Modeling relations

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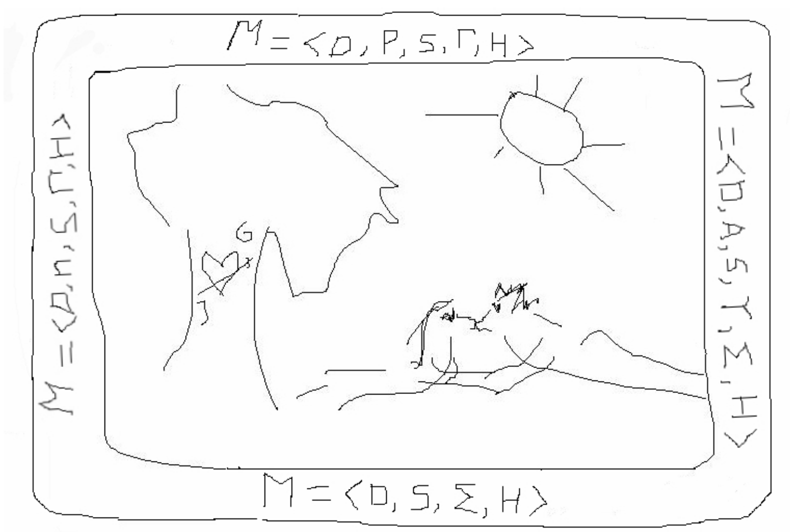

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Abstract

In ordinary representations of relations the order of the relata plays a structural role, but in the states of affairs or the facts themselves such an order often does not seem to be intrinsically present. An alternative way to represent relations makes use of positions for the arguments. This is no problem for the love relation, but for relations like the adjacency relation and cyclic relations, different assignments of objects to the positions can give exactly the same states. This is a puzzling situation. The question is, what is the ontological status of the positions, and, more generally, what is the internal structure of relations? In this thesis mathematical models for relations are developed, which provide more insight into the structure of relations "out there" in the real world.

The question of relations is one of the most important that arise in philosophy, as most other issues turn on it: monism and pluralism; the question whether anything is wholly true except the whole of truth, or wholly real except the whole of reality; idealism and realism, in some of their forms; perhaps the very existence of philosophy as a subject distinct from science and possessing a method of its own.

Bertrand Russell, *Logical Atomism*, 1924.

Preface

In January 2005, I asked Albert Visser for suggestions for a subject for my Master's Thesis. Albert came up with a variety of subjects and gave me some papers to read. One of the papers was 'Neutral Relations' of Kit Fine. I knew that Fine was one of Albert's favorite authors. The only thing I had read of him before was about his counterfactual with Nixon and a nuclear holocaust. I was curious about the paper 'Neutral Relations' and one evening I started reading it with high expectations. At the end of the evening I had read the first five pages, but I had the feeling that, although I understood all the words, I could just as well have looked all the time at a poem in Russian. What was the point Fine wanted to make and why did I seem to miss it completely?

I let the paper rest for a couple of days before I took it up again. At some point I started to see a little light, but I still found it difficult to grasp that the fundamental structure of relations could be an object of study. About a week later I had finished a first reading of the paper. I felt that I had a rudimentary understanding, but above all I was confused. Fine argued that it was wrong to think that all relations have argument-places as specific entries. But did this imply that for certain relations I should no longer use positional representations? For example, if A and B attract each other, would it be wrong to assign A and B to positions to represent this fact? If that would be the case, then that would mean that something was seriously wrong with my intuition of what relations are. This issue intrigued and irritated me so strongly, that I took a decision: I was going to write my thesis about modeling the structure of relations.

Now, half a year later, I think I obtained more insight in the ontological relation between states of affairs and their constituents. A main result of my study is that positions apparently are rather innocent creatures. More precisely, in this thesis I prove that each relation in a certain well-defined class of relations can adequately be modeled both with and without positions. This class is such that it includes all 'real' relations with a fixed and finite number of arguments that I could think of.

The theory presented in this thesis is for the most part rather technical. The models are all mathematical models, some of which are probably not so easy to understand. In addition, the abstractness of the subject is also not of much help in this respect. I would like to advise readers who want to get only a global idea of the results, to skip the proofs of the theorems and to skip the lemmas. I do not assume that the reader is familiar with the paper 'Neutral relations' of Fine, although it definitely might be helpful as background information. Moreover, it might radically change your ideas about relations.

Acknowledgements

First, I am grateful to Albert Visser, my thesis supervisor. During my study, Albert came with various stimulating ideas and suggestions that shaped and reshaped my view about different mathematical and philosophical issues. Sometimes it was just an article or book he recommended, sometimes a short remark or just a gesture. For my thesis, he initially put me on the right track by sketching his ideas about how permutation groups might play a role for modeling relations. In our regular meetings about my progress I was often surprised by his insight and feeling for the subject.

I am also especially grateful to Kit Fine for reviewing the final draft of this thesis. His detailed comments and suggestions were very valuable. They were not only helpful to clarify a number of difficult issues, but they also provide impetus for further research. In particular, for further research on substitution of occurrences, and for experimental research on fundamental aspects of the way we represent the world.

Further, I like to mention Rosja Mastop, who took a detailed look at my thesis. I thank him for his comments and the interesting discussions we had. I also like to mention Vincent van Oostrom. Vincent did not make suggestions, he only asked questions. Questions that pointed to spots where to search for solutions. A word of thanks I also owe to my sister Riëtte for linguistic improvements of the text.

To start studying again three years ago meant a big change in my life with Gitte and Mo. Before my study I was probably away too often, but in the last years it was rather the opposite. I want to thank them very much for allowing me to spend much time on my study. Lastly, I want to thank all others whose names I used in examples in this thesis for their interest.

Contents

Chapter 1

Introduction

When I say “Koos loves Marietje” do I express the same state of affairs as I would express if I said “Marietje is loved by Koos”? If you accept that states of affairs are “out there” in reality, you will probably affirm this. But then we have two ways to describe a single state of affairs. Which one is the better one? If we can’t say, then we might ask if there is a way to express the state of affairs in a neutral way.

This is the problem Kit Fine deals with in his paper ‘Neutral relations’ [Fin00]. Fine shows the inadequacy of what he calls the *standard view* on relations, according to which all relations hold of objects in a given order. In search of a better alternative, he first proposes a *positionalist view* on relations in which each relation comes with a set of positions. For example, for the amatory relation, the positions would be *Lover* and *Beloved*. Fine calls the positional view very natural and plausible, but he also regards this view as problematic. One of the objections stated by Fine is that under the positional view no relation can be symmetric. A second alternative proposed by Fine is his *antipositionalist view*, of which he claims that it combines the virtues of the standard view and the positionalist view.

Because, as Fine says, the positionalist view is so very natural, I have two basic questions: (1) Can the positionalist view in some way be saved from the objections raised by Fine? (2) For what kind of relations are positional representations still adequate, and for what kind are they not?

Of course, there is one more basic question to ask, namely: What is the relevance of this subject? Let me briefly comment on this. In our acting and ordinary understanding of the world a leading role is played by noticing resemblances and differences between things and events. Analytically investigating what the essence of relations is, might deepen our understanding of fundamental aspects of reality. As such it is an important metaphysical subject, not outdated, but definitely of the present.

In this thesis I will develop mathematical models for different views on relations that are not only of interest in themselves, but that also increase our insight into the adequacy of different conceptions of relations. In particular, I will define *positional models* that agree with the positionalist view, and an elegant type of models, the *substitution models*, which is inspired by Fine's antipositionalist view. I will prove that a natural subclass of the substitution models corresponds in a well-defined way to a subclass of the positional models. As a consequence, without any commitment to an ontology of positions, positional representations of relations are justified for at least a large class of relations, including all kind of symmetrical relations.

The structure of the thesis is as follows. In Chapter 2 the subject of this thesis is placed in a wider philosophical context. Further, a summary is given of the views on relations as distinguished by Fine, and Fine's objections against the positionalist view are criticized. Chapters 3 to 5 form the core of this thesis. In Chapter 3 types of mathematical models for relations are developed and in Chapter 4 these types are refined. I follow here a bottom-up approach: the strengths and weaknesses of different types of models are used as starting points for designing better ones. In Chapter 5 operations on models are defined and preservation properties are examined. In Chapter 6 we focus our attention on metaphysical aspects of the structure of relations. In Chapter 7 some of the insights obtained are applied to the identity relation and to Putnam's Twin Earth argument. I end in Chapter 8 with a recapitulation of the main argument, conclusions and suggestions for further inquiry.

One final note about the scope of this thesis. The relations to be discussed are the relations "out there" in the real world. Occasionally we will use the term 'real' relations to stress this point. If you like, you may regard the mathematical relations as a subclass of the 'real' relations. However, in our arguments we will freely use what is called 'relations' in set theory as tools.

Chapter 2

What are relations?

In this chapter, I will not give a comprehensive treatment of the philosophy of relations, but only highlight some aspects that I consider relevant background information for this thesis. I start with a short exposition of ideas developed by different philosophers about relations. Because the questions to be discussed in this thesis are applicable to a great variety of views on relations, these ideas are not critically reviewed. Also no attempt is made to give sharp definitions of the notions of objects, states of affairs and relations. In fact, the exact identity of these entities is not too important for the argument to be developed. The views on relations as distinguished by Fine are discussed separately. I will argue that his objections against the positionalist view do not hold.

2.1 About the nature of relations

What relations are, seems not so easy to explain. Very generally, we can say that relations are ways in which entities can stand to each other and to themselves. Familiar examples are the *love* relation, the *adjacency* relation, and the *betweenness* relation. Many people regard relations as an extremely important ingredient for understanding the resemblances and differences that we find among particulars, but there are also people who deny the existence of relations.

Relations constitute one of the central themes of metaphysics, the study of the fundamental structure of reality. Important metaphysical questions with

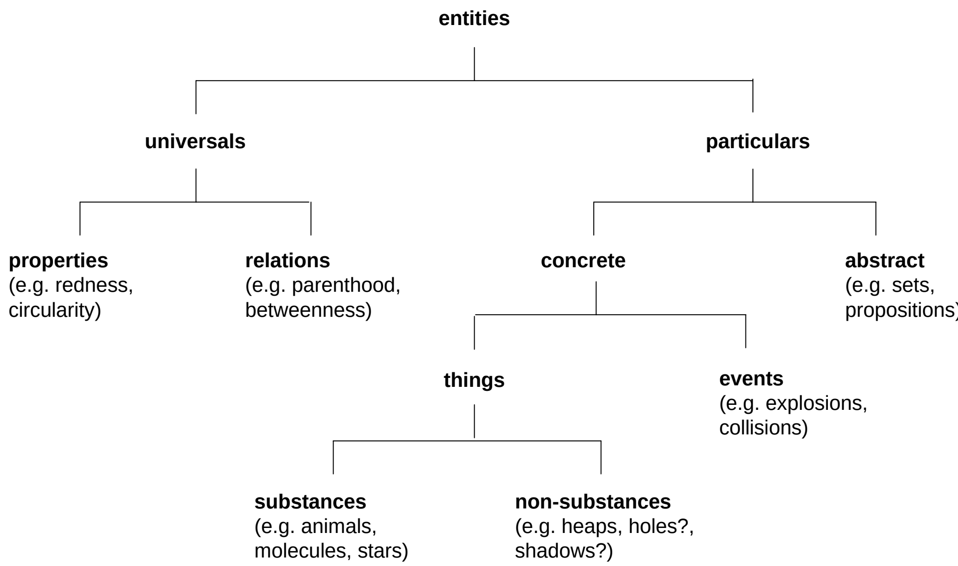


Figure 2.1: An illustrative, but controversial system of categories (taken from [Low02, 16]).

respect to relations are about their ontology, their location, their internal structure, their types, and the question whether relations really exist. We will briefly look at these issues.

Aristotle identified in his work the *Categories*, relations as one of the ten main categories, the irreducible kinds of being. Some of the other categories are substance, quality, and quantity. Aristotle called relations "things toward something" (ta pros ti). In a later work, the *Metaphysics*, Aristotle said about relations that "what is relative is least of all things a kind of entity or substance".

Aristotle's model of categories was enormously influential during the Middle Ages. According to Brower [Bro05], medieval thinkers regarded polyadic properties as something that can only exist in the mind. Also Leibniz asked himself where a relation which links two objects is located. He argued that it cannot be only in one of them and also not in a kind of void between them. Leibniz claims that relations are reducible to properties. According to Mertz [Mer96, 43], it was until the late nineteenth century that this view was almost universally held. Contrary to this view, relations are now often regarded as a separate subcategory of the universals, as shown in Figure 2.1.

Currently, there are many systems of categories. In some no universals occur, and in some also no relations. For example, for nominalists there are no universals, but *moderate* nominalists hold that there are properties and relations. For them properties and relations are particulars. These ‘particularized qualities’ are often called *tropes*.

If we assume that universals exist, then properties are often regarded as monadic universals and relations as polyadic universals. Properties can be seen as a limiting case of a universal. In the context of this thesis, we will not treat them separately. The relational models to be developed can also be used as models for properties.

Armstrong regards universals as state of affairs *types*. For Armstrong, a universal is “everything that is left in the state of affairs after the particular particulars involved in the state of affairs have been abstracted away in thought” [Arm97, 29]. In the next section we will discuss states of affairs in more detail.

The number of arguments that a relation can have is called its degree or its adicity. Whether there really are relations of a particular polyadicity may well be an empirical question. One can imagine that all ontologically fundamental relations are dyadic. One can even imagine that all properties are constructed out of relations. Also for Wittgenstein, what kind of objects relations are, is to be found out by empirical investigation, not by logic [Lok88, 2]. Wittgenstein, by the way, regards relations and properties as objects: “Auch Relationen und Eigenschaften etc. sind *Gegenstände*” [Wit79, 16.6.15].

Relations do not necessarily have a fixed adicity. For example, the relation ‘is surrounded by’ has a variable number of objects in its different instantiations. Armstrong [Arm97, 85] regards such relations as ‘second-class’ relations, since their different instantiations would differ in their “essential nature”. He considers it a truism that a universal is strictly identical in its different instantiations.

An important distinction, that goes back to Hume, is the distinction between *internal* and *external* relations. Internal relations are relations whose existence supervenes upon the existence of its terms. For example, the relation ‘greater than’ for numbers is an internal relation. External relations are simply those that are not internal. So, ‘being neighbors’ is an external relation. External relations are ontologically the more interesting ones, since they might add something to the “fundamental furniture of the universe”.

Bradley [Bra83, 21-29] disputes the existence of external relations. According to him the assumption that external relations exist, leads to an infinite regress. Let me briefly sketch his argument. Assume that Hein loves Els. Then this fact cannot come down to the existence of the constituents Hein, Els, and 'loves', because these constituents need to be related in a certain way. Bradley calls this the 'relating relation'. This gives a new fact. Now for this new fact we can apply the same reasoning as above to show that we also need another 'relating relation', and so on. Bradley claims that in this way we get a vicious infinite regress.

Many philosophers have contested Bradley's regress. Let me only mention that Frege avoided the regress by treating relations as objects that by themselves are incomplete ("unvolständig, ergänzungsbedürftig oder ungesättigt" [Fre75, 22]), and that can come together with arguments without further mediation. As a consequence, relations do not need additional relations to connect them to their arguments.

Of course, much more can be said about views on the nature of relations. However, a more thorough or systematic treatment is not really needed here, since my arguments in this thesis do not depend on a specific view on relations.

2.2 States of affairs

I will use the following terminology. A state of affairs is a *possible fact*, and a state of affairs that actually obtains or holds is a *fact*. Examples are:

- Carla's being older than Mieke.
- Pim and Rianne's being the parents of baby Mike.
- Loed's drawing of a picture and Riëtte's reading of this thesis.

States of affairs are related to propositions, but they are not the same as propositions. Facts are what make a proposition true or false. For example, the fact that Mo is laughing makes the proposition that Mo is laughing true. Russell says in [Rus86, 163]: "If I say 'It is raining', what I say is true in a certain condition of weather and is false in other conditions of weather. The condition of weather that makes my statement true (or false as the case may be), is what I should call a 'fact'."

When the term 'state of affairs' is used, one often refers to Wittgenstein. Wittgenstein uses the following related terms in the *Tractatus*: *Tatsache*, *Sachlage*, and *Sachverhalt*. 'Tatsache' can be translated as 'fact', but the translation of the other two terms is somewhat problematic. In the Ogden translation of the Tractatus, which was approved by Wittgenstein, Sachlage is translated as 'state of affairs', and Sachverhalt as 'atomic fact'. However, in the also commonly used Pears and McGuiness translation of the Tractatus, 'Sachlage' is translated as 'situation', and Sachverhalt as 'state of affairs'. According to Nelson [Nel99] almost all commentators on Wittgenstein have accepted the Pears and McGuiness translation as the better one, but Nelson gives strong arguments why the Ogden translation of *Tatsache* and *Sachlage* is the correct one.

Barwise & Perry [Bar83, 49] use the term 'situations' as a general term for both states of affairs and for events. In their terminology, states of affairs are static situations and events are more dynamic situations.

Another interesting view is of David Lewis. He considers states of affairs as sets of possible worlds [Lew02, 185]. We will come back to this view in Chapter 6, when we will discuss operations on states of affairs.

In Figure 2.1, states of affairs did not occur as a category. If we would like to assign them a fundamental role in a system of ontology, then a question is, where to put them? Are states of affairs abstract or concrete entities? According to one view, states of affairs are very similar to propositions and therefore abstract. But according to another view, facts, like that the cat is on the mat, are concrete entities.

According to Lowe [Low02, 385] the ontological status of facts is controversial and it is questionable whether they should be given a fundamental role in an ontological system. Armstrong, on the other hand, considers facts and states of affairs as the fundamental constituents of the world: "[T]he world, all there is, is a world of states of affairs" [Arm97, 1].

A difficult question is whether states of affairs can be *molecular*. Are there conjunctive, disjunctive, conditional, and negative states of affairs? Armstrong accepts only conjunctive states of affairs [Arm97, 35]. Russell is somewhat more reserved in his judgement. He does not think it is plausible that there are single disjunctive facts. Further, he says he is inclined to think there are negative facts, but he does "not say positively that there are, but that there may be" [Rus86, 186-187]. A related question is whether states of affairs can be *atomic*. It is conceivable that states of affairs are composed of molecular states all the way down. I will not discuss this issue here, but only like to remark that Russell considers facts expressed in “This is white” and “This is left of that” as atomic [Rus86, 176].

In all the relational models that will be developed in this thesis, states will be used. But the exact identity of these states is for the argument not so important. If you like, you may substitute for them ‘states of affairs’, ‘situations’, ‘circumstances’, ‘occasions’, ‘events’, or even ‘propositions’.

2.3 Symmetry in relations

It seems obvious that not all relations are symmetric: if I love you, then this does not mean that you also love me. Nevertheless, Cian Dorr [Dor04] argues that all relations are symmetric. If predicates do not necessarily correspond to relations, then it is not immediately clear that this claim has to be false. According to Dorr it is even an open question if any of our current predicates correspond to relations.

Dorr’s argument for the claim that all relations are symmetric is rather complex and I will not discuss it here. I will only mention three examples Dorr gives of how to interpret theories with non-symmetric predicates in theories with only symmetric predicates.

First, a simple example. Consider the non-symmetric predicate ‘is part of’ as it occurs in mereology. It is easy to see that this predicate can be defined in terms of the symmetric predicate ‘overlaps’: x is part of y iff whatever overlaps x overlaps y.

The next example is a tricky one. Let P_1xy be an arbitrary binary predicate. Then we can define a symmetric ternary predicate P_2xyz such that P_1xy iff P_2xyy. It is easy to generalize this to predicates of any adicity. To Dorr, this interpretation looks like cheating, and I agree with him.

A more complicated and more interesting example, is from Allen Hazen [Haz99]. He proved that (well-founded) set theory can be formulated in terms of the following two symmetric predicates:

- $x\mathsf{E}y \;:\Leftrightarrow\; x \in y$ or $y \in x$.
- $x\mathsf{O}y \;:\Leftrightarrow\; x$ and y have the same rank.

The *rank* of a set is defined as the least ordinal number greater than the rank of any member of the set.

If there is at least one urelement, then it can be proved that:

$$a \in b \;\Leftrightarrow\; a\mathsf{E}b \;\;\&\;\; \exists x\,[\,x\mathsf{O}b \;\;\&\;\; \forall y\,[\,y\mathsf{O}a \Rightarrow y\mathsf{E}x\,]].$$

If there are no urelements, then this equivalence is not true for $a = \{\emptyset\}$ and $b = \emptyset$. Hazen shows that the assumption of an urelement can be avoided by a more complicated equivalence. As a consequence, for ZF set theory an equivalent axiomatization can be given with as primitives only E and O.

In this thesis I will not assume that relations or "fundamental" relations have specific symmetry properties.

2.4 Views on relations as distinguished by Fine

In his paper "Neutral relations", Kit Fine presents three views on relations. He calls them the *standard view*, the *positionalist view* and the *antipositionalist view*. He rejects the first two views because of various problems to which they give rise. Of the antipositionalist view he claims that it combines the virtues of the first two views, and that it gives a better understanding of the essence of our idea of a relation. I will briefly describe here the views as presented by Fine.

2.4.1 Standard view

According to the standard view, objects in a relation always come in a given order. For example, we may say that the relation *loves* holds of a and b (in that order) just in case a loves b. A consequence of this view is that each binary relation has a *converse*. For example, the converse of the relation *on top of* is the relation *beneath*. The consequence that each relation has a converse, is a shortcoming of this view because of the following reason.

Suppose a block a is on top of a another block b. Then we have a state of affairs s that may be described as the state of a's being on top of b, but that may also be described as the state of b's being beneath a. If s is a genuine relational complex, i.e. a state consisting of a relation in combination with its relata, then there must be a *single* relation that can be correctly said to figure in the complex in combination with the given relata. We have no reason to choose either *on top of* or *beneath* for this relation. Whatever this relation is, it cannot have a converse. Therefore the standard view is objectionable from a metaphysical perspective.

If we consider relations as belonging to reality rather than to our representation of it, then the order of the arguments is to be attributed to our representations, not to the relation itself. It is an artefact of our language that it leads us to suppose that relations themselves must apply to arguments in a given order.

There is also a linguistic argument against the standard view. In a graphical language, the love predicate could be a heart-shaped body with a red and a black side. On the red side we write the name of the lover and on the black side the beloved one. The relation signified by the heart does not fit in with the standard view, since the sides of the heart are not ordered in a relevant sense.

2.4.2 Positionalist view

The positionalist view assumes that each relation has a fixed number of positions or argument-places, which are specific entities that belong to the relation. For example, for the love relation we have the position of *Lover* and of *Beloved*.

Under the standard view, we can picture a relation as an arrow with the ends of the arrow corresponding to the argument-places, but under the positionalist view, it is more appropriate to picture a relation as a solid body with holes of various shapes for the argument-places. For the love relation, we could have a cubical hole for the lover and a conical hole for the beloved.

Under positionalism, there is no meaningful notion of the converse of a relation. Since there is no intrinsic order in the positions, the positionalist view is a *neutral* or *unbiased* conception of relations.

There are two objections to the positionalist view. The first objection is an ontological one. With the positionalist view it is impossible to see if there is a more basic position-free account of relations. If there is no such account, then we could have to include positions among the "fundamental furniture of the universe". The second objection concerns *strictly symmetric* relations, like the adjacency relation. The state of a's being adjacent to b is the same as the state of b's being adjacent to a. But if a and b occupy distinct positions within a state, then switching the positions of a and b

cannot yield the same state. Therefore, under the positionalist view no strictly symmetric relations are possible.

2.4.3 Antipositionalist view

Under the positionalist view, the *completion* of a relation is the state we get from assigning objects to the argument-places of the relation. In this case it is a single-valued operation. However, the antipositionalist has no argument-places to which objects can be assigned. He takes completion as a *multivalued* operation, yielding a plurality of states for the different ways in which the relation might be completed by the objects. For example, the completion of the love relation by Don José and Carmen contains the state of Don José's loving Carmen and the state of Carmen's loving Don José.

We can picture a relation under the antipositionalist view as an unadorned body or "magnet". Different configurations may be formed by different ways of attaching relata to a given body.

To distinguish different states, the antipositionalist makes use of the idea that a state can be a completion of a relation *in the same manner* as another state. We say that a state s is a completion of a given relation $\Re$ by constituents $a_1, a_2, \ldots, a_m$ *in the same manner* as a state t is a completion of $\Re$ by constituents $b_1, b_2, \ldots, b_m$, if s can be obtained by simultaneously substituting $a_1, a_2, \ldots, a_m$ for $b_1, b_2, \ldots, b_m$ in t (and vice versa). We assume that if the a_i's are the same, then the corresponding b_i's are also the same (and vice versa).

So, for example, the state of Anthony's loving Cleopatra is a completion of the love relation by Anthony and Cleopatra in the same manner as the state of Abelard's loving Eloise is a completion of the love relation by Abelard and Eloise.

The antipositionalist view has certain advantages over the positionalist view: (1) It does not have the ontological problem of the positionalist view, for it has no argument-places, (2) it does not have problems with strictly symmetric relations, and (3) it can account for relations with variable polyadicity.

The notion of *co-mannered completion*, i.e. the notion of the relation *in the same manner*, should not be taken as primitive. We defined the relation *in the same manner* in terms of substitution. Thus we should see the notion of *co-mannered completion* as a special case of the more general notion of substitution.

The antipositionalist can reconstruct the notion of position in terms of *co-positionality*. We say that a in s is *co-positional* to b in t if s results from t by a substitution in which b goes into a (and vice versa). If the antipositionalist accepts the existence of symmetric relations, he cannot satisfactorily reconstruct the positionalist's account of position, because if constituents occupy different positions, then interchanging constituents will give a different state.

Under the standard conception, a relation applied to its relata in an absolute manner; Under the positionalist conception, a relation applied to its relata relative to the positions of the relation, but the positions themselves are an absolute notion; Under the antipositionalist conception, we have the relative notion of co-positionality. The antipositionalist has stripped the concept of a relation to its core.

2.5 Criticizing Fine's objections to positionalism

The first objection Fine raised against the positionalist view is that a "full-blooded commitment to an ontology of positions" does not match with our inclination to think that a position-free account of relations is possible. For me it is not clear whether we would *a priori* be strongly inclined to think that such an account is possible. But anyhow, Fine comes with a beautiful alternative account, the antipositionalist view.

According to Fine the antipositionalist can reconstruct positions, but I find his solution not very satisfying. Fine claims [Fin00, 29]: "Positions can then be taken to be the abstracts of constituents in relational complexes with respect to the relation co-positionality". The way I understand this, is that we would get for certain states of cyclic relations just one position. Another peculiarity of Fine's reconstruction of positions is that certain relations can have more positions than the maximum number of arguments an instance can have. Consider the love relation and the state s of Narcissus's loving Narcissus. By definition of co-positionality, Narcissus in s is only co-positional to other objects in states where they love themselves. This would give us three positions for the love relation instead of two. So, positions reconstructed in this way are in general not very similar to the positions of the positionalist. I find it somewhat confusing to call the reconstructed entities *positions*. Perhaps it would be better to call them *roles*.

The second objection of Fine against the positionalist view, is that, for symmetric relations it is contradictory to assume that distinct objects occupy different positions and that positions are preserved under substitution. However, in my view, this does *not* mean that positional representations are somewhat dubious. I even think that a positionalist might concede the argument of Fine, but respond that the positions of a relation only have a *mediating* role. Assigning arguments to positions *yields* states, but I see no reason to assume that the objects *occupy* these positions within the states. Perhaps we should picture a relation under this view not as a solid body, but as a *non-solid* body with holes of various shapes through which objects can be pushed *into* the body. I think this argument refutes the second objection of Fine to the positionalist view.

Finally, Fine argues that, if we accept symmetric relations, then the antipositionalist cannot give a satisfactory reconstruction of the positionalist's account of positions [Fin00, 32, footnote 22]. In his argument for this claim, Fine uses the supposition that according to a positionalist objects must occupy positions within the states. But if you drop this supposition – and as I argued, we have good reasons for doing so – then a satisfactory reconstruction of "normal" positions, as seen by the positionalist, is possible for a large class of relations including all kind of symmetric relations, as I will show in Section 4.6.

In response to these criticisms, Kit Fine said that in his paper 'Neutral relations' he was, for simplicity, ignoring the fact that substitution is properly done on occurrences, as is made clear in [Fin89]. With respect to the second paragraph of this section he remarked that if we use the notion of what it is for one occurrence of an individual to be co-positional with another occurrence, then we can avoid the difficulty over there being too many positions. Further, if positions are something to be occupied, then we cannot properly distinguish different positions within a cyclic relation.

With respect to the third paragraph of this section, Kit Fine said that he has no objection to a 'thin' notion of position (one which is not occupied) as such. But he does not think it is basic; exemplification or completion through thin positions must be understood in other terms. (I completely agree with him on this point.) Further, he remarked that he thinks his paper 'Neutral relations' was not clear on how both of his objections to positionalism are to positionalism as a *basic* account of what relations are.

In conclusion, I would say that these considerations give hope for a non-basic form of positionalism, but they still leave us with questions with respect to the adequacy of positional representations.

Chapter 3

Framing relations

3.1 The philosophy of modeling

Models can be useful as instruments of investigation. They not necessarily mirror phenomena in reality, but they can also offer a partial representation that abstracts from the real nature. The goal of creating such models is to learn something about the things represented. What is essential in this way of modeling, is to make a clear distinction between models and reality.

In modeling reality we try to establish an analogy between the model and certain aspects of reality. For this purpose various things can be used. For example, we can use tangible objects like spheres and rods to create chemistry models, or we can cut out wooden hearts as models for the love relation. But of course also more abstract models are possible.

In our attempts to get hold of the essence of the structure of relations we will use mathematical models. The advantage of using mathematical models is their precision and their great flexibility to formulate all kinds of subtleties. The price, however, is that they are for most of us harder to appreciate, because they often lack a simple visual form. In such cases it might be helpful to create in addition tangible models for explaining these abstract models.

An alternative for using models to learn more about the structure of relations, would be the use of an axiomatization of relations. I think this would certainly be a useful complementary approach in this case. However, in this thesis, we limit ourselves to models. The choice for models also seems more straightforward, because the questions we have are partially about representations for relations.

In the choice of the models presented in this chapter, I let myself be guided by the relational views as described by Fine. Limitations of the models should not automatically be interpreted as the models necessarily being erroneous or inadequate. On the contrary, abstracting from certain aspects helps to highlight aspects we like to concentrate on. An example of a limitation we will deliberately use, is that we make no use of typed domains of objects for our models. So, our models are not accurate for a relation like 'drinks', because "Mo drinks beer" corresponds in a natural way to a state, but "beer drinks Mo" does not. However, such refinements can easily be incorporated into the models without loss of the results to be discussed.

3.2 Coordinated models

We start with models in which the order of the relata is relevant. Predicates like "__ loves __" can perfectly be expressed in these models.

Definition 3.2.1. A *coordinated model* is a quintuple $\langle D, n, S, \Gamma, H\rangle$, where D is a (nonempty) set of objects, n, the degree of the model, is a natural number, S is a (nonempty) set of states, Γ is a function from D^n to S, and H is a subset of S representing the states that hold.[1]

For coordinated models, and for other models to be discussed, we call the tuple without H, the *frame* of the model or its *logical space*.

To model an intensional conception of relations, I include states of affairs as a constituent of the model. If you prefer an extensional conception of relations, you could instead consider stripped versions, like $\langle D, n, F\rangle$ with F a subset of D^n. In this thesis, however, for all relational models an intensional view will be entertained.

Because the arguments are ordered, binary coordinated models have converses, and, more generally, all coordinated models have *permutations*.

Definition 3.2.2. Let $\mathcal{M} = \langle D, n, S, \Gamma, H\rangle$ be a coordinated model, and let π be a permutation of $1, 2, \ldots, n$. Then define $\Phi(\pi)(\mathcal{M})$ as the model $\langle D, n, S, \Gamma', H\rangle$ such that

$$\forall f \in D^n\ [\Gamma'(f) = \Gamma(f \circ \pi)].$$

[1] Here D^n represents the set of functions $\{1, 2, \ldots, n\} \to D$.

We call $\Phi(\pi)(\mathcal{M})$ a *permutation of* $\mathcal{M}$. By abuse of notation, we will write $\pi(\mathcal{M})$ for $\Phi(\pi)(\mathcal{M})$.

Definition 3.2.3. Let $\mathcal{M}$ be a coordinated model. We say that $\mathcal{M}$ has *strict symmetry* if $\mathcal{M} = \pi(\mathcal{M})$ for some permutation $\pi \neq \mathsf{id}_{\{1,2,\ldots,n\}}$.

In Section 2.4.1 we mentioned Fine's objection against the coordinated view that, as a consequence of that view, each binary coordinated relation has a converse [Fin00, 2]. This may be translated to a shortcoming of coordinated models. If there is a single underlying relation, then we would like to give a neutral representation for it. The coordinated models obviously fail in this respect. We could of course take the class of permutations of a coordinated model as a neutral representation, but, as we will show in this thesis, there are simpler alternatives.

Remark 3.2.4. A different view on converses of relations is proposed by Timothy Williamson [Wil85]. Williamson argues that *all* relations have a converse, but that converses are always identical. His argument is that if a relation $\mathfrak{R}$ is distinct from its converse $\breve{\mathfrak{R}}$, then a relational expression cannot stand for $\mathfrak{R}$ rather than for $\breve{\mathfrak{R}}$. So, for Williamson all relations are neutral, and he will probably see no reason (apart from conventional ones) to prefer for any relation any specific coordinated model as a representation. On the other hand, Fine thinks that there are both neutral and biased relations. Therefore, he might find for certain relations certain coordinated models more adequate than their permutations.

3.3 Positional models

Instead of letting the order of the objects play a constitutive role for relations, we can assign objects to orderless *positions* or *argument-places*.

Definition 3.3.1. A *positional model* is a quintuple $\langle D, P, S, \Gamma, H\rangle$, where D is a (nonempty) set of objects, P is a finite set of positions, S is a (nonempty) set of states, Γ is a function from D^P to S, and H is a subset of S representing the states that hold.

The number of elements of P is called the *degree* or *"arity" of* the model. We denote it as $\mathsf{degree}_{\mathcal{M}}$.

Analogous to permutations of coordinated models we can define positional variants of positional models.

Definition 3.3.2. Positional models $\langle D, P, S, \Gamma, H\rangle$ and $\langle D', P', S', \Gamma', H'\rangle$ are *positional variants* if $D = D'$, there is a bijective mapping $\pi : P \to P'$ such that for each $f \in D^{P'}, \Gamma'(f) = \Gamma(f \circ \pi)$, $S = S'$, and $H = H'$.

The function Γ may map different elements of D^P to the same state. For example for the adjacency relation, Γ will be symmetric. In our analysis of the structure of different models the notion of *positional structure* will play a central role.

Definition 3.3.3. Let $\mathcal{M} = \langle D, P, S, \Gamma, H\rangle$ be a positional model. Then the *positional structure* $E_{\mathcal{M}}$ is defined as:

$$E_{\mathcal{M}} := \{(f, g) \mid \Gamma(f) = \Gamma(g)\}.$$

It is clear that $E_{\mathcal{M}}$ is an equivalence relation. We could regard the equivalence classes of $E_{\mathcal{M}}$ as positional configurations of states. To emphasize the significance of the positional configurations, we might alternatively define models with Γ as an injective function from $D^P/E_{\mathcal{M}}$ to S.

For coordinated models we discussed the problem with converses. This problem does not occur for positional models, since there is no order in the positions. However, Fine [Fin00, 16] raises two objections against the positional view, whose impact for the positional models need to be considered. The first one is an ontological objection, namely that if there is no position-free account of relational facts, then we might have to include positions among the "fundamental furniture of the universe". The second objection concerns strictly symmetric relations. Different assignments to positions may give identical completions.

The first objection will be met by presenting alternative models. The second objection I do not consider as a disqualification of positional models. What Fine convincingly showed is that it would be wrong to assume that the positions correspond one-to-one to some kind of entities in the complexes of relations, and that these entities are occupied by the constituents of the relation. But in the positional models the positions are not part of the states nor is it said that objects occupy positions. The positions in the positional models only have a kind of mediating function. Assigning objects to them *yields* states. In this thesis I will defend that positional models are most appropriate for representing (a large class of) relations.

3.3.1 Neutralizing relations

There is an obvious correspondence between coordinated models and positional models with numerical positions. Let me for the sake of clarity make the notion of correspondence precise.

Definition 3.3.4. A coordinated model $\langle D, n, S, \Gamma, H\rangle$ *corresponds to* a positional model $\langle D', P, S', \Gamma', H'\rangle$ if $D' = D$, $S' = S$, $H' = H$, and there is a bijective mapping $\mu : \{1, 2, \ldots, n\} \to P$ such that

$$\forall f \in D^P\, [\, \Gamma'(f) = \Gamma(f \circ \mu)\,].$$

Fine states that we can transform biased relations, i.e. relations in which the arguments are ordered, into unbiased ones by taking a "permutation class" of biased relations and by identifying each position of the unbiased relation with a function that takes each biased relation of the permutation class into a corresponding numerical position [Fin00, 15]. Let us try to translate this construction into set theoretic terms.

Let $\mathcal{M}$ be an n-ary coordinated model $\langle D, n, S, \Gamma, H\rangle$, and let $\mathsf{Perm}(n)$ denote the set of permutations of $\{1, 2, \ldots, n\}$. Define the permutation class Φ as:

$$\Phi := \{\pi(\mathcal{M}) \mid \pi \in \mathsf{Perm}(n)\}$$

and define P as:

$$P := \{F : \Phi \to \{1, 2, \ldots, n\} \mid \forall \mathcal{M} \in \Phi\, \forall \pi \in \mathsf{Perm}(n)\, [\, F(\pi(\mathcal{M})) = \pi(F(\mathcal{M}))\,]\}.$$

Lemma 3.3.5. *If $\mathcal{M}'$ is a permutation of $\mathcal{M}$, then the permutation classes of $\mathcal{M}$ and $\pi(\mathcal{M})$ are the same.*

Proof. From the definition of $\pi(\mathcal{M})$ it follows that $\pi'(\pi(\mathcal{M})) = (\pi' \circ \pi)(\mathcal{M})$. Therefore, if $\mathcal{M}' = \tau(\mathcal{M})$, then $\pi(\mathcal{M}') = \pi(\tau(\mathcal{M})) = (\pi \circ \tau)(\mathcal{M})$ and

$$\begin{aligned}
\pi(\mathcal{M}) &= \pi((\tau^{-1} \circ \tau)(\mathcal{M})) \\
&= (\pi \circ (\tau^{-1} \circ \tau))(\mathcal{M}) \\
&= ((\pi \circ \tau^{-1}) \circ \tau)(\mathcal{M}) \\
&= (\pi \circ \tau^{-1})(\tau(\mathcal{M})) \\
&= (\pi \circ \tau^{-1})(\mathcal{M}')
\end{aligned}$$

⊣

If $\mathcal{M}$ has no strict symmetry, i.e. $\mathcal{M} = \pi(\mathcal{M}) \Leftrightarrow \pi = \mathsf{id}_{\{1,2,\ldots,n\}}$, then P contains n elements. The set P is in that case perfectly suitable as the set of positions of the desired neutral model, as I will show in a moment. But if $\mathcal{M}$ has strict symmetry, then P contains less than n elements. For example, if $\mathcal{M}$ models the strictly symmetric binary relation, then P is empty. And if $\mathcal{M}$ is a ternary model with $\mathcal{M} = \pi(\mathcal{M})$ iff $\pi(1) = 1$, then P does not contain 3 functions – as we would like to have – but only 1. I will first give the solution for non-symmetric relations, and then consider the general case.

Assume $\mathcal{M}$ has no strict symmetry. Define $\overline{\Gamma}$ as a function from D^P to S by

$$\overline{\Gamma}(f) := \Gamma(f \circ \mu) \text{ with } \mu \in P^n \text{ such that } \forall i \in \{1, 2, \ldots, n\}\ [\,\mu(i)(\mathcal{M}) = i\,].$$

Then $\mathcal{N} = \langle D, P, S, \overline{\Gamma}, H \rangle$ is the desired neutral model. This claim is expressed more precisely in the next lemma.

Lemma 3.3.6. *If the coordinated model $\mathcal{M}$ has no strict symmetry, then $\mathcal{N} = \langle D, P, S, \overline{\Gamma}, H \rangle$ is a positional model corresponding to $\mathcal{M}$, and $\mathcal{N}$ is uniquely determined by the permutation class Φ of $\mathcal{M}$.*

Proof. We have to show three things: (1) $\mathcal{N}$ is a well-defined n-ary positional model; (2) In the definition of $\mathcal{N}$ no special role is played by any particular element of Φ; (3) $\mathcal{N}$ corresponds to $\mathcal{M}$.

1. We first prove that P contains exactly n elements. Consider

$$P' := \{F : \Phi \to \{1, 2, \ldots, n\} \mid \forall \pi \in \mathsf{Perm}(n)\ [F(\pi(\mathcal{M})) = \pi(F(\mathcal{M}))]\}.$$

We assumed that $\mathcal{M}$ has no strict symmetry, so $\pi(\mathcal{M}) = \pi'(\mathcal{M}) \Leftrightarrow \pi = \pi'$. Therefore P' has exactly n elements. It is clear that $P \subseteq P'$, since the conditions in the definition of P imply the conditions in the definition of P'. It is also the case that $P' \subseteq P$. For let F be an element of P' and $\mathcal{M}'$ be an element of Φ, say $\mathcal{M}' = \tau(\mathcal{M})$. Then, because

$$\begin{aligned} F(\pi(\mathcal{M}')) &= F(\pi(\tau(\mathcal{M}))) \\ &= F((\pi \circ \tau)(\mathcal{M})) \\ &= (\pi \circ \tau)(F(\mathcal{M})) \\ &= \pi(\tau(F(\mathcal{M}))) \\ &= \pi(F(\tau(\mathcal{M}))) \\ &= \pi(F(\mathcal{M}')) \end{aligned}$$

it follows that $\forall \mathcal{M} \in \Phi \, \forall \pi \in \mathsf{Perm}(n) \, [\, F(\pi(\mathcal{M}) = \pi(F(\mathcal{M}))\,]$. Therefore P contains n elements.

Now we prove that $\overline{\Gamma}$ is well-defined. Clearly, it is sufficient to show that there is exactly one function $\mu \in P^n$ such that $\forall i \in \{1, 2, \ldots, n\} \, [\, \mu(i)(\mathcal{M}) = i\,]$. It follows from the definition of P that $\forall F, F' \in P \, [\, F(\mathcal{M}) = F'(\mathcal{M}) \Rightarrow F = F'\,]$. Therefore, at most one function μ fulfills the condition. Because P contains n functions, we see that $\forall i \in \{1, 2, \ldots, n\} \, \exists F \, [\, F(\mathcal{M}) = i\,]$. So, there is also at least one function μ that fulfills the condition.

2. To prove that in the definition of $\mathcal{N}$ no special role is played by any particular element of Φ, we check the steps we took in the construction of $\mathcal{N}$. (1) We started with a coordinated model $\mathcal{M}$ and defined the permutation class $\overline{\Gamma}$ of $\mathcal{M}$. From Lemma 3.3.5 it follows that starting with a permutation of $\mathcal{M}$ would give the same permutation class. (2) We defined the positions P of $\mathcal{N}$. The definition of P is clearly independent of any element of Φ in a special way. (3) The last step was the definition of $\overline{\Gamma}$. We already showed that $\overline{\Gamma}$ is well-defined. Now we have to show that $\overline{\Gamma}$ does not depend in a special way on $\mathcal{M}$.

Assume $\mathcal{M}' := \langle D, n, S, \Gamma', H \rangle$ is another model in Φ. Consider the definition of $\overline{\Gamma}$. Let $\mu, \mu' \in P^n$ be such that $\forall i \; [\mu(i)(\mathcal{M}) = i \quad \& \quad \mu'(i)(\mathcal{M}') = i]$. To show that $\overline{\Gamma}$ does not depend in a special way on $\mathcal{M}$, we have to prove that $\Gamma'(f \circ \mu') = \Gamma(f \circ \mu)$.

Let π be such that $\mathcal{M}' = \pi(\mathcal{M})$. Then, by the definition of the permutation of a coordinated model, $\forall f \in D^n \; [\Gamma'(f) = \Gamma(f \circ \pi)]$. So, in particular we have, $\forall f \in D^P \; [\, \Gamma'(f \circ \mu') = \Gamma(f \circ \mu' \circ \pi)\,]$. Clearly, it is sufficient to prove that $\mu = \mu' \circ \pi$.

Because $\mu'(i)(\pi(\mathcal{M})) = i$ and $\mu'(i)(\pi(\mathcal{M})) = \pi(\mu'(i)(\mathcal{M}))$, it follows that $\mu'(i)(\mathcal{M}) = \pi^{-1}(i)$, for all i. So, $(\mu' \circ \pi)(i)(\mathcal{M}) = \mu'(\pi(i))(\mathcal{M}) = i$. Because there is exactly one $\overline{\mu} \in P^n$ such that $\forall i \in \{1, 2, \ldots, n\} \; [\, \overline{\mu}(i)(\mathcal{M}) = i\,]$, we see that $\mu = \mu' \circ \pi$.

3. To prove that $\mathcal{N}$ corresponds to $\mathcal{M}$, we have to show that there is a bijective mapping $\mu : \{1, 2, \ldots, n\} \to P$ such that $\forall f \in D^P[\overline{\Gamma}(f) = \Gamma(f \circ \mu)]$. We already proved that $\overline{\Gamma}(f) = \Gamma(f \circ \mu)$ with μ such that $\forall i[\mu(i)(\mathcal{M}) = i]$. So, it is sufficient to show that μ is bijective. Now μ is clearly injective, for $\forall i \in \{1, 2, \ldots, n\} \; [\, \mu(i)(\mathcal{M}) = i)\,]$. We already proved that P has n elements. So, we see that μ is also surjective. $\dashv$

Instead of starting with a coordinated model $\mathcal{M}$, we could also have started with a positional model. Then, as a first step, we could simply translate the given positional model to a corresponding coordinated model, and subsequently follow the construction as given above. This also gives a unique corresponding model. So, we have proved the following result.

Theorem 3.3.7. *For any coordinated or positional model $\mathcal{M}$ that has no strict symmetry, there is a corresponding positional model $\mathcal{N}$ that is uniquely determined by just the permutation class of $\mathcal{M}$.*

We could regard $\mathcal{N}$ as a canonical model. A nice feature of the construction of $\mathcal{N}$ is that it is completely in set theoretic terms.

Let us now face the general case in which $\mathcal{M}$ may have strict symmetry. Let Φ be again the permutation class of $\mathcal{M}$. Then it is simple to construct a corresponding positional model $\mathcal{N} = \langle D, P, S, \overline{\Gamma}, H \rangle$ in which all models in Φ are equally well represented:

1. Choose Π as a set of permutations of $\{1, 2, \ldots, n\}$ such that $\mathsf{id}_{\{1,2,\ldots,n\}} \in \Pi$ and $\forall \mathcal{M}' \, \exists! \pi \in \Pi \, [\, \pi(\mathcal{M}) = \mathcal{M}' \,]$.
2. Define $P := \{F : \Phi \to \{1, 2, \ldots, n\} \mid \forall \pi \in \Pi \, [\, F(\pi(\mathcal{M})) = \pi(F(\mathcal{M})) \,]\}$.
3. Define $\overline{\Gamma}$ as before, i.e. $\overline{\Gamma}(f) := \Gamma(f \circ \mu)$ with $\mu \in P^n$ such that $\forall i \in \{1, 2, \ldots, n\} \, [\, \mu(i)(\mathcal{M}) = i \,]$.

We can prove in a similar way as we did for the non-symmetric case, that $\mathcal{N}$ corresponds to $\mathcal{M}$. But now $\mathcal{N}$ is not uniquely determined by just the permutation class of $\mathcal{M}$. I have not yet found a construction that always gives a unique result. Fine [Fin00, 15, footnote 9] also mentions that there are certain complications. Actually, I am not so sure that for the general case it is possible to define a construction that produces a unique positional model, defined in normal set theoretic terms, in which all the models of Φ are equally well represented.

3.3.2 Positional models with variable polyadicity

Fine [Fin00, 22] claims that it is hard to see how any relation could be variable polyadic under the positional view. But in the following variant of the positional model, variable polyadicity can be handled in a natural way by allowing *open* positions.

Definition 3.3.8. A *positional model with variable polyadicity* is a quintuple $\langle D, P, S, \Gamma, H \rangle$, where D is a (nonempty) set of objects, P is a set of positions, S is a (nonempty) set of states, Γ is a partial function from V to S, with V the set of partial functions from P to D, and H is a subset of S representing the states that hold.

This definition is perhaps too general for models of 'real' relations. By not requiring that P is finite, and by only requiring that Γ is a partial function whose domain consists also of partial functions, we permit much freedom. In this thesis this kind of models will not be further discussed.

3.4 Substitution models

In this section we develop an elegant type of models, the *substitution models*, that agrees with the antipositionalist view. For a better appreciation of this model, we first develop a few other models that also reflect ideas of the antipositionalist view.

Model 1
As we saw in Section 2.4.3, under the antipositionalist view the completion of a relation is a *multivalued* operation. For example, the completion of the love relation by Fleur and Marco contains the state of Fleur's loving Marco and the state of Marco's loving Fleur. Further, because we have no positions, there is no reason to assume that we always have a fixed number of arguments. These ideas lead us to define Model 1 as:

$$\langle D, A, S, \Upsilon, H \rangle,$$

with D a (nonempty) set of objects, S a (nonempty) set of states, $A \subseteq \mathcal{P}(D)$ a set of argument sets, $\Upsilon : A \to \mathcal{P}(S)$ a completion operation, and $H \subseteq S$ a set representing the states that hold.

We assumed that A is a set of subsets of D, but we could of course generalize this to a set of multisets with elements in D. Then each element can stand for an occurrence of an object in a state. Such a multiset approach is also suggested by Fine [Fin00, 20].

Model 2
An obvious shortcoming of Model 1 is that it has no mechanism to distinguish between different completions of a set of constituents. We like to

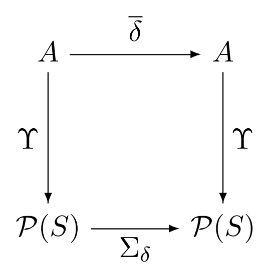
express in our models that objects of a state can be substituted by other objects, and that this gives another state. For example, we want to be able to express that in a state where Fleur loves Marco, we can replace Fleur and Marco by Gerard and Marion, respectively. By adding a substitution operation we get Model 2:

$$\langle D, A, S, \Upsilon, \Sigma, H \rangle,$$

with $D, A, S, \Upsilon,$ and H as in Model 1, and with $\Sigma : S \times D^D \to S$ such that:

1. Σ is a substitution function, i.e.
   (a) $\forall s \in S\, [\, \Sigma(s, \mathsf{id}_D) = s\,]$,
   (b) $\forall s \in S\, \forall \delta, \delta' \in D^D\, [\, \Sigma(s, \delta' \circ \delta) = \Sigma\, (\Sigma(s, \delta), \delta')\,]$,
2. $\forall \alpha \in A\, \forall \delta : D \to D\, [\, \overline{\Sigma}(\Upsilon(\alpha), \delta) = \Upsilon(\overline{\delta}(\alpha))\,]$. [2]

The first condition agrees with how we understand substitution. The intended interpretation of $\Sigma(s, \delta)$ is the state we get when we simultaneously substitute for each object d in the state s the object $\delta(d)$. The second condition states that for any $\delta : D \to D$ the following diagram commutes:



with Σ_δ defined by $\Sigma_\delta(S') := \overline{\Sigma}(S', \delta)$.

Model 2 could be criticized for not being very elegant. The interdependencies between Σ and Υ make the model rather complicated.

Model 3

We can simplify things if, instead of starting with a function Υ for the completion operation, we start with a function $\Delta : S \to \mathcal{P}(D)$ that gives for

[2] We denote lifted functions by overlining the function name. For example, $\overline{\Sigma}$ denotes the lifted function from $\mathcal{P}(S) \times D^D$ to $\mathcal{P}(S)$ defined by $\overline{\Sigma}(S', \delta) := \{\Sigma(s, \delta) \mid s \in S'\}$.

each state its constituents. We may do so if each s in S belongs to exactly one set $\Upsilon(\alpha)$. This results in Model 3:

$$\langle D, S, \Delta, \Sigma, H \rangle,$$

with $D, S,$ and H as in model 2, and $\Delta : S \to \mathcal{P}(D)$ such that:

1. Σ is a substitution function,
2. $\forall s \in S\, \forall \delta, \delta' \in D^D\, [\, \delta =_{\Delta(s)} \delta' \Rightarrow \Sigma(s, \delta) = \Sigma(s, \delta')\,]$,
3. $\forall s \in S\, \forall d \in \Delta(s)\, \exists \delta, \delta'\, [\, \delta =_{\Delta(s) - \{d\}} \delta' \quad \& \quad \Sigma(s, \delta) \neq \Sigma(s, \delta')\,]$.

Let us look at the conditions. The first condition says that Σ fulfills the two 'standard' substitution rules we defined in model 1. Condition 2 says that the result of a substitution is determined by what is substituted for the constituents of a state, and condition 3 says that for each constituent d of a state there is a substitution for which it makes a difference for the result which object is substituted for d. So, these conditions seem to agree with our idea of constituents of a state. Nevertheless, also this model looks a bit complicated. But we can simplify things further.

It is not difficult to see that for a given function Σ there can be at most one function Δ such that the conditions 2 and 3 are fulfilled. For suppose Δ and Δ' both satisfy conditions 2 and 3. Let $X := \Delta(s)$, and $Y := \Delta'(s)$. Suppose $\delta =_{X \cap Y} \delta'$. Define $\delta'' := \delta \restriction X \cup \delta' \restriction X^c$. Then $\delta =_X \delta''$ and $\delta' =_Y \delta''$. So, $\Sigma(s, \delta) = \Sigma(s, \delta')$. Therefore, Δ'' defined by $\Delta''(s) := \Delta(s) \cap \Delta'(s)$ also fulfills condition 2. Since Δ and Δ' fulfill condition 3, it follows immediately that $\Delta = \Delta''$ and $\Delta' = \Delta''$.

From the previous observation it follows that we could alternatively define a model $\langle D, S, \Sigma, H \rangle$ with the same strength as Model 3 by demanding that Σ is a substitution function, and by demanding that there exists a function Δ that fulfills conditions 2 and 3. However, models with a substitution function Σ and no further constraints might also be interesting for further study. Therefore we define:

Definition 3.4.1. A *substitution model* is a quadruple $\langle D, S, \Sigma, H \rangle$, where D is a (nonempty) set of objects, S is a (nonempty) set of states, Σ is a substitution function from $S \times D^D$ to S, i.e.:

1. $\forall s \in S\, [\, \Sigma(s, \mathsf{id}_D) = s\,]$,

2. $\forall s \in S\, \forall \delta, \delta' \in D^D\ [\, \Sigma(s, \delta' \circ \delta) = \Sigma\,(\Sigma(s,\delta), \delta')\,]$,

and H is a subset of S representing the states that hold.

For convenience, we will often write $s \cdot \delta$ for $\Sigma(s, \delta)$. Further, we will also often write $f \cdot g$ for $g \circ f$. With this notation, Σ is a substitution function iff for all $s \in S$ and for all $\delta, \delta' \in D^D$, $s \cdot \mathsf{id}_D = s$ and $s \cdot (\delta \cdot \delta') = (s \cdot \delta) \cdot \delta'$.

In terms of category theory we can make the following observations:

1. A substitution model is a category, with objects all states $s \in S$, and arrows all triples $\langle s, \delta, \Sigma(s, \delta)\rangle$.

2. The function $\Sigma^* : D^D \to S^S$ defined by $\Sigma^*(\delta)(s) := \Sigma(s, \delta)$ is a functor from the monoid D^D to the monoid S^S.

3. We could alternatively have defined a substitution model as a quadruple $\langle D, S, \Sigma^*, H\rangle$, where D is a (nonempty) set of objects, S is a (nonempty) set of states, and Σ^* is a functor from D^D to S^S.

The substitution models are in a sense more basic than all other models we discussed so far. Nevertheless, under apparently mild conditions, we can define for them the objects of a state and, as we will see in Chapter 4, also the notions of roles and in an indirect way of positions. It is good to keep in mind that here these notions are just technical ones. Their metaphysical significance will be further discussed in Chapter 6.

We will now define the objects or relata of a state. The idea is to define them as the objects for which it can make a difference for the resulting state which objects are substituted for them.[3] We have, however, to be a bit cautious in our formulation.

Definition 3.4.2. Let $\mathcal{M} = \langle D, S, \Sigma, H\rangle$ be a substitution model. Let s be a state in S and let X be a subset of D. Then we call X an *object-domain of s* if

$$\forall \delta, \delta'\ [\, \delta =_X \delta' \Rightarrow s \cdot \delta = s \cdot \delta'\,].$$

[3]This approach to define the objects of a state corresponds to a remark Fine made in note 15 of [Fin00, 26].

We define $\mathsf{Obj}_{\mathcal{M}}(s)$, the *objects of* s, as the smallest object-domain of s, if such a smallest set exists. If no smallest object-domain of s exists, then we leave $\mathsf{Obj}_{\mathcal{M}}(s)$ undefined.[4]

We will often write $\mathsf{Obj}(s)$ for $\mathsf{Obj}_{\mathcal{M}}(s)$.

Note that if X and X' are object-domains of s, then $X \cap X'$ is also an object-domain of s. Also note that because the number of objects may be different for different states, we get in a natural way variable polyadicity.

In the last definition, the condition 'if such a smallest set X exists' is needed, because it is not fulfilled for certain infinite simultaneous substitutions, as is shown in the next example.

Example 3.4.3. Let $\mathcal{M} = \langle D, S, \Sigma, H \rangle$ be a substitution model with D an infinite set of objects, S the set of subsets of D modulo a finite difference, i.e.

$$S := \{\, [X] \mid X \subseteq D\},$$

with $[X] := \{X' \subseteq D \mid (X - X') \cup (X' - X) \text{ is finite}\}$, and Σ defined by

$$[X] \cdot \delta := [\overline{\delta}(X)].$$

It is not difficult to see that for any infinite set $X \subseteq D$ no smallest object-domain of $[X]$ exists.

The next lemma gives a nice characterization of the objects of a state.

Lemma 3.4.4. *Let* $\mathcal{M} = \langle D, S, \Sigma, H \rangle$ *be a substitution model. For any* $s \in S$, *if* $\mathsf{Obj}(s)$ *is defined, then*

$$\mathsf{Obj}(s) = \{d \mid \exists \delta, \delta' : D \to D\ [\, \delta =_{D-\{d\}} \delta' \quad \& \quad s \cdot \delta \neq s \cdot \delta']\}.$$

Proof. Assume $\mathsf{Obj}(s)$ is defined. We will prove that

$$\mathsf{Obj}(s) \subseteq \{d\ [\exists \delta, \delta' : D \to D\ [\, \delta =_{D-\{d\}} \delta' \quad \& \quad s \cdot \delta \neq s \cdot \delta']\}.$$

The inclusion in the other direction is obvious.

Consider $d \in \mathsf{Obj}(s)$. Let δ, δ' be such that

$$\delta =_{\mathsf{Obj}(s)-\{d\}} \delta', \text{ and } s \cdot \delta \neq s \cdot \delta'.$$

[4]The object-domains of any state s form a (possibly degenerated) filter C_s. As suggested by Albert Visser, it might be possible to give a general definition of the objects of s as the ultrafilters extending C_s. The details of this approach are under development.

Further, let δ_1, δ'_1 be such that

$$\delta_1 =_{D-\{d\}} \delta, \quad \delta'_1 =_{D-\{d\}} \delta', \text{ and } \delta_1(d) = \delta'_1(d).$$

Then $\delta_1 =_{\mathsf{Obj}(s)} \delta'_1$. So, $s \cdot \delta_1 = s \cdot \delta'_1$. Because $s \cdot \delta \neq s \cdot \delta'$, it follows that $s \cdot \delta_1 \neq s \cdot \delta$ or $s \cdot \delta'_1 \neq s \cdot \delta'$. This proves the claim. $\dashv$

Also the next lemma is useful.

Lemma 3.4.5. *Let $\mathcal{M} = \langle D, S, \Sigma, H\rangle$ be a substitution model. If s is a state in S and δ is a function from D to D such that $\mathsf{Obj}(s)$ and $\mathsf{Obj}(s \cdot \delta)$ are defined, then*

$$\forall s \, \forall \delta \, [\, \mathsf{Obj}(s \cdot \delta) \subseteq \overline{\delta}(\mathsf{Obj}(s))\,].$$

Proof. Consider $d \in \mathsf{Obj}(s \cdot \delta)$. Then, by Lemma 3.4.4, there are functions δ' and δ'' such that $\delta' =_{D-\{d\}} \delta''$ and $s \cdot \delta \cdot \delta' \neq s \cdot \delta \cdot \delta''$. So, $\delta \cdot \delta' \neq_{\mathsf{Obj}(s)} \delta \cdot \delta''$. Because $\delta' =_{D-\{d\}} \delta''$, it follows that d is an element of $\overline{\delta}(\mathsf{Obj}(s))$. $\dashv$

For some substitution models we have $\mathsf{Obj}(s \cdot \delta) \subsetneqq \overline{\delta}(\mathsf{Obj}(s))$, as is shown in the next two examples. In the first example D is finite and in the second D is infinite.

Example 3.4.6. Let $\mathcal{M} = \langle D, S, \Sigma, H\rangle$ be a substitution model with D a finite set of two or more elements, $S := \{s_0, s_1\}$, and Σ such that

$$s \cdot \delta = s_0 \Leftrightarrow (s = s_0 \quad \& \quad \delta \text{ is bijective}).$$

Then $\mathsf{Obj}(s_0) = D$ and $\mathsf{Obj}(s_1) = \emptyset$, and for any δ that is not bijective we have $\mathsf{Obj}(s_0 \cdot \delta) \subsetneqq \overline{\delta}(\mathsf{Obj}(s_0))$.

Example 3.4.7. Let $\mathcal{M} = \langle D, S, \Sigma, H\rangle$ be a substitution model with domain $D := \omega$, the set of natural numbers, $S := S_0 \cup S_1$, with $S_0 := \{\emptyset\}$, and $S_1 := \{X \mid X \subseteq \omega\}$, and Σ defined by:

for $s \in S_0$, $s \cdot \delta := s$,

for $s \in S_1$, $s \cdot \delta := \overline{\delta}(s)$ if δ is injective on s; $s \cdot \delta := \emptyset$ otherwise.

It is easy to verify that $\mathcal{M}$ is indeed a substitution model, and that for all $s \in S$, we have $\mathsf{Obj}(s) = s$. But, we also see that for any function $\delta : D \to D$, that is not injective on $s \in S_1$, $\emptyset = \mathsf{Obj}(s \cdot \delta) \subsetneqq \overline{\delta}(\mathsf{Obj}(s))$.

Another interesting thing to note is that if $\mathsf{Obj}(s)$ is defined, then not necessarily $\mathsf{Obj}(s \cdot \delta)$ is defined as well. We show this in the next example, in which the model is a kind of merge of the model of the previous example and the model of Example 3.4.3.

Example 3.4.8. Let $\mathcal{M} = \langle D, S, \Sigma, H \rangle$ be a substitution model with domain $D := \omega$, $S := S_0 \cup S_1$, with

$$S_0 := \{\, [X] \mid X \subseteq \omega\}, \text{ and}$$

$$S_1 := \{X \mid X \subseteq \omega\},$$

with $[X] := \{X' \subseteq \omega \mid (X - X') \cup (X' - X) \text{ is finite}\}$, and Σ defined by:

$$\text{for } [X] \in S_0,\ [X] \cdot \delta := [\overline{\delta}(X)],$$

$$\text{for } s \in S_1,\ s \cdot \delta := \overline{\delta}(s) \text{ if } \delta \text{ is injective on } s;\ \ s \cdot \delta := [\overline{\delta}(s)] \text{ otherwise.}$$

It is easy to verify that $\mathcal{M}$ is indeed a substitution model. Further, for all $s \in S_1$, $\mathsf{Obj}(s)$ is defined, but for any function $\delta : D \to D$, that is not injective on s, $\mathsf{Obj}(s \cdot \delta)$ is not defined.

Remark 3.4.9. In Section 4.5 we will define *simple* substitution models. In these models the states are in a well-defined way connected. The models in the two previous examples do not fulfill this connectivity condition. However, if we restrict in the examples S_1 to infinite subsets of ω, then the models become simple substitution models and we can still make the same claims.

For each substitution model we can define its degree as a cardinal number.

Definition 3.4.10. Let $\mathcal{M} = \langle D, S, \Sigma, H \rangle$ be a substitution model. For a state s in S, we define the *degree of* s as:

$$\mathsf{degree}(s) := \mathsf{glb}\ \{|X| \mid X \text{ is an object-domain of } s\}.$$

The *degree of* $\mathcal{M}$ we define as:

$$\mathsf{degree}_{\mathcal{M}} := \mathsf{lub}\ \{\mathsf{degree}(s) \mid s \in S\}.$$

Here $|X|$ denotes as usual the cardinality of X, glb denotes the greatest lower bound, and lub denotes the least upper bound. Note that the degree of s and the degree of $\mathcal{M}$ always exist and are indeed cardinal numbers.

If the degree of a model is transfinite, then either (1) there is some state s for which $\mathsf{Obj}(s)$ is infinite, or (2) for all states s the set $\mathsf{Obj}(s)$ is finite, but the size of the sets $\mathsf{Obj}(s)$ is unbounded, or (3) there is some state s for which $\mathsf{Obj}(s)$ is not defined. In the last case all the object-domains of s are obviously infinite sets.

Note that if $\mathcal{M} = \langle D, S, \Sigma, H \rangle$ is a substitution model of finite degree and $|D| > \mathsf{degree}_{\mathcal{M}}$, then the simultaneous substitution of many objects can be defined in terms of single substitutions (cf. [Fin00, 2, note 15]).

In the remainder of this section, we will briefly discuss two more important notions, positions and manners of completions, and we will briefly consider more a more refined notion of substitution, namely substitution of occurrences.

At first sight it seems problematic to define positions in terms of the ingredients of substitution models. For suppose we would define a predicate $R(a, s, a', s')$ as "*a occupies in s the same position(s) as* a' *in* s'". Then it seems 'natural' that the definition will be such that $R(a, s, a', s')$ is true if for some $\delta : D \to D$ with $\delta(a) = a'$ we have $s \cdot \delta = s'$. Now suppose we have a model $\mathcal{M}$ with $s \cdot \delta = s$ for some state s with $\mathsf{Obj}(s) = \{a, b, c, d\}$, and

$$\delta = \{a \mapsto b, b \mapsto c, c \mapsto d, d \mapsto \ a\}.$$

Then, although $\delta(a) = b$, we would in general *not* like to say that a and b occupy the same position in s, for an interchange of a and b might result in a different state. In Section 4.6 we will see how in a natural way positions fit in with the so called *simple* substitution models.

As we said in Section 2.4.3, it is possible to distinguish between different states in S by making use of the notion of *manners of completion.* With this notion we can make explicit how states are interconnected. This allows us to uniquely identify individual states. We now give a precise definition.

Definition 3.4.11. Let $\mathcal{M} = \langle D, S, \Sigma, H \rangle$ be a substitution model. Let s, t be states in S for which $\mathsf{Obj}(s)$ and $\mathsf{Obj}(t)$ exist. Further, let I be some index set, and let $\sigma : I \to \mathsf{Obj}(s)$ and $\tau : I \to \mathsf{Obj}(t)$ be surjective functions. Then we say that s is the completion of σ *in the same manner* as t is the completion of τ, if

$\exists\delta\,[\,\sigma\cdot\delta=\tau \quad \& \quad s\cdot\delta=t\,]$, and vice versa, i.e.
$\exists\delta\,[\,\tau\cdot\delta=\sigma \quad \& \quad t\cdot\delta=s\,]$.

Further, we define the *manner of completion of* s *and* σ as the set of pairs (t,τ) for which s is the completion of σ in the same manner as t is the completion of τ.

If s is the completion of σ *in the same manner* as t is the completion of τ, we might imagine that the constituents $\sigma(i)$ and $\tau(i)$ occupy the same internal positions (if such things exist).

Note that the relation *in the same manner* is an equivalence relation. Also note that it is not true that Lieke's loving Henk is the completion by Lieke and Henk in the same way as Narcissus' loving Narcissus is the completion by Narcissus. If you think differently about this, you could consider to leave out the 'vice versa'-part in the definition.

So far we only considered *global* substitutions of objects in a state. However, the notion of occurrences of objects in a state definitely makes sense and also the idea of substitution done on individual occurrences. For example, in the state where Narcissus loves Narcissus we might substitute Narcissus in the role of lover by Echo, but leave Narcissus in the role of beloved untouched. Roles might be helpful to distinguish different occurrences, but there are also cases where an object a may have in a state s different occurrences with exactly the same role(s). This is for example the case for models for cyclic relations. An interesting question for further research is whether the notion of occurrences is something basic or whether we can understand it in terms of global substitution and the like.

Chapter 4

Subtypes of relational models

The models developed in the previous chapter might be too general for 'real' relations. For example, it might be that relations can only have certain limited forms of symmetry. In this chapter, subtypes of the positional models will be presented and a subtype for the substitution model. We will not discuss here the coordinated models. For these models similar subtypes can be given as for the positional models.

The subtypes of the positional models will be formulated in terms of their *positional structure*. Recall that in the previous chapter we defined the positional structure of a positional model $\langle D, P, S, \Gamma, H\rangle$ as the equivalence relation $E_{\mathcal{M}}$ on D^P for which $f\ E_{\mathcal{M}}\ g \Leftrightarrow \Gamma(f) = \Gamma(g)$. Insight in the possible positional structures for metaphysically meaningful relations might provide a better understanding of the essence of relations. For example, it would be interesting if the positional models of all or many metaphysically meaningful relations would turn out to have a rather simple positional structure.

We follow a kind of bottom-up approach. We start with a simple subtype, the role-based models. The obvious limitations of this subtype will necessitate a generalization, the permutation-based models, which are apparently also too limited. We will develop four subtypes of the positional models. Their inclusion relation is depicted in Figure 4.1.

For the substitutional model, we define one subtype, the simple substitution models. We will show that these models are intimately related with the pattern-based models. We will discuss metaphysical aspects in more detail in Chapter 6, but I want to remark already that the class of simple substitution models looks promising as an adequate class of models for all 'real' relations.

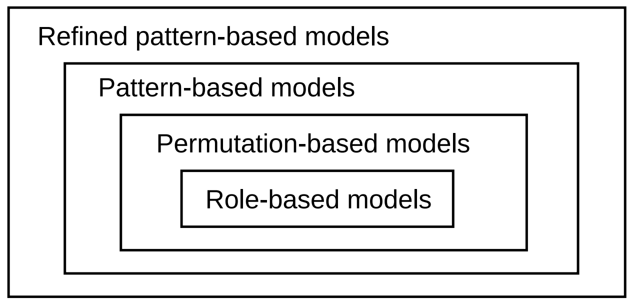


Figure 4.1: Inclusion relation for positional models

4.1 Role-based models

Positions can fulfill certain roles. In the positional model for the amatory relation one position fulfills the role of 'Lover' and the other position the role of 'Beloved'. In this case positions and roles coincide. However, there is no compelling reason why there would always be a one-to-one correspondence between positions and roles. On the contrary, it is natural to say that in the positional model for the adjacency relation the two positions fulfill the same role.

Definition 4.1.1. Let $\mathcal{M} = \langle D, P, S, \Gamma, H \rangle$ be a positional model. Let p, p' be elements of P. Then we say that p *has the same role as* p' if for some injective function $\sigma : P \rightarrowtail P$,

$$p' = \sigma(p) \quad \& \quad \forall f \in D^P \; [\, f \; E_{\mathcal{M}} \; (f \circ \sigma)\,].$$

We define $\mathsf{Roles}_{\mathcal{M}}$ as the set of equivalence classes of this relation, and $\mathsf{Role}(p)$ as the set of positions that have the same role as p.

Note that the relation *has the same role as* is an equivalence relation. Also note that if we did not require σ to be injective, then this would not necessarily be the case. For example, suppose

$$\Gamma(\begin{pmatrix} p_1 & p_2 \\ a & b \end{pmatrix}) = \Gamma(\begin{pmatrix} p_1 & p_2 \\ c & d \end{pmatrix}) \;\Leftrightarrow\; a = c.$$

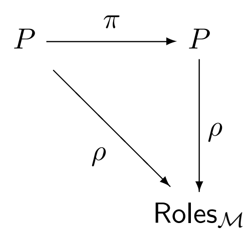
Then by requiring σ to be injective, p has the same role as p' iff $p = p'$. But if σ was not necessarily injective, then we would also get that p_2 has the same role as p_1, but not p_1 has the same role as p_2. So, then the relation *has the same role as* would not be symmetric.

We now define a type of positional models for which changing the positions of the objects does not change the corresponding state as long as the roles of objects are kept invariant.

Definition 4.1.2. Let $\mathcal{M} = \langle D, P, S, \Gamma, H \rangle$ be a positional model, and let $E_{\mathcal{M}}$ be its positional structure. We call $\mathcal{M}$ a *role-based model* if

$$E_{\mathcal{M}} = \{(f, f \circ \pi) \mid f \in D^P \quad \& \quad \pi \text{ is a role-preserving permutation}\},$$

where $\pi : P \to P$ *is a role-preserving permutation* if π is a bijective mapping for which the following diagram commutes:



with $\rho : P \to \mathsf{Roles}_{\mathcal{M}}$ such that $\forall p \in P\ [\,\rho(p) = \mathsf{Role}(p)\,]$.

Note that the role-preserving permutations form a group.

We defined roles within the context of individual models. In this respect they differ from *thematic roles* which apply to arguments of different predicates. For example, thematic roles like *agent*, *patient*, and *location* are used to classify arguments of natural language predicates. We will not study such kind of global roles in this thesis.

Certain strictly symmetric relations can be modeled by role-based models. For example, for the adjacency relation we can define a model $\mathcal{M} = \langle D, P, S, \Gamma, H \rangle$ with two positions, say *Next* and *Nixt*. If the function Γ is strictly symmetric, then $\mathcal{M}$ is clearly a role-based model with one role. However, roles do not solve all problems, as shown by Fine [Fin00, 17]. In particular for certain circular relations no role-based model is possible. Let me state Fine's argument in terms of roles.

Example 4.1.3. Let $\mathcal{M} = \langle D, P, S, \Gamma, H \rangle$ be a role-based model for which the states $\Gamma abcd$, $\Gamma bcda$, $\Gamma cdab$, and $\Gamma dabc$ are identical.[1] Then the model has just one role. But then also the state $\Gamma acbd$ is necessarily identical to $\Gamma abcd$. Therefore, a quaternary circular relation cannot be modeled by a role-based model.

[1]We use $\Gamma abcd$ as an abbreviation for $\Gamma(\begin{pmatrix} p_1 & p_2 & p_3 & p_4 \\ a & b & c & d \end{pmatrix})$, with p_1, p_2, p_3, and p_4 the positions of the model.

4.2 Permutation-based models

The roles function as a way to treat certain permutations of the positions as equivalent. As we already remarked, the role-preserving permutations form a permutation group. However, not every permutation group on the positions can be created with roles. This limitation makes the role-based models inadequate for certain cyclic relations. A natural generalization solving this problem can be obtained by identifying the permutation group of a positional model.

Definition 4.2.1. Let $\mathcal{M} = \langle D, P, S, \Gamma, H \rangle$ be a positional model, and let $E_{\mathcal{M}}$ be its positional structure. Then the *permutation group* $G_{\mathcal{M}}$ is defined by:

$$G_{\mathcal{M}} \;:=\; \{\pi : P \rightarrowtail P \mid \forall f\ [\, f\ E_{\mathcal{M}}\ (f \circ \pi)\ ]\}.$$

We call $\mathcal{M}$ a *permutation-based model* if

$$E_{\mathcal{M}} = \;\{(f, f \circ \pi) \mid f : P \to D \quad \& \quad \pi \in G_{\mathcal{M}}\}.$$

Circular relations can adequately be modeled with permutation-based models. For example, the circular relation in the previous section can be modeled by a model of which the permutation group is generated by

$$\begin{pmatrix} p_1 & p_2 & p_3 & p_4 \\ p_2 & p_3 & p_4 & p_1 \end{pmatrix}.$$

Unfortunately, also this type of models has shortcomings, in particular for certain relations of degree three and higher. Consider a relation $\Re$ in which $\Re abc$ represents the state that a loves b and b loves c. Then $\Re aba$ represents the same state as $\Re bab$, but aba is not a permutation of bab. This means that no permutation-based model for this relation is possible. In the next chapter we will see related shortcomings of the permutation-based models. Then, we will see that permutation-based models are not closed under identification of positions.

We could define a new type of positional models by combining the idea of role-based models and of permutation-based models. Namely, we could allow only role-preserving permutations that belong to a given permutation group. Or more precisely formulated, if ρ is a function from P to $\mathsf{Roles}_{\mathcal{M}}$ and G is a permutation group on P, then we could define a restricted set of admissible permutations as $\{\pi{\in}G \mid \rho \circ \pi = \rho\}$. Only this is no real extension of the permutation-based models, because the restricted set of admissible permutations forms again a group. Nevertheless, a combination of roles and a permutation group might be useful to give more insight in the specific structure of certain relations.

4.3 Pattern-based models

Our approach in identifying interesting types of positional models is to define their positional structure in terms of a structure that involves only their positions. For the permutation-based models we started with a permutation group on the positions P. As we have seen the permutation groups limit the class of positional models more than we would like. It seems a natural idea to investigate whether an elegant extension of the class of permutation-based models can be obtained by using a relation on P^P as a basis for the positional structure. This is what we are going to do in this section.

We will define so called *pattern-based models*. To make the definition easier to appreciate, we first give an example.

Example 4.3.1. Let $\Re$ be the previously discussed relation in which $\Re abc$ represents the state that a loves b, and b loves c. Let $\mathcal{M} = \langle D, P, S, \Gamma, H \rangle$ be a positional model for $\Re$, with $P = \{p_1, p_2, p_3\}$. $\mathcal{M}$ has a special kind of symmetry, namely $\Gamma aba = \Gamma bab$. As we already saw, because of this kind of symmetry, $\mathcal{M}$ is not a permutation-based model. But we can describe this symmetry in terms of the following pair of functions:

$$(\begin{pmatrix} p_1 & p_2 & p_3 \\ p_1 & p_2 & p_1 \end{pmatrix}, \begin{pmatrix} p_1 & p_2 & p_3 \\ p_2 & p_1 & p_2 \end{pmatrix}).$$

The interpretation of this pair of functions is that for any $a, b \in D$ we have

$$\Gamma(\begin{pmatrix} p_1 & p_2 & p_3 \\ a & b & a \end{pmatrix}) = \Gamma(\begin{pmatrix} p_1 & p_2 & p_3 \\ b & a & b \end{pmatrix}).$$

If we take the complete set of all pairs of functions of a similar form that are valid for $\mathcal{M}$, we get what we will call the *pattern* of $\mathcal{M}$. Because the positional structure of $\mathcal{M}$ can in this case be completely described by its pattern, we call the model a *pattern-based model.*

Now we give a formal definition of a pattern and a pattern-based model. Instead of defining a pattern as a relation on P^P, as we did in the previous example, we define it as a relation on $(2 \times P)^P$. For reasons that will become clear later on in this section, this choice is more convenient, but not necessary. We will show that we could have defined the same class of pattern-based models also in terms of a relation on Q^P with Q an arbitrary set with $|Q| \geq \mathsf{max}\{\,|P|, 2\,\}$.

Definition 4.3.2. Let $\mathcal{M} = \langle D, P, S, \Gamma, H\rangle$ be a positional model, and let $E_{\mathcal{M}}$ be its positional structure. Then the *pattern* $C_{\mathcal{M}}$ is defined by:

$$C_{\mathcal{M}} := \{(\sigma, \sigma') \mid \forall h : (2 \times P) \rightarrow D\;[\,(h \circ \sigma)\; E_{\mathcal{M}}\; (h \circ \sigma')\,]\}.$$

We call $\mathcal{M}$ a *pattern-based model* if

$$E_{\mathcal{M}} = \{(h \circ \sigma, h \circ \sigma') \mid h : (2 \times P) \rightarrow D \quad \& \quad \sigma\; C_{\mathcal{M}}\; \sigma'\,\}.$$

Remark 4.3.3. Instead of saying that $C_{\mathcal{M}}$ is a relation on $((2 \times P)^P$, we could of course also have said that $C_{\mathcal{M}}$ is a relation on $(P \oplus P)^P$, with $\oplus$ the disjoint union operator. Further, we can already see an argument for choosing the pattern to be relation on $(2 \times P)^P$, namely this choice guarantees that for every pair $f, g : P \rightarrow D$, there is always a 'factorization' $f = h \circ \sigma \quad \& \quad g = h \circ \sigma'$.

As we will see later, the pattern-based models are very interesting from a metaphysical perspective, since many, if not all, relations can be modeled by a pattern-based model. Clearly, not all positional models are pattern-based. Take for example a model $\mathcal{M}$ for which $\Gamma ab = \Gamma ba$, but *not* $\Gamma ac = \Gamma ca$. Then $\mathcal{M}$ is not pattern-based. For a full appreciation of the pattern-based models, we will look at them in this section in more detail from a technical perspective. The next lemmas state some basic properties of patterns and pattern-based models.

Lemma 4.3.4. *Let $\mathcal{M} = \langle D, P, S, \Gamma, H\rangle$ be a positional model. Then its pattern, $C_{\mathcal{M}}$ is an equivalence relation that respects substitution, i.e.*

$$\sigma\; C_{\mathcal{M}}\; \sigma' \Rightarrow \forall \tau : (2 \times P) \rightarrow (2 \times P)\;[\,(\tau \circ \sigma)\; C_{\mathcal{M}}\; (\tau \circ \sigma')].$$

Proof. That $C_{\mathcal{M}}$ is an equivalence relation, follows directly from the definition of a pattern and from the fact that the positional structure $E_{\mathcal{M}}$ is an equivalence relation.

To prove that $C_{\mathcal{M}}$ respects substitution, assume $\sigma\; C_{\mathcal{M}}\; \sigma'$. Let τ be a function from $(2\times P)$ to $(2\times P)$. Since $\forall f : (2\times P) \rightarrow D\;[\,(f\circ\sigma)\; C_{\mathcal{M}}\; (f\circ\sigma)\,]$, we have in particular $\forall f : (2 \times P) \rightarrow D\;[\,(f \circ \tau \circ \sigma)\; C_{\mathcal{M}}\; (f \circ \tau \circ \sigma)\,]$. Thus, by the definition of a pattern, $(\tau \circ \sigma)\; C_{\mathcal{M}}\; (\tau \circ \sigma')$. ⊣

Definition 4.3.5. Let $\mathcal{M} = \langle D, P, S, \Gamma, H \rangle$ be a positional model. Let $C_{\mathcal{M}}$ be its pattern. Define $E_{C_{\mathcal{M}}}$ as the relation on D^P *induced by* $C_{\mathcal{M}}$, i.e.:

$$E_{C_{\mathcal{M}}} \;:=\; \{(h \circ \sigma, h \circ \sigma') \mid h : (2 \times P) \to D \quad \& \quad \sigma \, C_{\mathcal{M}} \, \sigma'\}.$$

Lemma 4.3.6. *Let $\mathcal{M}$ be a positional model. Let $E_{\mathcal{M}}$ be its positional structure, let $C_{\mathcal{M}}$ be its pattern, and let $E_{C_{\mathcal{M}}}$ be the relation on D^P induced by $C_{\mathcal{M}}$. Then:*

- *$E_{C_{\mathcal{M}}}$ is an equivalence relation.*
- *$E_{C_{\mathcal{M}}} \subseteq E_{\mathcal{M}}$.*
- *$\mathcal{M}$ is pattern-based iff $E_{C_{\mathcal{M}}} = E_{\mathcal{M}}$.*

Proof. The proof follows immediately from the definitions. ⊣

The next lemma states a relation between $E_{\mathcal{M}}$ and $C_{\mathcal{M}}$ that is complementary to the relation given in the definition of pattern-based models.

Lemma 4.3.7. *Let $\mathcal{M} = \langle D, P, S, \Gamma, H \rangle$ be a positional model. Let $E_{\mathcal{M}}$ be its positional structure, and let $C_{\mathcal{M}}$ be its pattern. Then $\mathcal{M}$ is pattern-based iff*

$$E_{\mathcal{M}} = \; \{(f, g) \mid \forall \mu : D \to (2 \times P) \; [\; (\mu \circ f) \, C_{\mathcal{M}} \, (\mu \circ g) \;]\}.$$

Proof. We first show that for any positional model

$$E_{\mathcal{M}} \supseteq \; \{(f, g) \mid \forall \mu : D \to (2 \times P) \; [\; (\mu \circ f) \, C_{\mathcal{M}} \, (\mu \circ g) \;]\}.$$

Let f, g be such that $\forall \mu : D \to (2 \times P) \; [(\mu \circ f) \, C_{\mathcal{M}} \, (\mu \circ g)]$. Choose an injective function $\mu : (\mathsf{im}\ f \cup \mathsf{im}\ g) \rightarrowtail (2 \times P)$. Then $(\mu \circ f) \, C_{\mathcal{M}} \, (\mu \circ g)$. Because of the injectivity of μ, there is a function h such that $f = h \circ \mu \circ f$, and $g = h \circ \mu \circ g$. Thus, by the definition of a pattern, we have $f \, E_{\mathcal{M}} \, g$.

Now assume $\mathcal{M}$ is pattern-based. Further, assume $f \, E_{\mathcal{M}} \, g$. Then for some h, σ, σ', we have $f = h \circ \sigma$, $g = h \circ \sigma'$, and $\sigma \, C_{\mathcal{M}} \, \sigma'$. Then, as we showed in Lemma 4.3.4, $\forall \tau : (2 \times P) \to (2 \times P) \; [\tau \circ \sigma \, C_{\mathcal{M}} \, \tau \circ \sigma')]$. So, in particular, $\forall \mu : D \to (2 \times P) \; [(\mu \circ f) \, C_{\mathcal{M}} \, (\mu \circ g)]$.

To prove the other direction of the theorem, assume that for any f, g we have $f \, E_{\mathcal{M}} \, g \Rightarrow \forall \mu : D \to (2 \times P) \; [(\mu \circ f) \, C_{\mathcal{M}} \, (\mu \circ g)]$. Like before, we can choose an injective function $\mu : (\mathsf{im}\ f \cup \mathsf{im}\ g) \rightarrowtail (2 \times P)$. So, for some function h, we have $f = h \circ \mu \circ f$, and $g = h \circ \mu \circ g$. Further, if $f \; E_{\mathcal{M}} \; g$, then, by our assumption, $(\mu \circ f) \; C_{\mathcal{M}} \; (\mu \circ g)$. Thus, by looking at the definition of a pattern-based model, we see that $\mathcal{M}$ is pattern-based. $\dashv$

We now prove that the class of pattern-base models is indeed more general than the class of permutation-based models.

Theorem 4.3.8. *Permutation-based models* $\subsetneq$ *Pattern-based models.*

Proof. Let $\mathcal{M} = \langle D, P, S, \Gamma, H \rangle$ be a permutation-based model. Let $E_{\mathcal{M}}$ be its positional structure, let $G_{\mathcal{M}}$ be its permutation group, and let $C_{\mathcal{M}}$ be its pattern. Consider f and g in D^P. Suppose $f \; E_{\mathcal{M}} \; g$. Because $\mathcal{M}$ is permutation-based, we have $g = f \circ \pi$ for some $\pi \in G_{\mathcal{M}}$. Therefore, $\forall f' \; [\, f' \; E_{\mathcal{M}} \; (f' \circ \pi) \,]$. Define $\mathsf{emb} : P \to 2 \times P$ by $\mathsf{emb}(p) := (0, p)$. Then we obviously have $(\mathsf{emb}, \mathsf{emb} \circ \pi) \in C_{\mathcal{M}}$. So,

$$\exists (\sigma, \sigma') \in C_{\mathcal{M}} \; \exists h \; [\, f = h \circ \sigma \quad \& \quad g = h \circ \sigma' \,],$$

which proves that $\mathcal{M}$ is pattern-based.

With pattern based models certain symmetries can be modeled, that cannot be modeled with permutation-based models. As we saw in example 4.3.1, we can construct a pattern-based model for which $\Gamma aba = \Gamma bab$, with $a \neq b$. But no permutation-based model has this property, because if $\mathcal{M}$ is permutation-based, then $f \; E_{\mathcal{M}} \; g \Rightarrow \forall d \in D \; [\; |\{p \mid f(p) = d\}| = |\{p \mid g(p) = d\}| \;]$. $\dashv$

In the next example, we define another pattern-based model that is not permutation-based. For this model any assignment of objects to its positions yields the same state.

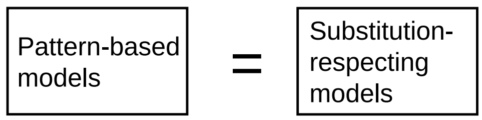
Example 4.3.9. Let $\mathcal{M} = \langle D, P, S, \Gamma, H \rangle$ be a positional model for which $\forall f, g \; [\Gamma(f) = \Gamma(g)]$. Then the pattern $C_{\mathcal{M}}$ contains all pairs of functions from P to $2 \times P$. Now let f, g be arbitrary functions from P to D. Then it is easy to see that $\exists (\sigma, \sigma') \in C_{\mathcal{M}} \; \exists h \; [\, f = h \circ \sigma \quad \& \quad g = h \circ \sigma' \,]$. So, $\mathcal{M}$ is pattern-based. But, if $|D| > 1$, then $\mathcal{M}$ is not permutation-based, because for any permutation-based model, we have $f \; E_{\mathcal{M}} \; g \Rightarrow \mathsf{im} \; f = \mathsf{im} \; g$.

The next definition is important, because as we will show, it gives an elegant characterization of the pattern-based models.

Definition 4.3.10. We say that a positional model $\mathcal{M}$ *respects substitution* if

$$\forall f, g\ [f\ E_{\mathcal{M}}\ g \Rightarrow \forall \delta : D \to D\ [\,(\delta \circ f)\ E_{\mathcal{M}}\ (\delta \circ g)\,]].$$

Theorem 4.3.11. *A positional model is a pattern-based model iff it respects substitution. In other words:*



Proof. Let $\mathcal{M} = \langle D, P, S, \Gamma, H \rangle$ be a positional model. Let $E_{\mathcal{M}}$ be its positional structure, and let $C_{\mathcal{M}}$ be its pattern.

Suppose that $\mathcal{M}$ is pattern-based and that $f\ E_{\mathcal{M}}\ g$. So, by the definition of a pattern-based model, for some $(\sigma, \sigma') \in C_{\mathcal{M}}$, $\exists h\ [\, f = h \circ \sigma \quad \& \quad g = h \circ \sigma'\,]$. By definition of a pattern, $\forall h : (2 \times P) \to D\ [\,(h \circ \sigma)\ E_{\mathcal{M}}\ (h \circ \sigma')\,]$. Thus, in particular, $\forall \delta : D \to D\ [\,(\delta \circ f)\ E_{\mathcal{M}}\ (\delta \circ g)\,]$. So, $\mathcal{M}$ respects substitution.

To prove the other direction of the theorem, consider f and g in D^P. There is an injection $j : \mathsf{im}\ f \cup \mathsf{im}\ g \to 2 \times P$. Let j^+ extend this injection to $D \to 2 \times P$. Find $h : 2 \times P \to D$ such that $h \circ j$ is the identical embedding emb of $\mathsf{im}\ f \cup \mathsf{im}\ g$ in D. We define $\sigma := j \circ f$, $\sigma' := j \circ g$. So we have

$$h \circ \sigma = h \circ j \circ f = \mathsf{emb} \circ f = f.$$

Similarly for σ' and g.

Now suppose that $\mathcal{M}$ respects substitution and that $f\ E_{\mathcal{M}}\ g$. It is sufficient to show that $\sigma\ C_{\mathcal{M}}\ \sigma'$. In other words, it is sufficient to show that for every $h' : 2 \times P \to D$, $h' \circ \sigma\ E_{\mathcal{M}}\ h' \circ \sigma'$. We have

$$h' \circ \sigma = h' \circ j \circ f = (h' \circ j^+) \circ f \;\; E_{\mathcal{M}} \;\; (h' \circ j^+) \circ g = h' \circ j \circ g = h' \circ \sigma'.$$

So, $\mathcal{M}$ is pattern-based. ⊣

Instead of defining a pattern as a relation on $(2 \times P)^P$, we could also have defined it as a relation on Q^P, with Q some set for which $|Q| \geq 2|P|$. However, this would not increase the expressive power of the models. This can be seen as follows. Suppose we have an *extended pattern-based model* with an ‘extended’ pattern $C^{+}_{\mathcal{M}}$ on Q^P, then a straightforward proof shows that this model respects substitution. But, by Theorem 4.3.11, the model is then also a pattern-based model. So, we win nothing with extended patterns.

Now the question might come up whether a ‘reduced’ pattern $C^{-}_{\mathcal{M}}$ on Q^P with $|Q| < 2|P|$ has the same expressive power. This is indeed the case, as the next theorem shows.

Theorem 4.3.12. *Let $\mathcal{M} = \langle D, P, S, \Gamma, H\rangle$ be a positional model for which $|P| \geq 2$. Let $E_{\mathcal{M}}$ be its positional structure, and let $C^{-}_{\mathcal{M}}$ be defined by:*

$$C^{-}_{\mathcal{M}} := \{(\sigma, \sigma') \mid \forall f : P \to D\, [\, (f \circ \sigma)\; E_{\mathcal{M}}\; (f \circ \sigma')\,]\}.$$

Then $\mathcal{M}$ is pattern-based iff

$$E_{\mathcal{M}} = \{(f, g) \mid \forall \mu : D \to P\; [\; (\mu \circ f)\; C^{-}_{\mathcal{M}}\; (\mu \circ g)\;]\}.$$

Proof. We stipulate a standard identification of P with $\{0\} \times P$. Then, from the definition of $C_{\mathcal{M}}$ and of $C^{-}_{\mathcal{M}}$ it follows immediately that for any positional model:

$$\forall \sigma, \sigma' : P \to P\, [\, \sigma\; C^{-}_{\mathcal{M}}\; \sigma' \Leftrightarrow\; \sigma\; C_{\mathcal{M}}\; \sigma'\;].$$

We will show that for any positional model:

$$\forall \mu : D \to P\; [(\mu \circ f)\; C_{\mathcal{M}}\; (\mu \circ g)] \;\Rightarrow\; \forall \mu : D \to (2 \times P)\; [(\mu \circ f)\; C_{\mathcal{M}}\; (\mu \circ g)].$$

Then the theorem follows from Lemma 4.3.7.

Let f, g be functions for which $\forall \mu : D \to P\; [(\mu \circ f)\; C_{\mathcal{M}}\; (\mu \circ g)]$, and let μ_0 be an arbitrary function from D to $(2 \times P)$. Define $m := |\mathsf{im}\ f \cup \mathsf{im}\ g|$. We will show by induction on m that $(\mu_0 \circ f)\; C_{\mathcal{M}}\; (\mu_0 \circ g)$.

Basis. Assume $m \leq |P|$. Then also $|\mathsf{im}\ (\mu_0 \circ f) \cup \mathsf{im}\ (\mu_0 \circ g))| \leq |P|$. So, we can choose an injective function $\tau : (\mathsf{im}\ (\mu_0 \circ f) \cup \mathsf{im}\ (\mu_0 \circ g)) \rightarrowtail P$. Because $(\tau \circ \mu_0 \circ f)\; C_{\mathcal{M}}\; (\tau \circ \mu_0 \circ g)$ and because $C_{\mathcal{M}}$ respects substitution, it follows that $(\mu_0 \circ f)\; C_{\mathcal{M}}\; (\mu_0 \circ g)$.

$m \Rightarrow m + 1$. Assume $|\mathsf{im}\ f \cup \mathsf{im}\ g| = m + 1 > |P|$. Then, because $|P| \geq 2$, we can define a function $\delta : D \to D$ such that

- $\delta \circ f = f$, and
- $|\mathsf{im}\ f \cup \mathsf{im}\ (\delta \circ g)| = m$, and

- $|\mathsf{im}\ (\delta \circ g) \cup \mathsf{im}\ g| \leq |P|$.

Because $\forall \mu : D \to P\ [(\mu \circ f)\ C_{\mathcal{M}}\ (\mu \circ g)]$, and because $\delta \circ f = f$, we have $\forall \mu : D \to P\ [(\mu \circ f)\ C_{\mathcal{M}}\ (\mu \circ \delta \circ g)]$. So, by the induction hypothesis, $(\mu_0 \circ f)\ C_{\mathcal{M}}\ (\mu_0 \circ \delta \circ g)$. Further, because $C_{\mathcal{M}}$ is an equivalence relation, we also have $\forall \mu : D \to P\ [(\mu \circ g)\ C_{\mathcal{M}}\ (\mu \circ \delta \circ g)]$. So, by the basis step, $(\mu_0 \circ g)\ C_{\mathcal{M}}\ (\mu_0 \circ \delta \circ g)$. By again using the fact that $C_{\mathcal{M}}$ is an equivalence relation, we see that $(\mu_0 \circ f)\ C_{\mathcal{M}}\ (\mu_0 \circ g)$. ⊣

Remark 4.3.13. If $\forall (f, g) \in E_{\mathcal{M}}\ [\ |\mathsf{im}\ f \cup \mathsf{im}\ g| \leq |P|\ ]$, then we can prove that $\mathcal{M}$ is pattern-based iff

$$\forall (f, g) \in E_{\mathcal{M}}\ \exists (\sigma, \sigma') \in C^{-}_{\mathcal{M}}\ \exists h\ [\ f = h \circ \sigma \quad \& \quad g = h \circ \sigma'\ ].$$

But, if for some $(f, g) \in E_{\mathcal{M}}$ we have $|\mathsf{im}\ f \cup \mathsf{im}\ g| > |P|$, then obviously this is *not* true. Therefore, for the general case it seems more natural to define pattern-based models in terms of $C_{\mathcal{M}}$.

We can also prove that $\mathcal{M}$ is pattern-based iff $E_{\mathcal{M}}$ is the smallest equivalence relation for which $\forall (\sigma, \sigma') \in C^{-}_{\mathcal{M}}\ \forall f : P \to D\ [(f \circ \sigma)\ E_{\mathcal{M}}\ (f \circ \sigma')]$. But also this characterization seems less convenient as the definition we gave for pattern-based models.

A further reduction of pattern to relations on Q^P with $|Q| < |P|$, is in general not possible without loss of expressive power. For example, if $\mathcal{M}$ models a cyclic relation, then it is easy to see that $|Q|$ has to be at least $|P|$.

The next example illustrates the relation between $C_{\mathcal{M}}$ and $C^{-}_{\mathcal{M}}$.

Example 4.3.14. Let $\mathcal{M} = \langle D, P, S, \Gamma, H \rangle$ be a positional model with $P = \{p_1, p_2\}$ and reduced pattern $C^{-}_{\mathcal{M}} = \{(\sigma, \sigma') \mid \sigma, \sigma' : P \to P\}$. Then, for any $a, b, c, d \in D$, we have

$$\Gamma(\begin{pmatrix} p_1 & p_2 \\ a & b \end{pmatrix}) = \Gamma(\begin{pmatrix} p_1 & p_2 \\ a & a \end{pmatrix}) = \Gamma(\begin{pmatrix} p_1 & p_2 \\ c & c \end{pmatrix}) = \Gamma(\begin{pmatrix} p_1 & p_2 \\ c & d \end{pmatrix}).$$

So, for any $f, g \in D^P$, we have $f\ E_{\mathcal{M}}\ g$. It follows that for the pattern $C_{\mathcal{M}}$ we have $C_{\mathcal{M}} = \{(\sigma, \sigma') \mid \sigma, \sigma' : P \to (2 \times P)\}$. This corresponds to what we showed in the proof of Theorem 4.3.12. Note that from the specification of $C_{\mathcal{M}}$ it directly follows that for any $f, g \in D^P$, $f\ E_{\mathcal{M}}\ g$.

In the next section we will discuss a variant of the pattern-based models with more expressive power.

4.4 Refined pattern-based models

On a first reading this section could be skipped, since the refined pattern-based models do no play a major role in the rest of this thesis.

With the pattern-based models as we defined them, it is not possible to model relations for which $\Re abc$ and $\Re cba$ represent the same state *iff* a and b are not equal. But we can model such relations by making a refinement in the definition of the pattern-based models. We can define *refined* patterns by considering *injective* functions from subsets of $(2 \times P)$ to D.

Definition 4.4.1. Let $\mathcal{M} = \langle D, P, S, \Gamma, H \rangle$ be a positional model, and let $E_{\mathcal{M}}$ be its positional structure. Then the *refined pattern* $C^{\diamond}_{\mathcal{M}}$ is defined by:

$$\begin{aligned} C^{\diamond}_{\mathcal{M}} := \; &\{(\sigma, \sigma') \mid \sigma, \sigma' : P \to 2 \times P \quad \& \\ &\forall h : (\mathsf{im}\ \sigma \cup \mathsf{im}\ \sigma') \rightarrowtail D\ [\,(h \circ \sigma)\ E_{\mathcal{M}}\ (h \circ \sigma')\ ]\}. \end{aligned}$$

We call $\mathcal{M}$ a *refined pattern-based model* if

$$E_{\mathcal{M}} = \; \{(h \circ \sigma, h \circ \sigma') \mid \sigma\ C^{\diamond}_{\mathcal{M}}\ \sigma' \quad \& \quad h : (\mathsf{im}\ \sigma \cup \mathsf{im}\ \sigma') \rightarrowtail D\}.$$

Remark 4.4.2. In this definition we take for h as domain $\mathsf{im}\ \sigma \cup \mathsf{im}\ \sigma'$, and not $2 \times P$, to account for the cases that $|D| < 2|P|$.

It is rather straightforward to formulate lemmas that are variants of the lemmas we gave for the regular pattern-based models. As might be expected, the proofs are also very similar. Therefore, we do not present these lemmas here.

Theorem 4.4.3. *Pattern-based models* $\subsetneqq$ *Refined pattern-based models.*

Proof. Let $\mathcal{M} = \langle D, P, S, \Gamma, H \rangle$ be a pattern-based model. Let $E_{\mathcal{M}}$ be its positional structure, let $C_{\mathcal{M}}$ be its pattern, and let $C^{\diamond}_{\mathcal{M}}$ be its refined pattern. Let f, g be arbitrary functions from P to D. Then, there are functions $\sigma, \sigma' : P \to D$, and an injective function $h : (\mathsf{im}\ \sigma \cup \mathsf{im}\ \sigma') \rightarrowtail D$ such that $f = h \circ \sigma$ and $g = h \circ \sigma'$. Now assume $f\ E_{\mathcal{M}}\ g$. Then, as we will show, $\sigma\ C_{\mathcal{M}}\ \sigma'$. Because obviously $C_{\mathcal{M}} \subseteq C^{\diamond}_{\mathcal{M}}$, it follows that $\mathcal{M}$ is refined pattern-based.

We now show that $\sigma\ C_{\mathcal{M}}\ \sigma'$. Because $(h \circ \sigma)\ E_{\mathcal{M}}\ (h \circ \sigma')$, we have functions h_1, σ_1, and σ'_1 such that $h \circ \sigma = h_1 \circ \sigma_1$, $h \circ \sigma' = h_1 \circ \sigma'_1$, and $\sigma_1\ C_{\mathcal{M}}\ \sigma'_1$.

It is easy to see that there is a function τ such that $h_1 = h \circ \tau$. Because h is injective, it follows that $\sigma = \tau \circ \sigma_1$ and $\sigma' = \tau \circ \sigma'_1$. Because, by Lemma 4.3.4, $C_{\mathcal{M}}$ respects substitution, we have $(\tau \circ \sigma_1)\ C_{\mathcal{M}}\ (\tau \circ \sigma'_1)$. Therefore, $\sigma\ C_{\mathcal{M}}\ \sigma'$.

To show that there are refined pattern-based model that are not pattern-based, consider a model $\mathcal{M}$ for which the positional structure $E_{\mathcal{M}}$ is such that $f\ E_{\mathcal{M}}\ g \Leftrightarrow |\mathsf{im}\ f| = |\mathsf{im}\ g|$. It is not difficult to see that $\mathcal{M}$ is a refined pattern-based model with $\sigma\ C^{\diamond}_{\mathcal{M}}\ \sigma' \Leftrightarrow |\mathsf{im}\ \sigma| = |\mathsf{im}\ \sigma'|$. But $\mathcal{M}$ is not pattern-based if the domain of $\mathcal{M}$ has at least 2 elements, because then $\mathcal{M}$ clearly does not respect substitution. ⊣

It is natural to refine also the notion of substitution and to introduce the notion of 'injective substitution'.

Definition 4.4.4. We say that a positional model $\mathcal{M}$ *respects injective substitution* if

$$\forall f, g\ [f\ E_{\mathcal{M}}\ g \Rightarrow \forall \delta : D \rightarrowtail D\ [\delta \circ f\ E_{\mathcal{M}}\ \delta \circ g]].$$

What makes the class of refined pattern-based models interesting is in particular the following refined version of a similar theorem for regular pattern-based models.

Theorem 4.4.5. *A positional model is a refined pattern-based model iff it respects injective substitution.*

Proof. The proof is similar to the proof of Theorem 4.3.11. Let $\mathcal{M} = \langle D, P, S, \Gamma, H\rangle$ be a positional model. Let $E_{\mathcal{M}}$ be its positional structure, and let $C_{\mathcal{M}}$ be its pattern.

Suppose $\mathcal{M}$ is a refined pattern-based model and that $f\ E_{\mathcal{M}}\ g$. So, for some $(\sigma, \sigma') \in C^{\diamond}_{\mathcal{M}}$, $\exists h : (\mathsf{im}\ \sigma \cup \mathsf{im}\ \sigma') \rightarrowtail D\ [\, f = h \circ \sigma \ \ \&\ \ g = h \circ \sigma'\ ]$. By definition of a refined pattern, $\forall h : (\mathsf{im}\ \sigma \cup \mathsf{im}\ \sigma') \rightarrowtail D\ [\,(h \circ \sigma)\ E_{\mathcal{M}}\ (h \circ \sigma')\,]$. Then, in particular, $\forall \delta : D \rightarrow D\ [\,(\delta \circ f)\ E_{\mathcal{M}}\ (\delta \circ g)\ ]$. So, $\mathcal{M}$ respects injective substitution.

To prove the other direction of the theorem, consider f and g in D^P. There is an injection $j : \mathsf{im}\ f \cup \mathsf{im}\ g \rightarrow 2 \times P$. Find $h : 2 \times P \rightarrow D$ such that $h \circ j$ is the identical embedding emb of $\mathsf{im}\ f \cup \mathsf{im}\ g$ in D. We define $\sigma := j \circ f$, $\sigma' := j \circ g$. So we have

$$h \circ \sigma = h \circ j \circ f = \mathsf{emb} \circ f = f.$$

Similarly for σ' and g.

Now suppose that $\mathcal{M}$ respects injective substitution and that $f\ E_{\mathcal{M}}\ g$. It is sufficient to show that $\sigma\ C^{\diamond}_{\mathcal{M}}\ \sigma'$. In other words, it is sufficient to show that for every injective function $h' : (\mathsf{im}\ \sigma \cup \mathsf{im}\ \sigma') \rightarrowtail D$, $h' \circ \sigma\ E_{\mathcal{M}}\ h' \circ \sigma'$. Let $\delta : D \rightarrowtail D$ be an injective extension of $h' \circ j$. We have

$$h' \circ \sigma = h' \circ j \circ f = \delta \circ f \ \ E_{\mathcal{M}} \ \ \delta \circ g = h' \circ j \circ g = h' \circ \sigma'.$$

So, $\mathcal{M}$ is refined pattern-based. $\dashv$

As for the regular pattern-based models, the expressive power does not increase by extending the refined patterns to binary relations on Q^P, with $|Q| \geq 2|P|$. The argument is similar. Suppose we defined refined extended pattern $C^{+\diamond}_{\mathcal{M}}$ on Q^P, then a straightforward induction shows that a *refined extended pattern-based model* respects injective substitution. But, by Theorem 4.4.5 the model is then also a refined pattern-based model.

To define types with more expressive power than the refined pattern-based models, we could generalize the definition of a refined pattern. We could define pattern-like structures on $(2 \times P)^P$ as follows:

$$C^{\circledast}_{\mathcal{M}} := \{(\sigma, \sigma') \mid \forall h : (\mathsf{im}\ \sigma \cup \mathsf{im}\ \sigma') \rightarrow D\ [\ \mathcal{R}(h) \Rightarrow (h \circ \sigma)\ E_{\mathcal{M}}\ (h \circ \sigma')\ ]\}.$$

If we define the predicate $\mathcal{R}$ by: $\mathcal{R}(h) :\Leftrightarrow \forall p, p'\ [\ p \neq p' \Rightarrow h(p) \neq h(p')\ ]$, then $C^{\circledast}_{\mathcal{M}}$ is the refined pattern of $\mathcal{M}$. But we could also define $\mathcal{R}$ by: $\mathcal{R}(h) :\Leftrightarrow \exists p, p'\ [\ p \neq p' \ \ \&\ \ h(p) = h(p')\ ]$. Or, if D is a partially ordered set, then we could define $\mathcal{R}$ by: $\mathcal{R}(h) :\Leftrightarrow \exists p\, \forall p'\ [\ h(p) \leq h(p')\ ]$. In principle we could use the full language of predicate logic to specify the positional structure of different types of models, but I don't expect that this extra power is needed for 'real' relations.

4.5 Simple substitution models

In Section 3.4 we defined a substitution model just as a tuple $\langle D, S, \Sigma, H \rangle$ with Σ a function from $S \times D^D$ to S, with two natural constraints for substitution, namely $\forall s \in S\ [\ s \cdot \mathsf{id}_D = s\ ]$, and $\forall s\, \forall \delta, \delta'\ [\ s \cdot (\delta \cdot \delta') = (s \cdot \delta) \cdot \delta'\ ]$. However, we did not demand any kind of connectivity between different states. This probably makes this class of models too general to serve as an adequate class for 'real' relations. In this section, we consider a subtype that might perhaps be the appropriate class of models for all 'real' relations. We call the models of this subtype the *simple* substitution models.

Definition 4.5.1. Let $\mathcal{M} = \langle D, S, \Sigma, H \rangle$ be a substitution model. We call $\mathcal{M}$ a *simple substitution model* if:

$$\forall s, s' \in S\, \exists s''\, [\, s'' \twoheadrightarrow s \quad \& \quad s'' \twoheadrightarrow s' \,],$$

with $s'' \twoheadrightarrow s$ defined by $\exists \delta\, [\, s'' \cdot \delta = s \,]$.

Note that it would be a too restrictive constraint to demand that for any s, s' we have $s \twoheadrightarrow s'$ or $s' \twoheadrightarrow s$, in other words, that $\twoheadrightarrow$ is linear. For suppose we have the previously discussed relation $\Re$ with $\Re abc$ representing the state that a loves b and b loves c. Then neither $\Re aab \twoheadrightarrow \Re abb$ nor $\Re abb \twoheadrightarrow \Re aab$. The connectivity constraint given in the definition of a simple substitution model is somewhat milder.

Recall that in Section 3.4 we defined the objects of a state as the objects that make a noticeable contribution in some transition from s to another state, and that we defined the degree of a substitution model as the least upper bound of the degrees of its states, with the degree of a state as the greatest lower bound of the cardinality of its object-domains. We have for simple substitution models of finite degree the following basic fact.

Lemma 4.5.2. *If $\mathcal{M} = \langle D, S, \Sigma, H \rangle$ is a simple substitution model of finite degree, then:*

$$\forall s \in S\, [\, |\mathsf{Obj}(s)| = \mathsf{degree}_{\mathcal{M}} \Leftrightarrow \forall s' \in S\, [\, s \twoheadrightarrow s' \,]].$$

Proof. Let s be a state such that $|\mathsf{Obj}(s)| = \mathsf{degree}_{\mathcal{M}}$, and let s' be an arbitrary state in S. Then, because $\mathcal{M}$ is a simple substitution model, there is a state s'' such that $s'' \twoheadrightarrow s$ and $s'' \twoheadrightarrow s'$. Now let δ be such that $s'' \cdot \delta = s$. Then, by Lemma 3.4.5, we have $\mathsf{Obj}(s) \subseteq \overline{\delta}(\mathsf{Obj}(s''))$. Because the size of $\mathsf{Obj}(s)$ is maximal, δ must be bijective from $\mathsf{Obj}(s'')$ to $\mathsf{Obj}(s)$. Since the relevant part of δ has domain $\mathsf{Obj}(s'')$, there is a function δ' such that $s \cdot \delta' = s''$. So, we see that $s \twoheadrightarrow s''$. Because we also have $s'' \twoheadrightarrow s'$, it follows that $s \twoheadrightarrow s'$.

To prove the other direction, let s be such that $\forall s' \in S\, [\, s \twoheadrightarrow s' \,]$. Let s' be a state for which $|\mathsf{Obj}(s')| = \mathsf{degree}_{\mathcal{M}}$. Let δ be such that $s \cdot \delta = s'$. Then, again by Lemma 3.4.5, we have $\mathsf{Obj}(s') \subseteq \overline{\delta}(\mathsf{Obj}(s))$. It follows immediately that $|\mathsf{Obj}(s)| = \mathsf{degree}_{\mathcal{M}}$. ⊣

The previous lemma is clearly in general not true if the degree of $\mathcal{M}$ is infinite. (Take for example $D := \omega$ and $S := \omega^\omega$.) A nice generalization of the lemma could probably be obtained if we would define occurrences of objects in states. Then, we might expect that for an arbitrary simple substitution model the states in which no individual occurs more than once are exactly the states s for which $\forall s' \in S\,[\, s \twoheadrightarrow s'\,]$.

We argued in Section 4.5 that reconstructing positions in a straightforward way seems in certain cases problematic for substitution models. But *roles* can be defined in a direct and satisfactory manner for simple substitution models of finite degree.

Definition 4.5.3. Let $\mathcal{M} = \langle D, S, \Sigma, H\rangle$ be a substitution model. Let a, b be elements of D, and s, t be elements of S. Then we say that *a in s has at least the same roles as b in t* if $\exists\delta\,[\, t\cdot\delta = s \quad \& \quad \delta(b) = a\,]$.

If $\mathcal{M}$ is a simple substitution model of finite degree, then for $s \in S$ with $|\mathsf{Obj}(s)| = \mathsf{degree}_\mathcal{M}$ and $a \in \mathsf{Obj}(s)$, we define the *role of a in s* as

$$\mathsf{Role}(s, a) := \{(t, b) \mid \exists\delta\,[\, t\cdot\delta = s \quad \& \quad \delta(b) = a\,].$$

Further, we define the *roles of $\mathcal{M}$* as

$$\mathsf{Roles}_\mathcal{M} := \{\mathsf{Role}(s, a) \mid |\mathsf{Obj}(s)| = \mathsf{degree}_\mathcal{M} \quad \& \quad a \in \mathsf{Obj}(s)\}.$$

For arbitrary $s \in S$, $a \in D$, we say that *a in s fulfills role* $\mathsf{Role}(t, b) \in \mathsf{Roles}_\mathcal{M}$ if $\exists\delta\,[\, t\cdot\delta = s \quad \& \quad \delta(b) = a\,]$.

It is easy to see that if $\mathcal{M}$ is an n-ary simple substitution model, than $\mathcal{M}$ has at most n roles. Objects sometimes fulfill more than one role in certain states. For example, if $\mathcal{M}$ models the amatory relation, then in the state where Narcissus loves himself, he fulfills both roles of the model. It is also possible that an n-ary model with $n > 1$, has only one role. This is for example the case for cyclic models. In the next section (Theorem 4.6.11) we show a natural correspondence between the roles of substitution models and the roles of positional models.

4.6 Relating positional – and substitution models

In this section we present a main results of this thesis. We will show how strongly related positional models are with substitution models. Each simple substitution model of finite degree turns out to correspond in a well-defined sense to a unique positional model, modulo positional variants. Metaphysically this result is quite interesting. It gives a justification of using positional models for all relations that can adequately be represented by simple substitution models of finite degree, without any commitment to an ontology of positions. In other words, the use of positional models for such relations does not force us to accept positions as fundamental entities, because we can treat positions as 'light' products of our own mind. In Chapter 6 these metaphysical aspects will be discussed in more detail.

Definition 4.6.1. Let $\mathcal{M} = \langle D, S, \Sigma, H \rangle$ be a substitution model, and let $\mathcal{N} = \langle D, P, S, \Gamma, H \rangle$ be a positional model. Then we say that $\mathcal{M}$ *corresponds to* $\mathcal{N}$ if:

1. $\forall f \in D^P \, \forall \delta : D \to D \, [\, \Gamma(f) \cdot \delta = \Gamma(f \cdot \delta) \,]$,
2. $\mathsf{im}\ \Gamma = S$, and
3. $\mathsf{degree}_{\mathcal{M}} = \mathsf{degree}_{\mathcal{N}}$.

The first condition states that for any $\delta : D \to D$ the following diagram commutes:

$$\begin{array}{ccc} D^P & \xrightarrow{\ \overline{\delta}\ } & D^P \\ \Gamma \Big\downarrow & & \Big\downarrow \Gamma \\ S & \xrightarrow[\ \Sigma_\delta\ ]{} & S \end{array}$$

where $\overline{\delta}$ is defined by $\overline{\delta}(f) := f \cdot \delta$, and Σ_δ is defined by $\Sigma_\delta(s) := s \cdot \delta$.

It is not difficult to see that the three conditions are independent of each other. A useful property following from condition 1 is stated in the next lemma.

Lemma 4.6.2. *Let* $\mathcal{M} = \langle D, S, \Sigma, H \rangle$ *and* $\mathcal{N} = \langle D, P, S, \Gamma, H \rangle$ *be corresponding models. Then*

$$\forall f : P \to D \, [\, \mathsf{Obj}(\Gamma(f)) \subseteq \mathsf{im}\ f \,].$$

Proof. Consider $f \in D^P$. Obviously,

$$\forall \delta, \delta' \, [\, \delta =_{\mathsf{im}\ f} \delta' \Rightarrow \Gamma(f \cdot \delta) = \Gamma(f \cdot \delta') \,].$$

So, by condition 1 of the definition of corresponding models,

$$\forall \delta, \delta' \, [\, \delta =_{\mathsf{im}\ f} \delta' \Rightarrow \Gamma(f) \cdot \delta = \Gamma(f) \cdot \delta' \,].$$

This means that $\mathsf{im}\ f$ is an object-domain of $\Gamma(f)$ (cf. Definition 3.4.2). Because $\mathsf{im}\ f$ is a finite set, there is also a smallest object-domain of $\Gamma(f)$. By definition, $\mathsf{Obj}(\Gamma(f))$ is this smallest object-domain. ⊣

From the three conditions of the definition of correspondence follows another useful property:

Lemma 4.6.3. *Let $\mathcal{M} = \langle D, S, \Sigma, H \rangle$ and $\mathcal{N} = \langle D, P, S, \Gamma, H \rangle$ be corresponding models. Then*

$$\forall f : P \rightarrowtail D \, [\, \mathsf{Obj}(\Gamma(f)) = \mathsf{im}\ f \,].$$

Proof. Consider an injective function f from P to D. Let $s \in S$ be a state such that $|\mathsf{Obj}(s)| = \mathsf{degree}_{\mathcal{M}}$. Then, by condition 2 of the definition of correspondence, there is a function g such that $\Gamma(g) = s$. Because f is injective, $g = f \cdot \delta$ for some δ. So, by Lemma 3.4.5, $\mathsf{Obj}(\Gamma(g)) \subseteq \overline{\delta}(\mathsf{Obj}(\Gamma(f)))$.

Because $|\mathsf{Obj}(\Gamma(g))| = \mathsf{degree}_{\mathcal{M}}$, it follows that $|\mathsf{Obj}(\Gamma(f))| = \mathsf{degree}_{\mathcal{M}}$. Thus, because $\mathsf{Obj}(\Gamma(f)) \subseteq \mathsf{im}\ f$ by Lemma 4.6.2, and because $\mathsf{degree}_{\mathcal{M}} = \mathsf{degree}_{\mathcal{N}}$ by condition 3, we see that $\mathsf{Obj}(\Gamma(f)) = \mathsf{im}\ f$. ⊣

Theorem 4.6.4. *For any substitution model $\mathcal{M}$ there is at most one positional model $\mathcal{N}$, modulo positional variants, that corresponds to $\mathcal{M}$.*

Proof. Assume that $\mathcal{N} = \langle D, P, S, \Gamma, H \rangle$ and $\mathcal{N}' = \langle D, P', S, \Gamma', H \rangle$ both correspond to $\mathcal{M}$. Choose $s \in S$ such that $|\mathsf{Obj}(s)| = \mathsf{degree}_{\mathcal{M}}$. By condition 2 of the definition of corresponding models, there are functions $f_1 \in D^P$ and $f'_1 \in D^{P'}$ such that $\Gamma(f_1) = \Gamma'(f'_1) = s$. Then, by Lemma 4.6.2, $\mathsf{Obj}(s) \subseteq \mathsf{im}\ f_1$ and $\mathsf{Obj}(s) \subseteq \mathsf{im}\ (f'_1)$. So, by condition 3 of the definition of corresponding models, $\mathsf{im}\ f_1 = \mathsf{im}\ f'_1$, and f_1, f'_1 are injective. Thus, there is a bijective mapping $\pi : P \to P'$ such that $f_1 = \pi \cdot f'_1$.

By condition 1 of the definition of corresponding models, it follows that $\forall \delta : D \to D \, [\, \Gamma'(f'_1 \cdot \delta) = \Gamma(f_1 \cdot \delta) \,]$. So, because $f_1 = \pi \cdot f'_1$, we see that

$\forall\delta : D \to D\ [\,\Gamma'(f'_1 \cdot \delta) = \Gamma(\pi \cdot f'_1 \cdot \delta)\,]$. Because f'_1 is injective, we have $\forall f' \in D^{P'}\ \exists\delta : D \to D\ [\,f' = f'_1 \cdot \delta\,]$. So, $\forall f' \in D^{P'}\ [\,\Gamma'(f') = \Gamma(f'_1 \cdot \delta)\,]$. Therefore, $\mathcal{N}$ and $\mathcal{N}'$ are positional variants. ⊣

Positional models, even pattern-based models, can have positions that play absolutely no role for the states assigned by the function Γ. We will call such positions *dummy positions*.

Definition 4.6.5. $\mathcal{N} = \langle D, P, S, \Gamma, H\rangle$ be a positional model. Then we call a position $p \in P$ a *dummy position* if

$$\forall f, g \in D^P\ [\,f =_{P-\{p\}} g \Rightarrow \Gamma(f) = \Gamma(g)\,].$$

In the next theorem we show that dummy positions do not occur in models corresponding with substitution models. Further we show that the corresponding positions models respect substitution. Recall that models that respect substitution are the models for which

$$\forall f, g\ [\,\Gamma(f) = \Gamma(g) \Rightarrow \forall\delta : D \to D\ [\,\Gamma(f \cdot \delta) = \Gamma(g \cdot \delta)\,]].$$

Also recall that the class of positional models that respect substitution is the class of pattern-based models.

Theorem 4.6.6. *Let* $\mathcal{M} = \langle D, S, \Sigma, H\rangle$ *be a substitution model, and let* $\mathcal{N} = \langle D, P, S, \Gamma, H\rangle$ *be a positional model. If* $\mathcal{M}$ *and* $\mathcal{N}$ *correspond, then:*

- $\mathcal{M}$ *is a simple substitution model,*
- $\mathcal{N}$ *respects substitution, and*
- $\mathcal{N}$ *has no dummy positions.*

Proof. First, we prove that $\mathcal{M}$ is a simple model. Let $f_0 \in D^P$ be such that $|\mathsf{im}\ f_0| = \mathsf{degree}_{\mathcal{N}}$. Such a function f_0 exists, since $|D| \geq \mathsf{degree}_{\mathcal{M}}$. So by condition 1 and 2 of the definition of corresponding models,

$$S = \mathsf{im}\ \Gamma = \{\Gamma(f_0 \cdot \delta) \mid \delta : D \to D\} = \{\Gamma(f_0) \cdot \delta \mid \delta : D \to D\}.$$

So, $\exists s\ [\,S = \{s \cdot \delta \mid \delta : D \to D\}$, which shows that $\mathcal{M}$ is a simple model.

Second, we prove that $\mathcal{N}$ respects substitution. Assume $\Gamma(f) = \Gamma(g)$. Then

$$\forall\delta : D \to D\ [\,\Gamma(f \cdot \delta) = \Gamma(f) \cdot \delta = \Gamma(g) \cdot \delta = \Gamma(g \cdot \delta)\,].$$

So, $\mathcal{N}$ respects substitution.

Finally, we prove that $\mathcal{N}$ has no dummy positions. Let p be a position in P, let $s \in S$ be such that $|\mathsf{Obj}(s)| = \mathsf{degree}_{\mathcal{M}}$, and let $f_1 \in D^P$ be such that $\Gamma(f_1) = s$. Then, by Lemma 4.6.2, $\mathsf{Obj}(s) \subseteq \mathsf{im}\ f_1$, and by condition 3 of the definition of correspondence, it follows that $\mathsf{Obj}(s) = \mathsf{im}\ f_1$ and that f_1 is injective. So, by Lemma 3.4.4,

$$\exists \delta, \delta' : D \to D\ [\ \delta =_{D-\{f_1(p)\}} \delta' \quad \& \quad s \cdot \delta \neq s \cdot \delta'\ ].$$

Thus by condition 1 of the definition of correspondence,

$$\exists \delta, \delta' : D \to D\ [\ \delta =_{D-\{f_1(p)\}} \delta' \quad \& \quad \Gamma(f_1 \cdot \delta) \neq \Gamma(f_1 \cdot \delta')\ ].$$

So, because f_1 is injective, it follows that p is no dummy position. ⊣

The next theorem shows that for simple substitution models, we can reconstruct indirectly the notion of positions in a satisfactory way.

Theorem 4.6.7. *For any simple substitution model $\mathcal{M}$ of finite degree there is exactly one positional model $\mathcal{N}$, modulo positional variants, that corresponds to $\mathcal{M}$.*

Proof. Assume $\mathcal{M}$ is $\langle D, S, \Sigma, H\rangle$. We construct a model $\mathcal{N} = \langle D, P, S, \Gamma, H\rangle$ that corresponds to $\mathcal{M}$ as follows:

1. Choose a state $s_0 \in S$ such that $|\mathsf{Obj}(s_0)| = \mathsf{degree}_{\mathcal{M}}$,
2. Define $P := \mathsf{Obj}(s_0)$,
3. Let f be an arbitrary element of D^P. Let f^+ extend f to $D \to D$. Define $\Gamma(f) := s_0 \cdot f^+$.

We first show that Γ is well-defined. We have to prove that $s_0 \cdot f^+$ is independent of the choice of f^+. But that is easy to see, since all extensions of f are identical on P and $P = \mathsf{Obj}(s_0)$.

To prove that $\mathcal{M}$ corresponds to $\mathcal{N}$, we now show that the three conditions of the definition of correspondence are fulfilled.

Condition 1 follows from:

$$\Gamma(f) \cdot \delta = s_0 \cdot f^+ \cdot \delta = \Gamma(f \cdot \delta).$$

Condition 2 follows from the following observations. From the definition of Γ it follows immediately that $\mathsf{im}\ \Gamma \subseteq S$. To prove that also $S \subseteq \mathsf{im}\ \Gamma$, let s be an arbitrary state in S. Then, by Lemma 4.5.2, for some δ we have $s = s_0 \cdot \delta$. Let f in D^P be such that δ extends f to $D \to D$. Then $\Gamma(f) = s_0 \cdot \delta = s$, which proves that $S \subseteq \mathsf{im}\ \Gamma$.

Condition 3 follows directly from steps 1 and 2 of the construction of $\mathcal{N}$.

This proves that $\mathcal{M}$ corresponds to $\mathcal{N}$. The uniqueness of $\mathcal{N}$, modulo positional variants, follows from Theorem 4.6.4. $\dashv$

The construction of the model $\mathcal{N}$ in Theorem 4.6.7 can be pictured as follows. Choose a state with a maximum number of objects. Draw circles around the objects of this state and give these circles unique labels. These labeled circles are the positions. If we substitute objects, we simply replace the objects in the circles. In some cases the positions of the objects will change, but not the state. As long as you do not consider the positions a genuine part of the states occupied by objects, there is nothing magical about this.

If we would allow positional models to have an infinite number of positions, then we could give a generalization of Theorem 4.6.7 for any simple substitution model with a state from which all other states can be derived.

We also have a converse of Theorem 4.6.7. Note that by Theorem 4.6.6 the positional model of Theorem 4.6.7 fulfills the conditions of the positional model of the next theorem.

Theorem 4.6.8. *Let $\mathcal{N} = \langle D, P, S, \Gamma, H\rangle$ be a positional model such that:*

- *$\mathcal{N}$ respects substitution,*
- $\mathsf{im}\ \Gamma = S$*,*
- $\mathsf{degree}_{\mathcal{N}} \leq |D|$*, and*
- *$\mathcal{N}$ has no dummy positions.*

Then there is exactly one substitution model that corresponds to $\mathcal{N}$.

Proof. Define $\mathcal{M} = \langle D, S, \Sigma, H\rangle$ by

$$\Gamma(f) \cdot \delta := \Gamma(f \cdot \delta).$$

Since $\mathcal{N}$ respects substitution, $\forall f, g\ [\Gamma(f) = \Gamma(g) \Rightarrow \Gamma(f \cdot \delta) = \Gamma(g \cdot \delta)]$. Therefore Σ is well-defined.

It is easy to see that $\mathcal{M}$ is a substitution model:

1. $\Gamma(f) \cdot \mathsf{id}_D = \Gamma(f \cdot \mathsf{id}_D) = \Gamma(f)$,
2. $\Gamma(f) \cdot (\delta \cdot \delta') = \Gamma(f \cdot \delta \cdot \delta') = \Gamma(f \cdot \delta) \cdot \delta'$,

We now show that $\mathcal{M}$ and $\mathcal{N}$ correspond. Condition 1 of the definition of corresponding models is trivially fulfilled by the definition of $\mathcal{M}$. Condition 2 is an explicit property of $\mathcal{N}$. So, we only have to verify condition 3.

Because condition 1 of the definition of corresponding models is fulfilled, we have $\forall f : P \to D\ [\,\mathsf{Obj}(\Gamma(f)) \subseteq \mathsf{im}\ f\,]$ (cf. Lemma 4.6.2). So, because $\mathsf{im}\ \Gamma = S$, it follows that $\mathsf{degree}_{\mathcal{M}} \leq \mathsf{degree}_{\mathcal{N}}$.

By the condition that $\mathsf{degree}_{\mathcal{N}} \leq |D|$, it follows that for some $f_0 \in D^P$ we have $|\mathsf{im}\ f_0| = \mathsf{degree}_{\mathcal{N}}$. Because f_0 is injective, and because $\mathcal{N}$ contains no dummy positions, it follows that $\mathsf{im}\ f_0 = \mathsf{Obj}(\Gamma(f_0))$. So, we also have $\mathsf{degree}_{\mathcal{N}} \leq \mathsf{degree}_{\mathcal{M}}$.

The uniqueness of $\mathcal{M}$ follows immediately from the fact that for any substitution model that corresponds to $\mathcal{M}$ we must have $\Gamma(f) \cdot \delta = \Gamma(f \cdot \delta)$ by condition 1 of the definition of corresponding models. ⊣

Remark 4.6.9. Note that if $\mathsf{degree}_{\mathcal{N}} > |D|$, then there cannot be a corresponding substitution model for $\mathcal{N}$, because the degree of a substitution model is at most $|D|$. Further, note that for $\mathcal{N}$ in the last theorem it is *not* necessarily true that $\forall f, g\ [\,\Gamma(f) = \Gamma(g) \Rightarrow \mathsf{im}\ f = \mathsf{im}\ g\,]$. For example, let $\mathcal{N}$ be a binary model with $|D| \geq 2$ and such that

$$\Gamma(f) = \Gamma(g) \Leftrightarrow (f = g \text{ or } \forall p, p' \in P\ [\,f(p) = f(p') \quad \& \quad g(p) = g(p')\,]).$$

Then $\mathcal{N}$ fulfills all the conditions of the model in the last theorem, but for completely different f and g we might have $\Gamma(f) = \Gamma(g)$. Also note that in the substitution model corresponding to this model, the set of objects of one of the states is empty. So, in particular for some f, $\mathsf{Obj}(\Gamma(f)) \subsetneq \mathsf{im}\ f$.

Remark 4.6.10. From the proof of Theorem 4.6.8 it follows that if $\mathcal{N}$ respects substitution and $\mathsf{im}\ \Gamma = S$, then there is a unique substitution model $\mathcal{M}$ such that the first two conditions of Definition 4.6.1 are fulfilled and $\mathsf{degree}_{\mathcal{M}} \leq \mathsf{degree}_{\mathcal{N}}$, but not necessarily $\mathsf{degree}_{\mathcal{M}} = \mathsf{degree}_{\mathcal{N}}$.

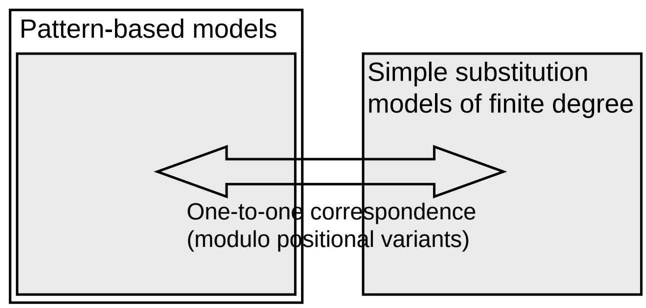


Figure 4.2: Relation between positional models and substitution models.

Theorem 4.6.7 and Theorem 4.6.8 together show that there is a fundamental one-to-one correspondence between the simple substitution models and a well-defined subclass of the positional models. The correspondence is depicted in Figure 4.2.

We could also define a weaker form of correspondence, namely by demanding that only for any *injective* function $\delta : D \rightarrowtail D$ the diagram in Definition 4.6.1 commutes. We can prove that if a substitution model corresponds in this weaker sense to a position-based model, then the position model is a refined pattern-based model. Unfortunately the weak correspondence is not a one-to-one correspondence: for one substitution model, there may be more than one weakly corresponding positional model, modulo positional variants.

In 4.1.1 we defined the roles of positional models, and in 4.5.3 we defined the roles of simple substitution models of finite degree. The next theorem shows that there is a natural one-to-one correspondence between the roles of corresponding models.

Theorem 4.6.11. *Let $\mathcal{M} = \langle D, S, \Sigma, H \rangle$ and $\mathcal{N} = \langle D, P, S, \Gamma, H \rangle$ be corresponding models. Let f_0 be an injective function from P to D. Then for any $p, p' \in P$, p has the same role as p' iff $f_0(p)$ in $\Gamma(f_0)$ has the same role as $f_0(p')$ in $\Gamma(f_0)$.*

Proof. Let us first recall a few definitions.

We say that *p has the same role as p'* if

$$\exists \sigma : P \rightarrowtail P \; [\, p' = \sigma(p) \quad \& \quad \forall f \in D^P \; [\, \Gamma(f) = \Gamma(\sigma \cdot f)\,]].$$

For a simple substitution model of finite degree, we defined for $s \in S$ with $|\mathsf{Obj}(s)| = \mathsf{degree}_{\mathcal{M}}$ and $a \in \mathsf{Obj}(s)$, the *role of a in s* as

$$\mathsf{Role}(s, a) := \{(t, b) \mid \exists \delta\, [\, t \cdot \delta = s \quad \& \quad \delta(b) = a\,].$$

Observe that if $\mathsf{Role}(s, a)$ is defined, then

$$\mathsf{Role}(s, a) = \mathsf{Role}(s', a') \Leftrightarrow \exists \delta : D \rightarrowtail D\, [\, s' = s \cdot \delta \quad \& \quad a' = \delta(a)\,].$$

Now suppose p has the same role as p'. So, for some $\sigma : P \rightarrowtail P$ we have $p' = \sigma(p)$ and $\forall f \in D^P\ [\, \Gamma(f) = \Gamma(\sigma \cdot f)\,]$. Because σ and f_0 are both injective, we have $\sigma \cdot f_0 = f_0 \cdot \delta$ for some injective function $\delta : D \to D$. So, $\Gamma(f_0) = \Gamma(f_0 \cdot \delta) = \Gamma(f_0) \cdot \delta$, and $f_0(p') = \delta(f_0(p))$. By Lemma 4.6.3, $|\mathsf{Obj}(\Gamma(f_0))| = \mathsf{degree}_{\mathcal{M}}$. This means that $\mathsf{Role}(\Gamma(f_0), f_0(p))$ is defined, and from the above observation it follows that

$$\mathsf{Role}(\Gamma(f_0), f_0(p)) = \mathsf{Role}(\Gamma(f_0) \cdot \delta, \delta(f_0(p))) = \mathsf{Role}(\Gamma(f_0), f_0(p')).$$

Conversely, suppose $\mathsf{Role}(\Gamma(f_0), f_0(p)) = \mathsf{Role}(\Gamma(f_0), f_0(p'))$. Then, for some $\delta : D \rightarrowtail D$, we have $\Gamma(f_0) \cdot \delta = \Gamma(f_0)$ and $\delta(f_0(p)) = f_0(p')$.

We claim that $\mathsf{im}\ f_0 = \overline{\delta}(\mathsf{im}\ f_0)$. By Lemma 3.4.5,

$$\mathsf{Obj}(\Gamma(f_0)) \subseteq \overline{\delta}(\mathsf{Obj}(\Gamma(f_0))).$$

Thus, because by Lemma 4.6.3, $\mathsf{Obj}(\Gamma(f_0)) = \mathsf{im}\ f_0$, we have $\mathsf{im}\ f_0 \subseteq \overline{\delta}(\mathsf{im}\ f_0)$. Since $\mathsf{im}\ f_0$ is a finite set, it follows that $\mathsf{im}\ f_0 = \overline{\delta}(\mathsf{im}\ f_0)$.

So, there is an injective function $\sigma : D \rightarrowtail D$ such that $f_0 \cdot \delta = \sigma \cdot f_0$. By condition 1 of the definition of corresponding models, $\Gamma(f_0) \cdot \delta = \Gamma(f_0 \cdot \delta)$. So, because $\Gamma(f_0) \cdot \delta = \Gamma(f_0)$ and $\delta(f_0(p)) = f_0(p')$, we have

$$\Gamma(f_0) = \Gamma(\sigma \cdot f_0) \text{ and } f_0(\sigma(p)) = f_0(p').$$

By the injectivity of f_0, we have $\sigma(p) = p'$. Also by the injectivity of f_0 and because $\mathcal{N}$ respects substitution, we have $\forall f \in D^P\ [\, \Gamma(f) = \Gamma(\sigma \cdot f)\,]$. This means that p has the same role as p'. ⊣

It may be observed that in this chapter the set H, the states that hold, did hardly any work. Therefore we could just as well, or perhaps better, have stated definitions and results in terms of the frames (i.e. logical spaces) of the models. The changes required are straightforward.

Chapter 5

Relational operations

In the previous chapters we have built relational models. The models serve as a kind of tools for understanding the essence of the structure of relations. As is the case for ordinary tools like hammers, we can also learn more about the adequacy of models by manipulating them. This is a good argument for studying operations on models of relations.

In this chapter we give definitions for various operations on relational models. We primarily look at operations from a technical perspective. Metaphysical questions like whether the negation of a relation is always a relation, will be postponed until Chapter 6.

We will define operations only for positional models and substitution models. From the operations for the positional models we can deduce operations for coordinated models in a trivial way. For the coordinated models we can define in addition a converse operation, and more generally permutation operations. But, because these operations are completely straightforward, we will not discuss them here.

5.1 Operations for positional models

For each of the operations to be defined, we can ask the following questions:

1. What are the roles, the permutation group, the pattern, and the refined pattern of the resulting model?

2. Which subtypes are preserved under the operation?

In the following subsections we take a detailed look at the permutation groups, the patterns and the refined patterns of the resulting models, and at the preservation properties.

5.1.1 Basic operations

What we should regard as basic operations is to a certain extent arbitrary. For example, we can define the disjoint disjunction of models in terms of disjoint conjunction and negation, but we could also define disjoint conjunction in terms of the disjoint disjunction and negation. Because in general there are from a metaphysical point of view probably less objections to conjunctions of relations, we will present disjoint conjunction and projective conjunctive projection as basic, and not their disjunctive counterparts. But no further meaning should be attached to the classification given. If you like, you can easily modify it. For example, if you reject negation as a valid operation for 'real' relations, but accept disjoint disjunction, then the last operation could defined analogously to the disjoint conjunction.

Disjoint conjunction

To express in a single state that Joost loves Janine and that Peter makes love to Francine, we have to combine states. To accomplish this, we simply(?) have to glue states together.

We will assume that we have a primitive mapping $\wedge$ from $S \times S'$ to a possibly new set of states $\overline{S}$:

$$\wedge : S \times S' \to \overline{S}.$$

The intended metaphysical interpretation of $s \wedge s'$ is a state (s *and* s'). There might be good reasons to claim that $\wedge$ is always symmetric, but we will not assume it here *a priori*. Questions with respect to the justification of this mapping will be discussed in Chapter 6.

Definition 5.1.1. Let $\mathcal{M} = \langle D, P, S, \Gamma, H \rangle$ and $\mathcal{M}' = \langle D, P', S', \Gamma', H' \rangle$ be positional models. Then the *disjoint conjunction* $\mathcal{M} \otimes \mathcal{M}'$ is $\langle D, \overline{P}, \overline{S}, \overline{\Gamma}, \overline{H} \rangle$ with:

- $\overline{P} := P \oplus P'$,

- $\overline{\Gamma}([f, f']) := \Gamma(f) \wedge \Gamma'(f')$,
- $\overline{H} := \{s \wedge s' \mid s \in H \quad \& \quad s' \in H'\}$.

In this definition $P \oplus P'$ denotes the disjoint union of P and P', i.e. $P \oplus P' := (P \times \{0\}) \cup (P' \times \{1\})$. Further, $[f, f']$ is defined by $[f, f'](p, 0) := f(p)$ and $[f, f'](p', 1) := f'(p')$. Note that f, f' can be retrieved from $[f, f']$.

If two positions in different models happen to be the same, then we consider this as an accidental coincidence to which no special meaning should be attached. Therefore, we are justified to define the positions of the disjunct conjunction in a rather arbitrary way. If, on the other hand, we would be serious about positions of different models being the same, then we would have to define the positions of the conjoined relation in a non-arbitrary way. It is not immediately clear how this should be done.

Lemma 5.1.2. *(1) If $\wedge$ is commutative, then the disjoint conjunction operation on positional models is commutative. (2) If $\wedge$ is associative, then the disjoint conjunction operation on positional models is associative.*[1]

Proof. The proof follows immediately from the definition of the disjoint conjunction of positional models. ⊣

In general, the positional structure of the disjoint conjunction cannot be deduced from only the positional structure of the operands. The mapping $\wedge$ also plays a role in this. We will describe the permutation group and the pattern for the case that $\wedge$ is injective. When $\wedge$ is not injective, then the permutation group, the pattern, and the refined pattern will be supersets of them.

Theorem 5.1.3. *Let $\mathcal{M} = \langle D, P, S, \Gamma, H\rangle$ and $\mathcal{M}' = \langle D, P', S', \Gamma', H'\rangle$ be positional models. Let $\overline{\mathcal{M}} = \langle D, \overline{P}, \overline{S}, \overline{\Gamma}, \overline{H}\rangle$ be the disjoint conjunction $\mathcal{M} \otimes \mathcal{M}'$. If $\wedge$ is injective on $S \times S'$, then we have for the permutation group $G_{\overline{\mathcal{M}}}$, the pattern $C_{\overline{\mathcal{M}}}$ and the refined-pattern $C^{\diamond}_{\overline{\mathcal{M}}}$:*

- $G_{\overline{\mathcal{M}}} = \{[\pi, \pi'] \mid \pi \in G_{\mathcal{M}} \quad \& \quad \pi' \in G_{\mathcal{M}'}\}$.
- $C_{\overline{\mathcal{M}}} = \{([\mu \circ \sigma, \mu' \circ \sigma'], [\mu \circ \tau, \mu' \circ \tau']) \mid \sigma\, C_{\mathcal{M}}\, \tau \quad \& \quad \sigma'\, C'_{\mathcal{M}}\, \tau' \quad \& \quad \mu : (2 \times P) \to (2 \times \overline{P}) \quad \& \quad \mu' : (2 \times P') \to (2 \times \overline{P})\}$.

[1] Strictly speaking, we have to add 'up to isomorphism', since the disjoint union of sets is commutative and associate up to isomorphism.

- $C^{\diamond}_{\overline{\mathcal{M}}} = \{([\mu \circ \sigma, \mu' \circ \sigma'], [\mu \circ \tau, \mu' \circ \tau']) \mid \sigma\ C^{\diamond}_{\mathcal{M}}\ \tau \quad \&\quad \sigma'\ C^{\diamond}_{\mathcal{M}'}\ \tau' \quad \&$
  $\mu : (2 \times P) \rightarrowtail (2 \times \overline{P}) \quad \& \quad \mu' : (2 \times P') \rightarrowtail (2 \times \overline{P})\}.$

Proof. The proof follows immediately from the definitions, and therefore we omit it here. ⊣

Note that the specification for the refined pattern $C^{\diamond}_{\overline{\mathcal{M}}}$ is similar to the specification of the pattern $C_{\overline{\mathcal{M}}}$. The only difference is that the function μ and μ' are injective. Let us now look at the preservation properties.

Theorem 5.1.4. *Let $\mathcal{M} = \langle D, P, S, \Gamma, H\rangle$ and $\mathcal{M}' = \langle D, P', S', \Gamma', H'\rangle$ be positional models. If $\wedge$ is injective on $S \times S'$, then:*

- *If $\mathcal{M}$ and $\mathcal{M}'$ are role-based, then $\mathcal{M} \otimes \mathcal{M}'$ is role-based.*
- *If $\mathcal{M}$ and $\mathcal{M}'$ are permutation-based, then $\mathcal{M} \otimes \mathcal{M}'$ is permutation-based.*
- *If $\mathcal{M}$ and $\mathcal{M}'$ are (refined) pattern-based, then $\mathcal{M} \otimes \mathcal{M}'$ is (refined) pattern-based.*

Proof. The proof follows immediately from the definitions. ⊣

If $\wedge$ is commutative, then clearly $\wedge$ is injective on $S \times S'$ only if $S \cap S' = \emptyset$. In many cases, however, S and S' will not be disjoint, and if $\wedge$ is not injective, then, in general, disjoint conjunction does not preserve any of the four types of positional models we identified, not even if $\wedge$ is commutative and *quasi-injective*, as defined in the next definition.

Definition 5.1.5. We call $\wedge$ *quasi-injective* if

$$\forall s_1, s_2, s_1', s_2'\ [s_1 \wedge s_2 = s_1' \wedge s_2' \rightarrow \{s_1, s_2\} = \{s_1', s_2'\}].$$

The next example shows that even if the models are role-based and $\wedge$ is quasi-injective, then their disjoint conjunction might not even be refined pattern-based.

Example 5.1.6. Let $\mathcal{M} = \langle D, P, S, \Gamma, H\rangle$ and $\mathcal{M}' = \langle D, P', S', \Gamma', H'\rangle$ be positional models. Let $\mathcal{M} \otimes \mathcal{M}' = \langle D, \overline{P}, \overline{S}, \overline{\Gamma}, \overline{H}\rangle$. Assume that for just four particular objects say a, b, c, d we have $\Gamma ab = \Gamma' ab$ and $\Gamma cd = \Gamma' cd$. Then, if $\wedge$ is commutative and quasi-injective, we have $\overline{\Gamma} abcd = \overline{\Gamma} cdab$, but for other objects, say a', b', c', d', it is not necessarily the case that $\overline{\Gamma}a'b'c'd' = \overline{\Gamma}c'd'a'b'$. This means that $\mathcal{M} \otimes \mathcal{M}'$ is not necessarily refined pattern-based, not even if Γ and Γ' are injective.

If $\wedge$ is commutative and quasi-injective, and $\mathcal{M}$ is role-based, then even the disjoint conjunction of $\mathcal{M}$ with itself is *not* necessarily role-based. However, $\mathcal{M}\otimes\mathcal{M}$ does preserve the other three types of positional models we identified, as one can easily check. Also for models that respect substitution in a *global* sense, we do have a positive result:

Theorem 5.1.7. *Let $\mathcal{M} = \langle D, P, S, \Gamma, H\rangle$ and $\mathcal{M}' = \langle D, P', S', \Gamma', H'\rangle$ be (refined) pattern-based models. If $\wedge$ is commutative and quasi-injective on $S \times S'$ and*

$$\forall f, g \, \forall \delta \, [\, \Gamma(f) = \Gamma'(g) \Rightarrow \Gamma(\delta \circ f) = \Gamma'(\delta \circ g)\,],$$

then $\mathcal{M} \otimes \mathcal{M}'$ is (refined) pattern-based.

Proof. By Theorem 4.3.11, it follows that if $\mathcal{M}$ and $\mathcal{M}'$ are pattern-based, then they respect substitution. It is easy to check that under the given conditions, also $\mathcal{M} \otimes \mathcal{M}'$ respects substitution. By again applying Theorem 4.3.11 it follows that $\mathcal{M} \otimes \mathcal{M}'$ is pattern-based. ⊣

Fixation

A simple operation is *fixing* the values assigned to some positions.

Definition 5.1.8. Let $\mathcal{M} = \langle D, P, S, \Gamma, H\rangle$ be a positional model, let $\overline{P}$ be a subset of P, and let f_0 be an element of $D^{P-\overline{P}}$. Then the *f_0-fixation of* $\mathcal{M}$ is $\langle D, \overline{P}, S, \overline{\Gamma}, H\rangle$ with:

- $\overline{\Gamma}(f) := \Gamma([f, f_0])$.

Compared to the simplicity of the definition of the fixation operation, the characterization of its pattern and refined pattern looks rather complicated.

Theorem 5.1.9. *Let $\mathcal{M} = \langle D, P, S, \Gamma, H\rangle$ be a positional model. Let $\overline{P}$ be a subset of P, and let f_0 be an element of $D^{P-\overline{P}}$. Let $\overline{\mathcal{M}} = \langle D, \overline{P}, S, \overline{\Gamma}, H\rangle$ be the f_0-fixation of $\mathcal{M}$. Then we have for the permutation group $G_{\overline{\mathcal{M}}}$, the pattern $C_{\overline{\mathcal{M}}}$ and the refined-pattern $C^{\diamond}_{\overline{\mathcal{M}}}$:*

- $G_{\overline{\mathcal{M}}} = \{\pi|_{\overline{P}} \mid \pi \in G_{\mathcal{M}} \quad \& \quad f_0 = f_0 \circ \pi|_{P-\overline{P}}\}$.
- $C_{\overline{\mathcal{M}}} = \{(\sigma|_{\overline{P}}, \tau|_{\overline{P}}) \in (2 \times \overline{P})^{\overline{P}} \times (2 \times \overline{P})^{\overline{P}} \mid \sigma\, C_{\mathcal{M}}\, \tau \quad \&$ $\exists h : 2 \times (P - \overline{P}) \to D\, [\, f_0 = h \circ \sigma|_{P-\overline{P}} \quad \& \quad f_0 = h \circ \tau|_{P-\overline{P}}\,]\}$.
- $C^{\diamond}_{\overline{\mathcal{M}}} = \{(\sigma|_{\overline{P}}, \tau|_{\overline{P}}) \in (2 \times \overline{P})^{\overline{P}} \times (2 \times \overline{P})^{\overline{P}} \mid \sigma\, C^{\diamond}_{\mathcal{M}'}\, \tau \quad \&$ $\exists h : X \rightarrowtail D\, [\, f_0 = h \circ \sigma|_{P-\overline{P}} \quad \& \quad f_0 = h \circ \tau|_{P-\overline{P}}\,]$ *with* $X = 2 \times (P - \overline{P}) \cap (\mathsf{im}\ \sigma|_{P-\overline{P}} \cup \mathsf{im}\ \tau|_{P-\overline{P}})\}$.

Proof. The proof follows immediately from the definitions. ⊣

As to be expected, the fixation operation has nice preservation properties.

Theorem 5.1.10. *Let $\mathcal{M} = \langle D, P, S, \Gamma, H\rangle$ be a positional model. Let $\overline{\mathcal{M}} = \langle D, \overline{P}, S, \Gamma, H\rangle$ be a fixation of $\mathcal{M}$. Then:*

- *If $\mathcal{M}$ is role-based, then $\overline{\mathcal{M}}$ is role-based.*
- *If $\mathcal{M}$ is permutation-based, then $\overline{\mathcal{M}}$ is permutation-based.*
- *If $\mathcal{M}$ is (refined) pattern-based, then $\overline{\mathcal{M}}$ is (refined) pattern-based.*

Proof. The proof follows immediately from the definitions. ⊣

Conjunctive projection

If we ignore the values attached to certain positions, then we get what we will call a *projection of* a model. Here we present the conjunctive projection. With this operation we can express in a single state that Maarten loves everybody.

We will assume that we have a primitive mapping $\bigwedge$ from the powerset of S to a possibly new set of states $\overline{S}$:

$$\bigwedge : \mathcal{P}(S) \to \overline{S}.$$

Definition 5.1.11. Let $\mathcal{M} = \langle D, P, S, \Gamma, H\rangle$ be a positional model, and let $\overline{P}$ be a subset of P. Then the $\overline{P}$*-conjunctive projection of* $\mathcal{M}$ is $\langle D, \overline{P}, \overline{S}, \overline{\Gamma}, \overline{H}\rangle$ with:

- $\overline{\Gamma}(f) := \bigwedge\{\Gamma(g) \mid g|_{\overline{P}} = f\}$,

- $\overline{H} := \{\bigwedge(S') \mid S' \subseteq H\}$.

The function $g|_{\overline{P}}$ is the restriction of g to the domain $\overline{P}$.

If $\bigwedge$ behaves as might reasonably be expected, then simultaneous projection gives the same result as sequential projection.

Lemma 5.1.12. *Let $\mathcal{M} = \langle D, P, S, \Gamma, H\rangle$ be a positional model. Let $\overline{P_1}$ and $\overline{P_2}$ be disjoint subsets of $\mathcal{M}$. If*

$$\forall S_1, S_2 \subseteq S\ [\ \bigwedge(S_1 \cup S_2) = \bigwedge(\bigwedge(S_1) \cup \bigwedge(S_2))\ ],$$

then the $\overline{P_1} \cup \overline{P_2}$-conjunctive projection of $\mathcal{M}$ is identical to the $\overline{P_1}$-conjunctive projection of the $\overline{P_2}$-conjunctive projection of $\mathcal{M}$.

Proof. The proof follows immediately from the definitions. ⊣

Sometimes it is more convenient to use hiding operations instead of projections. These operations can be defined in terms of each other.

Definition 5.1.13. Let $\mathcal{M} = \langle D, P, S, \Gamma, H\rangle$ be a positional model. Then the *$\overline{P}$-conjunctive hiding of $\mathcal{M}$*, denoted as $\bigwedge_{\overline{P}} \mathcal{M}$, is the $(P-\overline{P})$-conjunctive projection of $\mathcal{M}$.

The positional structure of the conjunctive projection depends not only on the positional structure of the operand, but also on the mapping $\bigwedge$. We will consider here in particular situations where $\bigwedge$ is injective.

Theorem 5.1.14. *Let $\mathcal{M} = \langle D, P, S, \Gamma, H\rangle$ be a positional model. Let $\overline{\mathcal{M}} = \langle D, \overline{P}, \overline{S}, \overline{\Gamma}, \overline{H}\rangle$ be a conjunctive projection of $\mathcal{M}$. If $\bigwedge$ is injective, then we have for the pattern $C_{\overline{\mathcal{M}}}$ and the refined-pattern $C^{\diamond}_{\overline{\mathcal{M}}}$:*

- $$\begin{aligned} C_{\overline{\mathcal{M}}} = \{(\overline{\sigma}, \overline{\tau}) \in (2 \times \overline{P})^{\overline{P}} \times (2 \times \overline{P})^{\overline{P}} \mid \\ \forall\sigma\, \exists\tau\ [\,\overline{\sigma} = \sigma|_{\overline{P}} \Rightarrow (\sigma\ C_{\mathcal{M}}\ \tau \quad \& \quad \overline{\tau} = \tau|_{\overline{P}})\,] \quad \& \\ \forall\tau\, \exists\sigma\ [\,\overline{\tau} = \tau|_{\overline{P}} \Rightarrow (\tau\ C_{\mathcal{M}}\ \sigma \quad \& \quad \overline{\sigma} = \sigma|_{\overline{P}})\,]\}. \end{aligned}$$

- $$\begin{aligned} C^{\diamond}_{\overline{\mathcal{M}}} = \{(\overline{\sigma}, \overline{\tau}) \in (2 \times \overline{P})^{\overline{P}} \times (2 \times \overline{P})^{\overline{P}} \mid \\ \forall\sigma\, \exists\tau\ [\,\overline{\sigma} = \sigma|_{\overline{P}} \Rightarrow (\sigma\ C^{\diamond}_{\mathcal{M}}\ \tau \quad \& \quad \overline{\tau} = \tau|_{\overline{P}})] \quad \& \\ \forall\tau\, \exists\sigma\ [\,\overline{\tau} = \tau|_{\overline{P}} \Rightarrow (\tau\ C^{\diamond}_{\mathcal{M}}\ \sigma \quad \& \quad \overline{\sigma} = \sigma|_{\overline{P}})\,]\}. \end{aligned}$$

Proof. The proof follows immediately from the definitions. ⊣

Even when $\bigwedge$ is injective, we cannot give a specification of the permutation group $G_{\overline{\mathcal{M}}}$ simply in terms of $\overline{P}$ and the permutation group of $\mathcal{M}$. It is clear that $G_{\overline{\mathcal{M}}} \supseteq \{\overline{\pi} : \overline{P} \rightarrowtail \overline{P} \mid \exists \pi \in G_{\mathcal{M}}\ [\overline{\pi} = \pi|_{\overline{P}}]\}$, but the permutation group might be larger, as the following example shows.

Example 5.1.15. Let $\mathcal{M} = \langle D, P, S, \Gamma, H\rangle$ be a positional model, with $D = \{a, b\}$, and $P = \{p_1, p_2, p_3\}$. Assume $G_{\mathcal{M}} = \mathsf{id}_P\}$. Then it is still possible that $\Gamma aba = \Gamma bab \neq \Gamma abb = \Gamma baa$. Now let $\overline{P}$ be $\{p_1, p_2\}$. Then for the conjunctive projection $\overline{\mathcal{M}} = \langle D, \overline{P}, \overline{S}, \overline{\Gamma}, \overline{H}\rangle$, we have $\overline{\Gamma} ab = \overline{\Gamma} ba$. Therefore, $\begin{pmatrix} p_1 & p_2 \\ (0, p_2) & (0, p_1) \end{pmatrix} \in G_{\overline{\mathcal{M}}}$, but we cannot deduce this from $G_{\mathcal{M}}$.

In the next theorem we consider the case that $\mathcal{M}$ is permutation-based.

Theorem 5.1.16. *Let $\mathcal{M} = \langle D, P, S, \Gamma, H\rangle$ be a permutation-based model. Let $\overline{\mathcal{M}} = \langle D, \overline{P}, \overline{S}, \overline{\Gamma}, \overline{H}\rangle$ be a conjunctive projection of $\mathcal{M}$. If $\bigwedge$ is injective and $|D| > |\overline{P}|$, then $\overline{\mathcal{M}}$ is permutation-based, and:*

- $G_{\overline{\mathcal{M}}} = \{\overline{\pi} : \overline{P} \rightarrowtail \overline{P} \mid \exists \pi \in G_{\mathcal{M}}\ [\overline{\pi} = \pi|_{\overline{P}}]\}$.

Proof. As we already remarked, it is clear that if $\mathcal{M}$ is an arbitrary positional model, then $G_{\overline{\mathcal{M}}} \supseteq \{\overline{\pi} : \overline{P} \rightarrowtail \overline{P} \mid \exists \pi \in G_{\mathcal{M}}\ [\overline{\pi} = \pi|_{\overline{P}}]\}$.

Suppose that $\mathcal{M}$ is permutation-based and that $\bigwedge$ is injective. Further, assume that $\overline{\Gamma}(f) = \overline{\Gamma}(g)$. Then it is not difficult to see that $\mathsf{im}\ f = \mathsf{im}\ g$. Because we also assumed that $|D| > |\overline{P}|$, it follows that for some $\pi \in G_{\mathcal{M}}$, we have $f = g \circ \pi|_{\overline{P}}$. ⊣

In the proof we used that $|D| > |\overline{P}|$. The following example shows that a certain bound is needed.

Example 5.1.17. Let $\mathcal{M} = \langle D, P, S, \Gamma, H\rangle$ be a permutation-based model with $D = \{a, b\}$, $P = \{p_1, p_2, p_3, p_4, p_5, p_6\}$, and $G_{\mathcal{M}}$ the permutation group generated by

$$\begin{pmatrix} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 \\ p_6 & p_1 & p_2 & p_3 & p_4 & p_5 \end{pmatrix}.$$

Let $\overline{P}$ be $\{p_1, p_2, p_3, p_4, p_5\}$ and let $\overline{\mathcal{M}} = \langle D, \overline{P}, \overline{S}, \overline{\Gamma}, \overline{H}\rangle$ be the $\overline{P}$-conjunctive projection of $\mathcal{M}$. Then $\overline{\Gamma} aaaab = \overline{\Gamma} baaaa$. However, there is no corresponding permutation in $G_{\overline{\mathcal{M}}}$, since $\overline{\Gamma} abbab = \overline{\Gamma}(f) \Rightarrow f = abbab$. Therefore, $\overline{\mathcal{M}}$ is not permutation-based.

For pattern-based models, we have again a positive result.

Theorem 5.1.18. *Let $\mathcal{M} = \langle D, P, S, \Gamma, H \rangle$ be a positional model. If $\bigwedge$ is injective, then:*

- *If $\mathcal{M}$ is (refined) pattern-based, then the conjunctive projections of $\mathcal{M}$ are (refined) pattern-based.*

Proof. It is easy to check that if $\mathcal{M}$ respects substitution, then the conjunctive projections of $\mathcal{M}$ also respect substitution. Then, by Theorem 4.3.11, it follows that if $\mathcal{M}$ is pattern-based, then the conjunctive projections of $\mathcal{M}$ are pattern-based.

Similarly, it is easy to check that if $\mathcal{M}$ respects injective substitution, then the conjunctive projections of $\mathcal{M}$ also respect injective substitution. Then, by Theorem 4.4.5, it follows that if $\mathcal{M}$ is refined pattern-based, then the conjunctive projections of $\mathcal{M}$ are also refined pattern-based. ⊣

Negation

> [...] I argued that there were negative facts, and it nearly produced a riot: the class would not hear of there being negative facts at all.
>
> Bertrand Russell, *The Philosophy of Logical Atomism*, 1918.

Not everybody accepts that the negation of a relation is a 'real' relation. Here we are not going to bother with possible objections. We concern ourselves here primarily with models. For the definition of the negation of a model, we have two options:

1. We keep the same model, except that we take for $\overline{H}$, the states that hold, the complement of H, or
2. We take for $\overline{S}$, the set of states, the negation of the states in S, and for $\overline{H}$ the negation of the complement of H.

The first option is the simpler one, but the second option is the more interesting form of negation. Consider again the love relation. To express in a single state that Mo does not love Tim, we need the negation of the love states themselves. Therefore, for the negation of a relation we get new states, namely the negation of the original states. If we want to let the negation of models correspond with the negation of relations, option 2 seems the more appropriate choice. We will call here option 2 simply the negation of a model. We now give a more precise definition.

For the negation we use a primitive mapping $\neg$ from S to a possibly new set of states $\overline{S}$:

$$\neg : S \to \overline{S}.$$

Definition 5.1.19. Let $\mathcal{M} = \langle D, P, S, \Gamma, H\rangle$ be a positional model. Then the *negation* $\neg\mathcal{M}$ is $\langle D, P, \overline{S}, \overline{\Gamma}, \overline{H}\rangle$ with:

- $\overline{\Gamma}(f) := \neg(\Gamma(f))$,
- $\overline{H} := \neg(S - H)$.

If the function $\neg : S \to \overline{S}$ is injective, then obviously the roles, the permutation group, the pattern and the refined pattern are the same for a model and for its negation. Also it will be obvious that if $\neg$ is injective, then negation preserves the four types of positional models we identified.

Identification

If we want to express in a model only the individuals who love themselves, we have to identify positions. The resulting model can be regarded as a model for a subrelation of the original relation.

Definition 5.1.20. Let $\mathcal{M} = \langle D, P, S, \Gamma, H\rangle$ be a positional model, and let $\sim$ be an equivalence relation on P. Then $\mathcal{M}/\sim$ is $\langle D, \overline{P}, S, \overline{\Gamma}, H\rangle$ with:

- $\overline{P} := P/\sim$,
- $\overline{\Gamma}(f) := \Gamma(f \circ \mathsf{proj})$, with $\mathsf{proj}(p) := [p]$.

Simultaneous identification gives essentially the same result as sequential identification:

Lemma 5.1.21. *Let $\mathcal{M} = \langle D, P, S, \Gamma, H\rangle$ be a positional model. Let $\sim_1$ and $\sim_2$ be equivalence relations on P. Let $\sim_3$ be the smallest equivalence* *relation containing* $\sim_1$ *and* $\sim_2$*. Then* $\mathcal{M}/\sim_3$ *is a positional variant of* $(\mathcal{M}/\sim_1)/\sim_2$.[2]

Proof. The proof is elementary. ⊣

The permutation group and the pattern of $\mathcal{M}/\sim$ can be described completely in terms of the permutation group respectively the pattern of $\mathcal{M}$. In the description of the pattern of $\mathcal{M}/\sim$, we use the following definition.

Definition 5.1.22. Let $\sim$ be an equivalence relation on P. Let σ be a function from P to $(2 \times P)$. Then we say that σ *respects* $\sim$ if

$$\forall p, p' \, [\, p \sim p' \Rightarrow \sigma(p) = \sigma(p')\,].$$

Further, if σ respects $\sim$, then the function $\sigma|_\sim : (P/\sim) \to (2 \times (P/\sim))$ is defined by:

$$\forall p \in P \, [\, \sigma|_\sim([p]) := (\mathsf{proj}_1(\sigma(p)), [\mathsf{proj}_2(\sigma(p))])\,].$$

Theorem 5.1.23. *Let* $\mathcal{M} = \langle D, P, S, \Gamma, H\rangle$ *be positional models, and let* $\sim$ *be an equivalence relation on* P*. Let* $\overline{\mathcal{M}} = \langle D, \overline{P}, S, \overline{\Gamma}, \overline{H}\rangle$ *be* $\mathcal{M}/\sim$*. Then we have for the pattern* $C_{\overline{\mathcal{M}}}$ *and the refined-pattern* $C^\diamond_{\overline{\mathcal{M}}}$*:*

- $C_{\overline{\mathcal{M}}} = \{(\sigma|_\sim, \tau|_\sim) \mid \sigma \; C_\mathcal{M} \; \tau \quad \&$ *$\quad \sigma, \tau$ both respect* $\sim\,]\}$.
- $C^\diamond_{\overline{\mathcal{M}}} = \{(\sigma|_\sim, \tau|_\sim) \mid \sigma \; C^\diamond_\mathcal{M} \; \tau \quad \&$ *$\quad \sigma, \tau$ both respect* $\sim\,]\}$.

Proof. The proof follows immediately from the definitions. ⊣

For the permutation group $G_{\overline{\mathcal{M}}}$, we cannot in general give a formulation in terms of the permutation group $G_\mathcal{M}$. On the basis of $G_\mathcal{M}$ we can only conclude that $G_{\overline{\mathcal{M}}} \supseteq \{\overline{\pi} \mid \exists \pi \in G_\mathcal{M} \; \forall p \in P \; [\, \overline{\pi}([p]) = [\pi(p)]\,]\}$, but the next example shows that $G_{\overline{\mathcal{M}}}$ can be a proper superset.

Example 5.1.24. Let $\mathcal{M} = \langle D, P, S, \Gamma, H\rangle$ be a positional model, with $P = \{p_1, p_2, p_3\}$. Suppose $G_\mathcal{M} = \{\mathsf{id}_P\}$. Further suppose that for any objects $a, b \in D$, we have $\Gamma aab = \Gamma bba$. Now if we identify p_1 with p_2, then for $\overline{\mathcal{M}}$ we have $\overline{\Gamma}ab = \overline{\Gamma}ba$, but this could not be deduced from $G_\mathcal{M}$.

[2]Strictly speaking, $\sim_2$ is not a relation on $P/\sim_1$, but we use here a natural identification of positions.

Let us now determine which types are preserved by the identification operation.

Theorem 5.1.25. *Let $\mathcal{M} = \langle D, P, S, \Gamma, H\rangle$ be a positional model, and let $\sim$ be an equivalence relation on P. Then:*

- *If $\mathcal{M}$ is (refined) pattern-based, then $\mathcal{M}/\sim$ is (refined) pattern-based.*

Proof. It is easy to check that if $\mathcal{M}$ respects substitution, then $\mathcal{M}/\sim$ also respects substitution. Then, by Theorem 4.3.11, it follows that if $\mathcal{M}$ is pattern-based, then $\mathcal{M}/\sim$ is pattern-based.

Similarly, it is easy to check that if $\mathcal{M}$ respects injective substitution, then $\mathcal{M}/\sim$ also respects injective substitution. Then, by Theorem 4.4.5, it follows that if $\mathcal{M}$ is refined pattern-based, then $\mathcal{M}/\sim$ is also refined pattern-based. ⊣

The role-based models and the permutation-based models are not closed under identification, as shown in the next example.

Example 5.1.26. Let $\mathcal{M} = \langle D, P, S, \Gamma, H\rangle$ be a role-based model for which $\Gamma abcd = \Gamma cdab$. For example, this is the case if $\Gamma abcd$ represents the state that a loves b and c loves d. Now if we identify the second and third position, then $\overline{\Gamma}(aba) = \overline{\Gamma}(bab)$, but for $a \neq b$, this situation cannot be modeled by a permutation-based model.

5.1.2 Compositions

With the operations we defined in the previous section we can compose a variety of operations. We will discuss here some of the more natural ones.

Disjoint disjunction

It seems easy to express in a single state that Riëtte loves Hans or Jos makes love to Jan. Although not everybody accepts that disjunction of relations is itself a relation, we have no problem in defining the operation in terms of positional models.

The disjoint disjunction operation $\oplus$ can be defined completely analogous to the 'disjoint conjunction' operation $\otimes$. The main difference is that instead of the mapping $\wedge$ we need a mapping $\vee : S \times S' \rightarrow \overline{S}$. The interpretation of $s \vee s'$ will be a state (s *or* s'). Further, the definition of $\overline{H}$, the states that hold, needs an adjustment:

$$\overline{H} := \{s \vee s' \in \overline{S} \mid s \in H \text{ or } s' \in H'\}.$$

All the results we gave for 'disjoint conjunction', can be translated directly to results for 'disjoint disjunction'. In particular, we get the same permutation groups, the same patterns, and the same preservation results.

There is also an alternative approach that does not need $\bigvee$ as a primitive operation. We could define the disjoint disjunction $\mathcal{M} \oplus \mathcal{M}'$ as:

$$\neg(\neg\mathcal{M} \otimes \neg\mathcal{M}').$$

If $\forall s \in S, s' \in S' \,[\, s \vee s' = \neg(\neg s \wedge \neg s')\,]$, then both approaches give the same results. However, you might perhaps have metaphysical objections to the negation of states. Then, of course, the second approach is no option.

Disjunctive projection

To express that Gitte loves somebody, we have to abstract from specific values of a position. We can do this with the 'disjoint disjunction' operation.

Like for the 'disjoint disjunction' we can define 'disjunctive projection' analogously to its conjunctive counterpart. If we do this, then we have to assume that the primitive mapping $\bigvee$ also maps subsets of the set of states S to some set of states. De definition of $\overline{H}$, the states for which the disjunctive projection holds, becomes:

$$\overline{H} := \{\bigvee(S') \mid S' \subseteq S \quad \& \quad \exists h \in H \,[\, h \in S' \,]\}.$$

We can also define disjunctive projection of a model $\mathcal{M}$ as the negation of the conjunctive projection of the negation of $\mathcal{M}$, i.e. as:

$$\neg(P\text{-conjunctive projection of } (\neg\mathcal{M})).$$

If $\forall S' \subseteq S \,[\bigvee(S') = \neg \bigwedge \neg(S')\,]$, then this gives the same result as when we define 'disjunctive projection' analogously to 'conjunctive projection'.

Conjunction and Disjunction

We can define 'conjunction' in terms of 'disjoint conjunction' and 'identification'. With this operation we can express that Eric loves Martha and makes loves to her.

Definition 5.1.27. Let $\mathcal{M} = \langle D, P, S, \Gamma, H\rangle$ and $\mathcal{M}' = \langle D', P', S', \Gamma', H'\rangle$ be positional models, and let I be a subset of $P \times P'$. Then the *conjunction* $\mathcal{M} \odot_I \mathcal{M}'$ is $(\mathcal{M} \otimes \mathcal{M}')/ \sim$, with $\sim$ the smallest equivalence relation on $P \oplus P'$ such that

$$(p, p') \in I \Rightarrow (p, 0) \sim (p', 1).$$

Lemma 5.1.28. *(1) If $\bigwedge$ is commutative, then the conjunction operation is commutative. (2) If $\bigwedge$ is associative, then the conjunction operation is associative.*[3]

Proof. Direct from the definition of conjunction and Lemma 5.1.2. ⊣

The specification of the permutation group and the pattern of a conjunction can be deduced from the lemmas and theorems we gave for the disjoint conjunction and identification. Also conditions under which the types of the models are preserved under conjunction can be derived from results we gave for the disjoint conjunction and identification.

Like we did for conjunction, we can define 'disjunction' in terms of 'disjoint disjunction' and 'identification'.

Hiding positions

To express in a single state that someone loves Margriet and makes love to her, we have to take a conjunction and a hiding operation. We can define the following variants of operations that hide positions:

1. Conjunction with conjunctive hiding,
2. Conjunction with disjunctive hiding,
3. Disjunction with conjunctive hiding,

[3]Strictly speaking, as for disjoint conjunction, we have to add 'up to isomorphism'.

4. Disjunction with disjunctive hiding.

Here we only give the definition for 'conjunction with conjunctive hiding'. The other operations can be defined in a similar way.

Definition 5.1.29. Let $\mathcal{M} = \langle D, P, S, \Gamma, H \rangle$ and $\mathcal{M}' = \langle D', P', S', \Gamma', H' \rangle$ be positional models, and let I be a subset of $P \times P'$. Then $\mathcal{M} \circ_I \mathcal{M}'$ is the $\overline{I}$-conjunctive hiding of the conjunction $\mathcal{M} \odot_I \mathcal{M}'$ with $\overline{I}$ defined by:

$$\overline{I} := \{\, [(p,0)] \mid p \in P \quad \& \quad \exists p' \in P' \, [\, (p,p') \in I \,]\}.$$

5.2 Operations for substitution models

For substitution models we can define operations that are to a certain extent similar to the operations for positional models. I will not discuss these operation in great detail, but I will concentrate on aspects that are connected with operations for positional models.

5.2.1 Basic operations

Conjunction

The first operation I call with some hesitation 'conjunction'. The reason for my hesitation is that the operation is not a direct counterpart of the conjunction for positional models. As we will see in Theorem 5.2.6 the conjunction of substitution models is under certain conditions rather the counterpart of the 'disjoint conjunction' of positional models.

Like before, we use a mapping $\wedge : S \times S' \to \overline{S}$, but now we also assume that the mapping is surjective and that the following constraint is fulfilled. If $\mathcal{M} = \langle D, S, \Sigma, H \rangle$ and $\mathcal{M}' = \langle D, S', \Sigma', H' \rangle$ are substitution models, then

$$s \wedge s' = t \wedge t' \Rightarrow \Sigma(s, \delta) \wedge \Sigma'(s', \delta) = \Sigma(t, \delta) \wedge \Sigma'(t', \delta).$$

We need this constraint to be able to define the substitution function $\overline{\Sigma}$ for the conjunction.

Definition 5.2.1. Let $\mathcal{M} = \langle D, S, \Sigma, H \rangle$ and $\mathcal{M}' = \langle D, S', \Sigma', H' \rangle$ be substitution models. Then the *conjunction* $\mathcal{M} \otimes \mathcal{M}'$ is $\langle D, \overline{S}, \overline{\Sigma}, \overline{H} \rangle$ with:

- $\overline{\Sigma}(s \wedge s', \delta) := \Sigma(s, \delta) \wedge \Sigma'(s', \delta)$,
- $\overline{H} := \{s \wedge s' \mid s \in H \quad \& \quad s' \in H'\}$.

It is not trivial that $\mathcal{M} \otimes \mathcal{M}'$ is a substitution model. So, we will prove this first.

Lemma 5.2.2. *If $\mathcal{M}$ and $\mathcal{M}'$ are substitution models, then $\mathcal{M} \otimes \mathcal{M}'$ is also a substitution model.*

Proof. Let $\mathcal{M} = \langle D, S, \Sigma, H \rangle$ and $\mathcal{M}' = \langle D, S', \Sigma', H' \rangle$ be substitution models. Then by the definition of a substitution model, we have:

$$\overline{\Sigma}(s \wedge s', \mathsf{id}_D) = \Sigma(s, \mathsf{id}_D) \wedge \Sigma'(s', \mathsf{id}_D) = s \wedge s'.$$

Further,

$$\begin{aligned}
\overline{\Sigma}(s \wedge s', \delta' \circ \delta) &= \Sigma(s, \delta' \circ \delta) \wedge \Sigma'(s', \delta' \circ \delta) \\
&= \Sigma(\Sigma(s, \delta), \delta') \wedge \Sigma'(\Sigma'(s', \delta), \delta') \\
&= \overline{\Sigma}(\Sigma(s, \delta) \wedge \Sigma'(s', \delta), \delta') \\
&= \overline{\Sigma}(\overline{\Sigma}(s \wedge s', \delta), \delta'),
\end{aligned}$$

which proves that $\overline{\Sigma}$ fulfills the condition for composition of substitutions. ⊣

Lemma 5.2.3. *(1) If $\wedge$ is commutative, then the conjunction operation on substitution models is commutative. (2) If $\wedge$ is associative, then the conjunction operation on substitution models is associative.*

Proof. The proof follows immediately from the definitions. ⊣

Theorem 5.2.4. *Let $\mathcal{M} = \langle D, S, \Sigma, H \rangle$ and $\mathcal{M}' = \langle D, S', \Sigma', H' \rangle$ be simple substitution models. If D is infinite or $\mathsf{degree}_{\mathcal{M}} + \mathsf{degree}_{\mathcal{M}'} \leq |D|$, then the conjunction $\mathcal{M} \otimes \mathcal{M}'$ is also a simple substitution model, with $\mathsf{degree}_{\mathcal{M} \otimes \mathcal{M}'} \leq \mathsf{degree}_{\mathcal{M}} + \mathsf{degree}_{\mathcal{M}'}$.*

Proof. We already proved in Lemma 5.2.2 that $\mathcal{M} \otimes \mathcal{M}'$ is a substitution model. Now we have to show that $\mathcal{M} \otimes \mathcal{M}'$ fulfills the connectivity constraint of a simple substitution model.

Let $s_1 \wedge s_1'$ and $s_2 \wedge s_2'$ be arbitrary states in $\overline{S}$. We claim that there are states $s_0 \in S$ and $s_0' \in S'$ and object-domains X_0 of s_0 in $\mathcal{M}$ and X_0' of s_0' in $\mathcal{M}'$ such that:

1. $s_0 \twoheadrightarrow_{\mathcal{M}} s_1$ and $s_0 \twoheadrightarrow_{\mathcal{M}} s_2$, and

2. $s'_0 \twoheadrightarrow_{\mathcal{M}'} s'_1$ and $s'_0 \twoheadrightarrow_{\mathcal{M}'} s'_2$, and

3. $X_0 \cap X'_0 = \emptyset$.

From this claim it follows immediately that $s_0 \wedge s'_0 \twoheadrightarrow_{\mathcal{M}\otimes\mathcal{M}'} s_1 \wedge s'_1$ and $s_0 \wedge s'_0 \twoheadrightarrow_{\mathcal{M}\otimes\mathcal{M}'} s_2 \wedge s'_2$, which proves that $\mathcal{M}$ is a simple substitution model.

To prove the claim, we first consider the case that D is finite and that $\mathsf{degree}_{\mathcal{M}} + \mathsf{degree}_{\mathcal{M}'} \leq |D|$. Since $\mathcal{M}$ is a substitution model, for some state $s_0 \in S$ condition 1 of the claim is fulfilled. Because $\mathcal{M}'$ is also a simple substitution model and because $\mathsf{degree}_{\mathcal{M}} + \mathsf{degree}_{\mathcal{M}'} \leq |D|$, we see that for some state $s'_0 \in S'$ condition 2 and 3 of the claim are also fulfilled with $X_0 := \mathsf{Obj}_{\mathcal{M}}(s_0)$ and $X'_0 := \mathsf{Obj}_{\mathcal{M}'}(s'_0)$.

Now, we consider the case that D is infinite. Note that in this case for states s in S, $\mathsf{Obj}_{\mathcal{M}}(s)$ may be undefined. Choose a state $s \in S$ and a state $s' \in S'$ such that:

1. $s \twoheadrightarrow_{\mathcal{M}} s_1$ and $s \twoheadrightarrow_{\mathcal{M}} s_2$, and

2. $s' \twoheadrightarrow_{\mathcal{M}'} s'_1$ and $s' \twoheadrightarrow_{\mathcal{M}'} s'_2$.

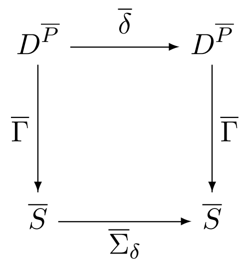
Because D is infinite, there is an injective function $\delta : D \oplus D \rightarrowtail D$. We may write δ as $[\delta_0, \delta'_0]$. Now define $s_0 := \Sigma(s, \delta_0)$ and $s'_0 := \Sigma(s, \delta'_0)$. Then $X_0 := \delta_0(D)$ is an object-domain of s_0 in $\mathcal{M}$ and $X'_0 := \delta'_0(D)$ is an object-domain of s'_0 in $\mathcal{M}'$. Clearly, the states s_0 and s'_0 and object-domains X_0 and X'_0 fulfill the claim.

Further, we see that $\forall s, s'\ [\,\mathsf{Obj}_{\mathcal{M}\otimes\mathcal{M}'}(s \wedge s') \subseteq \mathsf{Obj}_{\mathcal{M}}(s) \cup \mathsf{Obj}_{\mathcal{M}'}(s')\,]$. Therefore, $\mathsf{degree}_{\mathcal{M}\otimes\mathcal{M}'} \leq \mathsf{degree}_{\mathcal{M}} + \mathsf{degree}_{\mathcal{M}'}$. $\dashv$

Remark 5.2.5. If $\mathsf{degree}_{\mathcal{M}} + \mathsf{degree}_{\mathcal{M}'} > |D|$, then $\mathcal{M}\otimes\mathcal{M}'$ is not necessarily a simple substitution model. For example, if $\mathcal{M}$ is a simple substitution model with $D = \{a, b\}$, and $S = \{s_1, s_2, s_3, s_4\}$. Then, by Lemma 4.5.2, $\mathcal{M} \otimes \mathcal{M}$ can only be a simple substitution model if there is a state $\overline{s}$ such that $S = \{\Sigma(\overline{s}, \delta) \mid \delta : D \to D\}$. Because there are in D^D only four functions, this would mean that $S \wedge S$ has at most four states, but this is obviously not necessarily the case.

Theorem 5.2.6. *Let $\mathcal{M}$ and $\mathcal{M}'$ be substitution models, and let $\mathcal{N}$ and $\mathcal{N}'$ be positional models. If $\wedge$ is quasi-injective, and if $\mathcal{M}$ corresponds to $\mathcal{N}$, and $\mathcal{M}'$ corresponds to $\mathcal{N}'$, then $\mathcal{M} \otimes \mathcal{M}'$ corresponds to $\mathcal{N} \otimes \mathcal{N}'$.*

Proof. Let $\mathcal{N} = \langle D, P, S, \Gamma, H\rangle$ and $\mathcal{N}' = \langle D, P', S', \Gamma', H'\rangle$ and let $\mathcal{N} \otimes \mathcal{N}' = \langle D, \overline{P}, \overline{S}, \overline{\Gamma}, \overline{H}\rangle$. It is easy to check that if $\mathcal{M}$ corresponds to $\mathcal{N}$, and $\mathcal{M}'$ corresponds to $\mathcal{N}'$, then the following diagram commutes:



It is also easy to see that $\mathsf{im}\ \overline{\Gamma} = \overline{S}$. Further, because we assumed that $\wedge$ is quasi-injective, it follows that $\mathsf{degree}_{\mathcal{M}\otimes\mathcal{M}'} = \mathsf{degree}_{\mathcal{N}\otimes\mathcal{N}'}$. ⊣

Conjunctive power operation

We now define an operation that yields a substitution model whose states are conjunctions of subsets of S. We will call this operation the *conjunctive power operation.*

Like we did for the projection of positional models, we assume that we have a primitive mapping $\bigwedge : \mathcal{P}(S) \to \overline{S}$. In addition, we assume that $\bigwedge$ is surjective and that the following constraint is fulfilled. If $\mathcal{M} = \langle D, S, \Sigma, H\rangle$ is a substitution model, then

$$\bigwedge(S') = \bigwedge(T') \Rightarrow \bigwedge(\Sigma(S', \delta)) = \bigwedge(\Sigma(T', \delta)).$$

We need this constraint to be able to define the substitution function $\overline{\Sigma}$ for the conjunctive power operation.

Definition 5.2.7. Let $\mathcal{M} = \langle D, S, \Sigma, H\rangle$ be a substitution model. Then $\mathcal{P}_{\bigwedge}(\mathcal{M})$ is $\langle D, \overline{S}, \overline{\Sigma}, \overline{H}\rangle$ with:

- $\overline{\Sigma}(\bigwedge(S'), \delta) := \bigwedge(\Sigma(S', \delta))$,
- $\overline{H} := \{\bigwedge(S') \mid S' \subseteq H\}$.

Lemma 5.2.8. *If $\mathcal{M}$ is a substitution model, then $\mathcal{P}_{\bigwedge}(\mathcal{M})$ is also a substitution model.*

Proof. The proof follows immediately from the definitions. ⊣

For simple substitution models preservation properties of the conjunctive power operation are a bit less straightforward. If $\mathcal{M} = \langle D, S, \Sigma, H\rangle$ is a substitution model with S finite and a state s with $|\mathsf{Obj}(s)| \geq 2$, then $\mathcal{P}_{\bigwedge}(\mathcal{M})$ is not a simple substitution model. This can easily be seen as follows.

Define $S' := \{s \in S \mid |\mathsf{Obj}(s)| = \mathsf{degree}_{\mathcal{M}}\}$. If $\mathcal{P}_{\bigwedge}(\mathcal{M})$ is a simple substitution model, then for some S'', $S'' \twoheadrightarrow S$ and $S'' \twoheadrightarrow S'$. Because S is finite, $S'' = S$ and $S'' = S'$. However, $\mathcal{M}$ must have a state s' with $|\mathsf{Obj}(s')| < 2$. So, $S \neq S'$, and therefore $\mathcal{M}$ is not a simple substitution model.

If S is infinite and the degree of $\mathcal{M}$ is finite, we have a positive result:

Theorem 5.2.9. *Let $\mathcal{M} = \langle D, S, \Sigma, H\rangle$ be a simple substitution model. If S is infinite and $\mathsf{degree}_{\mathcal{M}}$ is finite, then $\mathcal{P}_{\bigwedge}(\mathcal{M})$ is also a simple substitution model.*

Proof. In the previous lemma we saw that $\mathcal{P}_{\bigwedge}(\mathcal{M})$ is a substitution model. Now let $\mathcal{P}_{\bigwedge}(\mathcal{M})$ be the model $\langle D, \overline{S}, \overline{\Sigma}, \overline{H}\rangle$, and assume that S is infinite and $\mathsf{degree}_{\mathcal{M}}$ is finite. We will show that $\mathcal{P}_{\bigwedge}(\mathcal{M})$ fulfills the connectivity constraint of a simple substitution model.

Choose a maximal set of states $S_0 \subseteq S$ such that

$$\forall s, s' \in S_0\, [\, \mathsf{Obj}(s) \cap \mathsf{Obj}(s') = \emptyset \quad \& \quad |\mathsf{Obj}(s)| = |\mathsf{Obj}(s')| = \mathsf{degree}_{\mathcal{M}}\,].$$

Such a maximal set exists by Zorn's Lemma. We claim that for an arbitrary subset S' of S, there is a surjective mapping from S_0 to S'. From this claim it follows almost immediately that for some δ, we have $\overline{\Sigma}(S_0, \delta) = S'$, which proves the connectivity constraint for $\mathcal{P}_{\bigwedge}(\mathcal{M})$.

To prove the claim, first observe that because $\mathcal{M}$ is a simple substitution model of finite degree, for every $s \in S$, the set of objects $\mathsf{Obj}(s)$ is finite, and the set of states $\{s' \in S \mid \mathsf{Obj}(s') = \mathsf{Obj}(s)\}$ is finite. It follows that there is a surjective mapping from D to S. It is also not difficult to see that there is a surjective mapping from S_0 to D. So, there is a surjective mapping from S_0 to S'. ⊣

Whether the last theorem would still be true if we dropped the assumption that $\mathsf{degree}_{\mathcal{M}}$ is finite, is a question that still needs to be answered.

Negation

The definition of negation is straightforward. Like for the positional models, we use a primitive mapping $\neg : S \to \overline{S}$. To define the substitution function $\overline{\Sigma}$ we assume that $\neg$ is a surjective function and that the following constraint is fulfilled:

$$\neg s = \neg t \;\Rightarrow\; \neg(\Sigma(s,\delta)) = \neg(\Sigma(t,\delta)).$$

Note that this constraint is fulfilled if the mapping $\neg : S \to \overline{S}$ is injective.

Definition 5.2.10. Let $\mathcal{M} = \langle D, S, \Sigma, H\rangle$ be a substitution model. Then the *negation* $\neg\mathcal{M}$ is $\langle D, \overline{S}, \overline{\Sigma}, \overline{H}\rangle$ with:

- $\overline{\Sigma}(\neg s, \delta) := \neg(\Sigma(s,\delta))$,
- $\overline{H} := \neg(S - H)$.

Theorem 5.2.11. *Let $\mathcal{M} = \langle D, S, \Sigma, H\rangle$ be a simple substitution model. Then $\neg\mathcal{M}$ is a simple substitution model.*

Proof. The proof follows immediately from the definitions. ⊣

Theorem 5.2.12. *Let $\mathcal{M}$ be a substitution model, and let $\mathcal{N}$ be a corresponding positional model. Then $\neg\mathcal{M}$ corresponds to $\neg\mathcal{N}$.*

Proof. The proof follows immediately from the definitions. ⊣

Restriction

A natural operation for substitution models is to restrict the states.

Definition 5.2.13. Let $\mathcal{M} = \langle D, S, \Sigma, H\rangle$ be a substitution model. Let $\overline{S}$ be a subset of S closed under substitution, i.e. $\forall s \in \overline{S}\, \forall \delta \in D^D\ [\, s \cdot \delta \in \overline{S}\,]$. Then the *$\overline{S}$-restriction of $\mathcal{M}$* is $\langle D, \overline{S}, \overline{\Sigma}, \overline{H}\rangle$ with:

- $\overline{\Sigma} := \Sigma\,|_{\overline{S}\times D^D}$,

- $\overline{H} := H \cap \overline{S}$.

The restriction of a simple substitution model is not necessarily simple. However, this does not necessarily mean that the simple substitution models are too limited as models for metaphysically meaningful relations. It might well be that certain restrictions do not make much sense metaphysically.

As a direct consequence of Lemma 4.5.2, we see that for a simple substitution model $\mathcal{M}$ of finite degree, a $\overline{S}$-restriction $\mathcal{M}$ is a simple substitution model iff $\exists s\ [\overline{S} = \{\Sigma(s, \delta) \mid \delta : D \to D\}]$. So, such restrictions can be defined in terms of just one state.

The next theorem shows the close relationship between the restriction operation and the identification operation for positional models.

Theorem 5.2.14. *Let $\mathcal{M}$ be a substitution model, and let $\mathcal{N}$ be a corresponding positional model. Then:*

- *Any restriction of $\mathcal{M}$ that is a simple substitution model with at least two states corresponds to a model $\mathcal{N}/\sim$.*
- *Any model $\mathcal{N}/\sim$ corresponds to a restriction of $\mathcal{M}$.*

Proof. By Theorem 4.6.6, $\mathcal{M}$ is a simple substitution model, and by the definition of correspondence, the degree of $\mathcal{M}$ is finite. Let $\overline{\mathcal{M}} = \langle D, \overline{S}, \overline{\Sigma}, \overline{H}\rangle$ be a restriction of $\mathcal{M}$. Then obviously the degree of $\overline{\mathcal{M}}$ is also finite. Now assume $\overline{\mathcal{M}}$ is a simple substitution model. Further assume that $\overline{\mathcal{M}}$ has at least two states. Then it follows that its degree is not zero.

Now let $s \in \overline{S}$ be a state such that $|\mathsf{Obj}(s)| = \mathsf{degree}_{\overline{\mathcal{M}}}$. Let $f \in D^P$ be such that $\Gamma(f) = s$ and $\mathsf{im}\ f = \mathsf{Obj}(s)$. Then, define the equivalence relation $\sim$ on P as:

$$p_i \sim p_j :\Leftrightarrow f(p_i) = f(p_j).$$

It is not difficult to see that $\overline{\mathcal{M}}$ corresponds to $\mathcal{N}/\sim$.

To prove the second part of the theorem, let $\mathcal{N}/\sim\ = \langle D, \overline{P}, \overline{S}, \overline{\Gamma}, \overline{H}\rangle$. Let $f \in D^{\overline{P}}$ be such that $|\mathsf{im}\ f| = \mathsf{degree}_{\mathcal{N}/\sim}$. Then define $\overline{\mathcal{M}}$ as the $\overline{T}$-restriction of $\mathcal{M}$ with:

$$\overline{T} := \{\Sigma(\Gamma(f), \delta) \mid \delta : D \to D\}.$$

It is not difficult to see that $\mathcal{N}/\sim$ corresponds to $\overline{\mathcal{M}}$. ⊣

5.2.2 Compositions

Disjunction

Completely analogous to 'conjunction', we can define 'disjunction'. Only instead of a mapping $\bigwedge$ we use a mapping $\bigvee$ that maps the product of sets of states, $S \times S'$, to some set of states $\overline{S}$. And as before, we could define the disjunction $M \oplus M'$ of substitution models M and M' as:

$$\neg(\neg\mathcal{M} \otimes \neg M').$$

All the results we gave for the 'conjunction' of substitution models, can be translated directly to results for the 'disjunction' of substitution models.

Disjunctive power operation

Analogous to the 'conjunctive power operation', we can define its disjunctive variant. Then we have to assume that $\bigvee$ maps subsets of S to some set of states. Or we could define the 'disjunctive power operation' in terms of the 'conjunctive power operation' and 'negation':

$$\neg(\mathcal{P}_{\bigwedge}(\neg\mathcal{M})).$$

5.3 Overview of preservation of subtypes

In the table below we indicate for the four subtypes of positional models we identified, whether or not the subtype is preserved under the operation. We will assume that the mappings $\wedge, \bigwedge, \vee, \bigvee$, and $\neg$ are injective.

Operations	Role-based	Permutation-based	Pattern-based	Refined pattern-based
Disjoint conjunction[1]	√	√	√	√
Fixation	√	√	√	√
Conjunctive projection[2]	√ [6]	√ [6]	√	√
Negation[3]	√	√	√	√
Identification	–	–	√	√
Disjoint disjunction[4]	√	√	√	√
Disjunctive projection[5]	√ [6]	√ [6]	√	√
Conjunction[1]	–	–	√	√
Disjunction[4]	–	–	√	√

Notes:

1. Makes use of $\wedge : (S \times S') \rightarrow \overline{S}$.
2. Makes use of $\bigwedge : \mathcal{P}(S) \rightarrow \overline{S}$.
3. Makes use of $\neg : S \rightarrow \overline{S}$.
4. Makes use of $\vee : (S \times S') \rightarrow \overline{S}$.
5. Makes use of $\bigvee : \mathcal{P}(S) \rightarrow \overline{S}$.
6. Preserves the subtype if the number of objects in the domain $D >$ the number of positions of the resulting model.

Chapter 6

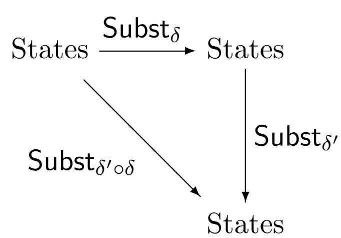
Metaphysical interpretations

6.1 Metaphysical principles for relations

Let me start by formulating a number of metaphysical principles concerning states of relations.

Substitution principles:

- Objects of any state can be substituted by other objects.
- Any substitution of objects in any state yields exactly one state of the same relation.
- Substitution is transitive, i.e. the following diagram commutes:



  with Subst_δ representing a substitution of objects.

Constituents principles:

- Every state has exactly one set of objects that forms its constituents.

- An object a belongs to a state iff for some substitution it makes a difference for the resulting state which object is substituted for a.

Connectivity principle:

- For any two states of a given relation there is a state of the same relation from which the first two states can both be obtained by substitution.

What supporting arguments can be given for these principles? It seems hard to give any conclusive arguments, since we have not given any clear definition of what relations are. Nevertheless, for several views on relations these principles might be acceptable, not in the least because they seem intuitively plausible, and because anything we commonly call a relation seems to satisfy the principles.

Let me briefly comment on the principles. I think that our intuitive understanding of substitution is in accordance with the substitution principles. Note that in the principles we only talk about substituting objects and not about substituting individual occurrences of an object by possibly different objects. The reason for not considering a more refined substitution mechanism is that otherwise we would have to make clear what exactly occurrences are. It might perhaps be possible to do this, but it would be an extra complication.

We might ask if the second substitution principle is perhaps not too strong and that it would be better to replace it by a weaker principle, namely:

- Any substitution of objects in any state yields *no more than* one state of the same relation.

This weaker principle might be a better choice if you consider the states of a relation as states of affairs. In Section 2.2, we said that a state of affairs is a *possible* fact. That Saddam loves Bush is a possible fact, but that $1 = 2$ is clearly not a possible fact. What is also impossible, I think, is that I am identical to Mo, my daughter. Such examples make clear that not every substitution yields a possible fact. However, if you regard states of relations as propositions or if you are willing to accept *impossible* states of affairs, then the stronger principle seems preferable.

Another point is that we can ask ourselves if there is not a more primitive operation in which substitution could be expressed. Fine [Fin00, 27] discusses the question whether substitution should be understood in terms of a structural operation. However, he considers the notion of substitution of a lower logical type. But even if Fine would be wrong in this respect, this would not make the substitution principles less credible.

What about the constituents principles? To a certain extent it is a matter of definition if you accept these principles. It might perhaps be defensible that a state can have two kind of objects, those that satisfy the second principle, and those that fulfill a kind of background role for the state. If you take this position, then this position is consistent with restricting the constituents principles to objects of the first kind.

The connectivity principle needs some explanation. Consider the state transition graph $\langle S, E \rangle$, with S the states of the relation and E the set of pairs (s, s') such that s' can be obtained from s by substitution. Perhaps, you might be tempted to think that for any relation this graph will be strongly connected, i.e. for any two states, one state can always be obtained from the other by a substitution. However, this is clearly not true. Take for example the state that there is a flight from New York to Amsterdam via New York, and the state that there is a flight from New York to New York via London. Then neither of these states can be obtained from the other by substituting objects. Only if we would allow substituting individual occurrences of objects, then it would be possible, but as we said before, we do not consider this more refined substitution mechanism here.

We get a very modest form of connectivity if we ignore the direction of the edges in the state transition graph and only demand that the graph is weakly connected. I have no conclusive arguments why this form of connectivity is not the appropriate one for relations. I even have no conclusive arguments for any form of connectivity of the state transition graph. In defense of the connectivity principle as formulated above, I can say that all 'real' relations with a fixed number of arguments I could think of, seem to satisfy this principle. But these relations also seem to satisfy a stronger assertion, namely that there is a state from which all states can be derived. What the correct connectivity principle is may depend on what we accept as relations.

If we accept variably polyadic relations, then at least for some of these relations the connectivity principle may fail, as Kit Fine pointed out. Take for example the relation *being in a line* where an object may have multiple occurrences. Then how could we get by substitution from a state of three objects in a line to a state with two objects in a line? I think that perhaps we need to consider as an additional basic operation the *elimination* of (occurrences of) objects from a state.

The connectivity principle may also fail if we would accept the existence of an amatory relation with Stéphane's loving Ilse and Ilse's being loved by Stéphane as *different* states. But I think we have good reasons to regard these states only as completions of different relations.

The principles might be taken as necessary criteria for relations with a fixed adicity, although probably some refinements will be needed if we would introduce typed domains for the objects. But also if one would argue that these principles do not apply to all such relations, then they could still be of interest. I claim that entities that satisfy the principles form a natural class that deserves further study, and that distinguishing this class might contribute to our understanding of how we represent the world. In the next section, we will relate these principles to our relational models.

6.2 Justifying positional representations

I do not claim that I have obtained a complete understanding of the essence of relations. I cannot even give a satisfying definition. The claims I will make in this section are modest in the sense that they are all conditional claims. They are only true for relations and other entities having certain properties. Whether all relations have these properties remains to be seen.

Definition 6.2.1. We call a relation $\Re$ that satisfies the substitution principles, the constituents principles, and the connectivity principle a *plain relation*.
If m is the maximum number of objects per state, then we call m the *degree of* $\Re$.

> **Claim 1.** Any plain relation can be modeled by a unique simple substitution model.

Note that we do not claim that the substitution model models all aspects of the relation. Because any substitution model of finite degree corresponds with a positional model (Theorem 4.6.7), we also have the following result.

Claim 2. Any plain relation of finite degree can be modeled by a positional model of the same degree that is unique, modulo positional variants.

The last claim gives a justification for using positional representations for a large class of relations. I consider this claim as the main result of this thesis. It is a modest result, but it is also very fundamental. Perhaps it looks completely trivial, but I don't think it is. The fact that we continually use positional representations for relations, is in itself no evidence for their validity.

What is also worth noting is that for a plain relation of finite degree, a positional model for it has exactly the same information content as a substitution model. From Theorems 4.6.7 and 4.6.8 it follows that there is a one-to-one correspondence between positional models for plain relations of finite degree and substitution models for such relations.

I do not claim that the positional model for a plain relation is unique in an absolute sense. The model is defined in terms of our definition of correspondence between substitution models and positional models. Other definitions of correspondence might give other positional models. But the given definition occurs to me as the most natural one.

For plain relations, substitution of objects in any state yields by definition exactly one state of the same relation. For the class of relations satisfying the weaker substitution principle that says that objects in any state yields *no more than* one state of the same relation, another kind of models might be more appropriate, namely substitution models with a partial function Σ and positional models with a partial function Γ. For these kind of models, a theory very similar to the one given in this thesis could be developed. In particular, we would get results similar to Claims 1 and 2.

Let us look again at Fine's ideas about positions. Fine claims that the antipositionalist can reconstruct the notion of positions within the confines of his theory [Fin00, 29]. In support of this claim, Fine defines positions in terms of *co-positionality*, where an object a in a state s is co-positional to b in t if t can be obtained from s by substituting b for a (and vice versa). However, I think it is confusing to call the resulting entities positions. They have more similarities with what we previously called *roles*. Fine also makes a second claim from which you might get the idea that there is something fundamentally wrong with positional models for relations [Fin00,

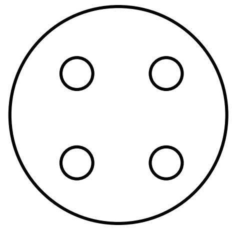


Figure 6.1: Could states have internal positions without identity?

32, footnote 22]: "[T]he antipositionalist cannot satisfactorily reconstruct the positionalist's account of position. But since the account is in error, this is no great loss." At first sight this seems to contradict my claim about the adequacy of positional models. But this is not necessarily true. According to Fine, a positionalist considers positions as entities occupied by objects within the states, but in the positional models as defined in this thesis, positions only have a mediating function.

Claim 2 says nothing about the ontological status of positions. But can we say something about the ontology of positions? Can we deny them a place in the "fundamental furniture of the universe"? I think we have no reason to grant them such an honorable place, but I also don't think that we have as yet a decisive argument why they cannot belong to this furniture. As long as you do not consider positions as things occupied by objects within the states, their fundamental existence seems hard to disprove. It is perhaps also not contradictory to claim that objects occupy a kind of *internal* positions within the states. For example, you could argue that the states of the adjacency relation have two internal positions, but that these positions have *no determinate identity*. The indeterminacy of the positions could be compared with the quantum-theoretic *ontic* indeterminacy of the electrons of a He atom (cf. [Low94]). For a cyclic relation the internal positions might be argued to be partially indiscernible like the unlabeled vertices of a regular mathematical polygon are (see Figure 6.1). Perhaps for arbitrary positional structures a similar defense can be given for a kind of internal (super)positions. But it is good to realize that the adequacy of positional models does not imply that the constituents of a state of affairs really need to occupy some kind of position. I think we can be perfectly happy with assigning to positions nothing more than the status of an innocent mental construction, *unless* we have reason to assume that there is a 'real' relation that can only be adequately modeled by a positional model that does *not* respect substitution. Then we still have something to explain.

Although the two claims in this section are about relations, they could be generalized to any entity that satisfies the principles of the previous section. Also for an extreme nominalist who does not believe in relations, substitution might still be a notion that makes sense, and he might find positional models useful. Substitution definitely has a wider scope of application than just states of affairs. I see no objection for applying substitution also to situations and propositions, and for using positional representations for them.

6.3 Epistemological aspects of relations

Substitution models seem to be more primitive than positional models. Then, why would we use positional models? I think there is one good reason for this. The strength of positional models (and of coordinated models) is that they provide us an ordered framework for all possible states, and the possibility to refer efficiently to each individual state. Natural languages like English obviously take advantage of this kind of representation. As far as I can see, almost any linguistic relational statement in English makes explicitly or implicitly use of positions.

With respect to determining the structure of relations we have the following result.

Claim 3. For any plain relation of finite degree the positional structure of the corresponding positional model can in principle be determined by a finite number of substitutions.

The claim is a direct consequence of the fact that positional models that correspond to substitution models are pattern-based models. The patterns have a finite number of elements and each element can be determined by just one substitution. I presuppose here that in principle one can determine whether a substitution results in the same state as the original state or in a different one. I don't know if this is true in practice. Also I do not know to what extent it is needed for a practical understanding of a relation to know explicitly or tacitly its positional structure.

How do we 'learn' relations? I have as yet no answer to this question. I have not investigated what kind of empirical research has been done on this subject. I think it would be very interesting to find answers to the following questions:

- Do we learn relations by substitution or by abstraction or by positional representations or via (hard-wired) processes with a completely different logic?
- In processing perception, do we use neutral rather than biased relations?
- How do small children learn relations?
- How do animals and other organisms learn relations?
- What is the role of language in learning relations?
- Do all natural languages use coordinated and positional representations of relations?
- Do we learn complex relations by applying operations like conjunction and projection to simple relations?

Answers to these questions might deepen our insight in the way we understand the world around us. In addition, it might suggest new learning programs or new ways for learning Artificial Intelligence systems to 'discover' and handle relations. For example, if we can implement a general notion of substitution in an AI system, then it might perhaps be possible to learn the system a variety of relations by examples.

6.4 Operations on relations

In Chapter 5 we considered operations on models from a technical perspective. We did not concern ourselves with the question whether these operations would have corresponding operations on relations. We introduced primitive mappings $\wedge$, $\bigwedge$, $\vee$, $\bigvee$ and $\neg$ that operate on states, but we said almost nothing about their nature. Now we will look at them from a metaphysical perspective. I will not give a complete analysis, but only discuss a few salient points.

Recall that we introduced in Chapter 5 a primitive mapping $\wedge$ from $S \times S'$ to a possibly new set of states $\overline{S}$. We used this mapping for the disjoint conjunction of positional models and for the conjunction of substitution models. A natural interpretation of this mapping is that a pair of states (s, s') is mapped to a state that is the *conjunction of* s and s'. The question is of course whether such complex states exist. Armstrong [Arm97, 35] accepts conjunctions of states of affairs as the only sort of *molecular* state of affairs. Disjunction and negation

For the conjunctive projection and the conjunctive power operation we assumed the existence of a mapping $\bigwedge$ from the powerset of S to a possibly new set of states $\overline{S}$. If S is infinite, then we have a complication, because the question is whether infinitely complex states are possible. It is interesting to note that, according to Lokhorst [Lok88, 37], Wittgenstein seems to have been uncertain about the maximum complexity states of affairs can have. In the Notebooks, Wittgenstein strongly rejects infinitely complex states [Wit79, 23.5.15]: "Trotzdem scheint nun der *unendlich* komplexe Sachverhalt ein Unding zu sein!", but later, in the Tractatus, he is perhaps less certain [Wit98, 4.2211]:

> Auch wenn die Welt unendlich komplex ist, so dass jede Tatsache aus unendlich vielen Sachverhalten besteht und jeder Sachverhalt aus unendlich vielen Gegenständen zusammengesetzt ist, auch dann müsste es Gegenstände und Sachverhalte geben.

For the negation operation on models we used a mapping $\neg$ from S to $\overline{S}$, and for the disjunction operations we used mappings $\vee$ and $\bigvee$. These mappings seem more controversial than $\wedge$ and $\bigwedge$. In particular, Armstrong denies the existence of negative and disjunctive states of affairs. Armstrong [Arm97, 26] has no objection against predicates like 'F or G' or 'not-F', but he denies that these are 'property-predicates'. According to him it is a truism that universals are strictly identical in their different instantiations. Now suppose a has property F and b has property G. Then, although 'F or G' applies to both, this does not mean that a and b have something in common. With respect to 'not-F' Armstrong suggests that all particulars that lack a certain property do not necessarily have an 'identical something'.

Armstrong's objections concern properties (monadic universals), but they can easily be generalized to relations. With respect to his arguments, I don't know whether they are really convincing. I even do not know how important his claims are. They do not impose a limitation on the kind of complex assertions we can make about states or situations using predicate logic. We could perhaps introduce the notion of 'pseudo-relations' as constructed entities that correspond to such complex assertions. Then the operations we defined on the relational models could be translated to operations on pseudo-relations.

As we said before, David Lewis [Lew02] regards states of affairs as sets of possible worlds. An advantage of this view, is that it gives a simple interpretation for the mappings on states:

$$\begin{aligned} s \wedge s' &:= s \cap s' \\ \bigwedge(S') &:= \bigcap S' \\ s \vee s' &:= s \cup s' \\ \bigvee(S') &:= \bigcup S' \\ \neg s &:= U - s, \text{ with } U \text{ the set of all possible worlds.} \end{aligned}$$

With this interpretation, commutativity and associativity of conjunction and disjunction are trivially fulfilled. With other kind of interpretations of the mappings on states, we probably also get these properties. However, we may also expect differences in structural properties. For example, suppose we have the state s that Mo is dancing, and the state s' that Mo is singing or not singing. Then s and $s \wedge s'$ are logically equivalent, and in Lewis's interpretation, they are the same state. But with an intensional conception of states of affairs, this is not necessarily true, since the states may be seen as structurally different.

Chapter 7

Two case studies

7.1 Exposing the identity relation

> Von *zwei* Dingen zu sagen, sie seien identisch, ist ein Unsinn, und von *Einem* zu sagen, es sei identisch mit sich selbst, sagt gar nichts.
>
> Ludwig Wittgenstein, *Tractatus Logico-Philosophicus*, 5.5303, 1918.

The identity relation is, despite its apparent simplicity, a source of metaphysical problems. Here we are not going to deal with subtle identity conditions, but with attempts to model the identity relation. We start with defining four substitution models for the identity relation. For all four models we assume that we have a set of objects D.

Model 1
Suppose we have for each subset X of D a unique state s_X. Then define $\mathcal{M}_1 = \langle D, S, \Sigma, H\rangle$ with:

- $S := \{s_X \mid X \subseteq D\}$,
- $\forall X \subseteq D\, \forall \delta : D \to D\, [\, s_X \cdot \delta := s_{\overline{\delta}(X)}\,]$,
- $H := \{s_X \mid X \subseteq D \quad \& \quad |X| = 1\}$.

Note that $\mathsf{degree}_{\mathcal{M}_1} = |\mathcal{P}(D)|$. Also note that $\mathcal{M}_1$ is not a simple substitution model, because $s_\emptyset$ is an 'isolated' state.

Model 2
Suppose we have for each unordered pair of objects $a, b \in D$ a unique state $s_{\{a,b\}}$. Then define $\mathcal{M}_2 = \langle D, S, \Sigma, H \rangle$ with:

- $S := \{s_{\{a,b\}} \mid a, b \in D\}$,
- $\forall a, b \in D\, \forall \delta : D \to D\, [\, s_{\{a,b\}} \cdot \delta := s_{\{\delta(a),\delta(b)\}}\,]$,
- $H := \{s_{\{a\}} \mid a \in D\}$.

Model 3
Suppose we have for each object $a \in D$ a unique state s_a. Then define $\mathcal{M}_3 = \langle D, S, \Sigma, H \rangle$ with:

- $S := \{s_a \mid a \in D\}$,
- $\forall a \in D\, \forall \delta : D \to D\, [\, s_a \cdot \delta := s_{\delta(a)}\,]$,
- $H := S$.

Model 4
Suppose we have two states, the identity state ID and the non-identity state DIF. Then define $\mathcal{M}_4 = \langle D, S, \Sigma, H \rangle$ with:

- $S := \{\mathsf{ID}, \mathsf{DIF}\}$,
- $\forall s \in S\, \forall \delta : D \to D\, [\, s \cdot \delta := s\,]$,
- $H := \{\mathsf{ID}\}$.

Note that $\mathsf{degree}_{\mathcal{M}_4} = 0$. Also note that $\mathcal{M}_4$ is not a simple substitution model, because there is no state from which both the state ID and the state DIF can be obtained by a substitution. Therefore it has no corresponding positional model.

The four models are not all conceivable models for the identity relation. For example, we could also make a model with for each $a \in D$ a state s_a, and one impossible state DIF. Or we could make a model with just one state, namely the identity state ID. This last model has a corresponding positional model with $P = \emptyset$.

Which model adequately represents the identity relation? An argument to choose model $\mathcal{M}_1$ is that it has the nice property that H is the set of singletons, which can be identified with the number 1, and that the set of unordered pairs can be identified with the number 2, etc. An argument to choose model $\mathcal{M}_2$ is that this is the only binary model of the models given, and that it models the smallest reflexive relation on D (provided $|D| > 1$). On the other hand, the fact that in $\mathcal{M}_1$ and $\mathcal{M}_2$ all states with more than one object are impossible states, might also be regarded as a deficiency. At one time I thought a preference for model $\mathcal{M}_3$ matched perfectly with what Kripke argues in [Kri03, 108]: "identity is just a relation between a thing and itself". However, as Kit Fine pointed out, I probably over-interpreted this remark of Kripke. According to Albert Visser, what Kripke means is on the level of holding and not on the level of logical space of the relation. Finally, model $\mathcal{M}_4$ could be defended by arguing that the state where a is identical to a is the same state as the state where b is identical to b. In conclusion, I think it is an open question how to find for the identity relation the right logical space.

A problem with all four models is that they say almost nothing. But perhaps this is also true of the identity relation itself. Or could it be that, as Wittgenstein said, there is no identity relation between objects? "Dass die Identität keine Relation zwischen Gegenständen ist, leuchtet ein" [Wit98, 5.5301].

In daily life, we frequently use identity statements. Such statements are generally accepted as meaningful and informative. But does this not imply that we should also accept that there is an identity relation between objects? I claim that this is not the case.

To prove that the identity relation is not needed for understanding identity statements, the reflexive-referential theory of Perry [Per01] might be helpful. Perry differentiates several layers of content for statements. Among others, statements have a *reflexive content*. For the statement "Hesperus = Phosphorus", the reflexive content is the following proposition:

> The object associated with 'Hesperus' by the conventions exploited by the use of this name in "Hesperus = Phosphorus" is the object associated with 'Phosphorus' by the conventions exploited by the use of 'Phosphorus' in "Hesperus = Phosphorus".

With reflexive contents correspond relations between *names*. To keep things simple, we assume that we have a set D of proper names that rigidly designate objects. Further, we assume that we have for each unordered pair of names $v, w \in D$ a unique state $s_{\{v,w\}}$. Now define the substitution model $\mathcal{N} = \langle D, S, \Sigma, H \rangle$ with:

- $S := \{s_{\{v,w\}} \mid v, w \in D\}$,
- $\forall v, w \in D \, \forall \delta : D \to D \, [\, s_{\{v,w\}} \cdot \delta := s_{\{\delta(v), \delta(w)\}} \,]$,
- $H := \{s_{\{v,w\}} \mid v$ and w designate the same object$\}$.

We could formalize this model further by taking for the domain D pairs (n, o) with n a name and o its designated object. Then we could define S and Σ as before, and H as:

$$H := \{s_{\{(n,o),(n',o)\}} \mid (n, o), (n', o) \in S\}.$$

This model makes clear that for understanding identity statements, we do not need an identity relation between objects.

Note that we did not treat the identity statement in a special way, as Frege did in §8 of his *Begriffsschrift*. There he treats "a = b" as a statement about the signs 'a' and 'b'. This is not what is done here. The identity statement has a reflexive content like any other statement.

According to Perry each statement has besides a reflexive content also a *referential content*. For an identity statement "Hesperus = Phosphorus" this would be the proposition:

Hesperus is **Phosphorus**

Here the names are written in boldface to indicate that Hesperus and Phosphorus *themselves* are the constituents of the proposition. So, because Hesperus is Phosphorus, this proposition *is* the proposition that **Hesperus** is **Hesperus**.

Let us assume that Perry's theory is correct and that the identity statement "Hesperus = Phosphorus" has as referential content the proposition that **Hesperus** is **Phosphorus**. Then does this mean that the identity relation lives here, on the level of referential content? I am not so sure about that. I do not see why the proposition that **Hesperus** is **Phosphorus** would imply that an identity relation between objects exists. I am afraid, the identity relation does not reveal its true identity that easily. But I think we can live without it.

7.2 Challenging Putnam's Twin Earth argument

Putnam's well-known Twin Earth argument [Put75] is about certain limitations of the contents of psychic states. So far, we did not say anything about psychic states. We considered states of relations, and it is not immediately clear what these states have to do with psychic states. Nevertheless, I think that notions developed in the previous chapters may shed some light on Putnam's argument, in particular on the claimed validity of one of his assumptions. I assume that the reader is familiar with Putnam's Twin Earth argument. Therefore, I will only briefly sketch the argument, and then zoom in on relational aspects of it.

Putnam argues that the following two assumptions are not compatible:

Assumption 1. Knowing the meaning of a term is just a matter of being in a certain psychological state.[1]

Assumption 2. The meaning of a term (in the sense of 'intension') determines its extension.

To show the incompatibility, Putnam uses the following example. Assume that there is a planet, Twin Earth, which is exactly like Earth, except that what Twin Earthians call 'water' is not H_20, but a different liquid XYZ. Under normal circumstances this liquid is indistinguishable from water. Let Oscar_1 be an Earthian English speaker and let Oscar_2 be his counterpart on Twin Earth. Assume Oscar_1 has exactly the same beliefs about water (H_20) as Oscar_2 has about 'water' (XYZ). Then Oscar_1 and Oscar_2 understand the term 'water' differently, although their psychic states (in a narrow sense) are the same. Therefore, the extension of water is *not* determined by psychic state. And so, the two assumptions we started with are not compatible.

In his argument, Putnam in fact also uses a third assumption, namely:

Assumption 3. Knowing the meaning I of a term A determines I and A.

Putnam gives the following argument for the correctness of this last assumption. If you know the meaning of A, then you also know that the meaning you have grasped is the meaning of A. Therefore if A and B are different

[1] As Albert Visser once remarked, it would be better to call these states not psychological states, but psychic states.

terms, then knowing the meaning of A is a different state from knowing the meaning of B. Likewise, if I_1 and I_2 are different meanings, then knowing that I_1 is the meaning of A is a different state from knowing that I_2 is the meaning of A. Although this argument may seem plausible, it is certainly not an analytic truth that grasping the meaning I of A implies that you know that you have grasped I and that what you have grasped is the meaning of A. In what follows, we will challenge the correctness of Assumption 3.

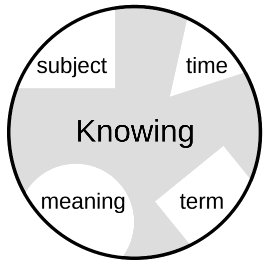
For Putnam psychological states are a special kind of states, where he defines a state as a two-place predicate whose arguments are an individual and a time. Putnam differentiates between psychological states in a wide sense and psychological states in a narrow sense. The psychological states in a narrow sense are the states permitted by methodological solipsism. The Twin Earth argument concerns psychological states in the narrow sense.

A property of states of relations is that they can hold. But this property is not true for Putnam's states, which are two-place predicates. However, I think it is not a problem to treat Putnam's states as *types* of states of relational models, i.e. as functions $\lambda x\, \lambda t\ [\, S(x,t)\,]$, where x stands for an individual and t for a time, and $S(x,t)$ is a relational state for any particular x and t. So, also psychological states will be regarded here as types of states of relational models.

What do models for the knowing-relation look like? What our models have in common is a domain of objects D, a set of states S, and a set H of states that hold. The domain D will contain objects like Oscar_1 and the term 'water'. In our models we have no typed domains for the objects. However, for the present argument this will not be an essential limitation. The states of S could be described in sentences like "Oscar_1 knows at time t that I_1 is the meaning of the term 'water' ".

A coordinated model $\langle D, n, S, \Gamma, H\rangle$ for the 'knowing'-relation corresponds directly with the predicate: "__ knows at time __ that __ is the meaning of __". Because, for our analysis, coordinated models do not have anything of importance to offer in comparison with positional models, we will ignore them in the sequel of our discussion.

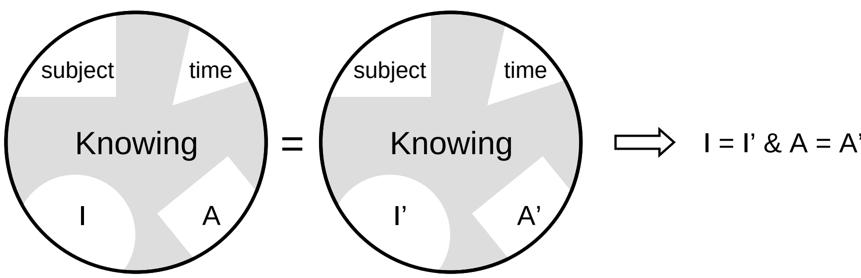
A plausible positional model $\mathcal{M}$ for the 'knowing'-relation is $\langle D, P, S, \Gamma, H\rangle$, with P the positions 'subject', 'time', 'meaning', and 'term'. Inspired by Fine's 'solid body' with holes for picturing positional relations [Fin00, 10], we picture $\mathcal{M}$ as follows:



If we denote $\Gamma(\begin{pmatrix} \text{subject} & \text{time} & \text{meaning} & \text{term} \\ X & T & I & A \end{pmatrix})$ as $\Gamma(X, T, I, A)$, then Assumption 3 would be true if

$$\lambda x\, \lambda t\, [\, \Gamma(x, t, I, A)\,] = \lambda x\, \lambda t\, [\, \Gamma(x, t, I', A')\,] \Rightarrow I = I' \quad \& \quad A = A'.$$

This condition can be depicted as:



Let us try to determine what kind of support we have for the validity of this condition. Since the knowing-relation has obviously no strict symmetry, the condition might perhaps look trivially true. However, in positional models it is possible that for some f and g with $\mathsf{im}\, f \neq \mathsf{im}\, g$ we have $\Gamma(f) = \Gamma(g)$. Even for positional models that correspond to substitution models such situations can occur, as we showed in Remark 4.6.9. But, as we showed in Theorem 4.6.6, it is also the case that if the positional model $\mathcal{M}$ corresponds to a substitution model – as is very reasonable to assume – then $\mathcal{M}$ respects substitution, i.e.

$$\forall f, g\ [\Gamma(f) = \Gamma(g) \Rightarrow \forall \delta : D \to D\ [\, \Gamma(\delta \circ f) = \Gamma(\delta \circ g)\,]].$$

This would mean that if $\Gamma(P, T, I, A) = \Gamma(P, T, I', A)$ with $I \neq I'$, then we could substitute any meaning I'' for I without changing the state. In other words, the meaning is completely irrelevant for the state. It seems clear that this cannot be correct for an adequate model for the 'knowing'-relation, and so we may want to conclude that Assumption 3 must be true. Yet, I don't think this argument is completely indisputable. The following objections could be made:

- There is no compelling reason to assume that an adequate positional model for the 'knowing'-relation corresponds to a substitution model.
- An adequate positional model for the 'knowing'-relation might perhaps not have four positions.
- It is not sure that an adequate positional model for the 'knowing'-relation is possible.

In itself, the last objection is not a real problem, since it could still be argued that the 'knowing'-relation can be adequately modeled by a substitution model. So, suppose we have a substitution model for the 'knowing'-relation. I don't think it is obvious that all states must have exactly one meaning and one term as objects. On what grounds could we exclude that certain states have more than one meaning or more than one term as objects? Even an infinite number of meanings $I, I', I'', \ldots$ for certain states might not be absurd. Because such considerations and objections seem hard to refute without making additional assumptions, my conclusion is that Assumption 3 is not true beyond any doubt.

The analysis we just gave of Putnam's Twin Earth argument reveals an interesting artefact of positional models. Let us assume that ontologically any state of affairs has a unique set of objects. Then this does not automatically imply that for any adequate positional model the objects of a state are always exactly the objects assigned to the positions of a positional model. It might happen that for certain states the objects assigned to the positions form a proper superset of the 'real' objects of the state. The 'extra' objects fulfill a kind of dummy role, since they have no effect on the state. That is, if we would substitute any other object on these positions, this would not change the state. I consider such dummy objects of a state as an artefact of positional models with a fixed number of positions. Whether such misleading situations also occur in practice in positional representations, I do not know.

Chapter 8

Conclusions

My aim in this thesis was to develop models for relations that would give a better understanding of the essence of relations. To what extent did I accomplish this goal? Let me recapitulate the main results. We developed mathematical models for the views on relations that Fine described in his paper “Neutral relations”. We proved that the *simple substitution models* correspond in a unique way with a subtype of the positional models (modulo positional variants). Further we argued that the simple substitution models adequately model a large class of relations, which we called the *plain relations*.

This result can be interpreted in two directions. It means support for the antipositionalist view. It shows that for plain relations the primitive notion of substitution has not less expressive power than the use of positions has. On the other hand, the positionalist can claim for the same class of relations that his use of positions is innocent compared with the use of substitutions.

One of the objections Fine raised against the positionalist view was that it could not handle symmetric relations. But, as I argued in Section 2.5, this is only an argument against a positionalist who would claim that objects *occupy* positions in relational states. If a positionalist only claims a mediating role for positions, then this argument of Fine does not apply.

Do we have arguments to prefer one view over the other? A strong argument for the antipositionalist view is that it is based on the very general notion of substitution, a primitive kind of operation. I don’t think that for a positional approach a similar claim can be made. The positionalist could point out that there are positional models that do not correspond to substitution models, but the converse is also true. Moreover, for positional models that do not correspond to substitution models it seems highly unlikely that they could adequately model 'real' relations. These models probably have no metaphysical significance at all for relations. In defense of a positional approach, it may be claimed that positional representations are very natural and practical. But that does not mean that they are very basic. In fact it is not an argument for a positionalist view, but only for the use of a certain representation. I conclude that the results of this thesis give extra support to the antipositionalist view of Fine, but that they also give a justification for the use of positional representations for relations.

The models and theory developed in this thesis can have more use than their contribution in reaching the conclusion we draw in the previous paragraph. The models can be useful tools for analyzing relations, and also for empirical research on how we 'learn' relations. With respect to learnability, we showed in Section 6.3 that the positional structure of plain relations can in principle be learned by a finite number of substitutions. Also further theoretical study of substitution models might be helpful to deepen our understanding of the structure of relations. We already mentioned a promising approach to define objects of states in substitution models in terms of ultrafilters.

I want to conclude with a remark about the approach of this thesis. I started with questions about the structure of 'real' relations. To find an answer to these questions, I developed mathematical models that highlighted certain aspects of relations. By playing with the models and studying properties of them, I got ideas for new models and I discovered some interesting connections between them. Subsequently, I could translate this insight back to characteristic properties of 'real' relations.

I think that this kind of approach might be fruitful also for other areas of metaphysics. It is sad that in metaphysics there is still hardly any consensus about almost anything. I think it only has any value to make ontological claims about our world if the claims are accompanied by very strong arguments. Developing and analyzing mathematical models or axiomatic systems might provide the right level of certainty in this respect. Doing physics without mathematics is hardly thinkable, but strangely enough, many seem to think that metaphysics can largely do without it. A reason to refrain from using rigorous mathematical methods for metaphysics might be that such methods are considered too difficult to handle for this subject. This might be true in some cases, but if we want to take metaphysics seriously, I see as yet no other way.

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