Prolegomena to The Categorical Study of Interpretations

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Abstract

In this paper, we provide basic facts about the category INT of interpretations. E.g., we give a characterization of its epimorphisms and we show that, modulo a small detail, its opposite category is regular and even coherent. We also study a salient subcategory, the category of direct interpretations.

Key words: interpretations, categories

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Contents

1 Introduction

In [Vis06], I studied several categories of interpretation. The main focus of that paper was on questions like the use of such categories to provide notions of sameness of theories, or to analyze Tarski's theorem on the undefinability of truth, or to explicate the notion of axiom scheme. In the present paper, I adress the more modest question of which salient properties from the literature two categories, to wit INT and $\mathsf{INT}_{\mathsf{dir}}$ possess. This question is asked without regard to possible applications. Thus, the contents of the paper should be characterized as being *basic groundwork*.

We provide a careful definition of our categories INT and $\mathsf{INT}_{\mathsf{dir}}$. We verify that the category has certain basic closure properties, like the existence of products and finite colimits. We show that, if we enrich the category with a bottom, the opposite of the resulting category is regular and even coherent.

We also study direct interpretations. These are unrelativized interpretations that send identity to identity. Direct interpretations are important, e.g. for the study of notions like sequentiality.

2 Basic Definitions

We consider Δ^1_b-axiomatized theories in predicate logic of finite signature with identity.[1] Interpretability will be *one-dimensional relative interpretability without parameters where identity is not necessarily translated as identity.* We will specify this notion below.

Officially, we demand that our theories have relational languages, i.e. that they are without function symbols or constants. However, we will be sloppy about this. Often we will speak about languages as if they have function symbols and constants. We assume implicitly that such function symbols and constants are eliminated using the well know translation algorithm of functional to relational languages.

There are some reasonable variants of interpretability, that we will not consider. In many cases, our results will immediately translate to analogous results for the variants. In some cases, there will be a real difference. The variants are the following. We could consider many-sorted theories, where we now restrict ourselves to the one sorted case. We could consider multidimensional interpretations, where an object of the interpreted theory is represented by a sequence of objects of the interpreting theory. For example, if we interpret a points-and-lines version of Euclidean Geometry into a points-only version, we will interpret a line as a pair of points. Finally, we could consider interpretations with parameters.[2]

To define an interpretation, we first need the notion of translation.

[1] The demand on the complexity of the axiom set is not as restrictive as it seems, since we often can diminish the complexity of the axiom set using versions of Craig's trick.

[2] There is not even full clarity about what the 'right' fully general definition of interpretation with parameters is. Closer investigation seems to be long overdue.

2.1 Relative Translations

Let Σ and Θ be finite signatures. A *relative translation* $\tau : \Sigma \to \Theta$ is given by a pair $\langle \delta, F \rangle$. Here δ is a Θ-formula representing the *domain* of the translation. We demand that δ contains at most v_0 free. The mapping F associates to each relation symbol R of Σ with arity n an Θ-formula $F(R)$ with variables among $v_0, \ldots, v_{n-1}$.

We will write: $v_0, \ldots, v_{n-1} : \delta$ for $\delta v_0 \wedge \cdots \wedge \delta v_{n-1}$. We translate Σ-formulas to Θ-formulas as follows:

- $(R(y_0, \cdots, y_{n-1}))^\tau := F(R)(y_0, \cdots, y_{n-1})$;
  here $F(R)(y_0, \cdots, y_{n-1})$ is our sloppy notation for:

  $$F(R)[v_0 := y_0, \cdots, v_{n-1} := y_{n-1}],$$

  the result of substituting the y_i for the v_i; we assume that some mechanism for α-conversion is built into our definition of substitution to avoid variable-clashes;
- $(\cdot)^\tau$ commutes with the propositional connectives;
- $(\forall y\, A)^\tau := \forall y\, (\delta(y) \to A^\tau)$;
- $(\exists y\, A)^\tau := \exists y\, (\delta(y) \wedge A^\tau)$.

Suppose τ is $\langle \delta, F \rangle$. Here are some convenient conventions and notations.

- We write δ_τ for δ and F_τ for F.
- We write R_τ for $F_\tau(R)$.
- We will always use '=' for the identity of a theory. In the context of translating, we will however switch to 'E'. So, E_τ is the translation of identity.
- We write $\vec{x} : \delta$ for: $\delta(x_0) \wedge \ldots \wedge \delta(x_{n-1})$.
- We write $\forall \vec{x} : \delta\; A$ for: $\forall x_0 \ldots \forall x_{n-1}\; (\vec{x} : \delta \to A)$.
- We write $\exists \vec{x} : \delta\; A$ for: $\exists x_0 \ldots \exists x_{n-1}\; (\vec{x} : \delta \wedge A)$.

We can compose relative translations as follows:

- $\delta_{\tau\nu} := (\delta_\nu \wedge (\delta_\tau)^\nu)$,
- $R_{\tau\nu} = \vec{v} : \delta_{\tau\nu} \wedge (R_\tau)^\nu$.

We write $\nu \circ \tau := \tau\nu$. Note that $(A^\tau)^\nu$ is provably equivalent in predicate logic to $A^{\tau\nu}$. The identity translation $\mathsf{id} := \mathsf{id}_\Theta$ is defined by:

- $\delta_{\mathsf{id}} := (v_0 = v_0)$,
- $R_{\mathsf{id}} := R(v_0, \ldots, v_{n-1})$.

Note that translations as defined here only have good properties modulo provable equivalence. E.g., $\delta_{\mathsf{id}\circ\mathsf{id}} = (v_0 = v_0 \wedge v_0 = v_0)$, which is not strictly identical to δ_{id}.

An translation τ is called *direct* if it is unrelativized (i.o.w., if its domain is given by $v_0 = v_0$), and if it sends identity to identity (i.o.w., $E_\tau = (v_0 = v_1)$).

2.2 Relative Interpretations

A translation τ supports a *relative interpretation* of a theory U in a theory V, if, for all U-sentences A, $U \vdash A \Rightarrow V \vdash A^\tau$. (Note that this automatically takes care of the theory of identity. Moreover, it follows that $V \vdash \exists v_0\ \delta_\tau$.) We will write $K = \langle U, \tau, V\rangle$ for the interpretation supported by τ. We write $K : U \to V$ for: K is an interpretation of the form $\langle U, \tau, V\rangle$. If M is an interpretation, τ_M will be its second component, so $M = \langle U, \tau_M, V\rangle$, for some U and V.

Par abus de langage, we write 'δ_K' for: δ_{τ_K}; 'P_K' for: P_{τ_K}; 'A^K' for: A^{τ_K}, etc.

Suppose T has signature Σ and $K : U \to V$, $M : V \to W$. We define:

- $\mathsf{id}_T : T \to T$ is $\langle T, \mathsf{id}_\Sigma, T\rangle$,
- $M \circ K : U \to W$ is $\langle U, \tau_M \circ \tau_K, W\rangle$.

We identify two interpretations $K, K' : U \to V$ if:

- $V \vdash \delta_K \leftrightarrow \delta_{K'}$,
- $V \vdash \delta_K \to (P_K \leftrightarrow P_{K'})$.

One can show that modulo this identification, the above operations give rise to a category INT of theories and interpretations.

A interpretation is *direct* iff its underlying translation is direct. The restriction of INT to direct interpretations is $\mathsf{INT}_{\mathsf{dir}}$.

Remark 2.1 A problem with our definition of our categories is the direction of the arrows. The direction that we have chosen feels like the natural one. Also, it was previously used e.g. in [Háj70] and [Vis06]. Moreover, it coheres with the extensive tradition in degrees of interpretability. On the other hand, it is opposite to the direction in boole algebras. *We* have the inconsistent theory on top, *they* have absurdum at the bottom. Also, with our direction, the MOD-functor that sends a theory to its class of models will be contravariant. Finally, as we will see in this paper, nice properties like regularity are enjoyed by the opposite category. ❑

We end this section with the definitions of a few important special morphisms and operations on morphisms.

- Suppose $U \subseteq V$, i.e. V extends U in the same language, say with signature Σ. We take $\mathcal{E}_{UV} := \langle U, \mathsf{id}_\Sigma, V\rangle$.
- Consider $K : U \to V$. We define $K^{-1}[V]$ as the theory with as theorems $\{A \mid V \vdash A^K\}$. We can find a suitably effective axiomatization of $K^{-1}[V]$ using Craig's trick. We define $\breve{K} := \langle K^{-1}[V], \tau_K, V\rangle$.

Note that K is equal to $\breve{K} \circ \mathcal{E}_{U,K^{-1}[V]}$. We will see that $\mathcal{E}_{U,K^{-1}[V]}$, $\breve{K}$ is an epi-mono factorization of K that is both an image factorization and a coimage factorization.

3 Basic facts about INT and $\mathsf{INT}_{\mathsf{dir}}$

In this section, we collect the basic facts about our categories. Part of the material is also found in [Vis06].

3.1 The Embedding of $\mathsf{INT}_{\mathsf{dir}}$ in INT

Let EMB be the embedding functor from $\mathsf{INT}_{\mathsf{dir}}$ in INT. We will show that EMB has a left adjoint DIR. Here is the definition. Consider a theory U. We take as signature of $V := \mathsf{DIR}(U)$, the signature of U enriched by two new symbols E^* and Δ. The theory V is the result of replacing '$=$' in the axioms of U by 'E^*' (including the axioms concerning identity provided by predicate logic), of adding the axioms of identity for the outer identity '$=$' for the full language, of relativizing the quantifiers in the axioms of U to Δ, and of adding, for each predicate symbol except '$=$', he following axiom.

- $\vdash \forall \vec{v}\, (P\vec{v} \to \vec{v} : \Delta)$.

We define $\eta : U \to V$ as follows:

- $\delta_\eta := \Delta$,
- $P_\eta := P$, $E_\eta := E^*$.

Suppose $K : U \to W$. We define a direct interpretation $M : V \to W$ as follows:

- $\Delta_M := \delta_K$,
- $P_M\vec{v} := (\vec{v} : \delta_K \wedge P_K\vec{v})$, $E^*_M v_0 v_1 := (v_0, v_1 : \delta_K \wedge E_K v_0 v_1)$.

We can easily verify that M is the unique direct interpretation that makes the following diagram commute.

$$\begin{tikzcd} & W \\ U \arrow[ur, "K"] \arrow[r, "\eta"'] & V \arrow[u, dotted, "M"'] \end{tikzcd}$$

These data determine the left adjoint DIR.

We end this subsection, by proving a useful lemma.

Lemma 3.1 *Let $V := \mathsf{DIR}(U)$. Let H be any V-formula not containing $=$. Consider any variables $\vec{y};\vec{z}$, such that $\vec{y}$ and $\vec{z}$ are disjoint and the free variables of H are among $\vec{y};\vec{z}$. Then there is a U-formula H_0 with free variables among $\vec{y}$, that $V \vdash (\vec{y} : \Delta \wedge \vec{z} : \Delta^c) \to (H \leftrightarrow H_0^\eta)$.*

Proof

The proof is by induction on H. First we treat the atomic case. Let $H := P\vec{x}$, where $\vec{x}$ is a subset of $\vec{y}, \vec{z}$. In case $\vec{x}$ is contained in $\vec{y}$ we take $H_0 := P\vec{x}$. In case, some x in $\vec{x}$ is in $\vec{z}$, we take $H_0 := \bot$. The cases of E^* and Δ are similar.

The cases of the propositional connectives are trivial.

Suppose $H\vec{x} = \forall u\ J(u, \vec{x})$. Then, H is equivalent to:

$$\forall u : \Delta\ J(u, \vec{x}) \wedge \forall u : \Delta^c\ J(u, \vec{x}).$$

Let $\vec{y};\vec{z}$ cover $\vec{x}$. We may assume that u does not occur in $\vec{y};\vec{z}$. Let $J_0(u, \vec{y})$ and $J_1\vec{y}$ be the formulas provided by the induction hypothesis for, respectively, $\vec{y}, u; \vec{z}$ and $\vec{y}; \vec{z}, u$. Then, we can take $H_0\vec{y} :\leftrightarrow \forall u\ J_0(u, \vec{y}) \wedge J_1\vec{y}$. ❑

Example 3.2 An interpretation $K : U \to V$ is *surjective* if, for every sentence A in the language of V, there is a sentence B in the language of U, such that $V \vdash A \leftrightarrow B^K$. In [Vis06] we proved that a morphism is surjective iff it is epi in INT_3. From the results of Subsection 3.4 it will follow that every epimorphism of INT is surjective. Here we give an example of a surjective morphism which is not epi in INT.

Consider U and $V := \mathsf{DIR}(U)$. Let W be the extension of V with the axiom:

- $\vdash x = y \leftrightarrow ((\neg \Delta x \wedge \neg \Delta y) \vee (\Delta x \wedge \Delta y \wedge xE^*y))$.

Let $\eta^+ := \mathcal{E}_{VW} \circ \eta$. We see that every W-formula is equivalent to a formula without $=$. Thus, by Lemma 3.1, η^+ is surjective. It is easily seen that, if U is consistent, η^+ is not an epimorphism. ❑

3.2 Isomorphisms

Isomorphism in INT is usually called *definitional equivalence* or *synonymy.* It is easily seen that isomorphisms are direct, so INT and $\mathsf{INT}_{\mathsf{dir}}$ have the same isomorphisms.

Consider a theory T with signature Σ. Let T' be a theory with signature Σ'. We say that T' is a *definitional extension* of T iff Σ' extends Σ and the axioms of T' are the axioms of T plus, for each predicate symbol P of $\Sigma' \setminus \Sigma$, an axiom of the form: $\vdash P\vec{x} \leftrightarrow A\vec{x}$, where A is in the language of T with at most $\vec{x}$ free. We have the following easy theorem.

Theorem 3.3 *Any definitional extension of a theory is definitionally equivalent to that theory.*

Note that it essential that we really pin down the meanings of the symbols in the extension of the signature. If we start with a theory and just extend the signature adding no extra axioms, the result will generally not be isomorphic to the original theory.

3.3 Monomorphisms

An interpretation $K : U \to V$ is *faithful* iff, for all U-sentences A, $U \vdash A \Leftrightarrow V \vdash A^K$. In [Vis06], we have shown that an interpretation is faithful iff it is a monomorphism in INT. It is easy to see that the proof also works for $\mathsf{INT}_{\mathsf{dir}}$. Thus, a direct interpretation is faithful iff it is a monomorphism in $\mathsf{INT}_{\mathsf{dir}}$.

In Subsection 4.2, we will show that all monomorphisms are cocovers (in both categories). In Subsection 4.4, we will show that all monomorphisms are regular (in both categories).

3.4 Epimorphisms

This subsection substantially improves the treatment in [Vis06]. We show that each epimorphism can be written as $J \circ \mathcal{E}$, where J is an isomorphism and $\mathcal{E}$ is a theory extension.

Lemma 3.4 *In* INT*, epimorphisms are direct.*

Proof

Suppose $K : U \to V$ is an epimorphism. We prove that K is direct. First suppose $V \nvdash \forall x\ \delta_K(x)$. Let V_0 be the following theory.

- The signature of V_0 is the signature of V expanded by a constant c and a binary predicate symbol E^*.
- V_0 is axiomatized by the axioms of V (including the axioms for identity) with $=$ replaced by E^*, plus the usual axioms for $=$, plus the axiom $\vdash \neg\,\delta'_K(c)$, where δ'_K is the result of replacing $=$ by E^* in δ_K, and the axiom $\vdash \exists y\ (cE^*y \wedge c \neq y)$.

Clearly, V_0 is consistent. We may now interpret V in V_0 via two interpretations M and N.

- $\delta_M v_0 := (v_0 = v_0)$, $v_0 E_M v_1 := v_0 E^* v_1$, $P_M \vec{v} := P\vec{v}$.
- $\delta_N v_0 := (v_0 \neq c)$, $v_0 E_N v_1 := v_0 E^* v_1$, $P_N \vec{v} := P\vec{v}$.

We clearly have that M and N are not equal and that $M \circ K$ is equal to $N \circ K$. A contradiction.

Next suppose that $V \vdash \forall x\ \delta_K(x)$ and $V \nvdash \forall x, y\ (xE_K y \to x = y)$. Let V_1 be the following theory.

- The signature of V_1 is the signature of V expanded by new constants c, d and e.
- The theory V_1 is axiomatized by the axioms of identity, plus the relativization of V to the domain δ given by $v_0 \neq c$, including axioms of the form $P\vec{v} \to \vec{v} : \delta$, plus the axioms $c \neq d$, $c \neq e$, $d \neq e$ and $dE'_K e$, where E'_K is the relativized variant of E_K.

It is easily seen that V_1 is consistent. We define two interpretations M and N of V into V_1 as follows.

- $\delta_M v_0 := (v_0 = v_0)$, $xE_M y := (x = y \vee (x = c \wedge y = d) \vee (x = d \wedge y = c))$, $P_M \vec{v} := \exists \vec{w}\ (\vec{w} E_M \vec{v} \wedge P\vec{w})$.
- $\delta_N v_0 := (v_0 = v_0)$, $xE_N y := (x = y \vee (x = c \wedge y = e) \vee (x = e \wedge y = c))$, $P_N \vec{v} := \exists \vec{w}\ (\vec{w} E_N \vec{v} \wedge P\vec{w})$.

We clearly have that M and N are not equal and that $M \circ K$ is equal to $N \circ K$. A contradiction.

We may conclude that K is direct. ❑

We can split any interpretation $K : U \to V$ into $\mathcal{E}_{U,K^{-1}[V]}$ and $\breve{K} : K^{-1}[V] \to V$. Note that $\breve{K}$ is faithful, and, hence, a monomorphism. Moreover, any $\mathcal{E}_{WZ}$ is trivially an epimorphism. Thus, we can factor K into an epi and a mono: $K = \breve{K} \circ \mathcal{E}_{U,K^{-1}[V]}$.

Theorem 3.5 *The following holds both in* INT *and in* $\mathsf{INT}_{\mathsf{dir}}$. *Suppose* $K : U \to V$. *The following are equivalent.*

1. *K is an epimorphism.*
2. *$\breve{K}$ is an isomorphism.*
3. $\mathsf{MOD}(K)$ *is injective.*

Here MOD *is the usual contravariant functor sending a theory to its class of models and an interpretation to the corresponding inner model construction.*

Proof

(1) $\Rightarrow$ (2). Suppose that $K : U \to V$ is epi. By Lemma 3.4, we know that K is direct. Suppose that U has signature Σ and that V has signature Θ. We arrange it so that Σ and Θ are disjoint, except that they share the identity symbol. Let Θ' be a copy of Θ, disjoint from Σ and Θ except again for the shared identity symbol. Let Z be the theory of signature $\Sigma \cup \Theta$ axiomatized by the axioms of V plus axioms of the form $\forall \vec{x}\, (P\vec{x} \leftrightarrow P_K\vec{x})$, for all P in Σ. Let Z' be the similar theory in signature $\Sigma \cup \Theta'$. Let be the theory in signature $\Sigma \cup \Theta \cup \Theta'$, with $W = Z \cup Z'$.

Let M and N be the two obvious interpretations of V in W.[3] Clearly, $M \circ K$ is equal to $N \circ K$. Since K is epi, we find that M is equal to N. This means that: $Z, Z' \vdash \forall \vec{x}\, (Q\vec{x} \leftrightarrow Q'\vec{x})$, for all corresponding Q, Q' in Θ, resp. Θ'. By Beth's Theorem, we may now conclude that, for each Q, there is a formula A_Q in Σ, such that $Z \vdash \forall \vec{x}\, (Q\vec{x} \leftrightarrow A_Q\vec{x})$. Thus, V and $K^{-1}[V]$ are both definitional extensions of each other. Let $L : V \to K^{-1}[V]$ be the direct interpretation based on $Q \mapsto A_Q$. It is easily seen that L is the inverse of $\breve{K}$.

(2) $\Rightarrow$ (3). This is trivial.

(3) $\Rightarrow$ (1). Suppose that $\mathsf{MOD}(K)$ is injective. Suppose that $M, N : V \to W$ and $M \circ K$ is equal to $N \circ K$. To obtain a contradiction, we assume that M is not equal to N. This implies that, for some model $\mathcal{M}$ of W, the models $\mathcal{N} := \mathsf{MOD}(M)(\mathcal{M})$ and $\mathcal{N}' := \mathsf{MOD}(N)(\mathcal{M})$ are different. (Note that the relevant notion of difference is numerical distinctness here.) On the other hand we have:

$$\begin{array}{rcl}
\mathsf{MOD}(K)(\mathcal{N}) & = & \mathsf{MOD}(K)(\mathsf{MOD}(M)(\mathcal{M})) \\
& = & \mathsf{MOD}(M \circ K)(\mathcal{M}) \\
& = & \mathsf{MOD}(N \circ K)(\mathcal{M}) \\
& = & \mathsf{MOD}(K)(\mathsf{MOD}(N)(\mathcal{M})) \\
& = & \mathsf{MOD}(K)(\mathcal{N}')
\end{array}$$

This contradicts the injectivity of $\mathsf{MOD}(K)$. ❑

We can consider the epimorphisms with codomain U as superobjects of U. These can be ordered by $\leq$, given by $M \leq N :\Leftrightarrow\ \exists K\ K \circ M = N$. The following theorem shows that the structure of the superobjects with $\leq$, is, modulo isomorphism, precisely the structure of the extensions of U in the same language with $\subseteq$.

Theorem 3.6 *Suppose that $M : U \to V$ and $N : U \to W$ are both epimorphisms. Then, the following are equivalent.*

a. For some $K : V \to W$, we have $K \circ M = N$.

b. $M^{-1}[V] \subseteq N^{-1}[W]$.

[3] The interpretations M and N are, in $\mathsf{INT}_{\mathsf{dir}}$, a cokernel pair of K.

Proof

$(a) \Rightarrow (b)$. It is easy to see that $M^{-1}[V] \subseteq (K \circ M)^{-1}[W]$.

$$
\begin{tikzcd}
V \arrow[rr, "K"] \arrow[d, shift right, "\hat{M}"'] & & W \arrow[d, shift right, "\hat{N}"'] \\
M^{-1}[V] \arrow[rr, "\mathcal{E}"] \arrow[u, shift right, "\breve{M}"'] & & N^{-1}[W] \arrow[u, shift right, "\breve{N}"'] \\
& U \arrow[ul, "\mathcal{E}"] \arrow[ur, "\mathcal{E}"'] &
\end{tikzcd}
$$

$(b) \Rightarrow (a)$. By Theorem 3.5, $\breve{M}$ has an inverse, say $\hat{M}$ and $\breve{N}$ has an inverse, say $\hat{N}$. So, we may take $K := \breve{N} \circ \mathcal{E} \circ \hat{M}$ ❑

More on superobjects in Section 5.

3.5 Initial Object

The category $\mathsf{INT}_{\mathsf{dir}}$ has as initial object the theory ID of pure identity. Note that ID is not *strict*: it may be codomain of an arrow that is not an isomorphism. In other words, $\mathsf{INT}_{\mathsf{dir}}$ has many non-initial, weak initial objects.

The category INT obviously has no initial object. This defect is caused by a design choice. We opted to work in ordinary predicate logic and not in free logic, thus excluding empty domains. For some purposes it is convenient to have an initial object. We will realize this by simply adding a formal object $\Box$ to INT, stipulating a unique arrow from $\Box$ to any theory and to $\Box$ itself, and adding the obvious compositions. Thus, we obtain the category $\mathsf{INT}(\Box)$. Note that if INT has limits of some kind then so has $\mathsf{INT}(\Box)$. Moreover, if INT has binary sums, then so does $\mathsf{INT}(\Box)$. Similarly for coequalizers.

We will show below that INT is closed under binary sums and coequalizers. It follows that $\mathsf{INT}(\Box)$ is closed under finite sums and coequalizers. By a well known theorem, we may conclude that $\mathsf{INT}(\Box)$ is closed under finite colimits. (See [Mac71], p109.)

Open Question 3.7 The category INT has many non-isomorphic *weak* initial objects. It seems to be a quite interesting question to characterize these modulo isomorphism. Also, are there properties in terms of the category that single out the pure predicate logics among the initial objects? ❑

3.6 Final Object

We allow the inconsistent theory in any signature. All these theories will be isomorphic in both categories and provide us with final objects. Note that final objects are *strict*: an arrow starting from a final object is an isomorphism.

3.7 The Cartesian Product

The cartesian product in INT will be called: $\otimes$. The reasons to prefer this notation to the more obvious $\times$, are as follows. (i) If we would write $\times$, we should also write $+$ for the sum. However, we use e.g. '$T + A$' for the theory T extended with the sentence A. So, this would introduce an unwanted ambiguity. (ii) No tensor products in sight in this area. The product of $\mathsf{INT}_{\mathsf{dir}}$ will be: $\boxtimes$.

The product has the same definition in both categories. The easiest way to introduce it is as follows. First we treat a special case. Consider theories U and V of the same signature such that $U \cup V$ is inconsistent. By a simple compactness argument, we find that there is an A such that $U \vdash A$ and $V \vdash \neg A$. We will call A, a *separating* formula for U and V. (Note: a separating formula is assigned to the *ordered* pair U, V.)

Let W be axiomatized by by the following sets of axioms: axioms of the form $A \to B$, where B is an axiom of U, and axioms of the form $\neg A \to C$, where C is an axiom of V. Clearly, $W = U \cap V$. Note that it follows that we can write U as $W + A$ and V as $W + \neg A$. So, any pair of mutually contradictory theories in the same language is finitely axiomatizable over their intersection.

We claim that W is the cartesian product of U and V in INT. The projections are the standard embeddings $\mathcal{E}_{WU}$ and $\mathcal{E}_{WV}$ witnessing the subtheory relation. Note that these embeddings are direct.

To show that $W = U \otimes V$ with the stated projections, we have to uniquely provide the dotted arrow that makes the following diagram commute.

$$\begin{tikzcd} & T \arrow[dl, "K"'] \arrow[d, dotted, "?"'] \arrow[dr, "M"] & \\ U & W \arrow[l, "\mathcal{E}"] \arrow[r, "\mathcal{E}"'] & V \end{tikzcd}$$

Suppose the translations associated with K, M are τ, ρ. Let A be the chosen separating sentence for U and V. We define a new translation $\tau\langle A\rangle\rho$ as follows.

- $\delta_{\tau\langle A\rangle\rho} := ((A \wedge \delta_\tau) \vee (\neg A \wedge \delta_\rho))$
- $P_{\tau\langle A\rangle\rho} := ((A \wedge P_\tau) \vee (\neg A \wedge P_\rho))$

The interpretation $N := K\langle A\rangle M : T \to W$ is the interpretation corresponding to $\tau\langle A\rangle\rho$. We take:

$$\begin{tikzcd}
& T \arrow[dl, "K"'] \arrow[d, dotted, "N"'] \arrow[dr, "M"] & \\
U & W \arrow[l, "\mathcal{E}"] \arrow[r, "\mathcal{E}"] & V
\end{tikzcd}$$

We may easily verify that $K\langle A\rangle M$ supports the unique morphism that makes our diagram commute. This also shows that equality is a congruence for $(\cdot)\langle A\rangle(\cdot)$. Thus, $(\cdot)\langle A\rangle(\cdot)$ can be viewed as an operation on arrows. Note that modulo equality $(\cdot)\langle\cdot\rangle(\cdot)$ preserves directness. In the context of $\mathsf{INT}_{\mathsf{dir}}$, the obvious modification will make it literally direct.

We defined the cartesian product for a special case. We extend the definition to arbitrary theories as follows. Let U and V be arbitrary with signatures Σ and Θ. Let $\Sigma \cup \Theta$ be the union of our signatures. To make sense of taking the union, it is best to consider the arities to be built in into the symbols, so that, say, a binary predicate P and a ternary predicate P are automatically counted as different predicates.

Consider definitional extensions U' and V' of U and V in the signature $\Sigma\cup\Theta$. In case $U'\cup V'$ is inconsistent, we take, as $U\otimes V$, the intersection $W' := U'\cap V'$. Our first projection becomes $\pi_0 := J \circ \mathcal{E}_{W'U'}$, where $J : U' \to U$ is the standard isomorphism associated with definitional extensions. Similarly, $\pi_1 := L \circ \mathcal{E}_{W'V'}$, where $L : V' \to V$ is an isomorphism. It is easy to see that we have defined a product in this way.

If $U' \cup V'$ is consistent, we extend the signature $\Sigma \cup \Theta$ with a fresh 0-ary predicate symbol P, and replace U' be the definitional extension $U' + P$ and V' by the definitional extension $V' + \neg P$. Now we may proceed as before.

Sometimes it is convenient to have a specific choice for our product. The default will be as follows. We take as new signature the union of the signatures of U and V plus a fresh 0-ary P. The axioms of the product are:

- $\vdash P \to A$, if $\vdash A$ is an axiom of U,
- $\vdash P \to \forall\vec{x}\, R\vec{x}$, if R is a predicate not in the signature of U,
- $\vdash \neg P \to B$, if $\vdash B$ is an axiom of V,
- $\vdash \neg P \to \forall\vec{x}\, S\vec{x}$, if S is a predicate not in the signature of V.

The projection π_0 is direct and sends P to $\top$ and R to $\top$, if R is not in the signature of U. The projection π_1 is direct and sends P to $\bot$ and S to $\top$, if S is not in the signature of V.

Remark 3.8 This remark presupposes familiarity with the results of [Vis06].

Consider $\mathsf{INT}_{\uparrow\mathsf{PA}}$, i.e. INT restricted to extensions of PA in the language of PA. We show that in this category, we do not generally have cartesian products. Consider the following diagram in $\mathsf{INT}_{\uparrow\mathsf{PA}}$.

$$\begin{tikzcd}
& \mathsf{PA} \arrow[dl, "\mathsf{id}"'] \arrow[d, dotted] \arrow[dr, "\mathsf{id}"] & \\
\mathsf{PA} & W \arrow[l, "\pi_0"] \arrow[r, "\pi_1"'] & \mathsf{PA}
\end{tikzcd}$$

Clearly, the π_i are retractions in INT. Moreover W is an extension of PA in the language of PA. It follows that the π_i are also retractions in hINT, aka INT_1, the category where we identify provably isomorphic arrows. We may now apply Corollary 9.4. of [Vis06] to conclude that $W \subseteq \mathsf{PA}$. Hence, $W = \mathsf{PA}$. More specifically, Theorem 9.2 of [Vis06] tells us that the π_i are both isomorphic to the identity interpretation. But, then, for any $M : U \to \mathsf{PA}$, $\pi_0 \circ M$ and $\pi_1 \circ M$ are isomorphic. Thus, our supposed product cannot possibly fulfill its role, as becomes clear by choosing K and L in the following diagram non-isomorphic.

$$\begin{tikzcd}
& \mathsf{PA} \arrow[dl, "K"'] \arrow[d, dotted, "M"] \arrow[dr, "L"] & \\
\mathsf{PA} & W \arrow[l, "\pi_0"] \arrow[r, "\pi_1"'] & \mathsf{PA}
\end{tikzcd}$$

Note that it also follows that $\mathsf{hINT}_{\uparrow\mathsf{PA}}$ has no cartesian product. $\square$

Note that the existence of a left adjoint to EMB implies that, if $\boxtimes$ exists, then $\otimes$ exists and $\otimes$ is equal to $\boxtimes$.

3.8 The Sum

In the present subsection we define sum in INT and $\mathsf{INT}_{\mathsf{dir}}$.[4] We will call the sum of INT: $\oplus$. The sum of $\mathsf{INT}_{\mathsf{dir}}$ will be called: $\boxplus$.

We first treat the case of $\mathsf{INT}_{\mathsf{dir}}$. Consider two theories U and V. Say, U has signature Σ_U and V has signature Σ_V. The sum $U \boxplus V$ is given as a theory W of signature Σ_W, where Σ_W is given as the disjoint union of Σ_U and Σ_V, *where we refrain from duplicating identity.* Let τ_U and τ_V be the obvious direct translations of the languages of U, respectively V into the language of W. We take W to be axiomatized by the following axioms.

[4] A number of details of the sum definition were studied by Spencer Gerhardt in the context of a small project. Specifically, he formulated the definition of sum in INT.

- $\vdash A^{\tau_U}$, for A a U-axiom,
- $\vdash B^{\tau_V}$, for B a V-axiom.

We consider the identity axioms as logical axioms, which are thus shared. The in-arrows are the interpretations based on τ_U and τ_V. It can be easily seen that this defines the sum in $\mathsf{INT}_{\mathsf{dir}}$.

We define $U \oplus V$ as $\mathsf{DIR}(U) \boxplus \mathsf{DIR}(V)$, extended by the following axioms.

- $\vdash \forall x\ (x : \Delta_U \vee x : \Delta_V)$,
- $\vdash x = y \leftrightarrow \forall z\ ((xE_U z \leftrightarrow yE_U z) \wedge (xE_V z \leftrightarrow yE_V z))$.

Note that, in the presence of the other axioms, the last axiom says that $=$ is the crudest congruence relation respecting all the all the predicates of W. The new in-arrows are the obvious ones. E.g., $\mathsf{in}_U = \mathcal{E}_{\mathsf{DIR}(U),W} \circ \mathsf{in}_{\mathsf{dir},\mathsf{DIR}(U)} \circ \eta_U$. The extra axioms are needed to insure the uniqueness condition for the sum. It is easy to check that $\oplus$ is the sum for the category INT.

3.9 Equalizers: a Counterexample

We show that neither INT nor $\mathsf{INT}_{\mathsf{dir}}$ has equalizers. In case of INT, this is trivial, since we may have $M, N : U \to V$, where V does not prove that the intersection of δ_M and δ_N is non-empty. However, this argument rests on the fact that we do not allow non-empty domains. We provide a counterexample, that works in both categories, where there are many K such that $M \circ K = N \circ K$, but there is no equalizer of M and N.

Let U be predicate logic in the language with a constant symbol 0, a unary function symbol S, a unary predicate symbol P and the identity symbol. Let V be a theory in the language with a constant 0, a unary function symbol S, two unary predicate symbols Q and R, and the identity symbol. The theory V is axiomatized by all axioms of the form $\vdash Q\underline{n} \leftrightarrow R\underline{n}$, where $\underline{0} := 0$, $\underline{n+1} := \mathsf{S}(\underline{n})$.

We define $M, N : U \to V$ as direct interpretations which preserve S and 0, with $P_M v_0 := Qv_0$ and $P_N v_0 := Rv_0$. Consider the signature of the language of predicate logic with just the identity symbol and one zero-ary predicate symbol S. Let Z be predicate logic in this language. Let $L_n : Z \to U$ be the direct interpretation that sends S to $P\underline{n}$. Note that $M \circ L_n = N \circ L_n$.

Suppose the pair M, N has an equalizer $K : W \to V$. Since, the direct interpretations L_n must factor through K, we find that K must be direct. Let O be any predicate symbol of W. Suppose $O_K \vec{v} = A\vec{v}$. We write $A(0, \mathsf{S}, P, \vec{v})$ to make the depence on symbols of the language visible. In this notation $A^M(0, \mathsf{S}, P, \vec{v}) = A(0, \mathsf{S}, Q, \vec{v})$ and $A^N(0, \mathsf{S}, P, \vec{v}) = A(0, \mathsf{S}, R, \vec{v})$. Since, K is an equalizer, we have $V \vdash \forall \vec{v}\ (O^N_M \vec{v} \leftrightarrow O^N_K \vec{v})$. Thus, (†) $V \vdash \forall \vec{v}\ (A(0, \mathsf{S}, Q, \vec{v}) \leftrightarrow A(0, \mathsf{S}, R, \vec{v}))$. Let us suppose that the proof only involves the axioms $\vdash Q\underline{i} \leftrightarrow R\underline{i}$, for $i = 0, \ldots, k-1$.

We extend the language of V with new propositional constants $p_0, \ldots, p_{k-1}$ and constants $\vec{c}$. We find from (†):

$$\bigwedge_{i<k}(Q\underline{i} \leftrightarrow p_i), A(0, \mathsf{S}, Q, \vec{c}) \vdash \bigwedge_{i<k}(R\underline{i} \leftrightarrow p_i) \to A(0, \mathsf{S}, R, \vec{c})$$

By applying interpolation and some familiar reasoning we find that, for some B,

$$U \vdash A(0, \mathsf{S}, P, \vec{v}) \leftrightarrow B(0, \mathsf{S}, P\underline{0}, \ldots, P(\underline{k-1})).$$

Let $k^\star$ be the maximum of the k corresponding to predicate symbols O of W. Since K is an equalizer, we have, for some $J : Z \to W$, $L_{k^\star} = K \circ J$. Thus, for some C,

$$U \vdash P(\underline{k}^\star) \leftrightarrow C(0, \mathsf{S}, P\underline{0}, \ldots, P(\underline{k^\star-1})).$$

By a simple model-theoretic argument, we see that this is impossible.

Open Question 3.9 Suppose we restrict $\mathsf{INT}_{\mathsf{dir}}$ to finitely axiomatized theories. Do we have equalizers in that case? ❑

3.10 Coequalizers

Suppose that, in INT, we have $M, N : U \to V$. Let W be V plus the following axioms.

- $\vdash \forall v\, (\delta_M v \leftrightarrow \delta_N v)$,
- $\vdash \forall \vec{v}\, (P_M \vec{v} \leftrightarrow P_N \vec{v})$.

It is easy to see that $\mathcal{E}_{VW}$ is a coequalizer for M, N. Similarly, we find coequalizers in $\mathsf{INT}_{\mathsf{dir}}$.

We see that coequalizers are, modulo isomophism, finite extensions. Conversely, each finite extension is a coequalizer. Consider $\mathcal{E} : V \to (V + A)$. Let U be predicate logic with just identity and a zero-ary predicate symbol P. Let $M : U \to V$ be the direct interpretation sending P to $\top$ and let $N : U \to V$ be the direct interpretation sending P to A. Clearly, $\mathcal{E}$ is the coequalizer of M, N.

3.11 Distributivity

We start with distributivity of plus over times in $\mathsf{INT}_{\mathsf{dir}}$. Consider theories U, V and W. We may find U' isomorphic to U, V' isomorphic to V and W' isomorphic to W, such that (i) the language of U' is disjoint, except for identity, from the languages of V' and W' and (ii) the languages of V' and W' are the same and (iii) $V' \cup W'$ is inconsistent.

Clearly, $U \boxplus (V \boxtimes W)$ is isomorphic to $U' \boxplus (V' \boxtimes W')$, which is $U' \cup (V' \cap W')$. (This last theory is a theory in the union of the languages of U' and V', W', with some appropriate axiomatization. Intersection and union of theories are supposed to be applied to theories-as-sets-of-theorems.) We easily see that $U' \cup (V' \cap W') = (U' \cup V') \cap (U' \cup W')$. Moreover, $(U' \cup V') \cap (U' \cup W')$ is

$(U' \boxplus V') \boxtimes (U' \boxplus W')$, since $U' \cup V'$ and $U' \cup W'$ are theories in the same language which are mutually contradictory. This last theory is isomorphic to $(U \boxplus V) \boxtimes (U \boxplus W)$. Inspection of the argument shows that the embedding from $U \boxplus (V \boxtimes W)$ in $(U \boxplus V) \boxtimes (U \boxplus W)$ can be taken to be the canonical one.

The case of distributivity of plus over times in INT is just a variant of the above argument. Note that we do not have full distributivity in $\mathsf{INT}(\Box)$.

For the case of distributivity of times over plus, in both categories, we can reproduce the standard lattice-theoretical reasoning which is used to prove one kind of distributivity from the other kind. We do not get an isomorphism in this way. We *do* get an arrow from $U \boxtimes (V \boxplus W)$ to $(U \boxtimes V) \boxplus (U \boxtimes W)$. Inspection of the hardware shows that the choice here is not unique. We can find such an arrow that witnesses the fact that the canonical arrow from $(U \boxtimes V) \boxplus (U \boxtimes W)$ to $U \boxtimes (V \boxplus W)$ is a split epi/retraction. Similarly for the case of $\otimes$.

There is a nice application of distributivity to compute:

$$T^n := \overbrace{T \otimes \cdots \otimes T}^{n\times} \text{ and } T^{[n]} := \overbrace{T \boxtimes \cdots \boxtimes T}^{n\times}.$$

We have: $T^n = \mathsf{EMB}(T^{[n]})$, so it is sufficient to compute $T^{[n]}$. By distributivity, we have: $T^{[n]} = T \boxplus \mathsf{ID}^{[n]}$. Now consider the category of finite boole algebras with the usual morphisms. With each finite boole algebra $\mathcal{B}$ we can associate a propositional theory $\mathsf{rep}(\mathcal{B})$ in the usual way by choosing a finite set of generators. We send $\mathcal{B}$ to the predicate logical theory $\mathsf{REP}(\mathcal{B}) := \mathsf{ID} + \mathsf{rep}(\mathcal{B})$ in the obvious signature corresponding to the chosen generators. It is easy to see that REP is a functor that preserves products. The finite boole algebras are dually isomorphic to the finite sets, which modulo isomorphism are just the natural numbers. If we take the boole algebra with 0 generators to be $\mathbb{T}$, we get:

$$n = \overbrace{1 + \cdots + 1}^{n\times} \mapsto \overbrace{\mathbb{T} \times \cdots \times \mathbb{T}}^{n\times} \mapsto \mathsf{ID}^{[n]}.$$

Here $\overbrace{\mathbb{T} + \cdots + \mathbb{T}}^{n\times}$ is the boole algebra with n atoms. Thus we can find T^n as follows. Find a formula ϕ that defines the boole algebra with n atoms in a language with propositional variables $p_0, \ldots, p_{k-1}$. Enrich the language of T with fresh 0-ary predicate synbols $P_0, \ldots, P_{k-1}$. then T^n is the theory in the extended language with $T^n = T + \phi(P_0, \ldots, P_{k-1})$. E.g., T^2 can be taken to be the theory T in the language of T extended by a fresh 0-ary predicate symbol P. (This illustrates that the extension of the signature plays a role independent of the formula ϕ!)

4 Image Factorization and Regularity

In this section, we show that both the category INT ($\mathsf{INT}_{\mathsf{dir}}$) and the category $\mathsf{INT}^{\mathsf{op}}$ ($\mathsf{INT}^{\mathsf{op}}_{\mathsf{dir}}$) have images. Moreover, we show that in $\mathsf{INT}^{\mathsf{op}}$ ($\mathsf{INT}^{\mathsf{op}}_{\mathsf{dir}}$) the image factorization is stable. This tells us that the category $(\mathsf{INT}(\Box))^{\mathsf{op}}$ is regular.

4.1 Image Factorization

Consider an arbitrary category $\mathcal{C}$. An morphism $f : a \to b$ has *an image factorization* iff $f = \mathsf{m}(f) \circ g$, where $\mathsf{m}(f) : \mathsf{im}(f) \to b$ is monic, such that, for every g' and h' with $f = h' \circ g'$ and h' is monic, there is a necessarily unique j with $j \circ g = g'$, $h' \circ j = \mathsf{m}(f)$. The category $\mathcal{C}$ *has images* iff every morphism has an image factorization.

$$\begin{tikzcd}
a \arrow[rr, "f"] \arrow[dr, "g"] \arrow[ddr, "g'"'] & & b \\
& \mathsf{im}(f) \arrow[ur, "\mathsf{m}(f)"'] \arrow[d, dashed, "j"] & \\
& c \arrow[uur, "h'"'] &
\end{tikzcd}$$

We first show that INT has an image factorization. Consider $K : U \to V$. We have $K = \breve{K} \circ \mathcal{E}_{U,K^{-1}[V]}$. Here $\breve{K}$ is monic. Consider any $L : U \to W$ and $M : W \to V$ such that $K = M \circ L$ and M is monic. We have $L = \breve{L} \circ \mathcal{E}_{U,L^{-1}[W]}$. Since, $M \circ \breve{L}$ is monic, it is faithful. Remember that $\tau_K = \tau_{\breve{K}}$, $\tau_{M \circ \breve{L}} = \tau_{M \circ L}$. So, V proves that $\tau_{\breve{K}}$ is equal to τ_K is equal to $\tau_{M \circ L}$ is equal to $\tau_{M \circ \breve{L}}$. It follows that $L^{-1}[W] = K^{-1}[V]$, $L = \breve{L} \circ \mathcal{E}_{U,K^{-1}[V]}$, $\breve{K} = M \circ \breve{L}$.

$$\begin{tikzcd}
U \arrow[rr, "K"] \arrow[dr, "\mathcal{E}"] \arrow[ddr, "L"'] & & V \\
& K^{-1}[V] \arrow[ur, "\breve{K}"'] \arrow[d, dashed, "\breve{L}"] & \\
& W \arrow[uur, "M"'] &
\end{tikzcd}$$

Note that the $\mathcal{E}$-arrows are direct. Moreover if K and L are direct, then so are $\breve{K}$ and $\breve{L}$. thus the result also holds in $\mathsf{INT}_{\mathsf{dir}}$.

We proceed to show that $\mathsf{INT}^{\mathsf{op}}$ has images. We translate the problem to INT. Consider $K : U \to V$. We have $K = \breve{K} \circ \mathcal{E}_{U,K^{-1}[V]}$. Here $\mathcal{E}_{U,K^{-1}[V]}$ is epi.

Consider any $L : U \to W$ and $M : W \to V$ such that $K = M \circ L$ and L is epi. As we have seen $\breve{L}$ is an isomorphism, say its inverse is N. Clearly $L^{-1}[W] \subseteq K^{-1}[U]$. We show that $P := \mathcal{E}_{L^{-1}[W],K^{-1}[V]} \circ N$ is the desired witnessing arrow. We have:

$$\begin{aligned}
\mathcal{E}_{U,K^{-1}[V]} &= \mathcal{E}_{L^{-1}[W],K^{-1}[V]} \circ \mathcal{E}_{U,L^{-1}[W]} \\
&= \mathcal{E}_{L^{-1}[W],K^{-1}[V]} \circ N \circ \breve{L} \circ \mathcal{E}_{U,L^{-1}[W]} \\
&= \mathcal{E}_{L^{-1}[W],K^{-1}[V]} \circ N \circ L
\end{aligned}$$

Note that the underlying translation of $M \circ \breve{L}$ is equal in V to the underlying translation of K. Hence $M \circ \breve{L} = \breve{K} \circ \mathcal{E}_{L^{-1}[W],K^{-1}[V]}$. We may conclude that $M = \breve{K} \circ \mathcal{E}_{L^{-1}[W],K^{-1}[V]} \circ N$.

$$\begin{tikzcd}
U \arrow[rr, "K"] \arrow[dr, "\mathcal{E}"] \arrow[ddr, "L"'] & & V \\
& K^{-1}[V] \arrow[ur, "\breve{K}"] & \\
& W \arrow[u, dashed, "P"] \arrow[uur, "M"'] &
\end{tikzcd}$$

Note that our witnessing arrow P is of the form $\mathcal{E} \circ N$, where N is an isomorphism. Since both $\mathcal{E}$-arrows and isomorphisms are direct, P is direct. Thus our result also holds in $\mathsf{INT}_{\mathsf{dir}}$.

4.2 Covers and Cocovers

We have that K is an epimorphism iff $\breve{K}$ is an isomorphism. This tells us that in INT ($\mathsf{INT}_{\mathsf{dir}}$) the epimorphisms are precisely the *covers*. Moreover, K is a monomorphism iff $\mathcal{E}_{U,K^{-1}[V]}$ is the identity. This means that in $\mathsf{INT}^{\mathsf{op}}$ ($\mathsf{INT}^{\mathsf{op}}_{\mathsf{dir}}$) the epimorphisms are precisely the covers. In other words, in INT ($\mathsf{INT}_{\mathsf{dir}}$), the monomorphisms are precisely the cocovers.

4.3 Stability

Next we show that in $\mathsf{INT}^{\mathsf{op}}$ the image factorization is *stable*. This means that images are stable under pullback. See [Jac99], p257. Our argument will also work for $\mathsf{INT}^{\mathsf{op}}_{\mathsf{dir}}$.

We translate stability to what it means in the opposite world of INT. Suppose the following diagram is a pushout.

$$\begin{tikzcd} U \arrow[r, "K"] \arrow[d, "L"'] & V \arrow[d, "\mathsf{in}_1^+"] \\ W \arrow[r, "\mathsf{in}_0^+"'] & Z \end{tikzcd}$$

We have to show that the following diagram is also a pushout.

$$\begin{tikzcd} U \arrow[r, "K"] \arrow[d, "\mathcal{E}"'] & V \arrow[d, "\mathcal{E}"] \\ L^{-1}[W] \arrow[r, "\widetilde{K}"'] & (\mathsf{in}_1^+)^{-1}[Z] \end{tikzcd}$$

The existence of $\widetilde{K}$ is guaranteed by the universal properties of the factorization. We can take Z to be $W \oplus V$ extended by equalizing axioms of the form:

- $\forall x\ ((\Delta_0(x) \wedge \delta_L^{\mathsf{in}_0}(x)) \leftrightarrow (\Delta_1(x) \wedge \delta_K^{\mathsf{in}_1}(x)))$
- $\forall \vec{x} : (\Delta_0 \cap \delta_L^{\mathsf{in}_0})\ (P_L^{\mathsf{in}_0}(\vec{x}) \leftrightarrow P_K^{\mathsf{in}_1}(\vec{x}))$

The in_i^+ are the in_i composed with the appropriate extensions. Computing the pushout of $\mathcal{E}_{U,L^{-1}[W]}$ and K directly, we see that we may take it to be $V + \{A^K \mid W \vdash A^L\}$. Thus, it is sufficient to show that, if $Z \vdash B^{\mathsf{in}_1^+}$, then $V + \{A^K \mid W \vdash A^L\} \vdash B$. We have a closer look at Z. This theory us axiomatized as follows.

P1) Axioms of the form $\vdash A^{\mathsf{in}_0}$, where A is an axiom of W.

P2) Axioms of the form $\vdash B^{\mathsf{in}_1}$, where B is an axiom of V.

P3) Axioms normalizing the predicates of W in Z: $\vdash Q_{\mathsf{in}_0}\vec{x} \to \vec{x} : \Delta_0$.

P4) Axioms normalizing the predicates of V in Z: $\vdash R_{\mathsf{in}_1}\vec{x} \to \vec{x} : \Delta_1$.

P5) Axioms concerning the domains: $\vdash \forall x\ (x : \Delta_0 \vee x : \Delta_1)$

P6) Axiom concerning outer identity:
$\vdash xEy \leftrightarrow \forall z\ ((xE_0z \leftrightarrow yE_0z) \wedge (xE_1z \leftrightarrow yE_1z))$.

P7) Equalizing axioms:

- $\vdash \forall x\ ((\Delta_0(x) \wedge \delta_L^{\mathsf{in}_0}(x)) \leftrightarrow (\Delta_1(x) \wedge \delta_K^{\mathsf{in}_1}(x)))$

- $\vdash \forall \vec{x} : (\Delta_0 \cap \delta_L^{\mathsf{in}_0})\ (P_L^{\mathsf{in}_0}(\vec{x}) \leftrightarrow P_K^{\mathsf{in}_1}(\vec{x}))$.

Suppose $Z \vdash B^{\mathsf{in}_1}$. Let Z_0 be the theory in the language of Z axiomatized by P1,2,3,4,7. By an easy model theoretical argument, we see that $Z_0 \vdash B^{\mathsf{in}_1}$. Our next step is to switch to another version of Z_0, say Z_1. We add to the language of Z_0 the predicates of U in a disjoint way, say P_{in_2}, plus a new unary predicate Δ_2. We axiomatize Z_1 as follows.

Q1) Axioms of the form $\vdash A^{\mathsf{in}_0}$, where A is an axiom of W.

Q2) Axioms of the form $\vdash B^{\mathsf{in}_1}$, where B is an axiom of V.

Q3) Axioms normalizing the predicates of W in Z_1: $\vdash Q_{\mathsf{in}_0}\vec{x} \to \vec{x} : \Delta_0$.

Q4) Axioms normalizing the predicates of V in Z_1: $\vdash R_{\mathsf{in}_1}\vec{x} \to \vec{x} : \Delta_1$.

Q5) Axioms normalizing the predicates of U in Z_1: $\vdash P_{\mathsf{in}_2}\vec{x} \to \vec{x} : \Delta_2$.

Q6) First set of equalizing axioms:

- $\vdash \forall x\ ((\Delta_0(x) \wedge \delta_L^{\mathsf{in}_0}(x)) \leftrightarrow \Delta_2(x))$,
- $\vdash \forall \vec{x} : \Delta_2\ (P_L^{\mathsf{in}_0}(\vec{x}) \leftrightarrow P^{\mathsf{in}_2}(x))$.

Q7) Second set of equalizing axioms:

- $\vdash \forall x\ ((\Delta_1(x) \wedge \delta_K^{\mathsf{in}_1}(x)) \leftrightarrow \Delta_2(x))$,
- $\vdash \forall \vec{x} : \Delta_2\ (P_K^{\mathsf{in}_1}(\vec{x}) \leftrightarrow P^{\mathsf{in}_2}(x))$.

We clearly have: $Z_1 \vdash B^{\mathsf{in}_1}$. It follows that $C \vdash (D \to B^{\mathsf{in}_1})$, where C is a conjunction of axioms from Q1,3,5,6 and D is a conjunction of axioms from Q2,4,5,7.

By the interpolation theorem, there is a J in the language of Δ_2 and the P_{in_2} such that $C \vdash J$ and $D, J \vdash B^{\mathsf{in}_1}$. By Lemma 3.1, there is an U-sentence I such that $\mathsf{Q5} \vdash I^{\mathsf{in}_2} \leftrightarrow J$.

By sending Δ_0 to $v_0 = v_0$, Q_{in_0} to Q, Δ_2 to δ_L, $P_{\mathsf{in}_2}(\vec{v})$ to $\vec{v} : \delta_L \wedge P_L(\vec{v})$, etc. we find: $W \vdash I^L$. Similarly, we have: $V + I^K \vdash B$. We are done.

4.4 Regularity

Consider the category $(\mathsf{INT}(\Box))^{\mathsf{op}}$. By the results of Section 3, this category has finite limits. By the results of this section this category has stable images. Hence, the category is regular. See e.g. [Jac99] or [GZ02] for further explanation of regular categories. Similarly, the category $\mathsf{INT}_{\mathsf{dir}}^{\mathsf{op}}$ is regular.

Some consequences of regularity will be spelled out in Section 5. We note one important consequence here: in a regular category a morphism is a cover iff it is a regular monomorphism. (A regular monomorphism is an equalizer.) Thus, in INT we will have that a morphism is mono iff it is a cocover iff it is a regular mono.

In Section 5, we will see that $(\mathsf{INT}(\Box))^{\mathsf{op}}$ $(\mathsf{INT}_{\mathsf{dir}}^{\mathsf{op}})$ is coherent.

5 Theory Extensions as Superobjects

For any category $\mathcal{C}$, we can define the associated category of subobjects $\mathsf{Sub}(\mathcal{C})$. See [Jac99]. Since we are living in a 'opposite world', we want to look at superobjects instead. Here are the basic definitions.

We define $\mathsf{Super}(\mathcal{C})$ as follows. The objects of $\mathsf{Super}(\mathcal{C})$ are the epimorphisms of $\mathcal{C}$ modulo the equivalence relation $\equiv$ defined by $f \equiv g$ iff, for some isomorphism h, $g = h \circ f$. A morphism in our new category $j : \alpha \to \beta$ is a morphism j of our old category such that $j \circ f = g \circ k$ for some f in α, g in β, and k. It is easy to see that, if we have such a triple f, g, k, then k is uniquely determined by f and g. Moreover, if $j : \alpha \to \beta$, then, for every $f' \in \alpha$, $g' \in \beta$, we have a k', such that $j \circ f' = g' \circ k'$. Identity and composition are directly induced by identity and composition in the original category. We easily verify that we have defined a category.

Consider the structure of all superobjects with domain a. We define $\alpha \leq \beta$ iff, for some $f \in \alpha$, $g \in \beta$ and h, we have $h \circ f = g$. Note that if we can find such a h for some pair of representatives, then we can find it for all pairs of representatives. Note also that h is automatically unique. We call this structure $\mathsf{super}(a)$. The structure $\mathsf{super}(a)$ is the *fiber* of the domain functor from $\mathsf{Super}(\mathcal{C})$ to $\mathcal{C}$.

We turn to the case of $\mathsf{INT}(\Box)$. We consider the following category $\mathsf{Ext}(\mathsf{INT}(\Box))$. The objects of this category are pairs $\langle T, U\rangle$ where U is an extension of T in the same language. A morphism $K : \langle T, U\rangle \to \langle V, W\rangle$ is a morphism $K : T \to V$ such that $\langle U, \tau_K, W\rangle : U \to W$. Theorems 3.5 and 3.6 show us that $\mathsf{Super}(\mathsf{INT}(\Box))$ is isomorphic to $\mathsf{Ext}(\mathsf{INT}(\Box))$.

It is well known that push-outs preserve epimorphisms. This gives rise to the push-out functor $K^\star := \mathsf{super}(K)$ between preorders of superobjects. This functor is given in the following diagram.

$$\begin{tikzcd}
U \arrow[r, "K"] \arrow[d, "\mathcal{E}"'] & V \arrow[d, "\mathcal{E}"] \\
U' \arrow[r, "K'"] & V + {U'}^K
\end{tikzcd}$$

Here $K' = \langle U', \tau_K, V' + {U'}^K\rangle$. It is easy to see that this does indeed define a functor. Note that $K^\star$ preserves meets and joins. We have the following result

Theorem 5.1 *Consider $K, M : U \to V$. We have $K^\star = M^\star$ iff, for all U-sentences A, $V \vdash A^K \leftrightarrow A^M$.*

Proof

Suppose $K^\star = M^\star$. Then,

$$V + A^K = V + (U + A)^K = V + (U + A)^M = V + A^M.$$

Conversely, suppose that, for all U-sentences A, $V \vdash A^K \leftrightarrow A^M$. Then,

$$V + {U'}^K = V + \{A^K \mid U' \vdash A\} = V + \{A^M \mid U' \vdash A\} = V + {U'}^M.$$

❑

Note that the above result means that the category INT_3, the category of interpretations modulo elementary equivalence, is definable in terms of INT.

A category is *coherent* if it is regular and the pullback functors preserve binary joins and it has a strict initial object. An initial object a is *strict* iff every $f : b \to a$ is an isomorphism. Since the $K^\star$ are the opposite variants of the pullback functors and since inconsistent theories are clearly strict final objects in INT ($\mathsf{INT}_{\mathsf{dir}}$), we find that $(\mathsf{INT}(\Box))^{\mathsf{op}}$ ($\mathsf{INT}^{\mathsf{op}}_{\mathsf{dir}}$) is coherent.

We have, for $K : U \to V$,

$$U' \subseteq K^{-1}[V'] \Leftrightarrow K^\star(U') \subseteq V'.$$

Thus, $K^{-1}[\cdot]$ is the right adjoint of $K^\star$. If we switch to the opposite category we see that $K^{-1}[\cdot]$ is a left adjoint and thus is the functor $\exists_K$.

Since $\mathsf{INT}^{\mathsf{op}}$ is regular, the functor $\exists_K$ the Beck-Chevalley condition. Translating this fact back to INT, we get the following result. Suppose the following diagram is a push-out.

$$\begin{tikzcd}
U \arrow[r, "K"] \arrow[d, "L"'] & V \arrow[d, "N"] \\
W \arrow[r, "M"'] & Z
\end{tikzcd}$$

We find that, for any $V' \supseteq V$, $W + (K^{-1}[V'])^L = M^{-1}[Z + {V'}^N]$.

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