Vardanyan's theorem for extensions of $I\Sigma_1$

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Abstract

Vardanyan's theorem states that the set of PA-valid formulas of quantified modal logic is Π^0_2-complete. This result also holds for extensions of $I\Sigma_1$ on the condition that inconsistency is not provable.

key words: Predicate Provability Logic

1 Introduction

Provability logic is a way of reducing difficult arithmetical theories to systems of modal logic. These systems are in many respects simpler and better understood. At the same time, they give full information about certain properties of Gödels predicate for provability. In case of propositional logic this reduction to modal systems is successfully done. The logic GL axiomatizes the class of sentences of propositional modal logic that are provable in PA under all substitutions of formulas of arithmetic.

The shift from propositional to predicate language leads to an essential increase of expressive power. New principles which are universally valid appear. So the question arises wether the same kind of axiomatizing can be done for predicate modal logic (QML). Vardanyan's theorem implies the negative answer. The set of always PA-provable formulas of QML is as undecidable as it is possible for it to be: Π^0_2-complete.

A generalization of this result is possible for a wide class of theories.[1] This article contains a proof for extensions of $I\Sigma_1$ under the assumption that $\Box\bot$ is not derivable. This is a minimal requirement for if inconsistency is derivable in the theory, Vardanyan's theorem trivially does not hold. For, suppose $\Box\bot$ is provable, then $\Box A$ is equivalent with $\top$ and the modal operator collapses. If that's the case, provability logic coincides with the ordinary predicate logic of the theory. In case of PA and $I\Sigma_1$ the set of valid predicate

[1] See A. Visser, *No Escape from Vardanyan's Theorem.* Utrecht University 2003.

formulas is just first order predicate logic, but this need not be the case for Σ^0_1-unsound theories[2].

Our proof originates with the method of [1, 7, 10] but it is adjusted wherever principles of PA are used that are not in general true for $I\Sigma_1$ extensions. The reader is assumed to be familiar with provability logic and especially with Boolos representation of Vardanyan's theorem in "The logic of provability" [Boo '93]. All lemmas referred to are taken from chapter 17 of this book and Boolos notational conventions are followed.

2 The theorem of Vardanyan for extensions of $I\Sigma_1$

In this section, we will investigate Vardanyan's theorem for extensions of $I\Sigma_1$, which we assume to be sufficiently sound.

2.1 Method

$I\Sigma_1$ is a weak variant of PA where the induction principle is just allowed for Σ^0_1 formulas. Vardanyan's proof makes use of the induction principle, namely in the proofs of lemmas 1, 2, 3 and 8 [Boo '93]. In all cases the proofs can be adjusted in such a way that Σ^0_1-induction is sufficient.

Our lemma 1 deals with lemmas 1, 2 and 3 [Boo '93] for $I\Sigma_1$. The following abbreviation is used:
$P(\sigma, x) := \mathsf{FinSeq}(\sigma) \wedge \mathsf{lh}(\sigma) \geq x+1 \wedge Z^*[\sigma(0)] \wedge \forall z < x\, S^*[\sigma(z), \sigma(z+1)]]$
So $P(\sigma, x)$ expresses that σ is a finite sequence with length $\geq x+1$ that fulfills some specific conditions. This σ takes care of $R(z, \sigma(z))$ with $z \leq x$. Note that $P(\sigma, x)$ need not be a recursive sentence, for Z^* and S^* can be quite complex, dependent on the realization $*$. To force recursiveness the formula D^* is used at once. The course of the proof is the following: $\{T\}^*$ and D^* are assumed and σ is treated as a parameter. The rest follows from induction. Equation 3 is proven as an example to illustrate our method of proof.

The proof of lemma 8 [Boo '93] makes use of Gödels β-function lemma. Lemma 2 state Gödels β-function lemma for provable recursive functions. This lemma is provable with Σ^0_1-induction and is used to prove lemma 4, our equivalent of lemma 8 [Boo '93] for $I\Sigma_1$.

2.2 Proof

Theorem 1 *The class of sentences which are always provable in sound extensions of $I\Sigma_1$ is Π^0_2-complete.*

[2] For more information, see R.E. Yavorsky, *First order logics of individual theories.*

Theorem 1 states Vardanyan's theorem for $I\Sigma_1$. We can follow the argument of Vardanyan replacing PA with $I\Sigma_1$, except for the lemmas proven below in $I\Sigma_1$.

Lemma 1

$$I\Sigma_1 \vdash \{T\}^* \wedge D^* \rightarrow \forall x \exists y\, R(x,y) \tag{1}$$

$$I\Sigma_1 \vdash \{T\}^* \wedge D^* \wedge R(x,y) \wedge E^*(y,y') \rightarrow R(x,y') \tag{2}$$

$$I\Sigma_1 \vdash \{T\}^* \wedge D^* \wedge R(x,y) \wedge R(x',y') \rightarrow [x = x' \leftrightarrow E^*(y,y')] \tag{3}$$

$$I\Sigma_1 \vdash \{T\}^* \wedge D^* \wedge R(x,y) \rightarrow [0 = x \leftrightarrow Z^*(y)] \tag{4}$$

$$I\Sigma_1 \vdash \{T\}^* \wedge D^* \wedge R(x,y) \wedge R(x',y') \rightarrow [\mathsf{s}x = x' \leftrightarrow S^*(y,y')] \tag{5}$$

$$\begin{aligned} I\Sigma_1 \vdash \{T\}^* \wedge R(x,y) \wedge R(x',y') \wedge R(x'',y'') \\ \rightarrow [x + x' = x'' \leftrightarrow A^*(y,y',y'')] \end{aligned} \tag{6}$$

$$\begin{aligned} I\Sigma_1 \vdash \{T\}^* \wedge D^* \wedge R(x,y) \wedge R(x',y') \wedge R(x'',y'') \\ \rightarrow [x \cdot x' = x'' \leftrightarrow M^*(y,y',y'')] \end{aligned} \tag{7}$$

$$I\Sigma_1 \vdash \{T\}^* \wedge R(x,y) \wedge y'\{<\}^* y \rightarrow \exists x' < x R(x',y') \tag{8}$$

Proof of lemma 1
All equations are provable with induction, we prove equation 3 as an example.

$\Rightarrow$: We have to prove the following formula in $I\Sigma_1$:
$\{T\}^* \wedge D^* \wedge R(x,y) \wedge R(x,y') \rightarrow E^*(y,y')$
Note that $R(x,y)$ and $R(x,y')$ guarantee the existence of a σ respectively a σ' sequence with $\sigma(x) = y$ and $\sigma'(x) = y'$. So it suffices to prove the formula below. We work in $I\Sigma_1$ and assume $\{T\}^*$ and D^*. This assumption makes E^*, S^* etcetera recursive in the theory, so induction is allowed.

To prove:
$P(\sigma,x) \wedge P(\sigma',x) \rightarrow E^*(\sigma(x),\sigma'(x)]$.

$x := 0$:
From $P(\sigma,0)$ and $P(\sigma',0)$ we derive $Z^*(\sigma(0))$ en $Z^*(\sigma'(0))$. Then we use an implication of $\{T\}^*$, namely the sentence $\forall z \forall z'[Z^*(z) \wedge Z^*(z') \rightarrow E^*(z,z')]$, to conclude that $E^*(\sigma(0),\sigma'(0))$.

$x := \mathsf{s}u$:
$P(\sigma,\mathsf{s}u)$ and $P(\sigma',\mathsf{s}u)$ give $S^*(\sigma(u),\sigma(\mathsf{s}u))$ and $S^*(\sigma'(u),\sigma'(\mathsf{s}u))$. Furthermore, $P(\sigma,u)$ and $P(\sigma',u)$ are derivable. The induction hypothesis gives $E^*(\sigma(u),\sigma'(u))$. Now we use the $\{T\}^*$ sentence

$\forall v \forall w \forall z \forall z'[E^*(v,w) \wedge S^*(v,z) \wedge S^*(w,z') \rightarrow E^*(z,z')]$
to conclude $E^*(\sigma(\mathsf{s}u), \sigma'(\mathsf{s}u))$

$\Leftarrow$: To prove in $I\Sigma_1$:
$\{T\}^* \wedge D^* \wedge R(x,y) \wedge R(x',y') \rightarrow [E^*(y,y') \rightarrow x = x']$
We use contradiction. Assume $E^*(y,y')$ and suppose $x \neq x'$, that is $x' > x$ of $x' < x$. Take $x' = x + z$ with $z \neq 0$ (in case $x' < x$ the proof is analogous). Consider the following formula:
$P(\sigma, x) \wedge P(\sigma', x+z) \rightarrow \sigma'(x+z)\{>\}^*\sigma(x)$
This Σ formula is provable in $I\Sigma_1$ under the assumption $\{T\}^* \wedge D^*$, we use induction on $z > 0$. Now we are done, because $R(x,y)$ and $R(x+z,y')$ guarantee the existence of a σ respectively a σ' sequence with $\sigma(x) = y$ and $\sigma'(x+z) = y'$. Beside, we use the fact that $\sigma'(x+z)\{>\}^*\sigma(x)$ implies $\neg E^*(\sigma(x), \sigma'(x+z))$. This is a consequence of $\{T\}^*$.

Proof of the formula above:
$z := 1$:
We have $P(\sigma', x+1)$, it follows that $P(\sigma', x)$. Now we use $P(\sigma, x)$ to derive $E^*(\sigma(x), \sigma'(x))$. See the proof above. Furthermore, remark that $P(\sigma', x+1)$ implies $S^*(\sigma'(x), \sigma'(x+1))$. We use the sentence
$\forall a \forall b \forall c[E^*(a,b) \wedge S^*(b,c) \rightarrow c\{>\}^*a]$
which is implied by the assumption $\{T\}^*$, to conclude $\sigma'(x+1)\{>\}^*\sigma(x)$.

$z := \mathsf{s}u$ $(u \neq 0)$:
$P(\sigma', x+\mathsf{s}u)$ implies $P(\sigma', x+u)$. Now the induction hypothesis is used to derive $\sigma'(x+u)\{>\}^*\sigma(x)$ from $P(\sigma, x)$ and $P(\sigma', x+u)$. Furthermore, note that $S^*(\sigma'(x+u), \sigma'(x+\mathsf{s}u))$ follows from $P(\sigma', x+\mathsf{s}u)$. At last, we use that $\{T\}^*$ implies: $\forall a \forall b \forall c[a\{>\}^*b \wedge S^*(a,c) \rightarrow c\{>\}^*b]$
and conclude $\sigma'(x+\mathsf{s}u)\{>\}^*\sigma(x)$.

Lemma 2 *Let h be a provable recursive function and take $\beta(a,b,i) = \mathsf{rm}\,(a, 1 + (i+1)\cdot b)$. Then:*
$I\Sigma_1 \vdash \forall k \exists a \exists b \forall i < k\,[\,\beta(a,b,i) = h(i)]$

Proof of lemma 2
This is Gödels β-lemma for provably recursive functions. It is a well known fact that this lemma is provable in $I\Sigma_1$

Lemma 3 *Let S' be defined in the following way, whereby $B(a,b,j,0) := \beta(a,b,i) = 0$ and $B(a,b,j,1) := \beta(a,b,i) = 1$.*
$\forall k \exists a,b \forall j < k\,[\langle \exists y < k[\mathsf{T}(j,j,y) \wedge U(y) = 0] \leftrightarrow B(a,b,j,0)\rangle \wedge$
$\langle \neg \exists y < k\,[\mathsf{T}(j,j,y) \wedge U(y) = 0] \leftrightarrow B(a,b,j,1)\rangle]$
Then $I\Sigma_1 \vdash S'$

Proof of lemma 3
This is an application of lemma 2 with h the function given below.

$$h(i) = \begin{cases} 0 & \text{if } \exists y < k[\mathsf{T}(i,i,y) \wedge U(y) = 0] \\ 1 & \text{if } \neg\exists y < k[\mathsf{T}(i,i,y) \wedge U(y) = 0] \end{cases} .$$

Notice that $h(i)$ is recursive, this in contrast with the function used by Boolos where the existential quantifier is not bounded. This is the crucial point in the proof of lemma 3 for $I\Sigma_1$

Lemma 4 $\mathsf{PA} \vdash \{T\}^* \wedge D^* \to \forall y \exists x\, R(x,y)$

Proof of lemma 4
We follow Vardanyans proof, except that we reason in $I\Sigma_1$. Note that S' is a theorem of $I\Sigma_1$ and assume that T implies S'. Suppose there exist a number k with the property that for all r: $\neg R(r,k)$. The sentence $\{S'\}^*$ tells us that there are numbers a and b such that for all $j\{<\}^*k$ the following holds.
$\exists y\{<\}^*k\,\{\mathsf{T}(j,j,y) \wedge U(y) = 0\}^* \leftrightarrow \{B\}^*(a,b,j,0)$ and
$\neg\exists y\{<\}^*k\,\{\mathsf{T}(j,j,y) \wedge U(y) = 0\}^* \leftrightarrow \{B\}^*(a,b,j,1)$
Consider the Turing machine μ as defined by Vardanyan. So, applied to any number i, μ begins by finding a number j such that $R(i,j)$ holds. Then he sends j to 1 if $\{B\}^*(a,b,j,0)$ and to 0 if $\{B\}^*(a,b,j,1)$. Notice that μ is totally defined, for equation 8 of lemma 1 gives $j\{<\}^*k$.

Let $C(i,0)$ and $C(i,1)$ be as defined by Vardanyan, so:
$C(i,0) := \exists l\,\mathsf{T}(i,i,l) \wedge U(l) = 0$ and $C(i,1) := \exists l\,\mathsf{T}(i,i,l) \wedge U(l) = 1$
Suppose $C(i,0)$ and $R(i,j)$. Then there is an y such that $\mathsf{T}(i,i,y)$ and $U(y) = 0$. Assume $\mathsf{T}(i,i,v) \wedge U(v) = 0$ en $R(v,w)$. Because of lemma 4 [Boolos '93] we have $\{\mathsf{T}(j,j,w) \wedge U(w) = 0\}^*$. Furthermore lemma 1 (equation 8) gives $w\{<\}^*k$. So $\exists y\{<\}^*k\,\{\mathsf{T}(j,j,y) \wedge U(y) = 0\}^*$. Now we use the sentence $\{S'\}^*$ to conclude $B^*(a,b,j,0)$ whereafter the definition of μ makes that $\mu(i) = 1$. Suppose $C(i,1)$ and $R(i,j)$. From $C(i,1)$ and lemma 4 [Boolos '93] we know that $\{C\}^*(j,1)$, now $\{T\}^*$ implies $\neg\{C\}^*(j,0)$, so $\neg\exists y\,[\{\mathsf{T}(j,j,y) \wedge U(y) = 0\}]^*$ and in particular $\neg\exists y\{<\}^*k\,[\{\mathsf{T}(j,j,y) \wedge U(y) = 0\}]^*$. From $\{S'\}^*$ and $j\{<\}^*k$, follows $\{B\}^*(a,b,j,1)$ and we can draw the conclusion that $\mu(i) = 0$.

Conclusion:
$C(i,0) \to \mu(i) = 1$ and
$C(i,1) \to \mu(i) = 0$.
From this we are able to derive a contradiction analoguos to the proof of Vardanyan as presented by Boolos.

3 Soundness conditions

This section deals with the soundness conditions. First it is shown that Σ^0_1-sound suffices, then we prove an even more general result.

3.1 Method

Again, we follow Vardanyan in connecting the set of always provable sentences with a arbitrary Π_2^0 set S. This is done in the assertion
$\forall * \mathsf{PA} \vdash \varphi_n^* \Leftrightarrow n \in S$. With
$\varphi_n := \{T\} \wedge D \wedge E \rightarrow \exists v \exists w\, (v\{<\}w \wedge \{Q_n\}(w) \wedge \forall z\, [\Box Gz \leftrightarrow \{H\}(v,z)])$
The problem appears in the left-right implication, a shift takes place from provability to truth, so a form of soundness is required. It's shown in theorem 2 that Σ_1^0-soundness suffices to prove Vardanyan's theorem, but the soundness conditions can be sharpened even further. Theorem 3 states that the only requirement is that inconsistency is not provable.

Vardanyan uses the soundness of PA to conclude ϕ_n^* from $\mathsf{PA} \vdash \phi_n^*$. Then he reasons outside the theory to get $\exists v(v > m \wedge \exists w[v < w \wedge Q_n(w)])$, whereafter $n \in S$ is true by definition. By reasoning in a arbitrary $I\Sigma_1$ extension U it follows that the sentence $\exists v(v > m \wedge \exists w[v < w \wedge Q_n(w)])$ is derivable in $U + \mathsf{con}(U)$. The argument proceeds in the following manner. Assume U is Σ_1^0-sound. The Σ_1^0-soundness of U gives the Σ_1^0-soundness of $U + \mathsf{Con}(U)$ and because of the fact that we deal with a Σ_1^0 sentence we can draw the conclusion that $\exists v(v > m \wedge \exists w[v < w \wedge Q_n(w)])$ is true. This proves theorem 2.

So Σ_1^0-soundness turns out to be sufficient to prove Vardanyan's theorem, the question remains wether it is also a necessary requirement. It appears to be not the case. First, note that we have a minimal soundness condition. If $\Box\bot$ is derivable in a theory U, then all formulas with the form $\Box A$ are equivalent with $\Box\bot$ (and so with $\top$). In that case, the modal operator looses it's significance and Vardanyan's theorem trivially does not hold. Theorem 3 goes to show that this sharpest possible condition is sufficient.

The φ_n chosen by Vardanyan requires Σ_1^0 reflection, with a small modification in the φ_n sentences this can be avoided. The idea is the following, define: $A := \exists v(v > m \wedge \exists w[v < w \wedge Q_n(w)])$
We use the FGH theorem to find a sentence B with the following properties: (1) B is provable in $U + \mathsf{Con}(U)$, (2) we have reflection for B, (3) A and B are equivalent. This proves the truth of A. The point is that although A and B are equivalent, they are not provably equivalent, that means $A \leftrightarrow B$ is not derivable in the theory. This sentence B is used to construct the new φ_n'.

3.2 Proof

Theorem 2 *Let U be a Σ_1^0-sound extension of $I\Sigma_1$. The class of sentences which are always provable in U is Π_2^0-complete.*

Given that U is a Σ_1^0-sound theory which extends $I\Sigma_1$ we have to prove $U \vdash \varphi_n^{*i}$ for all $*i \Rightarrow n \in S$ (the other direction is analogous to Vardanyan's proof). For $*i$ we chose the standard translation whereby $G(z)^{*i} := (z = i)$. So we start with the following assumption (with $*$ the standard translation).
$U \vdash T \wedge D^* \wedge E^* \rightarrow [\exists v \exists w [v < w \wedge Q_n(w)] \wedge \forall z(\mathsf{Bew}(z = i) \leftrightarrow H(v, z))]$.

Because U contains $I\Sigma_1$ and because the consistency of U implies $\Diamond\top$, we have $U + \mathsf{con}(U) \vdash T \wedge D^* \wedge E^*$. Thus we conclude:
$U + \mathsf{con}(U) \vdash \exists v \exists w [v < w \wedge Q_n(w) \wedge \forall z(\mathsf{Bew}(z = i) \leftrightarrow H(v, z))]$.

We use $\mathsf{con}(U) \vdash \mathsf{Bew}(z = i) \leftrightarrow (z = i)$ to get:
$U + \mathsf{con}(U) \vdash \exists v \exists w [v < w \wedge Q_n(w) \wedge \forall z(z = i \leftrightarrow H(v, z))]$

This holds for all i, so we have:
$U + \mathsf{con}(U) \vdash \bigwedge_{i \leq m+1} \exists v \exists w [v < w \wedge Q_n(w) \wedge \forall z(z = i \leftrightarrow H(v, z))]$

¿From this we derive:
$U + \mathsf{con}(U) \vdash \exists v_0 ... v_{m+1} [\bigwedge_{i \leq m+1} \exists w [v_i < w \wedge Q_n(w)] \wedge \bigwedge_{i,j \leq m+1; i \neq j} v_i \neq v_j]$

At last, we apply the pigeon hole principe to draw the conclusion:
$U + \mathsf{con}(U) \vdash \exists v_i > m \exists w [v_i < w \wedge Q_n(w)]$

We prove the Σ_1^0-soundness of $U + \mathsf{con}U$ using the fact that U is Σ_1^0-sound. Suppose that S is a Σ_1^0 sentence such that $U + \mathsf{con}(U) \vdash S$. We want to prove that S is true. Notice that $U + \mathsf{con}(U) \vdash S$ can be rewritten as $U \vdash S \vee \mathsf{Bew}(\bot)$. This is a Σ_1^0 sentence so we know from the Σ_1^0-soundness of U that it is a true sentence. At last, the consistency of U makes S true.

So the Σ_1^0 sentence $\exists v_i > m \exists w [v_i < w \wedge Q_n(w)]$ is derivable from φ_n^{*i} in $U + \mathsf{con}(U)$. The Σ_1^0-soundness of U proves the Σ_1^0-soundness of $U + \mathsf{con}(U)$. This suffices to establish that $\exists v_i > m \exists w [v_i < w \wedge Q_n(w)]$ is true. The rest of the proof is analogous to Vardanyan's. With this it is proven that Vardanyan's theorem holds for all extensions of $I\Sigma_1$ in the language of arithmetic which are Σ_1^0-sound.

Theorem 3 *Let U be an extension of $I\Sigma_1$ and suppose that inconsistency of U is not provable in U. Then, the class of sentences which are always provable in U is Π_2^0-complete.*

This theorem states Vardanyan's theorem for all theories which extends $I\Sigma_1$ under the sole restriction that $\Box_U \bot$ is not provable. The proof requires the construction of a sentence φ_n' such that:
$U \vdash \varphi_n'^{*}$ for all $* \Leftrightarrow \exists v(v > m \wedge \exists w[v < w \wedge Q_n(w)])$ for all m.
We use the FGH theorem to find a suitable φ_n'.

Lemma 5 *Consider a theory T and take $\Box := \Box_T$. Let A be a Σ^0_1 sentence and let B be a sentence such that $\mathsf{Q} \vdash B \leftrightarrow A \leq \Box B$. The sign '$\leq$' is defined as follows: Let $S = \exists x S_0(x)$ en $T = \exists x T_0(x)$ be Σ^0_1, then*
$S \leq T \leftrightarrow \exists x(S_0(x) \wedge \forall y < x \neg T_0(y))$. Note that Gödels diagonal lemma guarantees the existence of B and that B is also Σ^0_1. The FGH theorem states:

$EA + \mathsf{con}(T) \vdash (A \leftrightarrow B) \wedge (A \leftrightarrow \Box B)$.

Proof of lemma 5
For a proof of this theorem, see "Faith & Falsity: A study of Faithful Interpretations and False Σ^0_1 sentences" [Visser 2002, blz. 11].

Let $U^+ := U + \mathsf{con}(U)$. From the fact that U doesn't prove inconsistency, follows that U^+ is consistent. Now we apply the FGH theorem with $\Box := \Box_{U^+}$ and $A := \exists v(v > m \wedge \exists w[v < w \wedge Q_n(w)])$. We can write down $A \leq \Box B$ in full notation to show that B is provable equivalent with
$\exists v(v > m \wedge \exists w[v < w \wedge Q_n(w) \wedge \forall y \leq w \neg \mathsf{Proof}(y, B)])$.

(1) $A \Leftrightarrow B$. For, the FGH theorem gives $EA + \mathsf{con}(U^+) \vdash (A \leftrightarrow B)$, this means $EA \vdash (A \leftrightarrow B) \vee \neg \mathsf{con}(U^+)$. EA is a full sound theory, so EA provable sentences are certainly true. The consistency of U^+ gives $A \Leftrightarrow B$.

(2) $U^+ \vdash B \Rightarrow B$ is true. The FGH theorem provides for this reflection principe for B, as is proven below. To start with, note that if $U^+ \vdash B$ then there is a derivation for B in U^+, in other words $\Box_{U^+} B$ is true. The FGH theorem says $EA \vdash (A \leftrightarrow \Box B) \vee \neg \mathsf{con}(U^+)$. Again we use the consistency of U^+ and the soundness of EA, to argue that A is true. A and B are equivalent (See 1), so we conclude the truth of B.

Define:
$Q'_n(w) := Q_n(w) \wedge \forall y \leq w \neg \mathsf{Proof}(y, B)$ and
$\varphi'_n := \{T\} \wedge D \wedge E \rightarrow \exists v \exists w\, (v\{<\}w \wedge \{Q'_n\}(w) \wedge \forall z\, [\Box G(z) \leftrightarrow \{H\}(v, z)])$
(This sentence differs from Vardanyan's in using Q'_n instead of Q_n)

(3) $U \vdash \varphi'^*_n$ for all $*$, then $U^+ \vdash B$. To prove this, we first apply the reasoning of the proof of theorem 2 to the new sentence φ'_n. This gives us:
$U^+ \vdash \exists v(v > m \wedge \exists w[v < w \wedge Q'_n(w)])$. Our choice of B is such that Q proves the equivalence of B with the sentence mentioned above. Because U^+ is a much stronger theory then Q, this equivalence is also provable in U^+. This leads to $U^+ \vdash B$.

The sentence φ'_n sufficed to prove Vardanyan's theorem, that is to say that $U \vdash \varphi'^*_n$ for all $*$ if and only if $n \in S$.
$\Rightarrow$: Assume $U \vdash \varphi'^*_n$ for all $*$. From 3) we know $U^+ \vdash B$, then 2) gives that B is true. Finally we conclude A, and thus $n \in S$, from the equivalence of A and B as proven in 1).

$\Leftarrow$: Assume $n \in S$, that means A is true. From 1) we know the truth of B. From here we follow Vardanyan's argument with $Q'_n(w)$ instead of $Q_n(w)$, whereby we use the fact that $Q'_n(w)$ is Σ^0_1 just as $Q_n(w)$.

4 Conclusion

The theorem of Vardanyan states that the set of always provable predicate sentences of PA is Π^0_2-complete, it follows that no axiomatization is possible. The theorem holds for a wide range of theories. This article deals with $I\Sigma_1$ extensions. First, it is investigated wether the theory $I\Sigma_1$ is strong enough. This is done by adapting the proof of Vardanyan on all places where principles of PA are used that do not belong to $I\Sigma_1$. Second, the soundness conditions are determined. The crucial step is the use of the FGH theorem to construct an alternative sentence that fulfills the reflection principle. Vardanyan's theorem appears to hold for all $I\Sigma_1$ extensions on the condition that inconsistency is not provable. This result provides for the sharpest possible soundness condition for $I\Sigma_1$ extensions, for if inconsistency is provable, the modal operator collapses.

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