Parameter free induction and reflection

Lev D. Beklemishev*
Steklov Mathematical Institute
Vavilova 42, 117966 Moscow, Russia
e-mail: lev@bekl.mian.su

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Abstract

We give a precise characterization of parameter free Σ_n and Π_n induction schemata, $I\Sigma_n^-$ and $I\Pi_n^-$, in terms of reflection principles. This allows us to show that $I\Pi_{n+1}^-$ is conservative over $I\Sigma_n^-$ w.r.t. boolean combinations of Σ_{n+1} sentences, for $n \geq 1$. In particular, we give a positive answer to a question by R. Kaye, whether the provably recursive functions of $I\Pi_2^-$ are exactly the primitive recursive ones. We also obtain sharp results on the strength of bounded number of instances of parameter free induction in terms of iterated reflection.

1 Introduction

In this paper we shall deal with arithmetical theories containing Kalmar elementary arithmetic EA or, equivalently, $I\Delta_0+\mathrm{Exp}$. Σ_n and Π_n formulas are prenex formulas obtained from the bounded ones by n alternating blocks of similar quantifiers, starting from '$\exists$' and '$\forall$', respectively. $\mathcal{B}(\Sigma_n)$ denotes the class of boolean combinations of Σ_n formulas. Σ_n^{st} and Π_n^{st} denote the classes of Σ_n and Π_n sentences.

Parameter free induction schemata have been introduced and investigated by Kaye, Paris, and Dimitracopoulos [11], Adamowicz and Bigorajska [1], Ratajczyk [13], Kaye [10], and others. $I\Sigma_n^-$ is the theory axiomatized over EA by the schema of induction

$$A(0) \wedge \forall x\, (A(x) \rightarrow A(x+1)) \rightarrow \forall x A(x),$$

for Σ_n formulas $A(x)$ containing no other free variables but x, and $I\Pi_n^-$ is similarly defined.[1]

It is known that the schemata $I\Sigma_n^-$ and $I\Pi_n^-$ show a very different behaviour from their parametric counterparts $I\Sigma_n$ and $I\Pi_n$. In particular, for $n \geq 1$, $I\Sigma_n^-$ and $I\Pi_n^-$ are not finitely axiomatizable, and $I\Sigma_n^-$ is strictly stronger than $I\Pi_n^-$ (in fact, stronger than $I\Sigma_{n-1} + I\Pi_n^-$). Furthermore, it is known that $I\Sigma_n$ is a conservative extension of $I\Sigma_n^-$ w.r.t. Σ_{n+2} sentences, although $I\Sigma_n^-$ itself only has a $\mathcal{B}(\Sigma_{n+1})$ axiomatization [11].

In contrast, nontrivial conservation results for $I\Pi_n^-$, for $n > 1$, seem to have been unknown. In particular, it was unknown, if the provably total recursive functions of $I\Pi_2^-$ coincide with the primitive recursive ones (communicated by R. Kaye). The

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[1]This definition differs from the one in [11] in that we work over EA, rather than over the weaker theories $I\Delta_0$ or PA^-. Since $I\Sigma_1^-$ in the sense of [11] obviously contains EA, the two definitions are equivalent for $n \geq 1$ in Σ case, and for $n \geq 2$ in Π case.

case of $I\Pi_1^-$ (over PA^-) was essentially treated in [11], where the authors show that Π_2 consequences of that theory are contained in EA, cf also [6].

In this paper we prove that the provably total recursive functions of $I\Pi_2^-$ are exactly the primitive recursive ones. Moreover, we show that $I\Pi_{n+1}^-$ is conservative over $I\Sigma_n^-$ w.r.t. boolean combinations of Σ_{n+1} sentences ($n \geq 1$). We also obtain sharp results on the strength of bounded number of instances of parameter free induction in terms of iterated reflection.

The proofs of these results are based on a characterization of parameter free induction schemata in terms of reflection principles and (generalizations of) the conservativity results for local reflection principles obtained in [3] using methods of provability logic. In our opinion, such a relationship presents an independent interest, especially because this seems to be the first occasion when *local* reflection principles naturally arise in the study of fragments of arithmetic.

We shall also essentially rely on the results from [4] characterizing the closures of arbitrary arithmetical theories extending EA under Σ_n and Π_n induction *rules*. In fact, the results of this paper show that much of the unusual behaviour of parameter free induction schemata can be explained by their tight relationship with the theories axiomatized by induction rules.

2 Preliminaries

First, we establish some useful terminology and notation concerning rules in arithmetic (cf also [4]). We say that a *rule* is a set of *instances*, that is, expressions of the form

$$\frac{A_1, \ldots, A_n}{B},$$

where $A_1, \ldots, A_n$ and B are formulas. Derivations using rules are defined in the standard way; $T + R$ denotes the closure of a theory T under a rule R and first order logic. $[T, R]$ denotes the closure of T under *unnested applications* of R, that is, the theory axiomatized over T by all formulas B such that, for some formulas $A_1, \ldots, A_n$ derivable in T, $\frac{A_1,\ldots,A_n}{B}$ is an instance of R. $T \equiv U$ means that theories T and U are deductively equivalent, i.e., have the same set of theorems.

A rule R_1 is *derivable* from R_2 iff, for every theory T containing EA, $T + R_1 \subseteq T + R_2$. A rule R_1 is *reducible* to R_2 iff, for every theory T containing EA, $[T, R_1] \subseteq [T, R_2]$. R_1 and R_2 are *congruent* iff they are mutually reducible (denoted $R_1 \cong R_2$). For a theory U containing EA we say that R_1 and R_2 are *congruent modulo* U, iff for every extension T of U, $[T, R_1] \equiv [T, R_2]$.

Induction rule is defined as follows:

$$\text{IR:} \qquad \frac{A(0), \quad \forall x\,(A(x) \to A(x+1))}{\forall x A(x)}.$$

Whenever we impose a restriction that $A(x)$ only ranges over a certain subclass Γ of the class of arithmetical formulas, this rule is denoted Γ-IR. In general, we allow parameters to occur in A, however the following lemma holds (cf also [5]).

Lemma 2.1. *Π_n-IR is reducible to parameter free Π_n-IR. Σ_n-IR is reducible to parameter free Σ_n-IR.*

Proof. An application of IR for a formula $A(x, a)$ can obviously be reduced to the one for $\forall z A(x, z)$, and this accounts for the Π_n case.

On the other hand, if $A(x, y, a)$ is Π_{n-1}, then an application of Σ_n-IR for the formula $\exists y A(x, y, a)$ is reducible, using the standard coding of sequences available in EA, to the one for $\exists y \forall i \leq x A((i)_0, (y)_i, (i)_1)$ (cf Remark 4.1 in [5]), q.e.d.

Reflection principles, for a given r.e. theory T containing EA, are defined as follows. The *uniform* reflection principle is the schema

$$\mathrm{RFN}_T: \qquad \forall x\,(\mathrm{Prov}_T(\ulcorner A(\dot{x})\urcorner) \to A(x)), \quad A(x) \text{ a formula,}$$

where $\mathrm{Prov}_T(\cdot)$ denotes a canonical provability predicate for T. The *local* reflection principle is the schema

$$\mathrm{Rfn}_T: \qquad \mathrm{Prov}_T(\ulcorner A\urcorner) \to A, \quad A \text{ a sentence.}$$

Partial reflection principles are obtained from the above schemata by imposing a restriction that A belongs to one of the classes Γ of the arithmetic hierarchy (denoted $\mathrm{Rfn}_T(\Gamma)$ and $\mathrm{RFN}_T(\Gamma)$, respectively). It is known that, due to the existence of partial truthdefinitions, the schema $\mathrm{RFN}_T(\Pi_n)$ is equivalent to a single Π_n sentence over EA. In particular, $\mathrm{RFN}_T(\Pi_1)$ is equivalent to the consistency assertion Con_T for T. See [14, 12, 3] for some basic information about reflection principles.

We shall also consider the following *reflection rule*:

$$\mathrm{RR}(\Pi_n): \qquad \frac{P}{\mathrm{RFN}_{EA+P}(\Pi_n)}.$$

We let Π_m-$\mathrm{RR}(\Pi_n)$ denote the above rule with the restriction that P is a Π_m sentence. Main results (Theorems 1, 2 and 3) of [4] can then be reformulated as follows.

Proposition 2.1. 1. Π_n-IR $\cong \Pi_{n+1}$-$\mathrm{RR}(\Pi_n)$, for $n > 1$;

2. Π_1-IR $\cong \Pi_2$-$\mathrm{RR}(\Pi_1)$ (mod $I\Delta_0$+Supexp).

Proposition 2.2. 1. Σ_1-IR $\cong \Pi_2$-$\mathrm{RR}(\Pi_2)$;

2. Σ_n-IR $\cong \Pi_{n+1}$-$\mathrm{RR}(\Pi_{n+1})$ (mod $I\Sigma_{n-1}$), for $n > 1$.

Since $[EA, \Sigma_n\text{-IR}]$ contains $I\Sigma_{n-1}$, Statement 2 implies that the rules Π_{n+1}-$\mathrm{RR}(\Pi_{n+1})$ and Σ_n-IR are interderivable, for all $n \geq 1$.

3 Characterizing $I\Sigma_n^-$ and $I\Pi_n^-$ by reflection principles

Theorem 1. *For $n \geq 1$, over EA,*

1. $I\Sigma_n^- \equiv \{P \to \mathrm{RFN}_{EA+P}(\Pi_{n+1}) \mid P \in \Pi_{n+1}^{st}\}$;
2. $I\Pi_{n+1}^- \equiv \{P \to \mathrm{RFN}_{EA+P}(\Pi_{n+1}) \mid P \in \Pi_{n+2}^{st}\}$.

Proof. Both statements are proved similarly, respectively relying upon Propositions 2.2 and 2.1, so we shall only elaborate the proof of the first one. For the inclusion ($\subseteq$) we have to derive

$$A(0) \wedge \forall x\,(A(x) \to A(x+1)) \to \forall x A(x),$$

for each Σ_n formula $A(x)$ with the only free variable x. We let P denote the Π_{n+1} sentence (logically equivalent to) $A(0) \wedge \forall x\,(A(x) \to A(x+1))$. Then, by external induction on n it is easy to see that, for each n, $EA + P \vdash A(\bar{n})$. This fact is formalizable in EA, therefore

$$EA \vdash \forall x\,\mathrm{Prov}_{EA+P}(\ulcorner A(\dot{x})\urcorner). \qquad (1)$$

Denoting by T the theory axiomatized over EA by all formulas

$$Q \to \mathrm{RFN}_{EA+Q}(\Pi_{n+1})$$

such that Q is a Π_{n+1} sentence, we conclude that

$$\begin{aligned} T + P &\vdash \mathrm{RFN}_{EA+P}(\Pi_{n+1}) \\ &\vdash \forall x\,(\mathrm{Prov}_{EA+P}(\ulcorner A(\dot{x})\urcorner) \to A(x)) \\ &\vdash \forall x A(x), \quad \text{by (1).} \end{aligned}$$

It follows that $T \vdash P \to \forall x A(x)$, as required.

For the inclusion ($\supseteq$) we observe that for any Π_{n+1} sentence P the theory $I\Sigma_n^- + P$ contains $P + \Sigma_n$-IR by Lemma 2.1, and hence

$$I\Sigma_n^- + P \vdash \mathrm{RFN}_{EA+P}(\Pi_{n+1}),$$

by Proposition 2.2. It follows that

$$I\Sigma_n^- \vdash P \to \mathrm{RFN}_{EA+P}(\Pi_{n+1}),$$

q.e.d.

Remark 3.1. Statement 2 of the above theorem also holds for $n = 0$, with a similar proof, but only over $I\Delta_0 + \mathrm{Supexp}$ (cf Corollary 4.2 below).

4 Relativized provability and reflection

For $n \geq 1$, $\Pi_n(\mathbf{N})$ denotes the set of all true Π_n sentences. $\mathrm{True}_{\Pi_n}(x)$ denotes a canonical truthdefinition for Π_n sentences, that is, a Π_n formula naturally defining the set of Gödel numbers of $\Pi_n(\mathbf{N})$ sentences in EA. $\mathrm{True}_{\Pi_n}(x)$ provably in EA satisfies Tarski satisfaction conditions (cf [9]), and therefore, for every formula $A(x_1, \ldots, x_n) \in \Pi_n$,

$$EA \vdash A(x_1, \ldots, x_n) \leftrightarrow \mathrm{True}_{\Pi_n}(\ulcorner A(\dot{x}_1, \ldots, \dot{x}_n)\urcorner). \qquad (*)$$

Tarski's truth lemma $(*)$ is formalizable in EA, in particular,

$$EA \vdash \forall s \in \Pi_n^{st}\ \mathrm{Prov}_{EA}(s \dot{\leftrightarrow} \ulcorner \mathrm{True}_{\Pi_n}(\dot{s})\urcorner), \qquad (**)$$

where Π_n^{st} is a natural elementary definition of the set of Gödel numbers of Π_n sentences in EA. We also assume w.l.o.g. that

$$EA \vdash \forall x\,(\mathrm{True}_{\Pi_n}(x) \to x \in \Pi_n^{st}).$$

Let T be an r.e. theory containing EA. A provability predicate for the theory $T + \Pi_n(\mathbf{N})$ can be naturally defined, e.g., by the following Σ_{n+1} formula:

$$\mathrm{Prov}_T^{\Pi_n}(x) \ :=\ \exists s\,(\mathrm{True}_{\Pi_n}(s) \wedge \mathrm{Prov}_T(s \dot{\to} x)).$$

Lemma 4.1. *1. For each Σ_{n+1} formula $A(x_1, \ldots, x_n)$,*

$$EA \vdash A(x_1, \ldots, x_n) \to \mathrm{Prov}_T^{\Pi_n}(\ulcorner A(\dot{x}_1, \ldots, \dot{x}_n)\urcorner).$$

2. $\mathrm{Prov}_T^{\Pi_n}(x)$ satisfies Löb's derivability conditions in T.

Proof. Statement 1 follows from $(*)$. Statement 2 follows from Statement 1, Tarski satisfaction conditions, and is essentially well-known (cf [15]), q.e.d.

We define $\mathrm{Con}_T^{\Pi_n} := \neg\mathrm{Prov}_T^{\Pi_n}(\ulcorner 0 = 1\urcorner)$, and relativized reflection principles $\mathrm{RFN}_T^{\Pi_n}$ and $\mathrm{Rfn}_T^{\Pi_n}$ are similarly defined. For $n = 0$ all these schemata coincide, by definition, with their nonrelativized counterparts.

Lemma 4.2. *For all $n \geq 0$, $m \geq 1$, the following schemata are deductively equivalent over EA:*

1. $\mathrm{Con}_T^{\Pi_n} \equiv \mathrm{RFN}_T(\Pi_{n+1})$;
2. $\mathrm{Rfn}_T^{\Pi_n}(\Sigma_m) \equiv \{P \to \mathrm{RFN}_{T+P}(\Pi_{n+1}) \mid P \in \Pi_m^{st}\}$.

Proof. 1. Observe that, using $(**)$,

$$\begin{aligned} EA \vdash \neg\mathrm{Prov}_T^{\Pi_n}(\ulcorner 0 = 1\urcorner) &\leftrightarrow \neg\exists s\,(\mathrm{True}_{\Pi_n}(s) \wedge \mathrm{Prov}_T(s \dot{\to} \ulcorner 0 = 1\urcorner)) \\ &\leftrightarrow \forall s\,(\mathrm{Prov}_T(\dot{\neg} s) \to \neg\mathrm{True}_{\Pi_n}(s)) \\ &\leftrightarrow \forall s\,(\mathrm{Prov}_T(\ulcorner \neg\mathrm{True}_{\Pi_n}(\dot{s})\urcorner) \to \neg\mathrm{True}_{\Pi_n}(s)). \end{aligned}$$

The latter formula clearly follows from $\mathrm{RFN}_T(\Sigma_n)$, but it also implies $\mathrm{RFN}_T(\Sigma_n)$, and hence $\mathrm{RFN}_T(\Pi_{n+1})$, by $(*)$.

2. By formalized Deduction theorem,

$$EA \vdash \mathrm{Con}_{T+P}^{\Pi_n} \leftrightarrow \neg\mathrm{Prov}_T^{\Pi_n}(\ulcorner \neg P\urcorner). \tag{2}$$

Hence, over EA,

$$\begin{aligned} \mathrm{Rfn}_T^{\Pi_n}(\Sigma_m) &\equiv \{\mathrm{Prov}_T^{\Pi_n}(\ulcorner S\urcorner) \to S \mid S \in \Sigma_m^{st}\} \\ &\equiv \{P \to \neg\mathrm{Prov}_T^{\Pi_n}(\ulcorner \neg P\urcorner) \mid P \in \Pi_m^{st}\} \\ &\equiv \{P \to \mathrm{RFN}_{T+P}(\Pi_{n+1}) \mid P \in \Pi_m^{st}\}, \quad \text{by (2) and Statement 1,} \end{aligned}$$

q.e.d.

From this lemma and Theorem 1 we immediately obtain the following corollary.

Corollary 4.1. *For $n \geq 1$, the following schemata are deductively equivalent over EA:*

1. $I\Sigma_n^- \equiv \mathrm{Rfn}_{EA}^{\Pi_n}(\Sigma_{n+1})$;
2. $I\Pi_{n+1}^- \equiv \mathrm{Rfn}_{EA}^{\Pi_n}(\Sigma_{n+2})$.

Corollary 4.2. *Over $I\Delta_0 + \mathrm{Supexp}$,*

$$I\Pi_1^- \equiv \mathrm{Rfn}_{EA}(\Sigma_2) \equiv \mathrm{Rfn}_{I\Delta_0+\mathrm{Supexp}}(\Sigma_2).$$

Proof. This follows in essentially the same way from Lemma 4.2 and the proof of Theorem 1, where we rely upon Proposition 2.1 (2) rather than (1). Indeed, over $I\Delta_0 + \mathrm{Supexp}$,

$$\begin{aligned} I\Pi_1^- &\equiv \{P \to \mathrm{RFN}_{EA+P}(\Pi_1) \mid P \in \Pi_2^{st}\} \\ &\equiv \{P \to \mathrm{Con}_{EA+P} \mid P \in \Pi_2^{st}\} \\ &\equiv \{\mathrm{Prov}_{EA}(\ulcorner S\urcorner) \to S \mid S \in \Sigma_2^{st}\}. \end{aligned}$$

Since the formula expressing the totality of superexponentiation function is a Π_2 sentence, the schemata $\mathrm{Rfn}_{I\Delta_0+\mathrm{Supexp}}(\Sigma_2)$ and $\mathrm{Rfn}_{EA}(\Sigma_2)$ are also equivalent over $I\Delta_0 + \mathrm{Supexp}$, q.e.d.

5 Conservation results

The following theorem is the main result of this paper.

Theorem 2. *For any $n \geq 1$, $I\Pi^-_{n+1}$ is conservative over $I\Sigma^-_n$ w.r.t. $\mathcal{B}(\Sigma_{n+1})$ sentences.*

Proof. The result follows from Corollary 4.1 and the following relativized version of Theorem 1 of [3].

Theorem 3. *For any r.e. theory T containing EA and any $n \geq 0$, $T + \mathrm{Rfn}^{\Pi_n}_T$ is conservative over $T + \mathrm{Rfn}^{\Pi_n}_T(\Sigma_{n+1})$ w.r.t. $\mathcal{B}(\Sigma_{n+1})$ sentences.*

Proof. The proof of this theorem makes use of a purely modal logical lemma concerning Gödel-Löb provability logic **GL** (cf e.g. [7, 15]). Recall that **GL** is formulated in the language of propositional calculus endowed by a unary modal operator $\Box$. The expressions $\Diamond\phi$ and $\Box^+\phi$ are standard abbreviations for $\neg\Box\neg\phi$ and $\phi \wedge \Box\phi$, respectively. Axioms of **GL** are all instances of propositional tautologies in this language together with the following schemata:

L1. $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$;

L2. $\Box\phi \to \Box\Box\phi$;

L3. $\Box(\Box\phi \to \phi) \to \Box\phi$.

Rules of **GL** are *moduls ponens* and $\phi \vdash \Box\phi$ (necessitation).

By an *arithmetical realization* of the language of **GL** we mean any function $(\cdot)^*$ that maps propositional variables to arithmetical sentences. For a modal formula ϕ, $(\phi)^*_T$ denotes the result of substitution for all the variables of ϕ the corresponding arithmetical sentences and of translation of $\Box$ as the provability predicate $\mathrm{Prov}_T(\ulcorner\cdot\urcorner)$. Under this interpretation, axioms L1, L2 and the necessitation rule can be seen to directly correspond to the three Löb's derivability conditions, and axiom L3 is the formalization of Löb's theorem. It follows that, for each modal formula ϕ, $\mathbf{GL} \vdash \phi$ implies $T \vdash (\phi)^*_T$, for every realization $(\cdot)^*$ of the variables of ϕ. The opposite implication, for the case of a Σ_1 sound theory T, is also valid; this is the content of the important *arithmetical completeness theorem* for **GL** due to Solovay (cf [7]).

For us it will also be essential that **GL** is sound under the interpretation of $\Box$ as a *relativized* provability predicate. For an arithmetical realization $(\cdot)^*$, we let $(\phi)^*_{T+\Pi_n(N)}$ denote the result of substitution for all the variables of ϕ the corresponding arithmetical sentences and of translation of $\Box$ as $\mathrm{Prov}^{\Pi_n}_T(\ulcorner\cdot\urcorner)$. The following lemma is a corollary of Lemma 4.1.

Lemma 5.1. *If $\mathbf{GL} \vdash \phi$, then $T \vdash (\phi)^*_{T+\Pi_n(N)}$, for every arithmetical realization $(\cdot)^*$ of the variables of ϕ.*

The opposite implication, that is, the arithmetical completeness of **GL** w.r.t. the relativized provability interpretation, due to G. Boolos, is also well-known (cf [15]). Yet, below we do not use this fact.

The following crucial lemma is a modification of a similar lemma in [3].

Lemma 5.2. *Let modal formulas Q_i be defined as follows:*

$$Q_0 := p, \qquad Q_{i+1} := Q_i \vee \Box Q_i,$$

where p is a propositional variable. Then, for any variables $p_0, \ldots, p_m$,

$$\mathbf{GL} \vdash \Box^+(\bigwedge_{i=0}^{m}(\Box p_i \to p_i) \to p) \to (\bigwedge_{i=0}^{m}(\Box Q_i \to Q_i) \to p).$$

Proof. Rather than exhibiting an explicit proof of the formula above, we shall argue semantically, using a standard Kripke model characterization of **GL**.

Recall that a *Kripke model* for **GL** is a triple $(W, R, \Vdash)$, where

1. W is a finite nonempty set;
2. R is an irreflexive partial order on W;
3. $\Vdash$ is a forcing relation between elements (*nodes*) of W and modal formulas such that

$$\begin{aligned} x \Vdash \neg\phi &\iff x \nVdash \phi, \\ x \Vdash (\phi \to \psi) &\iff (x \nVdash \phi \text{ or } x \Vdash \psi), \\ x \Vdash \Box\phi &\iff \forall y \in W\,(xRy \Rightarrow y \Vdash \phi). \end{aligned}$$

Theorem 4 on page 95 of [7] (originally proved by Segerberg) states that a modal formula is provable in **GL**, iff it is forced at every node of any Kripke model of the above kind. This provides a useful criterion for showing provability in **GL**.

Consider any Kripke model $(W, R, \Vdash)$ in which the conclusion $(\bigwedge_{i=0}^{m}(\Box Q_i \to Q_i) \to p)$ is false at a node $x \in W$. This means that $x \nVdash p$ and $x \Vdash \Box Q_i \to Q_i$, for each $i \leq m$. An obvious induction on i then shows that $x \nVdash Q_i$ for all $i \leq m+1$, in particular, $x \nVdash Q_{m+1}$.

Unwinding the definition of Q_i we observe that in W there is a sequence of nodes

$$x = x_{m+1} R x_m R \ldots R x_0$$

such that, for all $i \leq m+1$, $x_i \nVdash Q_i$. Since R is irreflexive and transitive, all x_i's are pairwise distinct. Moreover, it is easy to see by induction on i that, for all i,

$$\mathbf{GL} \vdash p \to Q_i.$$

Hence, for each $i \leq m+1$, $x_i \nVdash p$.

Now we notice that each formula $\Box p_i \to p_i$ can be false at no more than one node of the chain $x_{m+1}, \ldots, x_0$. Therefore, by Pigeon-hole Principle, there must exist a node z among the $m+2$ nodes x_i such that

$$z \Vdash \bigwedge_{i=0}^{m}(\Box p_i \to p_i) \wedge \neg p.$$

In case z coincides with $x = x_{m+1}$ we have

$$x \nVdash \bigwedge_{i=0}^{m}(\Box p_i \to p_i) \to p.$$

In case $z = x_i$, for some $i \leq m$, we have xRz by transitivity of R, and thus

$$x \nVdash \Box(\bigwedge_{i=0}^{m}(\Box p_i \to p_i) \to p).$$

This shows that the formula in question is forced at every node of any Kripke model; hence it is provable in **GL**, q.e.d.

Lemma 5.3. *For any $n \geq 0$ and any theory T, the following schemata are deductively equivalent over EA:*

$$\mathrm{Rfn}_T^{\Pi_n}(\Sigma_{n+1}) \equiv \mathrm{Rfn}_T^{\Pi_n}(\mathcal{B}(\Sigma_{n+1})).$$

Proof: As in [3], using Lemma 4.1, q.e.d.

Now we complete our proof of Theorems 2 and 3. Assume $T+\mathrm{Rfn}_T^{\Pi_n} \vdash A$, where A is a $\mathcal{B}(\Sigma_{n+1})$ sentence. Then there are finitely many instances of relativized local reflection that imply A, that is, for some arithmetical sentences $A_0, \ldots, A_m$, we have

$$T \vdash \bigwedge_{i=0}^{m} (\mathrm{Prov}_T^{\Pi_n}(\ulcorner A_i \urcorner) \to A_i) \to A.$$

Since relativized provability predicates satisfy Löb's derivability conditions, we also obtain

$$T \vdash \mathrm{Prov}_T^{\Pi_n}(\ulcorner \bigwedge_{i=0}^{m} (\mathrm{Prov}_T^{\Pi_n}(\ulcorner A_i \urcorner) \to A_i) \to A \urcorner).$$

Considering an arithmetical realization $(\cdot)^*$ that maps the variable p to the sentence A and p_i to A_i, for each i, by Lemma 5.2 we conclude that

$$T \vdash \bigwedge_{i=0}^{m} (\mathrm{Prov}_T^{\Pi_n}(\ulcorner B_i \urcorner) \to B_i) \to A,$$

where B_i denote the formulas $(Q_i)^*_{T+\Pi_n(N)}$. Now we observe that, if $A \in \mathcal{B}(\Sigma_{n+1})$, then for all i, $B_i \in \mathcal{B}(\Sigma_{n+1})$. Hence

$$T + \mathrm{Rfn}_T^{\Pi_n}(\mathcal{B}(\Sigma_{n+1})) \vdash A,$$

which yields Theorem 3 by Lemma 5.3. Theorem 2 follows from Theorem 3 and the observation that the schema $\mathrm{Rfn}_{EA}^{\Pi_n}(\Sigma_{n+2})$ corresponding to $I\Pi_{n+1}^-$ is actually weaker than the full $\mathrm{Rfn}_{EA}^{\Pi_n}$, q.e.d.

It is obvious, e.g., since $I\Sigma_1^-$ contains $EA + \Sigma_1$-IR, that all primitive recursive functions are provably total recursive in $I\Sigma_1^-$ and $I\Pi_2^-$. Moreover, since $I\Sigma_1^-$ is contained in $I\Sigma_1$, by a well-known result of Parsons, any provably total recursive function of $I\Sigma_1^-$ is primitive recursive. The following corollary strengthens this result and gives a positive answer to a question by R. Kaye.

Theorem 4. *Provably total recursive functions of $I\Pi_2^-$ are exactly the primitive recursive ones.*

Proof: follows from $\mathcal{B}(\Sigma_2)$ conservativity of $I\Pi_2^-$ over $I\Sigma_1^-$, q.e.d.

Remark 5.1. Observe that $I\Sigma_1 + I\Pi_2^-$ (unlike each of these theories taken separately) has a wider class of provably total recursive functions than the primitive recursive ones. This follows, e.g., from the fact that

$$I\Sigma_1 + I\Pi_2^- \vdash \mathrm{RFN}_{I\Sigma_1}(\Pi_2)$$

by Theorem 1 (since $I\Sigma_1$ is equivalent to a Π_3 sentence).

Remark 5.2. Perhaps somewhat more naturally, conservation results for relativized local reflection principles can be stated modally within a certain bimodal system **GLB** due to Japaridze, with the operators $\Box$ and $\boxdot$, that describes the joint behaviour of the usual and the relativized provability predicate (cf [7]). Using a suitable Kripke model characterization of **GLB**, one can semantically prove that

$$\mathbf{GLB} \vdash \Box(\bigwedge_{i=0}^{m} (\boxdot p_i \to p_i) \to p) \to \Box(\bigwedge_{i=0}^{m} (\boxdot Q_i \to Q_i) \to p),$$

where the formulas Q_i are now understood w.r.t. the modality $\boxdot$, and this yields Theorem 3 almost directly.

6 Axiomatization results

The characterization of parameter free induction in terms of reflection principles (Theorem 1) actually reveals other interesting information about these schemata.

The following theorem, which is a corollary of the relativized version of another conservation result for local reflection principles (due, essentially, to Goryachev [8]), gives a characterization of Π_{n+1} consequences of $I\Sigma_n^-$ and $I\Pi_{n+1}^-$. For the case of $I\Sigma_n^-$ a related charcterization of provably total recursive functions is given in [1, 13]. On the other hand, the paper [11] also contains a related conservation result for $I\Pi_1^-$ w.r.t. Π_1 sentences ($I\Pi_1^-$ is formulated over PA^-).

Let T be an r.e. theory containing EA. For a fixed $n \geq 1$, we define a sequence of theories $(T)_i^n$ as follows:

$$(T)_0^n := T; \quad (T)_{i+1}^n := (T)_i^n + \mathrm{RFN}_{(T)_i^n}(\Pi_n); \quad (T)_\omega^n := \bigcup_{i\geq 0}(T)_i^n.$$

Theorem 5. *1. Let U_m be a theory axiomatized over EA by* arbitrary *m instances of $I\Pi_{n+1}^-$, $n \geq 1$. Then U_m is Π_{n+1} conservative over $(EA)_m^{n+1}$.*

2. There exist particular m instances U_m of $I\Pi_{n+1}^-$ (in fact, of $I\Sigma_n^-$) such that U_m contains $(EA)_m^{n+1}$.

Proof. The proof relies on the fact that our characterization of parameter free induction schemata in terms of reflection principles respects the number of instances of these schemata.

Lemma 6.1. *For every instance B of $I\Pi_{n+1}^-$ there is a Π_{n+2} sentence P such that $P \to \mathrm{RFN}_{EA+P}(\Pi_{n+1})$ implies B over EA. Vice versa, for every such P there is an instance B of $I\Pi_{n+1}^-$ such that $EA + B$ proves $P \to \mathrm{RFN}_{EA+P}(\Pi_{n+1})$.*

Proof: follows directly from our proof of Theorem 1. For the 'vice versa' part we employ Proposition 2.1 (1) stating that

$$[EA + P, \Pi_{n+1}\text{-IR}] \vdash \mathrm{RFN}_{EA+P}(\Pi_{n+1}),$$

and the fact that any finite number of *unnested* applications of Π_{n+1}-IR can be obviously merged into a single one, q.e.d.

Remark 6.1. A similar statement holds for $I\Sigma_n^-$, but only over $I\Sigma_{n-1}$. In general one seems to need $m+1$ instances of $I\Sigma_n^-$ in order to derive m instances of the corresponding reflection schema (the first one is used to derive $I\Sigma_{n-1}$).

Let $\bot$ denote the boolean constant 'falsum'.

Lemma 6.2. $\mathbf{GL} \vdash \Box^+\neg\bigwedge_{i=0}^m(\Box p_i \to p_i) \to \Box^{m+1}\bot$.

Proof. By Lemma 5.2 we have

$$\mathbf{GL} \vdash \Box^+(\bigwedge_{i=0}^m(\Box p_i \to p_i) \to p) \to (\bigwedge_{i=0}^m(\Box Q_i \to Q_i) \to p).$$

Then, substituting in the above formula $\bot$ for p, observe that

$$\mathbf{GL} \vdash Q_i(p/\bot) \leftrightarrow \Box^i\bot,$$

and therefore

$$\mathbf{GL} \vdash \bigwedge_{i=0}^m(\Box Q_i(p/\bot) \to Q_i(p/\bot)) \leftrightarrow \neg\Box^{m+1}\bot,$$

q.e.d.

Now we complete the proof of Theorem 5. For the first part, Lemma 6.1 implies that U_m is contained in a theory axiomatized by m instances of relativized local reflection, say $\mathrm{Prov}^{\Pi_n}_{EA}(\ulcorner A_i \urcorner) \to A_i$, for $i < m$. Let A be a Π_{n+1} sentence such that $EA + U_m \vdash A$. Then we have

$$EA \vdash \neg A \to \neg \bigwedge_{i=0}^{m-1} (\mathrm{Prov}^{\Pi_n}_{EA}(\ulcorner A_i \urcorner) \to A_i)$$

and, by Löb's derivability conditions,

$$EA \vdash \mathrm{Prov}^{\Pi_n}_{EA}(\ulcorner \neg A \urcorner) \to \mathrm{Prov}^{\Pi_n}_{EA}(\ulcorner \neg \bigwedge_{i=0}^{m-1} (\mathrm{Prov}^{\Pi_n}_{EA}(\ulcorner A_i \urcorner) \to A_i)\urcorner).$$

By Lemma 6.2 we then obtain

$$\begin{aligned} EA \vdash (\neg \Box^m \bot)^*_{EA+\Pi_n(N)} &\to (A \vee \neg \mathrm{Prov}^{\Pi_n}_{EA}(\ulcorner \neg A \urcorner)) \\ &\to A, \end{aligned}$$

by Lemma 4.1 (1). Statement 1 of Lemma 4.2 implies that, for all i,

$$(T)^{n+1}_i \vdash (\neg \Box^i \bot)^*_{EA+\Pi_n(N)},$$

therefore $(EA)^{n+1}_m \vdash A$, which shows the first claim of Theorem 5.

For the second claim, we use a result from [4] stating that m times iterated uniform Π_{n+1} reflection principle over EA is derivable by m nested applications of Π_{n+1}-IR, and even Σ_n-IR, that is,

$$(EA)^{n+1}_m \subseteq [\ldots[EA, \Sigma_n\text{-IR}], \ldots, \Sigma_n\text{-IR}] \quad (m \text{ times}).$$

(For the case of Π_{n+1}-IR this statement obviously follows from Proposition 2.1. For the case of Σ_n-IR we need an additional observation that $[EA, \Sigma_n\text{-IR}] \equiv I\Sigma_{n-1}$.) Hence, particular m instances U_m of $I\Sigma^-_n$ can be found such that $EA + U_m$ contains $(EA)^{n+1}_m$, q.e.d.

Remark 6.2. The first statement of Theorem 5 is also valid for $n = 0$, but over $I\Delta_0 + \mathrm{Supexp}$ rather than EA. A proof is similar, using Corollary 4.2. For EA a similar characterization can be obtained using bounded cut-rank provability a là Wilkie and Paris [16], cf also [4].

Corollary 6.1. *For $n \geq 1$, $I\Pi^-_{n+1}$ and $I\Sigma^-_n$ are Π_{n+1} conservative extensions of $(EA)^{n+1}_\omega$.*

The following corollary was proved model-theoretically in [11].

Corollary 6.2. *For $n \geq 1$, neither $I\Sigma^-_n$, nor $I\Pi^-_{n+1}$ is finitely axiomatizable.*

This corollary can be strengthened by using the following two generalizations of Theorem 5. They both are proved in essentially the same way as Theorem 5, so we omit their proofs.

Theorem 6. *Let T be an extension of EA by finitely many Π_{n+2} sentences, $n \geq 1$. Then*

1. *For any m instances U_m of of $I\Pi^-_{n+1}$, $T + U_m$ is Π_{n+1} conservative over $(T)^{n+1}_m$.*

2. *There exist particular m instances U_m of $I\Pi^-_{n+1}$ such that $T+U_m$ contains $(T)^{n+1}_m$.*

Theorem 7. *Let T be an extension of EA by finitely many Π_{n+1} sentences, $n \geq 1$. Then*

1. *For any m instances U_m of of $I\Sigma^-_n$, $T+U_m$ is Π_{n+1} conservative over $(T)^{n+1}_m$.*
2. *There exist particular $m+1$ instances U_{m+1} of $I\Sigma^-_n$ such that $T+U_{m+1}$ contains $(T)^{n+1}_m$.*

Corollary 6.3. *No consistent extension of $I\Pi^-_{n+1}$ by Π_{n+2} sentences is finitely axiomatizable.*

Proof. Suppose, on the contrary, that there is such an extension. We may assume w.l.o.g. that it has the form $T+U_m$, for some m instances U_m of $I\Pi^-_{n+1}$, where T is a finite Π_{n+2} axiomatized extension of EA. Then, by Theorem 6, Π_{n+1} consequences of $T+U_m$ coincide with those of $(T)^{n+1}_m$, yet

$$T + I\Pi^-_{n+1} \vdash \mathrm{RFN}_{(T)^{n+1}_m}(\Pi_{n+1}).$$

The latter formula is Π_{n+1} and unprovable in $(T)^{n+1}_m$, q.e.d.

Corollary 6.4. *No consistent extension of $I\Sigma^-_n$ by Π_{n+1} and Σ_{n+1} sentences is finitely axiomatizable.*

Proof. Suppose, on the contrary, that there is such an extension. We may assume w.l.o.g. that it has the form $T+U_m+S$, for some m instances U_m of $I\Sigma^-_n$, S a Σ_{n+1} sentence, and T a finite Π_{n+1} axiomatized extension of EA. Then, by Theorem 7, Π_{n+1} consequences of $T+U_m$ are contained in those of $(T)^{n+1}_m$, in particular, if $T+U_m \vdash \neg S$, then $(T)^{n+1}_m \vdash \neg S$. Moreover, this fact is formalizable in T (to see this it is convenient to apply Lemma 5.2 once again), so that $\mathrm{Con}_{(T)^{n+1}_m+S}$ implies Con_{T+U_m+S} and $\mathrm{Con}_{T+I\Sigma^-_n+S}$, by our choice of U_m. Yet, by Theorem 7,

$$T + I\Sigma^-_n \vdash \mathrm{RFN}_{(T)^{n+1}_m}(\Pi_{n+1}),$$

therefore

$$T + I\Sigma^-_n + S \vdash \mathrm{Con}_{(T)^{n+1}_m+S},$$

contradicting Gödel's second incompleteness theorem, q.e.d.

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