Veblen hierarchy in the context of provability algebras

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Abstract

We study an extension of Japaridze's polymodal logic **GLP** with transfinitely many modalities and develop a provability-algebraic ordinal notation system up to the ordinal Γ_0.

In the papers [1, 2] a new algebraic approach to the traditional proof-theoretic ordinal analysis was presented based on the concept of *graded provability algebra*. The graded provability algebra of a formal theory T is its Lindenbaum boolean algebra equipped with additional unary operators $\langle n \rangle$ mapping a sentence φ to the sentence $n\text{-}\mathsf{Con}(\varphi)$ expressing that $T + \varphi$ is n-consistent. The n-consistency operators, together with their dual n-provability operators, satisfy a particular modal logic **GLP** described by G. Japaridze (see [3]). In this framework, an ordinal notation system up to the ordinal ϵ_0 naturally emerges from the closed fragment of **GLP**. This allows for a transparent proof-theoretic analysis of Peano arithmetic PA, including a characterization of its class of provably total computable functions and a consistency proof à la Gentzen. More generally, it yields a characterization of provable Π_n-sentences of PA, for any $n \geq 1$, in terms of iterated reflection principles (Schmerl's theorem) and leads to an interesting combinatorial independent principle (see [2]).

It appears to be a natural project to extend this approach to theories stronger than PA. The next stage are theories of the strength of predicative analysis whose proof-theoretic ordinal is the Feferman–Schütte ordinal Γ_0. In this paper we make the first step in this direction and present a construction of *autonomous expansions* of provability algebras that leads to an ordinal notation system up to Γ_0. We postpone the proof-theoretic analysis of concrete theories to a later paper. Here we stay within the context of modal logic and ordinal notation systems. As a side result we obtain a normal form theorem for the closed fragment of an extension of Japaridze's logic **GLP**. A similar result for **GLP** itself is due to Ignatiev [5].

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1 GLP and its arithmetical interpretation

We first introduce and study a variant of Japaridze's polymodal logic **GLP** in a language with transfinitely many modalities.

Let $\mathcal{L}_\Lambda$ be the language of propositional polymodal logic with modalities $[x]$ labelled by ordinals x from a subclass $\Lambda \subseteq \mathrm{On}$. As usual, $\langle x\rangle\varphi$ abbreviates $\neg[x]\neg\varphi$. The system $\mathbf{GLP}_\Lambda$ is an analog of Japaridze's logic **GLP** in this language:

Axioms: (i) Boolean tautologies;

(ii) $[x](\varphi \to \psi) \to ([x]\varphi \to [x]\psi)$;

(iii) $[x]([x]\varphi \to \varphi) \to [x]\varphi$;

(iv) $[x]\varphi \to [y][x]\varphi$, for $x \leq y$.

(v) $\langle x\rangle\varphi \to [y]\langle x\rangle\varphi$, for $x < y$.

(vi) $[x]\varphi \to [y]\varphi$, for $x \leq y$;

Rules: modus ponens, $\varphi \vdash [x]\varphi$.

The original Japaridze's system **GLP** is just $\mathbf{GLP}_\omega$. The logic given by Axioms (i)–(v) and the same inference rules was isolated by Ignatiev [5] and is denoted $\mathbf{GLP}^-_\Lambda$ in this paper. If $\Lambda = \mathrm{On}$ we omit the subscript Λ, so **GLP** and $\mathbf{GLP}^-$ actually mean $\mathbf{GLP}_{\mathrm{On}}$ and $\mathbf{GLP}^-_{\mathrm{On}}$. It is easy to see that in the presence of Axiom (vi), Axiom (iv) becomes redundant.

Given a sufficiently strong arithmetical theory T, a provability interpretation for $\mathbf{GLP}_\omega$ (see [2]) is given by reading the modal formula $[n]\varphi$ as the sentence "φ is provable from T together with all true Π_n-sentences."

For ordinals $x > \omega$ the logic $\mathbf{GLP}_x$ can be interpreted in formal theories in which the hyperarithmetical hierarchy up to level x can be defined. The intuitive meaning of the formula $[y]\varphi$ is then "φ is provable from T together with all true hyperarithmetical sentences of level y."

We do not want to give precise definitions of the appropriate theories in this paper. However, we mention that the construction of hyperarithmetical hierarchy is explicit in various systems of *ramified analysis* (see [4, 8, 7]) and second order theories like $(\Pi^0_1\text{-}\mathsf{CA})_{<x}$ and ATR.

2 Normal forms for words

We study the closed (or letterless) fragment of **GLP**. The systems $\mathbf{GLP}_\Lambda$ are very similar to $\mathbf{GLP}_\omega$. A normal form theorem for closed formulas of $\mathbf{GLP}_\omega$ was obtained in [5]. Since in every formula only finitely many modalities can occur, essentially the same theorem holds in any $\mathbf{GLP}_\Lambda$. However, to make the paper self-contained, we present a short proof of this theorem here along the lines of [2]. Our treatment also simplifies the one in [5] for the case $\mathbf{GLP}_\omega$.

Let S denote the class of all words in the alphabet On, including the empty word ϵ. S_x will denote the class of all words in the alphabet

$$\Lambda = \{y \in \mathrm{On} : x \leq y\}.$$

We shall reserve Greek letters $\alpha, \beta, \gamma, \dots$ for words, and Latin letters $x, y, z \dots$ for ordinals. To each element $\alpha = x_1 x_2 \dots x_k$ of S we associate its *modal interpretation*, that is, the closed modal formula

$$\langle x_1 \rangle \langle x_2 \rangle \cdots \langle x_k \rangle \top, \tag{1}$$

We do not distinguish between the word α and formula (1). We also identify ϵ with $\top$.

Below we use $\vdash$ to denote provability in **GLP**. We write $\alpha \sim \beta$ if $\vdash \alpha \leftrightarrow \beta$. $\alpha = \beta$ means graphical identity.

For each x there is an ordering $<_x$ on S defined by

$$\alpha <_x \beta \iff \vdash \beta \rightarrow \langle x \rangle \alpha.$$

It is immediately seen that $<_x$ is transitive. One can show in two different ways that it is irreflexive. In view of Axiom (iii), $\alpha <_x \alpha$ iff $\vdash \alpha \rightarrow \langle x \rangle \alpha$ iff $\vdash \neg \alpha$. One way to show that this is impossible is to appeal to the provability interpretation of α and to see that α must be a true sentence. Another way would be to produce a Kripke model for **GLP** at some world of which α is true.

We shall later see that $<_x$ is well-founded and our task will be to determine its ordinal for logics of the form $\mathbf{GLP}_y$, for $y > x$.

We notice an obvious 'shifting' property of these orderings. Let $x \uparrow \alpha$ be the result of replacing in α every ordinal y by $x + y$. Clearly, all axioms of **GLP** are stable under this mapping, so

$$\alpha <_0 \beta \quad \Rightarrow \quad x \uparrow \alpha <_x x \uparrow \beta. \tag{2}$$

The converse implication also holds, and in fact $x \uparrow \cdot$ is an isomorphism of $(S_0, <_0)$ onto $(S_x, <_x)$ (see Corollary 7 below). The inverse mapping will be denoted $x \downarrow \cdot$.

Width $w(\alpha)$ of a word $\alpha \in S$ is the number of different letters occurring in it. We shall often define functions on words by induction on their width. The length of α is denoted $|\alpha|$ and $\min(\alpha)$ denotes the smallest ordinal occurring in α.

Some of the elements of S are pairwise equivalent, so we first define a subclass $NF \subset S$ of *normal forms*.

- ϵ and any word of width 1 belongs to NF.
- Assume $w(\alpha) > 1$ and let $x = \min(\alpha)$. Then graphically $\alpha = \alpha_1 x \cdots x \alpha_k$, where all α_i do not contain x and hence $w(\alpha_i) < w(\alpha)$ for $1 \leq i \leq k$. Then $\alpha \in NF$ iff all $\alpha_i \in NF$ and, for all $1 \leq i < k$, $\alpha_{i+1} \not<_{x+1} \alpha_i$. (Note that $\alpha_i \in S_{x+1}$.)

Lemma 1 *(i) If $x < y$, then* $\mathbf{GLP}^- \vdash (\langle y\rangle\varphi \wedge \langle x\rangle\psi) \leftrightarrow \langle y\rangle(\varphi \wedge \langle x\rangle\psi)$*;*

(ii) If $x < y$, then $\mathbf{GLP}^- \vdash (\langle y\rangle\varphi \wedge [x]\psi) \leftrightarrow \langle y\rangle(\varphi \wedge [x]\psi)$*;*

(iii) If $\alpha \in S_{x+1}$, then $\mathbf{GLP}^- \vdash \alpha \wedge x\beta \leftrightarrow \alpha x\beta$*.*

(iv) If $\alpha_1, \alpha_2 \in S_{x+1}$ and $\alpha_1 \sim \alpha_2$, then $\alpha_1 x\beta \sim \alpha_2 x\beta$.

Proof. Statements (i) and (ii) essentially follow from Axioms (v) and (iv), respectively. Statement (iii) follows by repeated application of (i). Statement (iv) follows from (iii). ⊠

Lemma 2 *Let $\alpha = \alpha_1 x\alpha_2 x \cdots x\alpha_k$, where all $\alpha_i \in S_{x+1}$. If $\alpha_1 >_{x+1} \alpha_2$, then*

$$\alpha \sim \alpha_1 x\alpha_3 x \cdots x\alpha_k.$$

Proof. Let $\beta = \alpha_3 x \cdots x\alpha_k$. Since $\alpha_1, \alpha_2 \in S_{x+1}$, using Lemma 1 (iii) we obtain

$$\begin{aligned}
\mathbf{GLP}^- \vdash \alpha = \alpha_1 x\alpha_2 x\beta \quad &\leftrightarrow \quad \alpha_1 \wedge x\alpha_2 x\beta \\
&\rightarrow \quad \alpha_1 \wedge x\beta, \quad \text{by Axiom (iv)} \\
&\rightarrow \quad \alpha_1 x\beta.
\end{aligned}$$

On the other hand, if $\vdash \alpha_1 \rightarrow \langle x+1\rangle\alpha_2$, then

$$\begin{aligned}
\vdash \alpha_1 x\beta \quad &\leftrightarrow \quad \alpha_1 \wedge x\beta \\
&\rightarrow \quad \langle x+1\rangle\alpha_2 \wedge x\beta \\
&\rightarrow \quad \langle x+1\rangle(\alpha_2 \wedge x\beta) \\
&\rightarrow \quad \langle x+1\rangle\alpha_2 x\beta \\
&\rightarrow \quad x\alpha_2 x\beta, \quad \text{by Axiom (vi).}
\end{aligned}$$

Hence,

$$\begin{aligned}
\vdash \alpha_1 x\beta \quad &\leftrightarrow \quad \alpha_1 \wedge x\beta \\
&\leftrightarrow \quad \alpha_1 \wedge x\alpha_2 x\beta \\
&\leftrightarrow \quad \alpha_1 x\alpha_2 x\beta.
\end{aligned}$$

Therefore, $\alpha_1 x\beta \sim \alpha_1 x\alpha_2 x\beta$. ⊠

Proposition 3 *Every word $\alpha \in S$ can be brought into an equivalent normal form, that is, there is an $\alpha' \in NF$ such that $\alpha' \sim \alpha$.*

Proof. The word α' can be constructed by induction on the length of α. Write α in the form $\alpha_1 x\alpha_2 x \cdots x\alpha_k$ with all $\alpha_i \in S_{x+1}$ and $x = \min(\alpha)$. By the induction hypothesis we may assume the word $\alpha_2 x \cdots x\alpha_k$ to be in a normal form. Secondly, using Lemma 1 (iv) we may also assume that $\alpha_1 \in NF$. Hence, if $\alpha_1 \not>_{x+1} \alpha_2$, α is already in a normal form. Otherwise, apply Lemma 2 and bring the word $\alpha_1 x\alpha_3 x \cdots x\alpha_k$ to a normal form using the induction hypothesis. ⊠

Proposition 4 *Any two normal forms $\alpha, \beta \in S_x$ are $<_x$-comparable, that is,*

$$\alpha <_x \beta \text{ or } \beta <_x \alpha \text{ or } \beta = \alpha. \qquad (*)$$

Proof. Without loss of generality, we may assume that x occurs in α or β, hence $x = \min(\alpha\beta)$. (If $x < \min(\alpha\beta)$ the claim only becomes weaker.) In view of (2), we may also assume that $x = 0$ (otherwise, consider $x{\downarrow}\alpha$ and $x{\downarrow}\beta$).

We reason by induction on $w(\alpha\beta)$. For unary words the claim is obvious, so we consider the case that $w(\alpha\beta) > 1$.

As before, α and β can be written in the form

$$\alpha = \alpha_k 0 \alpha_{k-1} 0 \cdots 0 \alpha_1, \qquad \beta = \beta_m 0 \beta_{m-1} 0 \cdots 0 \beta_1,$$

where all α_i and β_j do not contain 0. By the induction hypothesis we obtain

$$\alpha_1 <_1 \beta_1 \text{ or } \beta_1 <_1 \alpha_1 \text{ or } \beta_1 = \alpha_1.$$

Claim. If $\alpha_1 <_1 \beta_1$, then $\alpha <_0 \beta$. Symmetrically, if $\alpha_1 >_1 \beta_1$, then $\alpha >_0 \beta$.

We only prove the first part. Let $\overline{\alpha}_i = \alpha_i 0 \cdots 0 \alpha_1$. We prove by induction on i that $\overline{\alpha}_i <_1 \beta_1$, for all $i \leq k$. It is obvious that $\beta_1 \leq_0 \beta$ and $<_1$ is stronger than $<_0$, so the Claim will follow.

Notice that $\vdash \alpha_{i+1} 0 \overline{\alpha}_i \leftrightarrow (\alpha_{i+1} \wedge 0\overline{\alpha}_i)$. On the other hand, $\beta_1 >_1 \alpha_{i+1}$ by the transitivity of $<_1$ and because $\alpha \in NF$. By the induction hypothesis we have $\beta_1 >_1 \overline{\alpha}_i$, hence

$$\begin{aligned} \vdash \beta_1 &\to \langle 1 \rangle \alpha_{i+1} \wedge \langle 0 \rangle \overline{\alpha}_i \\ &\to \langle 1 \rangle (\alpha_{i+1} \wedge 0\overline{\alpha}_i) \\ &\to \langle 1 \rangle \alpha_{i+1} 0 \overline{\alpha}_i, \end{aligned}$$

which proves the induction step.

Continuing the proof of Proposition 4 from the Claim we can conclude that the disjunction $(*)$ can only be false if $\alpha_1 = \beta_1$. In this case we have to compare α_2 and β_2 using the induction hypothesis again. Assume w.l.o.g. that $\alpha_2 <_1 \beta_2$. Then we have $\alpha_2 0 \alpha_1 <_1 \beta_2 0 \beta_1$, because

$$\vdash \beta_2 \wedge 0\alpha_1 \to \langle 1 \rangle (\alpha_2 \wedge 0\alpha_1).$$

Following the proof of the Claim we then obtain $\overline{\alpha}_i <_1 \beta_2 0 \beta_1 \leq_0 \beta$, for all $i > 1$. It follows that in this case $\alpha <_0 \beta$. Using the symmetry, the only remaining case is that both $\alpha_1 = \beta_1$ and $\alpha_2 = \beta_2$, and the reasoning can be continued. If $\alpha \neq \beta$, at the end we come to the situation when one of the two words, say α, is a proper end segment of the other. Then obviously $\beta >_0 \alpha$. ⊠

From the proof of Proposition 4 one can extract the following recursive comparison algorithm. Given $\alpha, \beta \in S$ consider three cases:

1. $x = \min(\alpha) < \min(\beta)$. Then write α in the form $\alpha = \alpha_2 x \alpha_1$ with $\alpha_1 \in S_{x+1}$ and $<_{x+1}$-compare α_1 with β.

2. $x = \min(\beta) < \min(\alpha)$. This is symmetrical.

3. $x = \min(\beta) = \min(\alpha)$. Write $\alpha = \alpha_2 x \alpha_1$ and $\beta = \beta_2 x \beta_1$ with $\alpha_1, \beta_1 \in S_{x+1}$. $<_{x+1}$-compare α_1 with β_1. If $\alpha_1 = \beta_1$, $<_x$-compare α_2 with β_2.

This can be considered as a definition by recursion on $|\alpha| + |\beta|$, because in all cases $|\alpha_i| < |\alpha|$ and $|\beta_i| < |\beta|$, for $i = 1, 2$.

We also infer the following corollaries.

Corollary 5 *The normal form of a word is graphically unique.*

Corollary 6 *For any $\alpha, \beta \in S_x$, either $\vdash x\alpha \to x\beta$ or $\vdash x\beta \to x\alpha$ and this can be effectively decided.*

Corollary 7 *$(S, <_0)$ is isomorphic to $(S_x, <_x)$:*

$$\alpha <_0 \beta \iff x \uparrow \alpha <_x x \uparrow \beta.$$

Proof. If $\alpha \not<_0 \beta$, then $\alpha \sim \beta$ or $\beta <_0 \alpha$. Then we have $x \uparrow \alpha \sim x \uparrow \beta$ or $x \uparrow \beta <_x x \uparrow \alpha$. In both cases $x \uparrow \alpha <_x x \uparrow \beta$ would contradict the irreflexivity of $<_x$. ⊠

It also follows from Proposition 4 that on S_x the orderings $<_x$ and $<_0$ coincide.

Corollary 8 *For all $\alpha, \beta \in S_x$, $\alpha <_x \beta$ iff $\alpha <_0 \beta$.*

Proof. If $\alpha <_x \beta$ clearly $\alpha <_0 \beta$. Conversely, assume $\alpha, \beta \in S_x$, $\alpha <_0 \beta$ and $\alpha \not<_x \beta$. Then by Proposition 4, $\beta <_x \alpha$ or $\alpha \sim \beta$. So, in the first case we have $\alpha <_0 \beta <_0 \alpha$. In the second case, obviously $\alpha <_0 \beta \sim \alpha$, both cases contradicting the irreflexivity of $<_0$. ⊠

Lemma 9 *For all $\alpha, \beta \in S_x$ there is an (effectively constructible) $\gamma \in S_x$ such that $\vdash \gamma \leftrightarrow (\alpha \wedge \beta)$.*

Proof. We reason by induction on the width of $\alpha\beta$. Without loss of generality assume that $x = \min(\alpha\beta)$. We can write α and β in the form $\alpha = \alpha_1 x \alpha'$ and $\beta = \beta_1 x \beta'$ with $\alpha_1, \beta_1 \in S_{x+1}$. We then have

$$\alpha \wedge \beta \sim \alpha_1 \wedge x\alpha' \wedge \beta_1 \wedge x\beta'.$$

From Corollary 6 we know that either $\vdash x\alpha' \to x\beta'$ or $\vdash x\beta' \to x\alpha'$. Assume $x\alpha'$ is stronger. By the induction hypothesis we can find a $\gamma_1 \in S_{x+1}$ such that $\gamma_1 \sim \alpha_1 \wedge \beta_1$. Therefore

$$\begin{aligned} \alpha \wedge \beta &\sim \alpha_1 \wedge \beta_1 \wedge x\alpha' \\ &\sim \gamma_1 \wedge x\alpha' \\ &\sim \gamma_1 x\alpha', \end{aligned}$$

which has the required form. ⊠

3 Normal forms for arbitrary closed formulas of GLP

Here we prove that an arbitrary closed formula of **GLP** is equivalent to a boolean combination of words and give a decision procedure for the closed fragment of **GLP**. However, for the construction of the ordinal notation system this result is not needed, so the reader only interested in ordinal notations can skip this section.

Lemma 10 *Suppose $\alpha, \alpha_1, \ldots, \alpha_k \in S_x$. Then there is $\beta \in S_x \cup \{\bot\}$ such that*

$$\beta \sim \langle x \rangle (\alpha \wedge \textstyle\bigwedge_i \neg\alpha_i).$$

Proof. We consider two cases.

CASE 1. For some $i, \vdash \alpha \to \alpha_i$. Then $(\alpha \wedge \bigwedge_i \neg\alpha_i) \sim \bot$ and hence $\langle x \rangle (\alpha \wedge \bigwedge_i \neg\alpha_i) \sim \bot$.

CASE 2. For all i, $\nvdash \alpha \to \alpha_i$. Consider any α_i. By Lemma 1 $\alpha \wedge \alpha_i$ is equivalent to a word $\gamma \in S_x$. By Proposition 4, one of the three cases holds: $\gamma \sim \alpha$, $\vdash \gamma \to \langle x \rangle \alpha$, or $\vdash \alpha \to \langle x \rangle \gamma$.

By our assumption the first case is impossible. The third case is impossible, because $\vdash \gamma \to \alpha$, so one would get $\vdash \alpha \to \langle x \rangle \alpha$, contradicting the irreflexivity of $<_x$.

So, we conclude that the second case must hold, that is,

$$\vdash \alpha \wedge \alpha_i \to \langle x \rangle \alpha.$$

This holds for all i, so

$$\vdash \alpha \wedge \textstyle\bigvee_i \alpha_i \to \langle x \rangle \alpha.$$

Hence, we obtain

$$\begin{array}{rcl} \vdash [x](\alpha \to \bigvee_i \alpha_i) & \leftrightarrow & [x](\alpha \to (\alpha \wedge \bigvee_i \alpha_i)) \\ & \to & [x](\alpha \to \langle x \rangle \alpha) \\ & \to & [x]\neg\alpha \\ & \to & [x](\alpha \to \bigvee_i \alpha_i). \end{array}$$

So, $\langle x \rangle (\alpha \wedge \bigwedge_i \neg\alpha_i) \sim \langle x \rangle \alpha$. ⊠

Lemma 11 *Let $\varphi(\alpha_1, \ldots, \alpha_k)$ be a boolean combination of words $\alpha_1, \ldots, \alpha_k$. Then $\langle x \rangle \varphi$ is equivalent to a boolean combination of words.*

Proof. Every nonempty α_i can be represented in the form $\alpha_i = \alpha'_i y_i \alpha''_i$, where $\alpha'_i \in S_x$ and $y_i < x$. We have $\alpha_i \sim (\alpha'_i \wedge y_i \alpha''_i)$. This means that we can w.l.o.g. assume that, for every i, either $\alpha_i \in S_x$ or α_i begins with a letter $y_i < x$.

Write φ in disjunctive normal form, that is, $\varphi \sim \bigvee_l \varphi_l$, where $\varphi_l = \bigwedge_i \pm\alpha_i$. We have $\langle x \rangle \varphi \sim \bigvee_l \langle x \rangle \varphi_l$, so it is sufficient to bring $\langle x \rangle \varphi_l$ to the required form.

Lemma 1 enables us to use the following identities, for $y < x$:

$$\langle x \rangle (\psi \wedge \langle y \rangle \theta) \sim (\langle x \rangle \psi \wedge \langle y \rangle \theta),$$
$$\langle x \rangle (\psi \wedge \neg\langle y \rangle \theta) \sim (\langle x \rangle \psi \wedge \neg\langle y \rangle \theta).$$

It follows that

$$\vdash \langle x\rangle \bigwedge_i \pm\alpha_i \leftrightarrow (\bigwedge_{i\notin I} \pm\alpha_i \wedge \langle x\rangle \bigwedge_{j\in I} \pm\alpha_j),$$

where I consists of those i for which $\alpha_i \in S_x$ (hence, all the other α_i begin with some $y_i < x$).

Now we are in a position to apply Lemma 10, which yields that $\langle x\rangle \bigwedge_{i\in I} \pm\alpha_i$ is equivalent to a word (or falsity). ⊠

Corollary 12 *Every closed formula of* **GLP** *is equivalent to a boolean combination of words.*

Corollary 13 *Let Λ be a recursive well-ordering. There is an effective decision algorithm for the closed fragment of* $\mathbf{GLP}_\Lambda$.

Proof. We have to decide if $\vdash \varphi(\alpha_1, \ldots, \alpha_n)$, for a given boolean combination φ of words $\alpha_1, \ldots, \alpha_n$. Write φ in *conjunctive* normal form. The provability of φ is equivalent to the provability of every conjunct, so it is sufficient to test the provability of the formulas of the form $\bigvee_i \pm\alpha_i$. Since the set of words modulo **GLP** is closed under conjunction, we can simplify this to a formula of the form

$$\alpha \to \bigvee_j \alpha_j,$$

where α accumulates all negatively occurring words.

We claim: $\vdash \alpha \to \bigvee_j \alpha_j$ iff $\vdash \alpha \to \alpha_j$, for some j. This is similar to the proof of Lemma 10, where the implication from right to left is obvious.

Suppose $\nvdash \alpha \to \alpha_j$, and let β_j be a word equivalent to $\alpha \wedge \alpha_j$. By Proposition 4 we obtain $\vdash \beta_j \to \langle 0\rangle\alpha$. This holds for all j, so

$$\vdash \alpha \wedge \bigvee_j \alpha_j \to \langle 0\rangle\alpha.$$

Hence, if $\vdash \alpha \to \bigvee_j \alpha_j$, then $\vdash \alpha \to \langle 0\rangle\alpha$, which is impossible.

Finally, we can test if $\vdash \alpha \to \alpha_j$ by bringing the words α and $\alpha \wedge \alpha_j$ to a normal form and testing if they coincide. ⊠

As another corollary we obtain the following statement.

Corollary 14 *Let φ, ψ be any closed formulas of* **GLP**. *Then* $\vdash \langle 0\rangle\varphi \to \langle 0\rangle\psi$ *or* $\vdash \langle 0\rangle\psi \to \langle 0\rangle\varphi$.

Proof. By Lemma 10 both $\langle 0\rangle\varphi$ and $\langle 0\rangle\psi$ are equivalent to some words or falsity. The case of falsity or truth immediately validates one of the implications. We may also assume that both words begin with 0, because a formula of the form $\langle 0\rangle\varphi$ does not imply any other (nonempty) word. Of any two words beginning with 0 one of the two implies the other, by Corollary 6. ⊠

4 Well-foundedness of the orderings $<_x$ on words

Here we shall give a simple but not very constructive proof that the ordering $<_0$ of the set *NF* of normal forms is well-founded. In the following sections we shall compute its order type for the set of words in the logic $\mathbf{GLP}_x$, for $x \in \mathrm{On}$. The formulas obtained will then allow for a more constructive well-foundedness proof. This makes the present proof superfluous, except that it allows for a clear derivation of these formulas which otherwise would appear rather unmotivated.

Theorem 1 *The ordering $<_0$ on NF is well-founded.*

Proof. We use the idea of a minimal sequence coming from the proof of Kruskal theorem [6].

Suppose there is an infinite decreasing chain $\gamma_1 >_0 \gamma_2 >_0 \cdots$ of words in *NF*. We can assume that this chain is minimal in the following sense. γ_1 has minimal length of all the words such that there is an infinite decreasing chain starting from γ_1. γ_2 has minimal length of all the words such that there is an infinite decreasing chain starting from γ_1, γ_2. Etcetera.

We can also assume that 0 occurs in some γ_k. Otherwise, if x is the minimal letter occurring in some of the words γ_i, consider the decreasing sequence $\delta_i := x \downarrow \gamma_i$, for $i < \omega$.

Let $\gamma_k = \alpha_k 0 \beta_k$ with $\beta_k \in S_1$. We can also write every γ_i, for $i > k$, in the form $\gamma_i = \alpha_i 0 \beta_i$, assuming that $\beta_i \in S_1$ and the part $\alpha_i 0$ can be empty. By the recursive definition of the ordering we obviously have: $\beta_k \geq_0 \beta_{k+1} \geq_0 \ldots$.

For the sequence $\beta_k \geq_0 \beta_{k+1} \geq_0 \ldots$ there are two possibilities.

1. There is an infinite strictly decreasing subsequence $\beta_k >_0 \beta_{k_1} >_0 \beta_{k_2} >_0 \ldots$. Then the chain $\gamma_1 >_0 \cdots >_0 \gamma_{k-1} >_0 \beta_k >_0 \beta_{k_1} >_0 \beta_{k_2} >_0 \cdots$ is strictly decreasing. This contradicts the minimality of the length of γ_k.

2. The sequence β_i stabilizes at some stage s, that is, $\beta_s = \beta_{s+1} = \cdots$. Then by the recursive definition of $<_0$ the chain $\gamma_1 >_0 \cdots >_0 \gamma_{s-1} >_0 \alpha_s >_0 \alpha_{s+1} >_0 \alpha_{s+2} >_0 \cdots$ is strictly decreasing. This contradicts the minimality of the length of γ_s.

Both possibilities lead to a contradiction. ⊠

As an immediate corollary we also conclude that the orderings $(S_x \cap NF, <_x)$ for any x are well-founded.

5 The Veblen hierarchy

Recall the standard definition of the Veblen hierarchy (see [8]). Given a class $X \subseteq \mathrm{On}$ let en_X denote its enumerating function. Let X' denote the class of fixed points of en_X, that is, $X' = \{x \in \mathrm{On} : \mathrm{en}_X(x) = x\}$. Define by transfinite induction on x the so-called critical classes:

$$\begin{aligned} \mathrm{Cr}_0 &= \{\omega^{1+y} : y \in \mathrm{On}\} \\ \mathrm{Cr}_{x+1} &= (\mathrm{Cr}_x)' \\ \mathrm{Cr}_x &= \bigcap_{y<x} \mathrm{Cr}_y, \quad \text{if } x \text{ is a limit ordinal.} \end{aligned}$$

Let φ_x be the enumerating function of Cr_x. In particular, $\varphi_0(y) = \omega^{1+y}$ and φ_1 enumerates the fixed points of φ_0, that is, $\varphi_1(y) = \epsilon_y$.

Our definition of Cr_0 and φ_0 deviates slightly from the standard one, because we start counting with ω, not with 1. However, this does not change the definitions of Cr_x for $x > 0$.

It is easy to verify that for all x the classes Cr_x are closed and unbounded, and that the functions φ_x are continuous.

The least ordinal x such that $x \in \mathrm{Cr}_x$ is the Feferman-Schütte ordinal Γ_0. It can also be characterized as the limit of the sequence $\varphi_0(0), \varphi_{\varphi_0(0)}(0), \ldots$, in other words, as the first ordinal closed under the operation $x \mapsto \varphi_x(0)$.

6 Representation of the critical classes

In this section we shall derive formulas for the order types of segments of $<_0$. The same formulas can then be used to establish the well-foundedness of $<_0$.

Let $o_x(\alpha)$ denote the order type of α in the ordering $(S_x, <_x)$. $o(\alpha)$ is short for $o_0(\alpha)$. Obviously, we have $o_x(\alpha) = o(x \downarrow \alpha)$. For a set $X \subseteq S_x$ we also let $o_x(X) = \{o_x(\alpha) : \alpha \in X\}$.

Lemma 15 *(i)* $o(0^n) = n$, *for any* n.

(ii) If $\alpha = \alpha_1 0 \ldots 0 \alpha_n$, *where all* $\alpha_i \in S_1$ *and not all of them empty, then* $o(\alpha) = \omega^{o_1(\alpha_n)} + \cdots + \omega^{o_1(\alpha_1)}$.

Proof. We only prove (ii). Firstly, bringing α to the normal form does not change either $o(\alpha)$ or the value of the expression on the right hand side, so we may assume α to be in the unique normal form, in particular, $\alpha_1 \leq_1 \alpha_2 \leq_1 \ldots \leq_1 \alpha_n$.

Secondly, the ordering $(S \cap NF, <_0)$ is isomorphic to the lexicographic ordering of such sequences $(\alpha_n, \ldots, \alpha_1)$ of elements of $(S_1 \cap NF, <_1)$. Therefore, the mapping $f(\alpha) = \omega^{o_1(\alpha_n)} + \cdots + \omega^{o_1(\alpha_1)}$ is order-preserving:

$$\alpha <_0 \beta \Rightarrow f(\alpha) < f(\beta).$$

It is also obvious that f is onto, hence an isomorphism between $(S_0 \cap NF, <_0)$ and On. (Here we use the fact that o_1 is an isomorphism from $(S_1 \cap NF, <_1)$ to On.) ⊠

Define $S_x^+ = S_x \setminus \{\epsilon\}$.

Lemma 16 *If f enumerates $o(S^+_x)$ and g enumerates $o(S^+_y)$, then $o(S^+_{x+y})$ is enumerated by $f \circ g$.*

Proof. We have: $S^+_{x+y} \subseteq S^+_x \subseteq S^+_0$. Look at the logic $\mathbf{GLP}_\Lambda$ with $\Lambda = \{y : y \geq x\}$. As noted above, this system is isomorphic to **GLP** by the function mapping every modality $\langle y \rangle$ to $\langle x + y \rangle$. Now, $g(z)$ is the z-th element of $o(S^+_y)$, hence of $o_x(S^+_{x+y})$. Similarly, $f(u)$ is the u-th element of $o_x(S^+_x)$. Hence, $f(g(z))$ is the z-th element of $o(S^+_{x+y})$. ⊠

Now we use these lemmas to prove the following theorem.

Theorem 2 $o(S^+_{\omega^x}) = \mathrm{Cr}_x$.

Proof. Transfinite induction on x. The basis of the induction, $o(S^+_1) = \mathrm{Cr}_0 = \{\omega^{1+y} : y \in \mathrm{On}\}$, follows from Lemma 15.

Let us now prove that $o(S^+_{\omega^{x+1}}) = \mathrm{Cr}_{x+1}$. By the induction hypothesis we have

$$o(S^+_{\omega^x}) = \mathrm{Cr}_x = \{\varphi_x(y) : y \in \mathrm{On}\}.$$

Lemma 16 yields that $o(S^+_{\omega^x+\omega^x})$ is enumerated by $\varphi_x \circ \varphi_x$, that is, by $y \mapsto \varphi_x(\varphi_x(y))$. Similarly,

$$o(S^+_{\omega^x \cdot n}) = \{\varphi_x^{(n)}(y) : y \in \mathrm{On}\}.$$

However, the set

$$C := \bigcap_{n>0} \{\varphi_x^{(n)}(y) : y \in \mathrm{On}\} = \mathrm{Cr}_{x+1}.$$

Indeed, every $z \in \mathrm{Cr}_{x+1}$ is a fixed point of φ_x, hence $\varphi_x^{(n)}(z) = z$, for all n, that is, $z \in C$. In the converse direction, if $z \in C$, consider the sequence z_n such that $\varphi_x^{(n)}(z_n) = z$. We have $z_0 \geq z_1 \geq z_2 \geq \cdots$, because φ_x is monotone. Hence, there must exist an n such that $z_n = z_{n+1}$. Then

$$\varphi_x^{(n+1)}(z_n) = z = \varphi_x^{(n)}(z_n),$$

that is $\varphi_x(z) = z$, q.e.d.

Finally, if x is a limit ordinal, $\omega^x = \sup\{\omega^y : y < x\}$. Accordingly, we have

$$S^+_{\omega^x} = \bigcap_{y<x} S^+_{\omega^y} = \bigcap_{y<x} \mathrm{Cr}_y = \mathrm{Cr}_x.$$

This completes the proof. ⊠

7 Calculating ordinals

From the previous result we can derive a formula for the function $o(\alpha)$. If $x \leq z$, let $-x + z$ denote the unique ordinal y such that $x + y = z$.

Lemma 17 *Assume $\alpha \neq \epsilon$ and $x = \omega^{x_1} + \cdots + \omega^{x_k}$ in Cantor normal form, $x > 0$. Then*

$$o(x \uparrow \alpha) = \varphi_{x_1}(\ldots(\varphi_{x_k}(-1 + o(\alpha)))\ldots).$$

Proof. By the previous lemma and Theorem 2, $\varphi_{x_1}(\ldots(\varphi_{x_k}(y))\ldots)$ is the y-th element of $o(S_x^+) = o(\{x \uparrow \alpha : \alpha \in S_0^+\})$. However, the enumeration of $o(S_0^+)$ starts with 0 and hence the place of α in the enumeration is $-1 + o(\alpha)$. ⊠

Example 18 If $\alpha \neq \epsilon$, then

1. $o(1 \uparrow \alpha) = \varphi_0(-1 + o(\alpha)) = \omega^{1+(-1+o(\alpha))} = \omega^{o(\alpha)}$.
2. $o(2 \uparrow \alpha) = \varphi_0(\varphi_0(-1 + o(\alpha)) = \omega^{\omega^{o(\alpha)}}$.
3. $o(\omega \uparrow \alpha) = \varphi_1(-1 + o(\alpha))$.
4. $o(\omega\omega) = o(\omega \uparrow 00) = \varphi_1(-1 + 2) = \epsilon_1$.
5. $o(\omega + \omega) = \varphi_1(\varphi_1(-1 + o(0))) = \varphi_1(\varphi_1(0)) = \epsilon_{\epsilon_0}$.

A combination of Lemma 15 and Lemma 17 provides a recursive calculation procedure for the ordinal of any $\alpha \in S$ by recursion on the width. If 0 occurs in α use Lemma 15. Otherwise, find $x = \min(\alpha)$, find $o(x \downarrow \alpha)$ using Lemma 15 and then apply Lemma 17 to compute $o(\alpha)$. We illustrate this by two examples.

Example 19

$$\begin{aligned} o(\omega 2\omega) &= \varphi_0(\varphi_0(-1 + o(\omega 0\omega))) \\ &= \omega^{\omega^{o(\omega 0\omega)}} \\ &= \omega^{\omega^{\omega^{-1+o(\omega)}+\omega^{-1+o(\omega)}}} \\ &= \omega^{\omega^{\epsilon_0+\epsilon_0}} \end{aligned}$$

Example 20

$$\begin{aligned} o((\omega+1)(\omega+\omega)) &= \varphi_1(\varphi_0(-1 + o(0\omega))) \\ &= \varphi_1(\omega^{\epsilon_0+1}) \\ &= \epsilon_{\omega^{\epsilon_0+1}} = \epsilon_{\epsilon_0\cdot\omega} \end{aligned}$$

Notice that a more constructive way to prove the well-foundedness of $<_0$ would be to show that the mapping o defined by the above recursive procedure is order-preserving.

8 An ordinal notation system up to Γ_0

The provability algebraic view suggests the following notion of *autonomous expansion* of provability algebras. The structures considered will be free 0-generated algebras, in other words, Lindenbaum algebras of the closed fragments of the logics $\mathbf{GLP}_\Lambda$. The construction will be an expansion in the sense that the languages of the algebras will grow. It will also be an extension in the sense that more elements will be added to the structure at each step.

We start with the ordinal 0 and consider the free 0-generated provability algebra with the only modality $\langle 0 \rangle$. The ordering $<_0$ on the set of words provides an ordinal notation system[1] for all ordinals $< \omega$.

Next we consider the free 0-generated provability algebra with modalities labelled by ordinals up to ω (or rather, by their notations given by the previous algebra). This is, essentially, the graded provability algebra for $\mathbf{GLP}_\omega$ and it provides an ordinal notation system for ϵ_0.

We further consider the algebra with modalities labelled by ordinals up to ϵ_0, etc. At each step the previously constructed ordinal notations x are used to define new operators $\langle x \rangle$ which allow to freely generate more elements of the algebra and, thus, more ordinal notations.

In this way we obtain an ordinal notation system up to the least ordinal closed under the following operation F: given an ordinal x consider the free 0-generated provability algebra with modalities labelled by ordinals up to x and compute the order type $F(x)$ of the ordering $(NF, <_0)$. We call the least ordinal closed under F the first *modally inaccessible* ordinal.

From the previous section we obtain that the first modally inaccessible ordinal is Γ_0.

Corollary 21 *The least ordinal x such that $F(x) = x$ is Γ_0.*

Proof. If $x < \Gamma_0$, then so is ω^x. The free 0-generated algebra with modalities labelled by ordinals $< \omega^x$ provides an ordinal notation system for all ordinals up to $\min(o(S^+_{\omega^x}))$, that is, has order type $\varphi_x(0) < \Gamma_0$.

In the other direction, it is easy to see that any ordinal $x < \Gamma_0$ cannot be closed under F, because $x \neq \varphi_x(0)$. ⊠

The process of autonomous expansion naturally leads to a system of ordinal notation up to Γ_0. One can describe this system as follows.

Ordinal notations are essentially the balanced bracket expressions. More formally, the set of notations Γ is defined using the primitive symbols `(` and `)` by the following rules:

1. $\epsilon \in \Gamma$ (empty expression).
2. If $\alpha, \beta \in \Gamma$, then $(\alpha)\beta \in \Gamma$.

Intuitively, the brackets are just the usual modal operator brackets $\langle \cdot \rangle$, where one allows the modalities to be labelled by modal formulas themselves (obtained at some previous stage). We omit the symbols $\top$ everywhere, so only the brackets remain.

Formally, we define a translation $* : \Gamma \to S$ in such a way that an element $\alpha \in \Gamma$ will denote the ordinal $o(\alpha^*)$. We set

$$\epsilon^* = \epsilon, \quad ((\alpha)\beta)^* = o(\alpha^*)\beta^*.$$

[1] In this trivial case ordinal notations are just sequences of zeros.

For $\alpha \in \Gamma$ we also define $o^*(\alpha) := o(\alpha^*)$.

Example 22 1. $\texttt{()}^* = o(\epsilon) = 0$, hence $o^*(\texttt{()}) = 1$.

2. $\texttt{()()}$ translates to 00 and denotes 2.

3. $\texttt{(())}^* = o(\texttt{()}^*) = o(0) = 1$, so $o^*(\texttt{(())}) = o(1) = \omega$.

4. $\texttt{((()())()()(()))}^* = 2001$, so $\texttt{(()())()()(())}$ denotes $o(2001) = o(2) = \omega^\omega$.

5. $\texttt{((()))}^* = o(\texttt{(())}^*) = o(1) = \omega$, hence $\texttt{((()))}$ denotes $o(\omega) = \epsilon_0$.

One can also view the elements of Γ as ordered trees. ϵ is a single node tree and $\texttt{(}\alpha\texttt{)}\beta$ is obtained by planting the tree α immediately above the root and to the left of the rest of the tree β. The bracket expression corresponding to a tree can be obtained by a leftmost path search walk through the tree and writing $\texttt{(}$ when going up and $\texttt{)}$ when going down, respectively. In this way, e.g., ϵ_0 is represented by a linear tree of height 3.

Some notations from Γ denote the same ordinal, but there is a unique normal form theorem. Recursively, one defines $\texttt{(}\alpha\texttt{)}\beta$ to be in normal form, if so are α, β and the word $o(\alpha^*)\beta^* \in NF$ in the sense of Section 2. Let $NF(\Gamma)$ denote the set of all normal forms in Γ.

Lemma 23 *If $\alpha, \beta \in \Gamma$ are in normal form and $o^*(\alpha) = o^*(\beta)$, then $\alpha = \beta$.*

Proof. Induction on the depth of nesting of brackets in α and β. By definition of the normal form, both $\alpha^*, \beta^* \in NF$. Besides, we have $o(\alpha^*) = o(\beta^*)$, so $\alpha^* = \beta^*$. Write α in the form $\texttt{(}\alpha_1\texttt{)} \cdots \texttt{(}\alpha_k\texttt{)}$. Graphical equality $\alpha^* = \beta^*$ yields that β has the form $\texttt{(}\beta_1\texttt{)} \cdots \texttt{(}\beta_k\texttt{)}$ with $o(\alpha_i^*) = o(\beta_i^*)$, for all i. Hence, by the induction hypothesis $\alpha_i = \beta_i$, for all i, that is, $\alpha = \beta$. ⊠

For $\alpha, \beta \in \Gamma$ define

$$\alpha <_0 \beta \iff \alpha^* <_0 \beta^*.$$

From the above lemma we obtain the following corollary.

Proposition 24 *The ordering $(NF(\Gamma), <_0)$ is a well-ordering of order type Γ_0.*

Proof. We have to prove that $o^* : NF(\Gamma) \to \Gamma_0$ is an isomorphism. Lemma 23 implies that this mapping is injective. The mapping is order-preserving by the definition of $<_0$ and since the function $o : NF \to \mathrm{On}$ is an embedding:

$$\alpha <_0 \beta \iff \alpha^* <_0 \beta^* \iff o(\alpha^*) < o(\beta^*).$$

The mapping o^* is surjective by Corollary 21. ⊠

References

[1] L.D. Beklemishev. Provability algebras and proof-theoretic ordinals, I. Logic Group Preprint Series 208, University of Utrecht, March 2001. `http://preprints.phil.uu.nl/lgps/`.

[2] L.D. Beklemishev. The Worm principle. Logic Group Preprint Series 219, University of Utrecht, March 2003. `http://preprints.phil.uu.nl/lgps/`.

[3] G. Boolos. *The Logic of Provability.* Cambridge University Press, Cambridge, 1993.

[4] S. Feferman. Systems of predicative analysis. *The Journal of Symbolic Logic*, 29:1–30, 1964.

[5] K.N. Ignatiev. On strong provability predicates and the associated modal logics. *The Journal of Symbolic Logic*, 58:249–290, 1993.

[6] C.St.J.A. Nash-Williams. On well-quasi-ordering finite trees. *Proc. Cambridge Phil. Soc.*, 59:833–835, 1963.

[7] U.R. Schmerl. A proof-theoretical fine structure in systems of ramified analysis. *Archive for Mathematical Logic*, 22:167–186, 1982.

[8] K. Schütte. *Proof Theory.* Springer-Verlag, Berlin, Heidelberg, New York, 1977.