrak{A}$ belongs to $\mathsf{S}\,\mathsf{Rd}_T\,\mathsf{Q}_n$.

For the converse direction it suffices to prove that $\mathsf{S}\,\mathsf{Rd}_T\,\mathsf{Q}_n$ is a subclass of MA_n, since the latter class is closed under taking subalgebras. Hence, assume that $\mathfrak{A}$ itself is the Tarski reduct of some Q_n-algebra $\mathfrak{B}$. By Lemma 4.10 there is some Q_n-frame $\mathfrak{F}$ such that

$$\mathfrak{B} \rightarrowtail \mathfrak{Cm}\,\mathfrak{F}.$$

From this it is immediate that

$$\mathfrak{A} = \mathfrak{Rd}_T\,\mathfrak{B} \rightarrowtail \mathfrak{Rd}_T\,\mathfrak{Cm}\,\mathfrak{F},$$

while from Lemma 4.12.4 it follows that

$$\mathfrak{Rd}_T\,\mathfrak{Cm}\,\mathfrak{F} \simeq \mathfrak{Cm}\,\mathfrak{F}^A.$$

Since $\mathfrak{F}^A$ belongs to AF_n by Lemma 4.12.2, it is then immediate by the definitions that $\mathfrak{A}$ belongs to MA_n. This proves that $\mathsf{Rd}_T\,\mathsf{Q}_n$ is indeed a subclass of MA_n. QED

Finally, a nice fact concerning the family of varieties $(\mathsf{MA}_n)_{n\in\omega}$ is that its intersection is precisely the variety RRA:

$$\bigcap \mathsf{MA}_n \;=\; \mathsf{RRA}, \tag{4}$$

cf. [Maddux 1983]. Now suppose that we define Q_ω as the class of Q-type algebras $\mathfrak{A}$ such that for each n, its *n-reduct* $\mathfrak{A}_n = (A,\cdot,+,-,0,1',Q^{kl}_n)_{k,l<n}$ is in Q_n. It follows from (4) and Theorem 3 that

$$\mathsf{RRA} = \mathsf{S}\,\mathsf{Rd}_T\,\mathsf{Q}_\omega.$$

But it also holds that

$$\mathsf{RRA} = \mathsf{S}\,\mathsf{Rd}_T\,\mathsf{RQ} = \mathsf{S}\,\mathsf{Rd}_T\,\mathsf{Q},$$

as a rather straightforward argument will show. This raises the obvious question whether in fact $\mathsf{Q}_\omega = \mathsf{Q}$. Looking at the definition of Q_ω this seems unlikely: the 'network amalgamation' condition Q9 of Q seems far stronger than the conditions Q9* of Q_n. We leave this matter for further research.

References

[Andréka et alii 1991] H. Andréka, J.D. Monk and I. Németi (eds.), *Algebraic Logic; Proceedings of the 1988 Budapest Conference on Algebraic Logic*, North-Holland, Amsterdam, 1991,

[Goldblatt 1989] R. Goldblatt, "Varieties of Complex Algebras", *Annals of Pure and Applied Logic*, **38** (1989) pp. 173–241.

[Hirsch & Hodkinson 199?] R. Hirsch and I. Hodkinson, "Step by step — building representations in algebraic logic", *Journal of Symbolic Logic*, **62** (1997) pp. 225–279.

[Hirsch & Hodkinson 1997] R. Hirsch and I. Hodkinson, "Relation algebras with n-dimensional relational bases", manuscript, 1997.

[Jipsen 1993] P. Jipsen, "Discriminator varieties of Boolean algebras with residuated operators", in: [Rauszer 1993], pp. 239–252.

[Jónsson 1991] B. Jónsson, "The theory of binary relations", in: [Andréka et alii 1991], pp. 245–292.

[Jónsson & Tarski 1951] B. Jónsson and A. Tarski, "Boolean algebras with operators, Part 1". *American Journal of Mathematics*, **73** (1951), pp. 891-939.

[Kahl 1996] W. Kahl, *Algebraische Termgraphersetzung mit gebundenen Variabelen*, Herbert Utz Verlag Wissenschaft, 1996. (Disseration of the Fakultät für Informatik, Universität der Bundeswehr München).

[Maddux 1982] R.Maddux, "Some varieties containing relation algebras", *Transactions of the American Mathematical Society*, bf 272(1982), number 2, pp. 501-526.

[Maddux 1983] R.Maddux, "A sequent calculus for relation algebras", *Annals of Pure and Applied Logic*, **25**(1983), pp. 73-101.

[Maddux 1992] R. Maddux, "Relation algebras of every dimension", *Journal of Symbolic Logic*, **57** (1992) pp. 1213 – 1229.

[Németi & Andréka 1991] I. Németi and H. Andréka, "On Jónsson's clones of operations on binary relations", in: [Andréka et alii 1991], pp. 431–442.

[Nishimura 1980] H. Nishimura, "Interval logics with applications to study of tense and aspect in English", *Publications of the Research Institute for Mathematical Sciences, Kyoto University*, **16** (1980), pp. 417–459.

[Rauszer 1993] C. Rauszer (ed.), "Algebraic methods in logic and computer science", Banach Center Publications Volume 28, Polish Academy of Sciences, Warsaw, 1993.

[de Rijke & Venema 1995] M. de Rijke and Y. Venema, "Sahlqvist's theorem for Boolean algebras with operators, with an application to cylindric algebras", *Studia Logica*, **54** (1995), pp. 61–78.

[Venema 1991] Y. Venema, "Relational Games", in: [Andréka et alii 1991], pp. 695 – 718.