Operational Semantics of Term Rewriting with Priorities

Jaco van de Pol

Utrecht University, Dept. of Philosophy
Heidelberglaan 8, 3584 CS, Utrecht
e-mail: jaco@phil.ruu.nl

Abstract

We study the semantics of term rewriting systems with rule priorities (PRS), as introduced in [1]. Three open problems posed in that paper are solved, by giving counter examples. Moreover, a class of executable PRSs is identified. A translation of PRSs into transition system specifications (TSS) is given. This translation introduces negative premises. We prove that the translation preserves the operational semantics.

Contents

1 Introduction **2**

2 Term rewriting with rule priorities **3**
2.1 Definition and semantics 3
2.2 Fixed points 6
2.3 An executable class of PRSs 8
2.4 Counter examples to open questions 11

3 Transition system specifications **14**
3.1 Universal negative premises in TSSs 15
3.2 Translation of PRSs into TSSs 18

4 Operational semantics of PRSs **21**
4.1 Sound and complete vs. well supported model 22
4.2 Fixed points and complete specifications 24

A Just supported is not enough **27**

1 Introduction

Motivation. In [1], term rewriting with rule priorities has been introduced. A *priority rewrite system* (PRS) extends an ordinary term rewriting system (TRS) with a partial order on the rules. The main idea is to resolve a conflict between two rules by giving priority to the largest rule. One may hope that by ordering the rules of a non-confluent TRS, a confluent PRS can be obtained (i.e. a system in which each reduction eventually gives the same result). Indeed, some results of this kind are known.

The above motivation of the priority mechanism can be seen as an implementation issue: priorities drastically decrease the amount of non-determinism involved in term rewriting.

The second motivation evolves from a specification point of view. The priority mechanism adds expressive power. We mention two points only: (1) In a signature containing the booleans, an equality predicate for an arbitrary sort can be specified by two rules only (see Ex. 3.10). This cannot be done in ordinary TRSs. The other indication is: (2) The one step reduction relation of PRSs is not decidable in general.

These motivations justify the mathematical study of the priority mechanism itself. In this paper, we will not be concerned with restrictions on the rules, the partial order or the reduction strategy. Such restrictions can be fruitful, but form a different topic. We mention the following restrictions: specificity order on rules; left linear rules; leftmost/innermost reduction or a lazy strategy; operator-constructor discipline. See e.g [6, 4, 5] for various results obtained by making such restrictions.

Contribution. The semantics of a PRS is not straightforward. The reason is that the question whether a certain rule may be applied, cannot be answered by syntactically matching the rules of higher priority (cf. Ex. 2.3). It is even the case that not every PRS will have a semantics.

In [1], a PRS is called meaningful if it has a so called *unique sound and complete rewrite set*. A certain monotonic operator is associated to a PRS, which reaches its least and greatest fixed points at some closure ordinal α. It has been proved that in case these fixed points coincide, the PRS is meaningful. It has also been shown that if the PRS is bounded, then the least and greatest fixed points are equal. Three open questions concerning this fixed point construction were posed:

(I) Is the associated monotonic operator always continuous?

(II) Is the closure ordinal α always finite?

(III) Is coincidence of the least and the greatest fixed point a necessary condition for being meaningful?

We solve these questions in a negative way, i.e. by giving a counter example to each of them (Sect. 2.4). We also give a sufficient condition for decidability of the one-step reduction relation. This can be used to identify a subclass of executable PRSs (Sect. 2.3). In particular, the one step reduction relation of the PRS is decidable, if the underlying TRS is strongly normalizing.

In Sect. 3.2, we give a translation of a PRS into a transition system specification (TSS) with negative premises [2, 7]. Such a specification can be seen as an inductive definition with negative premises. Such definitions are not always meaningful. We show (Thm. 4.5, 4.9) that the operational semantics is preserved under this translation. Another application of TSS theory to term rewriting occurs in [3].

This translation relates the semantics of priority rewriting given in [1] with general techniques to deal with negation in operational semantics and logic programming (for references to logic programming we refer to [7]). It also explains the negative answer to the third of the open questions. Finally, it opens the way to combining priorities with positive/negative conditions.

2 Term rewriting with rule priorities

In Sect. 2.1 and 2.2, we shortly recapitulate the definitions and some theory on priority rewrite systems (PRSs). These sections are based on [1]; only Ex. 2.5 is new. In Sect. 2.3 we identify a subclass of executable PRSs and Sect. 2.4 contains counter examples to some open questions posed in [1].

2.1 Definition and semantics

We assume a signature Σ of the form $(\mathcal{F}, \mathcal{V})$. Here $\mathcal{F}$ is a set of function symbols with fixed arities, $\mathcal{V}$ is an infinite set of variables. Sets of (open) terms $\mathcal{T}(\mathcal{F}, \mathcal{V})$ and closed terms $\mathcal{T}(\mathcal{F})$ are defined as usual. $\mathrm{FV}(s)$ denotes the free variables occurring in term s. A substitution is a finite function from variables to terms.

Definition 2.1. [1, Def. 2.5]

1. *A* rule *is a pair of terms, written $l \to r$, such that l is not a variable and the variables of r occur in l.*

2. *A* term rewriting system *(TRS) is a pair (Σ, R), with R a set of rules.*

3. *A* priority rewrite system *(PRS) is a triple $(\mathcal{R}, >)$, with $\mathcal{R}$ a TRS, and $>$ a partial order on the rules of $\mathcal{R}$.*

In examples, the priority ordering will be denoted by arrows. We call $\mathcal{R}$ the *underlying TRS* of a PRS $(\mathcal{R}, >)$.

Definition 2.2. *Let PRS $\mathcal{P} = (\mathcal{R}, >)$ be given.*

1. *Let r be a rule in $\mathcal{P}$. An r-*rewrite *(written $s \to^{r} t$) is a closed instance of r.*

2. *Let R be a set of rewrites. The closure of R under closed context is denoted by $\longrightarrow_R$. The reflexive transitive closure of $\longrightarrow_R$ is denoted by $\twoheadrightarrow_R$. With $s \twoheadrightarrow_R^{\text{int}} t$ we denote an* internal *reduction, i.e. a reduction where each contracted redex is not at the root.*

3. *We write $\longrightarrow_{\mathcal{R}}$, $\twoheadrightarrow_{\mathcal{R}}$ and $\twoheadrightarrow_{\mathcal{R}}^{\text{int}}$ in case we work in the underlying TRS. (i.e. the reductions may use all rewrites).*

The priorities are used to indicate preference of one rule above another. In this way a conflict between two rules can be resolved. So not every rewrite is *enabled*. A rewrite is only enabled, if it is not blocked by a rule of higher priority. Let us first look at an example.

Example 2.3. [1, Ex. 2.1]

$r1:$	$P(0)$	$\to$	0
$r2:$	$P(S(x))$	$\to$	x
$\downarrow$ $r3:$	$x+0$	$\to$	x
$r4:$	$x+y$	$\to$	$S(x+P(y))$

The rewrite $x+0 \to^{r4} S(x+P(0))$ is blocked, because $r3$ takes precedence. However, also $x+(P(S(0))) \to^{r4} S(x+P(P(S(0))))$ should be blocked by $r3$, because *eventually*, $P(S(0))$ becomes 0. The correct reduction is: $x+(P(S(0))) \to^{r2} x+0 \to^{r3} x$. ⊠

As Ex. 2.3 shows, the definition of the reduction relation induced by a PRS is not straightforward. The rewrite $s+t \to^{r4} S(s+P(t))$ is enabled only, if $t \not\twoheadrightarrow 0$. So in the definition of the one step reduction relation, the *negation* of the more step reduction relation occurs. This explains the following definition.

Definition 2.4. [1, Def. 2.8, 2.9]

1. *Let* $x = s \to^{r} t$ *be a rewrite, and* R *be a set of rewrites for a PRS* $(\mathcal{R}, >)$. R *is an* obstruction *for* x *(written* $x \triangleleft R$*) if there is a rewrite* $s' \to^{r'} t'$ *with* $r' > r$ *and a reduction* $s \twoheadrightarrow_{R}^{\text{int}} s'$*, using* precisely all *rewrites in* R.
2. *Let* R *be a set of rewrites for a PRS* $\mathcal{P}$. *A rewrite* x *is* correct *with respect to* R*, if there is no obstruction* $O \subseteq R$ *for* x.
3. R *is* sound *if all its rewrites are correct w.r.t.* R.
4. R *is* complete *if it contains all rewrites that are correct w.r.t.* R.
5. $\mathcal{P}$ *is* meaningful *if it has a unique sound and complete rewrite set. This set is the* semantics *of* $\mathcal{P}$.

In [1] an example of a PRS is given that doesn't have a sound and complete rewrite set (see ExampleA.2), as well as a PRS that has more than one sound and complete rewrite set. Neither of them is meaningful by Def. 2.4.5. The following example will also play a rôle in Sect. 2.4.

Example 2.5. Consider the following PRS $\mathcal{P}$ with a constant a and a unary function symbol f:

$$\downarrow \begin{array}{lcl} f(a) & \to & f(f(f(a))) \\ f(x) & \to & a \end{array}$$

We write $f^n(a)$ for the n-fold application of f to a. Note that all closed terms are of the form $f^n(a)$. We claim that $\mathcal{P}$ is meaningful, because the following set is the unique sound and complete rewrite set for it:

$$R := \{ f(a) \to f^3(a) \} \cup \{ f^{2m+2}(a) \to a \mid m \geq 0 \} \ .$$

Completeness: The only rewrites not in R are of the form $f^{2m+1}(a) \to a$, for some $m \geq 0$, but these are not correct with respect to R, because $f^{2m+1}(a) \twoheadrightarrow_{R}^{\text{int}}$

$f(a)$. (If $m = 0$ in 0 steps, if $m > 0$ in one step). So R contains all rewrites that are correct w.r.t. itself.

Soundness: Note that if s has an even number of f-symbols, and $s \twoheadrightarrow_R t$, then t has an even number of f-symbols too. So for no m we have $f^{2m+2}(a) \twoheadrightarrow_R^{\text{int}} f(a)$, hence all rewrites of R are correct w.r.t. itself.

Uniqueness: Let S be a sound and complete rewrite set. By completeness S contains $f(a) \to f^3(a)$, so for all $m \geq 0$, $f(a) \twoheadrightarrow_S f^{2m+1}(a)$. Assume towards a contradiction, that S contains $f^{2m+1}(a) \to a$ for some $m \geq 0$. Then $f(a) \twoheadrightarrow_S a$; hence also $f^{2m}(a) \twoheadrightarrow_S a$. Now by soundness, $f^{2m+1}(a) \to a$ is not in S: contradiction. This shows $S \subseteq R$.

Vice versa, let $x \in R$, then x is correct w.r.t. R (soundness of R), hence also correct w.r.t. the subset S and, by completeness of S, $x \in S$. Hence $S = R$, proving uniqueness. ⊠

2.2 Fixed points

In Ex. 2.5 a rewrite set was given in advance and then checked for soundness and correctness. We want of course a method to compute this set by means of successive approximations. This is the aim of this section.

Definition 2.6. [1, Def. 2.13, 3.2]

1. *Let R be a set of rewrites of PRS $\mathcal{P}$. Then the* closure *of R, written $R^{\mathcal{P}}$ consists of all rewrites that are correct w.r.t. R.*

2. *Put $\mathbf{T}_{\mathcal{P}}(R) := (R^{\mathcal{P}})^{\mathcal{P}}$.*

Lemma 2.7. [1, Lemma 2.14] *Let R be a set of rewrites for PRS $\mathcal{P}$.*

1. *R is sound $\iff R \subseteq R^{\mathcal{P}}$*

2. *R is complete $\iff R \supseteq R^{\mathcal{P}}$*

3. *$R \subseteq S \Rightarrow R^{\mathcal{P}} \supseteq S^{\mathcal{P}}$*

Combining 1 and 2 of this lemma, we see that we need a unique fixed point of the closure map $()^{\mathcal{P}}$. Unfortunately, this map is not monotonic, but antitonic, as seen from the last part of the lemma. But then the operation $\mathbf{T}_{\mathcal{P}}$ is monotonic, so we can compute its least and greatest fixed points. Consider the following construction, parameterized by an arbitrary PRS $\mathcal{P}$. (Here and in the sequel, α ranges over arbitrary ordinals and λ over limit ordinals; m and n range over finite ordinals.)

Definition 2.8.[1, Def. 3.3]

$$
\begin{array}{rcl@{\qquad\qquad}rcl}
\mathbf{T}_{\mathcal{P}}\!\uparrow\!0 & := & \varnothing, & \mathbf{T}_{\mathcal{P}}\!\downarrow\!0 & := & \varnothing^{\mathcal{P}} \\
\mathbf{T}_{\mathcal{P}}\!\uparrow\!(\alpha+1) & := & \mathbf{T}_{\mathcal{P}}(\mathbf{T}_{\mathcal{P}}\!\uparrow\!\alpha) & \mathbf{T}_{\mathcal{P}}\!\downarrow\!(\alpha+1) & := & \mathbf{T}_{\mathcal{P}}(\mathbf{T}_{\mathcal{P}}\!\downarrow\!\alpha) \\
\mathbf{T}_{\mathcal{P}}\!\uparrow\!\lambda & := & \bigcup_{\alpha<\lambda}(\mathbf{T}_{\mathcal{P}}\!\uparrow\!\alpha) & \mathbf{T}_{\mathcal{P}}\!\downarrow\!\lambda & := & \bigcap_{\alpha<\lambda}(\mathbf{T}_{\mathcal{P}}\!\downarrow\!\alpha)
\end{array}
$$

Proposition 2.9. [1, Thm. 3.5] *For all PRSs $\mathcal{P}$ and ordinals α,*

1. $(\mathbf{T}_{\mathcal{P}}\!\uparrow\!\alpha)^{\mathcal{P}} = \mathbf{T}_{\mathcal{P}}\!\downarrow\!\alpha$.
2. $(\mathbf{T}_{\mathcal{P}}\!\downarrow\!\alpha)^{\mathcal{P}} = \mathbf{T}_{\mathcal{P}}\!\uparrow\!(\alpha+1)$.

Proposition 2.10. [1, Prop. 3.8] *For all PRSs $\mathcal{P}$ and ordinals α,*

1. $\mathbf{T}_{\mathcal{P}}\!\uparrow\!\alpha$ *is sound.*
2. $\mathbf{T}_{\mathcal{P}}\!\downarrow\!\alpha$ *is complete.*
3. *If R is sound and complete, then* $\mathbf{T}_{\mathcal{P}}\!\uparrow\!\alpha \subseteq R \subseteq \mathbf{T}_{\mathcal{P}}\!\downarrow\!\alpha$.

Proof: 1 and 2 are proved in [1]. Part 3 is not explicitly mentioned there, although it is needed in the following corollary.

Assume that R is sound and complete, then $R = R^{\mathcal{P}}$ by Prop. 2.7 (1 and 2), hence $\mathbf{T}_{\mathcal{P}}(R) = R$. With induction to α, we prove that $\mathbf{T}_{\mathcal{P}}\!\uparrow\!\alpha \subseteq R$:

- $\mathbf{T}_{\mathcal{P}}\!\uparrow\!0 = \varnothing \subseteq R$
- If $\mathbf{T}_{\mathcal{P}}\!\uparrow\!\alpha \subseteq R$, then as $\mathbf{T}_{\mathcal{P}}$ is monotonic, we have $\mathbf{T}_{\mathcal{P}}\!\uparrow\!\alpha+1 = \mathbf{T}_{\mathcal{P}}(\mathbf{T}_{\mathcal{P}}\!\uparrow\!\alpha) \subseteq \mathbf{T}_{\mathcal{P}}(R) = R$
- If $\mathbf{T}_{\mathcal{P}}\!\uparrow\!\alpha \subseteq R$ for all $\alpha < \lambda$, then also $\mathbf{T}_{\mathcal{P}}\!\uparrow\!\lambda = \bigcup_{\alpha<\lambda}\mathbf{T}_{\mathcal{P}}\!\uparrow\!\alpha \subseteq R$.

Then by Prop. 2.7.3, $\mathbf{T}_{\mathcal{P}}\!\downarrow\!\alpha = (\mathbf{T}_{\mathcal{P}}\!\uparrow\!\alpha)^{\mathcal{P}} \supseteq R^{\mathcal{P}} = R$. ⊠

Because the operation $\mathbf{T}_{\mathcal{P}}$ is monotonic, it has a least fixed point, which is reached at some ordinal. We define the *closure ordinal* of a PRS $\mathcal{P}$ as the first α such that $\mathbf{T}_{\mathcal{P}}\!\uparrow\!\alpha = \mathbf{T}_{\mathcal{P}}\!\uparrow\!(\alpha+1)$. Note that for the closure ordinal also $\mathbf{T}_{\mathcal{P}}\!\downarrow\!\alpha = \mathbf{T}_{\mathcal{P}}\!\downarrow\!(\alpha+1)$. In this way we find the least and the greatest fixed points for the closure map $()^{\mathcal{P}}$. We now have the following corollary

Corollary 2.11. [1, Corollary 3.9] *Let* α *be the closure ordinal of a PRS* $\mathcal{P}$. *If* $\mathbf{T}_{\mathcal{P}}{\uparrow}\alpha = \mathbf{T}_{\mathcal{P}}{\downarrow}\alpha$, *then* $\mathcal{P}$ *is meaningful and* $\mathbf{T}_{\mathcal{P}}{\uparrow}\alpha$ *is its semantics.*

Ex. 2.21 — which can be read independently of the next section — shows, that the condition of the corollary is not a necessary one.

2.3 An executable class of PRSs

In this section, we will prove that locally finite PRSs have a closure ordinal at most ω. Consequently, for bounded PRSs with finitely many rules, the rewrite relation is computable. This answers a question put in [1], by giving a class of PRSs that is *executable*.

We first define some relevant properties of PRSs, in terms of the underlying TRS.

Definition 2.12.

1. *A TRS is* strongly normalizing *if all reduction sequences are finite.*
2. *A possibly infinite reduction sequence* $s_0 \longrightarrow s_1 \longrightarrow \cdots$ *is* bounded *if there exists an* n *such that for all* i, $|s_i| \leq n$. *(Here* $|s|$ *denotes the length of a term* s *in symbols).*
3. *A TRS* $\mathcal{R}$ *is* bounded *if all reductions sequences in* $\mathcal{R}$ *are bounded.*
4. *A TRS is* locally finite *if for all* s, *the set* $\{t \mid s \twoheadrightarrow t\}$ *is finite.*
5. *A PRS is bounded (locally finite) if its underlying TRS is.*

An easy syntactic check for boundedness is that all rules are "non-duplicating" and "non-length-increasing". The first property holds if the multiset of variables on the right hand side is contained in the multiset of variables on the left. A rule is non-length-increasing if its right hand side contains not more symbols than its left hand side. (One can even assign weights to the function symbols). The existence of a recursive path order also implies boundedness, as strong normalization is stronger than boundedness. None of these syntactic conditions is necessary, however.

Proposition 2.13. *Let* $\mathcal{P}$ *be a bounded PRS with closure ordinal* α. *Then* $\mathbf{T}_{\mathcal{P}}{\uparrow}\alpha = \mathbf{T}_{\mathcal{P}}{\downarrow}\alpha$.

Proof: This follows immediately from Prop. 3.11 and 3.14 in [1]. ⊠

We strengthen this to Prop. 2.16, by proving that in case the set of rules is finite, then α is at most ω. We first need Prop. 2.14, relating the properties defined above, and the auxiliary Lemma 2.15. As a corollary we prove that bounded PRSs with finitely many rules are executable (Thm. 2.18).

Proposition 2.14. *Let $\mathcal{R} = (\Sigma, R)$ be a TRS.*

1. *If $\mathcal{R}$ is strongly normalizing, then $\mathcal{R}$ is bounded.*
2. *If $\mathcal{R}$ is locally finite, then $\mathcal{R}$ is bounded.*
3. *If R is finite and $\mathcal{R}$ is bounded, then $\mathcal{R}$ is locally finite.*

Proof:

1. Given a sequence $s_0 \longrightarrow s_1 \longrightarrow \cdots$, we can take the length of the largest term in it, as the sequence must be finite.
2. Given a sequence $s_0 \longrightarrow s_1 \longrightarrow \cdots$, we can take the length of the largest term in the finite set $\{t \mid s_0 \twoheadrightarrow t\}$.
3. Suppose $\mathcal{R}$ is bounded and finite; let s be given. Put $V = \{t \mid s \twoheadrightarrow_{\mathcal{R}} t\}$ and $E := \twoheadrightarrow_{\mathcal{R}} \cap V \times V$. Consider (V, E) as an s-rooted graph.
   (a) (V, E) is finitely branching because $\longrightarrow_{\mathcal{R}}$ is. This is because the set of rules is finite, hence every term contains only finitely many redexes.
   (b) All acyclic paths in (V, E) are finite. This is because each path corresponds with a reduction sequence in $\twoheadrightarrow_{\mathcal{R}}$. By boundedness, all terms in this sequence are shorter than n for some n. Furthermore, these terms are built from a finite set of function symbols: those occurring in s or in the finite set of rules. So there are only finitely many different terms on each path.

   Now (a) and (b) imply that V is finite. To see this, we apply König's Lemma on an acyclic subgraph of (V, E) that covers all nodes in V. To obtain such a graph, we proceed as follows.

   For $t \in V$, let $d(t)$ denote the distance from the root s to t. Define $D \subseteq E$ as $\{(r, t) \mid d(r) + 1 = d(t)\}$. Then D is acyclic by construction. For each $t \in V$, we have sD^*t as can be shown by induction on $d(t)$.

$\boxtimes$

Lemma 2.15. *Let $\mathcal{P}$ be a locally finite PRS. Let R be a rewrite set for it, containing x. Put $V := \bigcup\{O \mid x \triangleleft O\}$. Then V is finite.*

Proof: Because $\mathcal{P}$ is locally finite, the set $\{t \mid s \twoheadrightarrow t\}$ is finite for each s. Each term has finitely many subterms, so the set $\{t \mid \exists C, r. s \twoheadrightarrow r \wedge C[t] = r\}$ is also finite for each s.

Now each $a \to b$ in V is in an obstruction, so for some context C, we have $\mathrm{lhs}(x) \twoheadrightarrow^{\mathrm{int}}_R C[a] \longrightarrow_{\mathcal{R}} C[b]$. Hence V is finite. $\boxtimes$

Proposition 2.16. *If $\mathcal{P}$ is a locally finite PRS then its closure ordinal is at most ω.*

Proof: It is enough to prove that $\mathbf{T}_{\mathcal{P}}{\uparrow}\omega \supseteq \mathbf{T}_{\mathcal{P}}{\uparrow}(\omega+1)$. Consider $x \in \mathbf{T}_{\mathcal{P}}{\uparrow}(\omega+1) = (\mathbf{T}_{\mathcal{P}}{\downarrow}\omega)^{\mathcal{P}}$. Put $V := \bigcup\{O \mid x \triangleleft O\}$. Because x is correct w.r.t. $\mathbf{T}_{\mathcal{P}}{\downarrow}\omega$, there is no obstruction of x entirely in $\mathbf{T}_{\mathcal{P}}{\downarrow}\omega$, so we can find a set $W \subseteq V$, such that $W \cap \mathbf{T}_{\mathcal{P}}{\downarrow}\omega = \varnothing$ and for each obstruction O of x, $W \cap O \neq \varnothing$. By Lemma 2.15, V is finite, so W is finite too. Therefore, there exists a n, such that $W \cap \mathbf{T}_{\mathcal{P}}{\downarrow}n = \varnothing$. But then $x \in \mathbf{T}_{\mathcal{P}}{\uparrow}(n+1)$, so $x \in \mathbf{T}_{\mathcal{P}}{\uparrow}\omega$. $\boxtimes$

Corollary 2.17. *If $\mathcal{P}$ is a locally finite PRS, then $\mathbf{T}_{\mathcal{P}}{\uparrow}\omega$ is its semantics.*

Proof: By Lemma 2.13 $\mathcal{P}$ has a semantics, which must have been reached at ω by Prop. 2.16. $\boxtimes$

Theorem 2.18. *Let $\mathcal{P}$ be a bounded PRS with finitely many rules. Then $\mathcal{P}$ is executable.*

Proof: By Prop. 2.14.3, $\mathcal{P}$ is locally finite, hence (by the previous corollary) the semantics of $\mathcal{P}$ is $\mathbf{T}_{\mathcal{P}}{\uparrow}\omega$. So given a closed term s, we have to compute the set $\{t \mid \mathbf{T}_{\mathcal{P}}{\uparrow}\omega \vDash s \longrightarrow t\}$.

This is done by generating all successors t of s in the underlying TRS, and then testing whether the rewrite x used to obtain t is enabled, i.e. whether $x \in \mathbf{T}_{\mathcal{P}}{\uparrow}\omega$. Note that if so, then it is contained in $\mathbf{T}_{\mathcal{P}}{\uparrow}n$ for some finite n already. Otherwise, it is not in $\mathbf{T}_{\mathcal{P}}{\downarrow}\omega$ either, hence it is outside $\mathbf{T}_{\mathcal{P}}{\downarrow}n$ for some finite n already. So we consider the sequence $\mathbf{T}_{\mathcal{P}}{\uparrow}0, \mathbf{T}_{\mathcal{P}}{\downarrow}0, \mathbf{T}_{\mathcal{P}}{\uparrow}1, \mathbf{T}_{\mathcal{P}}{\downarrow}1, \ldots$ until we find an n, with $x \in \mathbf{T}_{\mathcal{P}}{\uparrow}n$ or $x \notin \mathbf{T}_{\mathcal{P}}{\downarrow}n$.

We still need to prove that for all finite n, it is decidable whether $s \to^r t$ is in $\mathcal{P}^n(\varnothing)$ (the n-fold application of $()^{\mathcal{P}}$). This is proved with induction to n. For $n = 0$, the answer is clearly NO. Now suppose that for some n, $\mathcal{P}^n(\varnothing)$ is decidable. Let some rewrite $s \to^r t$ be given. It is in $\mathcal{P}^{n+1}(\varnothing)$ if and only if it is correct w.r.t. $\mathcal{P}^n(\varnothing)$. This is the case if and only if there is no rewrite $s' \to^{r'} t'$ with $r' > r$ and $\mathcal{P}^n \vDash s \twoheadrightarrow^{\text{int}} s'$. This can be tested by generating all terms reachable from s using $\twoheadrightarrow^{\text{int}}_{\mathcal{R}}$ (there are only finitely many because $\mathcal{P}$ is locally finite), and test whether the used rewrites are in $\mathcal{P}^n$, which is decidable by induction hypothesis. ⊠

2.4 Counter examples to open questions

In [1, p. 297] three open questions concerning the mapping $\mathbf{T}_{\mathcal{P}}$ are posed

(I) Is the mapping $\mathbf{T}_{\mathcal{P}}$ always continuous, instead of only monotonic?

(II) Is the closure ordinal of each PRS finite?

(III) Is the condition of Corollary 2.11 necessary? That is, does every meaningful PRS $\mathcal{P}$ with closure ordinal α, satisfy $\mathbf{T}_{\mathcal{P}}{\uparrow}\alpha = \mathbf{T}_{\mathcal{P}}{\downarrow}\alpha$?

We have found counter examples to each of these questions. First, Ex. 2.19 provides for a finite PRS, with closure ordinal ω. This is a counter example to (II). It is easy to extend this example in order to find a closure ordinal beyond ω (Ex. 2.20). This refutes (I), because if $\mathbf{T}_{\mathcal{P}}$ were continuous, the closure ordinal would be at most ω. Finally, we show that for the PRS $\mathcal{P}$ of Ex. 2.5, $\mathbf{T}_{\mathcal{P}}{\uparrow}\alpha \neq \mathbf{T}_{\mathcal{P}}{\downarrow}\alpha$ for any α (see Ex. 2.21), although it is meaningful, as we already showed. This answers (III) negatively.

Example 2.19. Let $\mathcal{P}$ be the following PRS:

$$
\begin{array}{|rcl|}
\hline
\downarrow\; f(b) & \to & c \\
f(x) & \to & g(x) \\
g(a) & \to & b \\
g(f(x)) & \to & g(x) \\
\hline
\end{array}
$$

Note that the TRS underlying $\mathcal{P}$ is strongly normalizing, so it is bounded. By Lemma 2.16, the closure ordinal is at most ω. From claim 2 below, it follows that the closure ordinal is not finite, so it must be ω. This gives a negative answer to open question (II) at the beginning of this section.

Claim 1: *Let R be an arbitrary set of rewrites. $R^{\mathcal{P}}$ contains $f^{n+1}(a) \to g(f^n(a))$ if and only if $R^{\mathcal{P}} \models f^{n+1}(a) \twoheadrightarrow b$.*

Proof: $\Rightarrow$ is clear, because we have:

$$f^{n+1}(a) \to g(f^n(a)) \to g(f^{n-1}(a)) \twoheadrightarrow g(a) \to b$$

$\Leftarrow$: Suppose there were a reduction from $f^{n+1}(a)$ in $R^{\mathcal{P}}$ that doesn't start with $g(f^n(a))$. The first step must be an innermost application of the second rule. We have to reduce the topmost f at some later point. So the reduction has the following form (for some m, $k = n - m - 1$ and z):

$$f^{n+1}(a) \longrightarrow f^{m+1}(g(f^k(a))) \twoheadrightarrow^{\text{int}} f(z) \to g(z) \twoheadrightarrow b.$$

Inspection of the rules of $\mathcal{P}$ reveals that the total number of b- c- and g-symbols cannot decrease during rewriting. But then the reduction above cannot exist, because $g(z)$ contains at least 2 such symbols, so it can never reduce to b.

Claim 2: *For all m, the rewrite $f^{2m+1}(a) \to g(f^{2m}(a))$ is in $\mathbf{T}_{\mathcal{P}}{\uparrow}(m+1)$, but not yet in $\mathbf{T}_{\mathcal{P}}{\uparrow}m$.*

Proof: Induction to m. Base case: because a is a normal form, $\mathbf{T}_{\mathcal{P}}{\downarrow}0 \models a \not\twoheadrightarrow b$, so $\mathbf{T}_{\mathcal{P}}{\uparrow}1$ contains $f(a) \to g(a)$; $\mathbf{T}_{\mathcal{P}}{\uparrow}0 = \varnothing$.

Induction step: assume the claim holds for m, then (by Claim 1)

$$\mathbf{T}_{\mathcal{P}}{\uparrow}(m+1) \models f^{2m+1}(a) \twoheadrightarrow b \qquad \text{and} \qquad \mathbf{T}_{\mathcal{P}}{\uparrow}m \models f^{2m+1}(a) \not\twoheadrightarrow b,$$

hence $f^{2m+2}(a) \to g(f^{2m+1}(a))$ is not contained in $\mathbf{T}_{\mathcal{P}}{\downarrow}(m+1)$, but it is in $\mathbf{T}_{\mathcal{P}}{\downarrow}m$. Therefore (Claim 1),

$$\mathbf{T}_{\mathcal{P}}{\downarrow}(m+1) \models f^{2m+2}(a) \not\twoheadrightarrow b \qquad \text{and} \qquad \mathbf{T}_{\mathcal{P}}{\downarrow}m \models f^{2m+2} \twoheadrightarrow b.$$

Therefore, $f^{2m+3}(a) \to g(f^{2m+2}(a))$ is contained in $\mathbf{T}_{\mathcal{P}}{\uparrow}(m+2)$, but this rewrite is not in $\mathbf{T}_{\mathcal{P}}{\uparrow}(m+1)$, so the claim holds for $m+1$. ⊠

The idea of this example is, that $f^m(a)$ can be reduced to b for odd m only. These reductions block the reductions for even m. The system is constructed in such a way, that the larger m becomes, the later we decide whether $f^m(a)$ reduces to b. Because the system is bounded, we cannot go beyond ω. The only way to go beyond ω uses a non-bounded system. As the proof of Prop. 2.16 reveals, we need a term with infinitely many possible reducts. Only at stage ω, the system may know that none of these is really reached. This is the idea of the following example:

Example 2.20. Extend $\mathcal{P}$ of Ex. 2.19 with the following rules:

$$\boxed{\begin{array}{lcl} h(x) & \to & f(x) \\ h(x) & \to & h(h(h(x))) \end{array}}$$

We will show that $\mathbf{T}_{\mathcal{P}}{\uparrow}\omega \neq \mathbf{T}_{\mathcal{P}}{\uparrow}(\omega+1)$, by showing that the latter contains the rewrite $f(h(f(a))) \to g(h(f(a)))$, but the former doesn't. Note that Claim 1 and 2 still hold for the extended system, because the proofs remain valid for the new $\mathcal{P}$.

$$\begin{array}{lll} & \text{for all } m,\ h(x) \twoheadrightarrow f^{2m+1}(x) & \text{(induction to } m) \\ \Longrightarrow & \text{for all } m,\ h(f(a)) \twoheadrightarrow f^{2m+2}(a) & \\ \Longrightarrow & \text{for all } m,\ \mathbf{T}_{\mathcal{P}}{\downarrow}m \vDash h(f(a)) \twoheadrightarrow f^{2m+2}(a) \twoheadrightarrow b & \text{(see proof Claim 2)} \\ \Longrightarrow & \text{for all } m,\ \mathbf{T}_{\mathcal{P}}{\uparrow}m+1 \nvDash f(h(f(a))) \to g(h(f(a))) & \\ \Longrightarrow & \mathbf{T}_{\mathcal{P}}{\uparrow}\omega \nvDash f(h(f(a))) \to g(h(f(a))). & \text{(defined as union)} \end{array}$$

On the other hand, $\mathbf{T}_{\mathcal{P}}{\uparrow}(\omega+1) \vDash f(h(f(a))) \to g(h(f(a)))$ by Prop. 2.9 and the claim below.

Claim 3: $\mathbf{T}_{\mathcal{P}}{\downarrow}\omega \vDash h(f(a)) \not\twoheadrightarrow b$.

Proof: Any reduction of $h(f(a)) \twoheadrightarrow b$ would have the following form:

$$h(f(a)) \overset{\text{I}}{\twoheadrightarrow} f(z) \overset{\text{II}}{\to} g(z) \overset{\text{III}}{\twoheadrightarrow} b.$$

We will show that II is not a rewrite in $\mathbf{T}_{\mathcal{P}}{\downarrow}\omega$.

Because the number of b-, c- and g-symbols cannot decrease during (III), z may not contain one of these symbols, hence (I) uses only the two h-rules. Therefore, z consists of an odd number of f and h symbols, applied to a, so $z \twoheadrightarrow f^{2m+1}(a)$ for some m. Using Claim 2 above we get a reduction

$$\mathbf{T}_{\mathcal{P}}{\uparrow}(m+1) \vDash z \twoheadrightarrow f^{2m+1}(a) \to g(f^{2m}(a)) \twoheadrightarrow b.$$

Then $\mathbf{T}_{\mathcal{P}}{\downarrow}(m+1) \nvDash f(z) \to g(z)$. So step II above is indeed absent in $\mathbf{T}_{\mathcal{P}}{\downarrow}\omega$, because this is defined as the intersection of all the $\mathbf{T}_{\mathcal{P}}{\downarrow}m$. ☒

One might have the idea to reduce the number of rules in the previous example, by identifying f, g and h. In this way, one more or less gets Ex. 2.5. We showed that this system has a unique sound and complete rewrite set, hence it is meaningful. However, contrary to the examples before, for this system the least and greatest fixed points don't coincide. This solves the third open question.

Example 2.21. Let $\mathcal{P}$ be the PRS of Ex. 2.5. We have:

$$
\begin{array}{ll}
\mathbf{T}_{\mathcal{P}}{\uparrow}0 = \varnothing & \mathbf{T}_{\mathcal{P}}{\downarrow}0 = \{ f(c) \to f^3(c)\} \cup \{ f^{n+2}(c) \to c \mid n \in \mathbb{N}\} \\
\mathbf{T}_{\mathcal{P}}{\uparrow}1 = \{ f(c) \to f^3(c)\} & \mathbf{T}_{\mathcal{P}}{\downarrow}1 = \mathbf{T}_{\mathcal{P}}{\downarrow}0 \\
\mathbf{T}_{\mathcal{P}}{\uparrow}2 = \mathbf{T}_{\mathcal{P}}{\uparrow}1 &
\end{array}
$$

The crux of this system is that, although $\mathbf{T}_{\mathcal{P}}{\downarrow}0 \not\models f(c) \to c$, the reduction $f(c) \to f^3(c) \to c$ is still present. Therefore, every closed term reduces to c in $\mathbf{T}_{\mathcal{P}}{\downarrow}0$. Clearly, the closure ordinal of this system is 2, but the least and greatest fixed points are not equal. ⊠

3 Transition system specifications

Not every PRS is meaningful (Def. 2.4). The reason is that a rewrite $f(r) \to s$ is enabled if a certain reduction $r \twoheadrightarrow t$ is *not* present. However, one of these steps may involve the original question, whether $f(r) \to s$ is enabled or not. In [1] this problem is solved by asking for a unique sound and complete rewrite set. A fixed point construction was given to compute the semantics. We showed (Ex. 2.21) that this is not a complete method. For some meaningful PRSs the meaning cannot be obtained by this fixed point construction.

In this section, we put the priority mechanism in a wider context. We will present a translation from PRSs into transition system specifications (TSSs). This opens the way to use existing work on operational semantics of TSSs with negative premises [2, 7]. It will turn out (Sect. 4) that the PRS-semantics coincides with the operational semantics of the TSSs obtained by our translation.

In this way, the PRS-semantics gets a broader basis. The translation also shows, that the discrepancy between "meaningful" and the fixed point construction is structural. The translation also provides a basis to extend PRSs with positive and negative conditions, because all these features can be expressed in transition systems.

A PRS can be translated to a TSS in a smooth and intuitive way. The addition rules from Ex. 2.3 can be translated into:

$$
\frac{}{x + 0 \to x} \qquad \frac{y \not\twoheadrightarrow 0}{x + y \to S(x + P(y))}
$$

This is not the complete specification, because we also need rules for the context- and transitive closure of $\to$. (See Def. 3.9). Although this example illustrates the main idea, it simplifies matters too much. The following example is more representative:

$$\boxed{\begin{array}{rcl} \downarrow \; Zero?(S(y)) & \to & F \\ Zero?(x) & \to & T \end{array}}$$

These rules translate into

$$\frac{}{Zero?(S(y)) \to F} \qquad \frac{\forall y.\, x \not\twoheadrightarrow S(y)}{Zero?(x) \to T}$$

The second rule contains a universal quantifier in the premise. The second rule is enabled if there is no y, such that x reduces to $S(y)$. This falls out of the scope of the usual format for negative literals in TSS theory.

In Sect. 3.1, we recapitulate some TSS-theory, taken from [7]. On the fly, the format for negative literals will be generalized slightly. In Sect. 3.2 the translation of priorities into negative premises will be given. In Sect. 4 the connection with the PRS-semantics is established.

3.1 Universal negative premises in TSSs

We assume a a signature Σ of the form $(\mathcal{F}, \mathcal{L}, \mathcal{V})$. Here $\mathcal{F}$ is a set of function symbols with fixed arities, $\mathcal{V}$ is an infinite set of variables. Sets of (open) terms $\mathcal{T}(\mathcal{F}, \mathcal{V})$ and closed terms $\mathcal{T}(\mathcal{F})$ are defined as usual. $\mathrm{FV}(s)$ denotes the free variables occurring in term s. Furthermore, $\mathcal{L}$ is a set of relation symbols. These occur as names of transitions.

Definition 3.1. *(Literals and Rules)*

1. *A* positive literal *is of the form* $s \to^a t$. Negative literals *are of the form* $\forall \vec{z}.\, s \not\to^a t$. *Here* $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$, $\to^a \in \mathcal{L}$ *and* $\vec{z} = \mathrm{FV}(t) - \mathrm{FV}(s) \subset \mathcal{V}$. *A literal is* closed *if it contains no free variables (the* $\vec{z}$ *are considered bound). We let* K *and* L *range over arbitrary literals.* H *and* J *denote sets of literals;* N *is reserved for sets of negative literals.*

2. *Closed literals* $s \to^a t$ *and* $\forall \vec{z}.\, s \not\to^a r$ deny each other, *if there is a substitution* σ *with* $\mathsf{dom}(\sigma) = \vec{z}$, *such that* $r\sigma \equiv t$. *We write* $K \lightning L$ *if* K *and* L *deny each other. Moreover,* $H \lightning J$ *means that a literal from* H *denies one from* J.

3. *A* rule *is of the form* $\frac{H}{L}$*, with* L *a positive literal (the conclusion) and* H *a set of literals (the premises). We often write* L *for* $\frac{\varnothing}{L}$*. A rule is* positive *if all premises are positive.*

4. *A* transition system specification *(TSS) is a set of rules.*

The form of negative literals has been generalized in order to capture priorities. On the other hand, we can dispose of negative literals of the form $s \not\rightarrow^{a}$, because they are subsumed by $\forall z.\, s \not\rightarrow^{a} z$.

Literals, rules and complete TSSs are interpreted by transition relations. These are defined as sets of triples, but can alternatively be seen as families of binary relations (for each relation symbol a relation). It is defined below, when a transition relation is a model for a TSS. Of course, we can only speak about the meaning of a TSS, if there is a way to choose between different models. In case a TSS has only positive rules, the least model is a very natural choice. This model only contains the transitions that are really forced by the rules. This notion is formalized below as positive provability.

Definition 3.2. *(Models)*

1. *A* transition relation R *is a set of triples of the form* $s \rightarrow^{a} t$*, where* s *and* t *are closed terms, and* $\rightarrow^{a}$ *is a relation symbol.*

2. *A positive closed literal* L holds in R*, if* $L \in R$*. A negative closed literal* L *holds in* R*, if there is no* $K \lightning L$ *that holds in* R*. An open literal holds, if all its closed instances hold. A set of literals* H *holds in* R *if each literal in* H *holds in* R*.*

3. *A rule* $\frac{H}{L}$ *holds in* R *if for each closed instance of* H *that holds in* R*, the corresponding instance of* L *holds in* R *too.*

4. *A transition relation* R *is a* model *of a TSS* T*, if each rule from* T *holds in* R*.*

5. *We write* $R \vDash L$ *for* L *holds in* R*. Similar for rules and sets of literals or rules.*

Definition 3.3. *(Positive provability) Given a TSS* T*,* positive provability *(written* $\vdash_{+}^{T}$ *or* $\vdash_{+}$ *for short) is inductively defined by the following two clauses:*

1. *For any literal* L*,* $\{L\} \vdash_+ L$*.*

2. *If* $\frac{H}{L}$ *is an instance of a rule from* T*, and for all* $K \in H$*,* $H_K \vdash_+ K$*, then* $\bigcup_{K \in H} H_K \vdash_+ L$*.*

We write $\vdash_+^T L$ *for* $\varnothing \vdash_+^T L$*. A closed (negative) literal* L *is* refutable *from* T*, if* $\vdash_+^T K$ *for some positive literal* K ↯ L*.*

The following “deduction lemma” and “soundness lemma” will be useful in the sequel.

Lemma 3.4. *If* $H \vdash_+ L$ *and for all* $K \in H$*,* $H_K \vdash_+ K$*, then* $\bigcup_{K \in H} H_K \vdash_+ L$*.*

Proof: Induction on the proof of L from the H. ⊠

Lemma 3.5. *If* R *is a model for TSS* T*, and* $H \vdash_+^T L$*, then* $R \vDash \frac{H}{L}$*.*

Proof: Induction over the proof of L from H. ⊠

If a TSS T consists of positive rules only, it can be viewed as a (simultaneous) inductive definition of a certain labeled transition relation. The relation contains exactly those literals that are provable from T. If T contains negative premises in addition, it is not so clear which transition relation is defined. A TSS may even be refused, because it is meaningless. In [7] up to 11 different solutions for this problem are summarized and compared. Two of these are important for our purpose.

The first one gives a minimality criterion that transition relations should satisfy. The intuition is that positive literals are true only if they are forced somehow. A negative literal may be assumed true, as soon as this is consistent. This intuition is made formal by the notion *stable model*. A TSS is meaningful, if there is a unique stable model.

The other method has a proof theoretic flavour. The definition of *positive proof* is extended with a proof rule for deriving *negative* literals. In the second approach, a TSS is meaningful, if each literal is either provable or refutable. Unfortunately, these solutions don’t coincide.

Definition 3.6. *(Stable transition relations,* cf. [7, Def. 6]*)*
A transition relation R *is* well supported *for a TSS* T*, if for each* $s \to^a t$ *in* R*, there is a set* N *of negative literals, such that* $N \vdash_+^T s \to^a t$ *and* $R \vDash N$*.* R *is called* stable *if it is a well supported model.*

In [7, Prop. 5], it is proved that T has a unique stable model if and only if it has a least stable model. In case this exists, it can serve as the semantics of T.

We now recapitulate the second method, which adds a new proof rule in order to derive negative information. We dropped the possibility to start with assumptions, because this is not needed. For technical reasons, provability is restricted to closed literals.

Definition 3.7. *(Well-supported proof,* cf. [7, Def. 9]*) Given a TSS T,* well supported provability *($\vdash^T_{\mathrm{ws}}$ or $\vdash_{\mathrm{ws}}$ for short) is defined inductively by the following two clauses:*

1. *If $\frac{H}{L}$ is a closed instance of a rule from T, and for all $K \in H$, $\vdash_{\mathrm{ws}} K$, then $\vdash_{\mathrm{ws}} L$.*
2. *Let L be a negative closed literal. If for any $K \lightning L$ and set of negative closed literals N such that $N \vdash_+ K$ we can find an M such that $M \lightning N$ and $\vdash_{\mathrm{ws}} M$, then $\vdash_{\mathrm{ws}} L$.*

A TSS T is complete*, if for each closed transition $s \rightarrow^a t$, either $\vdash^T_{\mathrm{ws}} s \rightarrow^a t$ or $\vdash^T_{\mathrm{ws}} s \not\rightarrow^a t$.*

The second rule has a "negation as failure" flavour: if every attempt to prove a denial K of L fails (because it needs hypotheses N that are in conflict with some M that has been proved already), L may be considered valid. Note that in case no rule matches a transition $s \rightarrow t$, then the condition of the second clause is vacuously true, so $\vdash_{\mathrm{ws}} s \not\rightarrow t$ holds.

Proposition 3.8. [7, Prop. 10, 11] *Let T be a TSS.*

1. *$\vdash^T_{\mathrm{ws}}$ is consistent.*
2. *If $\vdash^T_{\mathrm{ws}} L$ then $R \vDash L$ for all well supported models R of T.*

3.2 Translation of PRSs into TSSs

In this section, we give a translation of a PRS $\mathcal{P}$ to $\mathsf{TSS}(\mathcal{P})$. Without loss of generality, we make two assumptions about $\mathcal{P}$. The first is that different rules have disjoint variables. This can always be reached by renaming the variables. The second assumption is, that for each inhabited arity m in $\mathcal{P}$, there is an m-ary function symbol that doesn't occur in the rules, denoted by $\langle\_\rangle_m$. This can always be achieved by adding new function symbols. This is to avoid $\twoheadrightarrow^{\text{int}}$ as a relation symbol.

Definition 3.9.

1. *Let* $\Sigma = (\mathcal{F},\mathcal{V})$. *Put* $\mathsf{TSS}(\Sigma) = (\mathcal{F},\mathcal{V},\mathcal{L})$, *where* $\mathcal{L} := \{\rightarrow, \longrightarrow, \twoheadrightarrow\}$.
2. *Let* $\mathcal{P} = (\mathcal{R},>)$ *be given. Let* $x = f(\vec{s}) \rightarrow t$ *be a rule in* $\mathcal{R}$. *Define*
$$\mathsf{TSS}(x) = \frac{\{\forall \vec{z}.\, \langle\vec{s}\rangle \not\twoheadrightarrow \langle\vec{r}\rangle \mid f(\vec{r}) \rightarrow t' > x \text{ and } \vec{z} = \mathrm{FV}(\vec{r})\}}{x}$$
3. *Depending on* Σ, *a set of rules* F *is defined, consisting of*
$$\frac{x_i \longrightarrow y}{f(x_1,\cdots,x_i,\cdots,x_n) \longrightarrow f(x_1,\cdots,y,\cdots,x_n)}\ \text{(C2)}$$
$$\frac{x \rightarrow y}{x \longrightarrow y}\ \text{(C1)} \qquad \frac{}{x \twoheadrightarrow x}\ \text{(T1)} \qquad \frac{x \longrightarrow y \quad y \twoheadrightarrow z}{x \twoheadrightarrow z}\ \text{(T2)}$$
Rule C2 *is present for each function symbol* f, *including* $\langle\_\rangle_m$, *and for each* $1 \leq i \leq \mathsf{arity}(f)$.
4. *Let* $\mathcal{P} = (\mathcal{R},>)$ *with* $\mathcal{R} = (\Sigma,R)$ *be given. Define* $\mathsf{TSS}(\mathcal{P}) = (\mathsf{TSS}(\Sigma), R')$, *where* $R' = \{\mathsf{TSS}(r) \mid r \in R\} \cup F$.

Example 3.10. Let $\mathcal{P}$ be the following PRS [1, Example 4]:

$$\boxed{\begin{array}{l} \big\downarrow \begin{array}{lcl} Eq(x,x) & \rightarrow & T \\ Eq(x,y) & \rightarrow & F \end{array} \end{array}}$$

The TSS associated to $\mathcal{P}$ has the following rules:

$$\frac{}{Eq(x,x) \rightarrow T} \qquad \frac{\forall z.\, \langle x,y\rangle \not\twoheadrightarrow \langle z,z\rangle}{Eq(x,y) \rightarrow F} \qquad \frac{x \rightarrow y}{x \longrightarrow y}$$

$$\frac{x \longrightarrow y}{Eq(x,z) \longrightarrow Eq(y,z)} \qquad \frac{x \longrightarrow y}{Eq(z,x) \longrightarrow Eq(z,y)} \qquad \frac{x \longrightarrow y}{\langle x,z\rangle \longrightarrow \langle y,z\rangle}$$

$$\frac{x \longrightarrow y}{\langle z,x\rangle \longrightarrow \langle z,y\rangle} \qquad \frac{}{x \twoheadrightarrow x} \qquad \frac{x \longrightarrow y \quad y \twoheadrightarrow z}{x \twoheadrightarrow z}$$

⊠

Any transition relation for $\mathsf{TSS}(\mathcal{P})$ can be seen as a triple of binary relations (R,C,T), where R interprets $\rightarrow$, C interprets $\longrightarrow$ and T interprets $\longrightarrow\!\!\!\!\rightarrow$. Any rewrite set R gives rise to a transition relation $(R, \longrightarrow_R, \longrightarrow\!\!\!\!\rightarrow_R)$. Note that if (R,C,T) is a transition relation for $\mathsf{TSS}(\mathcal{P})$, then R is not necessarily a rewrite set for $\mathcal{P}$ (i.e. a set of closed rule instances), nor is it always the case that $C = \longrightarrow_R$ and $T = \longrightarrow\!\!\!\!\rightarrow_R$.

The adequacy of the translation above is shown by the following technical lemmata, which are also key lemmata in subsequent sections.

Lemma 3.11. *Let $\mathcal{P}$ be a PRS, and R a set of $\mathcal{P}$-rewrites. For any closed terms s and t, we have:*

$$s \rightarrow t \text{ is a correct rewrite w.r.t. } R \iff \text{there are } N,\ r \text{ and } \sigma, \text{ such that:}$$

1. *$\frac{N}{r}$ is a rule of $\mathsf{TSS}(\mathcal{P})$; and*
2. *$r^\sigma = s \rightarrow t$; and*
3. *N^σ holds in $\longrightarrow\!\!\!\!\rightarrow_R$.*

Proof: $\Longrightarrow$: for some r, we have that $s \rightarrow^r t$ is correct w.r.t. R. Take σ as required by (2), and let $\mathsf{TSS}(r)$ be $\frac{N}{r}$, then (1) clearly holds. Assume that $s = f(\vec{l})$, then $N^\sigma = \{\forall \vec{z}.\langle \vec{l} \rangle \not\longrightarrow\!\!\!\!\rightarrow \langle \vec{a} \rangle \mid f(\vec{a}) \rightarrow b > r \text{ with free variables } \vec{z}\}$.

We now have the following equivalence:

$$\begin{array}{ll} & s \rightarrow^r t \text{ is correct w.r.t. } R \\ \iff & s \not\longrightarrow\!\!\!\!\rightarrow_R^{\text{int}} a^\tau \text{ for any rule } a \rightarrow b > r \text{ and closed substitution } \tau \\ \iff & \langle \vec{l} \rangle \not\longrightarrow\!\!\!\!\rightarrow_R \langle \vec{a} \rangle^\tau \text{ for any rule } f(\vec{a}) \rightarrow b > r \text{ and closed } \tau \\ \iff & \forall \vec{z}.\langle \vec{l} \rangle \not\longrightarrow\!\!\!\!\rightarrow_R \langle \vec{a} \rangle \text{ for any rule } f(\vec{a}) \rightarrow b > r \text{ with free variables } \vec{z} \\ \iff & N^\sigma \text{ holds in } \longrightarrow\!\!\!\!\rightarrow_R. \end{array}$$

This equivalence also suffices to prove the $\Longleftarrow$ direction of the lemma. ⊠

Lemma 3.12. *Let PRS $\mathcal{P}$ be given, define $T = \mathsf{TSS}(\mathcal{P})$, let N be a set of negative premises. Then for any terms s and t we have*

1. *$N \vdash_+^T s \rightarrow t \iff \frac{N}{s \rightarrow t}$ is an instance of some rule in T.*
2. *$N \vdash_+^T s \longrightarrow t \iff$ there are context $C[]$ and closed terms l and r such that*

(a) $N \vdash^T_+ l \to r$; and

(b) $C[l] = s$ and $C[r] = t$.

3. $N \vdash^T_+ s \twoheadrightarrow t \iff$ *there exist* n, $\vec{s}$ *and* $\vec{N}$, *such that*

   (a) $s = s_0$, $t = s_n$ and $N = \bigcup \vec{N}$; and

   (b) for all $0 \leq i < n$, $N_i \vdash^T_+ s_i \longrightarrow s_{i+1}$.

Proof: $\Longrightarrow$ All three statements are proved with induction to $\vdash^T_+$, taking into account the rules in T

1. The only applicable rules in T are of the form $\frac{M}{x}$, with M completely negative. Elements of M can only be positively proved as assumptions.

2. If C1 is applied in the last step, l and r can be read of and we have an empty context. If C2 is applied, we use the induction hypothesis and enlarge the context.

3. If T1 is applied in the last step, we take $n = 0$. Otherwise T2 is applied, which delivers s_1; the induction hypothesis yields $s_2, \dots s_n$.

$\Longleftarrow$

1. We have a one-step proof.

2. Using C1 and C2, we find $l \to r \vdash^T_+ s \longrightarrow t$. By Prop. 3.4, $N \vdash^T_+ s \longrightarrow t$.

3. Using T1 and T2, we find $\{s_i \longrightarrow s_{i+1} \mid 0 \leq i < n\} \vdash^T_+ s \twoheadrightarrow t$. By Prop. 3.4, $N \vdash^T_+ s \twoheadrightarrow t$ follows.

$\boxtimes$

4 Operational semantics of PRSs

We now want to establish a link between the PRS-semantics and the semantics that comes with transition systems. The comparison is made possible by our translation. Indeed, there is a quite remarkable connection. We will show (Thm. 4.5) that the sound and complete rewrite sets for $\mathcal{P}$ coincide with the well supported models of $\mathsf{TSS}(\mathcal{P})$.

First it is proved that a rewrite set is complete for $\mathcal{P}$ if and only if it is a model for $\mathsf{TSS}(\mathcal{P})$. In the same way, soundness and well-supportedness are tightly related.

In Sect. 4.2, we will also establish a link between complete TSSs and the fixed point construction for PRSs. It will turn out (Thm. 4.9) that $\mathsf{TSS}(\mathcal{P})$ is a complete specification if and only if the least and greatest fixed points of the operator $()^{\mathcal{P}}$ coincide.

4.1 Sound and complete vs. well supported model

R is complete if it contains all correct rewrites w.r.t. itself. Therefore, a rewrite is present whenever the negative premises connected to it are true. This in turn means that the rules of the associated TSS are true, hence the rewrite set is a model. Hence, a rewrite set is complete for $\mathcal{P}$ if and only if it is a model for $\mathsf{TSS}(\mathcal{P})$.

Proposition 4.1. *R is a complete rewrite set of a PRS $\mathcal{P}$ if and only if $(R, \longrightarrow_R, \twoheadrightarrow_R)$ is a model of $\mathsf{TSS}(\mathcal{P})$.*

Proof: Let $\mathcal{M} = (R, \longrightarrow_R, \twoheadrightarrow_R)$. $\Longrightarrow$: Rules C1, C2, T1 and T2 clearly hold in $\mathcal{M}$. Now let some other rule, $\frac{N}{r}$ be given. Assume that $\mathcal{M} \models N^{\sigma}$, for some substitution σ. Then by Lemma 3.11, r^{σ} is correct w.r.t. R, hence by completeness, $r^{\sigma} \in R$, so $\mathcal{M} \models r^{\sigma}$. So $\mathcal{M} \models \frac{N}{r}$. Now $\mathcal{M}$ is a model of $\mathsf{TSS}(\mathcal{P})$, because all rules hold in it.

$\Longleftarrow$: Let $s \to^r t$ be correct w.r.t. R. By Lemma 3.11, the negative premises of the corresponding instance of $\mathsf{TSS}(r)$ are true in $\twoheadrightarrow_R$. As $\mathcal{M}$ is a model, the consequence of $\mathsf{TSS}(r)$ holds in R, so $s \to t$ in R. Hence R is complete. ⊠

Now we will show that the sound rewrite sets coincide with the well supported models. The intuition is, that in a sound rewrite set, all rewrites are correct, so the negative premises connected with them are true. The latter forms the basic idea of well-supportedness.

Proposition 4.2. *Let $\mathcal{P}$ be a PRS and R a rewrite set for it. If R is sound, then $(R, \longrightarrow_R, \twoheadrightarrow_R)$ is a well supported transition relation for $\mathsf{TSS}(\mathcal{P})$.*

Proof: Let x be a rewrite in R. By soundness, x is correct w.r.t. R. Hence by Lemma 3.11 and 3.12.1, we find a negative set of premises N that holds in $\twoheadrightarrow_R$, with $N \vdash_+ x$.

If $s \longrightarrow_R t$, then we find l, r, $C[]$ such that $C[l] = s$, $C[r] = t$ and $l \to r$ in R. By the previous argument, we find a set N of negative premises that hold in $\twoheadrightarrow_R$, such $N \vdash_+ l \to r$. By Lemma 3.12.2, $N \vdash_+ s \longrightarrow t$.

If $s \twoheadrightarrow_R t$, we find $n, s = s_0, s_1, \ldots, s_n = t$, such that $s_i \longrightarrow_R s_{i+1}$ for each $0 \leq i < n$. By the previous argument, we find for each i a set N_i of negative premises that hold in R, such that $N_i \vdash_+ s_i \longrightarrow_R s_{i+1}$. Lemma 3.12.3 yields $\bigcup_{0 \leq i < n} N_i \vdash_+ s \twoheadrightarrow t$. ⊠

Proposition 4.3. *Let $\mathcal{P}$ be a PRS. If $(R, \longrightarrow_R, \twoheadrightarrow_R)$ is a well supported transition relation for* $\mathsf{TSS}(\mathcal{P})$*, then R is a sound rewrite set for $\mathcal{P}$.*

Proof: Assume xRy. By well-supportedness, there is a set N of negative premises that hold in $\twoheadrightarrow_R$, such that $N \vdash_+ x \to y$. By Lemma 3.12.1, $\frac{N}{x \to y}$ is an instance of a rule in $\mathsf{TSS}(\mathcal{P})$. By Lemma 3.11, $x \to y$ is a correct rewrite w.r.t. R. ⊠

Propositions 4.2 and 4.3 together show that for transition relations of the form $(R, \longrightarrow_R, \twoheadrightarrow_R)$, soundness and well-supportedness coincide. To guarantee that a well supported transition relation has this particular form, we in addition need that it is a model.

Lemma 4.4. *Let a PRS $\mathcal{P}$ be given. Any well supported model of* $\mathsf{TSS}(\mathcal{P})$ *is of the form $(R, \longrightarrow_R, \twoheadrightarrow_R)$ for some rewrite set R.*

Proof: Let (R, C, T) be a well supported model of $\mathsf{TSS}(\mathcal{P})$. If $x \in R$, then for some set of negative premises N, $N \vdash_+ x$. By Prop. 3.12.1, $\frac{N}{x}$ is an instance of a rule from $\mathsf{TSS}(\mathcal{P})$. Then x is a rewrite of $\mathcal{P}$.

Assume sCt. By well-supportedness, there is a set of negative premises N such that $N \vdash_+ s \longrightarrow t$ and N holds in (R, C, T). By Lemma 3.12.2, we find a context $D[]$, and l and r such that $D[l] = s$, $D[r] = t$ and $N \vdash_+ l \to r$. Because (R, C, T) is a model, and using Lemma 3.5, $l \to r$ holds in R, hence $s \longrightarrow_R t$.

Now assume sTt. By well-supportedness we find a set N of negative premises, such that $N \vdash_+ s \twoheadrightarrow t$, and N holds in (R, C, T). By Lemma 3.12.3, we find an n, $\vec{s}$ and $\vec{N}$, with $s_0 = s$ and $s_n = t$, and such that for each $0 \leq i < n$, $N \supseteq N_i \vdash_+ s_i \longrightarrow s_{i+1}$. Using Lemma 3.5, we have that $s_i C s_{i+1}$, hence $s_i \longrightarrow_R s_{i+1}$ for all these i, hence $s \twoheadrightarrow_R t$.

Conversely, if $s \longrightarrow_R t$ then sCt, because (R, C, T) is a model of C1 and C2. In the same way, by T1 and T2 it is made sure that from $s \twoheadrightarrow_R t$, we can infer sTt. ⊠

For the latter lemma, we really need that the transition relation is *well* supported. There exists a less restrictive notion of supportedness, but in Appendix A we give an example showing that this is not enough.

We are now able to state the main theorem of this section. The theorem says that the PRS-semantics can be expressed in terms of theory on models of TSSs.

Theorem 4.5. *Let $\mathcal{P}$ be a PRS. The following two statements are equivalent:*

1. *$\mathcal{P}$ has a unique sound and complete rewrite set.*

2. *$\mathsf{TSS}(\mathcal{P})$ has a least well supported model.*

Proof: Any sound and complete rewrite set R for $\mathcal{P}$ yields a well supported model $(R, \longrightarrow_R, \longrightarrow\!\!\!\!\!\rightarrow_R)$ for $\mathsf{TSS}(\mathcal{P})$, by Lemma 4.2 and 4.1. Conversely, each well supported model is of the form $(R, \longrightarrow_R, \longrightarrow\!\!\!\!\!\rightarrow_R)$, where R is a sound and complete rewrite set, by Lemma 4.4, 4.3 and 4.1.

By [7, Prop. 5], a least well supported model is uniquely well supported. Now the theorem follows. ⊠

4.2 Fixed points and complete specifications

Recall that the fixed point construction of Sect. 2.2 iterates the function $()^{\mathcal{P}}$, which assigns to each rewrite set R the set of correct rewrites. We had a series $\mathbf{T}_{\mathcal{P}}{\uparrow}\alpha$ iterating $()^{\mathcal{P}}$ an even number of times, starting with $\varnothing$, and $\mathbf{T}_{\mathcal{P}}{\downarrow}\alpha$, iterating $()^{\mathcal{P}}$ odd times, starting with $\varnothing^{\mathcal{P}}$. See Def. 2.8 for the exact definition.

The least and greatest fixed points of $()^{\mathcal{P}}$ always exist. This section is devoted to the proof that these fixed points coincide if and only if $\mathsf{TSS}(\mathcal{P})$ is a complete transition system specification. We have to relate truth in $\mathbf{T}_{\mathcal{P}}{\uparrow}\alpha$ and $\mathbf{T}_{\mathcal{P}}{\downarrow}\alpha$ with provability of positive and negative literals.

In Prop. 4.7 we show that for any α, $\mathbf{T}_{\mathcal{P}}{\uparrow}\alpha$ only contains information that is provable. This is proved simultaneously with the fact that only refutable transitions are outside $\mathbf{T}_{\mathcal{P}}{\downarrow}\alpha$. But first we need a lemma, relating $\mathbf{T}_{\mathcal{P}}{\downarrow}\alpha$ with $\mathbf{T}_{\mathcal{P}}{\uparrow}\alpha$.

Lemma 4.6. *Let $\mathcal{P}$ be a PRS and α an ordinal. For all closed terms x and y we have: if $\mathbf{T}_{\mathcal{P}}{\downarrow}\alpha \vDash x \not\twoheadrightarrow y$ then for each set of negative premises N with $N \vdash_+ x \rightarrow y$, there exists a literal K, such that $K \lightning N$ and $\mathbf{T}_{\mathcal{P}}{\uparrow}\alpha \vDash K$.*

Proof: $N \vdash_+ x \to y$ implies that $\frac{N}{x \to y}$ is a rule instance (Lemma 3.12.1). Assume $\mathbf{T}_{\mathcal{P}}{\downarrow}\alpha \models x \not\to y$. Then by Prop. 2.9.1, $x \to y$ is not correct w.r.t. $\mathbf{T}_{\mathcal{P}}{\uparrow}\alpha$. Hence $\frac{N}{x \to y}$ has a false negative premise (by Lemma 3.11), say $\forall\vec{z}.\, l \not\twoheadrightarrow r$. Because it is false, we find a substitution σ, such that $\mathbf{T}_{\mathcal{P}}{\uparrow}\alpha \models l \twoheadrightarrow r^{\sigma}$. This yields the required K. ⊠

Proposition 4.7. *Let PRS $\mathcal{P}$ with $\mathsf{TSS}(\mathcal{P}) = T$ and ordinal α be given. Let s and t be closed terms; let r be a term with free variables $\vec{z}$. Then we have:*

1. *If $\mathbf{T}_{\mathcal{P}}{\uparrow}\alpha \models s \to t$ then $\vdash^{T}_{\mathrm{ws}} s \to t$,*
2. *If $\mathbf{T}_{\mathcal{P}}{\uparrow}\alpha \models s \twoheadrightarrow t$ then $\vdash^{T}_{\mathrm{ws}} s \twoheadrightarrow t$,*
3. *If $\mathbf{T}_{\mathcal{P}}{\downarrow}\alpha \models s \not\to t$ then $\vdash^{T}_{\mathrm{ws}} s \not\to t$,*
4. *If $\mathbf{T}_{\mathcal{P}}{\downarrow}\alpha \models \forall\vec{z}.\, s \not\twoheadrightarrow r$, then $\vdash^{T}_{\mathrm{ws}} \forall\vec{z}.\, s \not\twoheadrightarrow r$.*

Proof: Simultaneous induction on α. We first prove that for fixed α, we have $(1) \Rightarrow (2) \Rightarrow (3)$ and $(2) \Rightarrow (4)$. Using these implications, we inductively prove (1).

$(1) \Rightarrow (2)$ Assume $\mathbf{T}_{\mathcal{P}}{\uparrow}\alpha \models s \twoheadrightarrow t$. The rewrites used in this reduction are in $\mathbf{T}_{\mathcal{P}}{\uparrow}\alpha$, hence provable by (1). Then also $\vdash_{\mathrm{ws}} s \twoheadrightarrow t$, by using C1, C2, T1 and T2.

$(2) \Rightarrow (3)$ Assume $\mathbf{T}_{\mathcal{P}}{\downarrow}\alpha \models s \not\to t$. We get $\vdash_{\mathrm{ws}} s \not\to t$, by using the second clause of Def. 3.7 as follows: Let $N \vdash_+ s \to t$, for some set of negative premises N. By Lemma 4.6, we find a literal $K \lightning N$, with $\mathbf{T}_{\mathcal{P}}{\uparrow}\alpha \models K$, hence by (2), $\vdash_{\mathrm{ws}} K$. Hence $\vdash_{\mathrm{ws}} s \not\to t$.

$(2) \Rightarrow (4)$ Define $L = \forall\vec{z}.\, s \not\twoheadrightarrow r$. Assume $\mathbf{T}_{\mathcal{P}}{\downarrow}\alpha \models L$. Any literal denying L is of the form $s \twoheadrightarrow r^{\sigma}$ for some closed substitution σ. Let such σ and an arbitrary set of negative literals N such that $N \vdash_+ s \twoheadrightarrow r^{\sigma}$, be given. By Lemma 3.12.3, there are $\vec{N}$ and $\vec{s}$, such that $N_i \vdash_+ s_i \longrightarrow s_{i+1}$, $s_0 = s$ and $s_n = r^{\sigma}$. Because $\mathbf{T}_{\mathcal{P}}{\downarrow}\alpha \models s \not\twoheadrightarrow r^{\sigma}$, there is a j such that $\mathbf{T}_{\mathcal{P}}{\downarrow}\alpha \models s_j \not\longrightarrow s_{j+1}$. By Lemma 3.12.2, we find C, a and b, such that $N_j \vdash_+ a \to b$, $C[a] = s_j$ and $C[b] = s_{j+1}$. Clearly, $\mathbf{T}_{\mathcal{P}}{\downarrow}\alpha \models a \not\to b$. Now by Lemma 4.6, we get a literal K such that $\mathbf{T}_{\mathcal{P}}{\uparrow}\alpha \models K$ and $K \lightning N_j$, hence also $K \lightning N$. By (2), $\vdash_{\mathrm{ws}} K$. Therefore, $\vdash_{\mathrm{ws}} s \not\twoheadrightarrow t$.

Now we prove (1) by ordinal induction on α. By the implications above, we may also use the induction hypothesis of (4).

0: $\mathbf{T}_{\mathcal{P}}\uparrow 0 = \varnothing$, and $\varnothing \vDash s \to t$ is impossible.

$\alpha+1$**:** If $\mathbf{T}_{\mathcal{P}}\uparrow(\alpha+1) \vDash s \to t$, then $s \to t$ is correct w.r.t. $\mathbf{T}_{\mathcal{P}}\downarrow\alpha$ (Prop. 2.9.2). Hence for some rule instance $\frac{N}{s\to t}$, we have $\mathbf{T}_{\mathcal{P}}\downarrow\alpha \vDash N$ (Prop. 3.11). By the induction hypothesis for (4), $\vdash_{\mathrm{ws}} N$. Now by clause 1 of Def. 3.7, $\vdash_{\mathrm{ws}} s \to t$.

λ (a limit ordinal)**:** If $\mathbf{T}_{\mathcal{P}}\uparrow\lambda \vDash s \to t$, then for some $\alpha < \lambda$, $\mathbf{T}_{\mathcal{P}}\uparrow\alpha \vDash s \to t$. By the induction hypothesis for (1), $\vdash_{\mathrm{ws}} s \to t$.

⊠

The next proposition serves as the converse of the previous one. It expresses that the provable transitions hold in fixed points of $T_{\mathcal{P}}$ and that refutable transitions are not correct w.r.t. them.

Proposition 4.8. *Let PRS $\mathcal{P}$ with $\mathsf{TSS}(\mathcal{P}) = T$ be given. Let a rewrite set R be given, with $(R^{\mathcal{P}})^{\mathcal{P}} = R$. Then we have for all closed terms s and t, and for all terms r with free variables $\vec{z}$:*

1. *If $\vdash^{T}_{\mathrm{ws}} s \to t$ then $R \vDash s \to t$.*
2. *If $\vdash^{T}_{\mathrm{ws}} s \twoheadrightarrow t$ then $R \vDash s \twoheadrightarrow t$.*
3. *If $\vdash^{T}_{\mathrm{ws}} s \not\to t$ then $R^{\mathcal{P}} \vDash s \not\to t$.*
4. *If $\vdash^{T}_{\mathrm{ws}} \forall\vec{z}.\, s \not\twoheadrightarrow r$ then $R^{\mathcal{P}} \vDash \forall\vec{z}.\, s \not\twoheadrightarrow r$.*

Proof: Simultaneous induction on the definition of $\vdash_{\mathrm{ws}}$.

1. Assume $\vdash_{\mathrm{ws}} s \to t$. The last step in this proof is an application of $\frac{N}{s\to t}$, for some N with $\vdash_{\mathrm{ws}} N$. By induction hypothesis (4), $R^{\mathcal{P}} \vDash N$. By Lemma 3.11, $s \to t$ is correct w.r.t. $R^{\mathcal{P}}$, hence it is in $(R^{\mathcal{P}})^{\mathcal{P}} = R$.
2. Assume $\vdash_{\mathrm{ws}} s \twoheadrightarrow t$. The last step in this proof is an application of T1 or T2. If T1 was applied, $s = t$, and $R \vDash s \twoheadrightarrow s$ holds. If T2 is applied, then there exists an r with $\vdash_{\mathrm{ws}} s \to r$ and $\vdash_{\mathrm{ws}} r \twoheadrightarrow t$. By induction hypothesis for (1) and (2), $R \vDash s \to r$ and $R \vDash r \twoheadrightarrow t$, hence $R \vDash s \twoheadrightarrow t$.

3. Assume $\vdash_{\mathrm{ws}} s \not\to t$. We have to show that $s \to t$ is not correct w.r.t. R. By Lemma 3.11 it suffices to show that for any rule instance $\frac{N}{s \to t}$, N doesn't hold in R. Let such N be given; then $N \vdash_{+} s \to t$. Because the last step of the proof $\vdash_{\mathrm{ws}} s \not\to t$ uses the second clause of Def. 3.7, there exists some K with $K \lightning N$ and $\vdash_{\mathrm{ws}} K$. By induction hypothesis (2), $R \vDash K$. Therefore $R \nvDash N$.

4. The argument is similar as for (3). Assume $\vdash_{\mathrm{ws}} \forall \vec{z}.\, s \not\twoheadrightarrow r$. Assume towards a contradiction that for some σ, $R^{\mathcal{P}} \vDash s \twoheadrightarrow r^{\sigma}$. Let I be the set of rewrites that are used in this reduction; clearly $R^{\mathcal{P}} \vDash I$. Let N be the union of the negative premises associated to the rewrites in I. Then (using the rules of $\mathsf{TSS}(\mathcal{P})$) we have $N \vdash_{+} s \twoheadrightarrow r^{\sigma}$. Because the last step in the proof $\vdash_{\mathrm{ws}} \forall \vec{z}.\, s \not\twoheadrightarrow r$ uses the second clause of Def. 3.7, there exists some K, such that $K \lightning N$ and $\vdash_{\mathrm{ws}} K$. By the induction hypothesis (2), $R \vDash K$, hence $R \nvDash N$, hence some $L \in I$ is not correct w.r.t. R. But then $R^{\mathcal{P}} \nvDash I$; contradiction. Therefore, $R^{\mathcal{P}} \nvDash s \twoheadrightarrow r^{\sigma}$ for each substitution σ, so $R^{\mathcal{P}} \vDash \forall \vec{z}.\, s \not\twoheadrightarrow r$.

$\boxtimes$

We are now able to prove that $T_{\mathcal{P}}$ has a unique fixed point if and only if the specification of $\mathsf{TSS}(\mathcal{P})$ is complete.

Theorem 4.9. *Let $\mathcal{P}$ be a PRS, with closure ordinal α. Then $\mathbf{T}_{\mathcal{P}}\!\uparrow\!\alpha = \mathbf{T}_{\mathcal{P}}\!\downarrow\!\alpha$ if and only if $\mathsf{TSS}(\mathcal{P})$ is complete.*

Proof: $\Longrightarrow$: For each rewrite $s \to t$, either $\mathbf{T}_{\mathcal{P}}\!\uparrow\!\alpha \vDash s \to t$ or $\mathbf{T}_{\mathcal{P}}\!\uparrow\!\alpha \nvDash s \to t$. In the first case, $\vdash_{\mathrm{ws}} s \to t$ by Prop. 4.7.1. In the second case, $\mathbf{T}_{\mathcal{P}}\!\downarrow\!\alpha \vDash s \not\to t$, as we may assume $\mathbf{T}_{\mathcal{P}}\!\uparrow\!\alpha = \mathbf{T}_{\mathcal{P}}\!\downarrow\!\alpha$. By Prop. 4.7.3, $\vdash_{\mathrm{ws}} s \not\to t$.

$\Longleftarrow$: Note that for the closure ordinal α, $\mathbf{T}_{\mathcal{P}}\!\uparrow\!\alpha = ((\mathbf{T}_{\mathcal{P}}\!\uparrow\!\alpha)^{\mathcal{P}})^{\mathcal{P}}$, so Prop. 4.8 is applicable. Note also that $\mathbf{T}_{\mathcal{P}}\!\uparrow\!\alpha \subseteq \mathbf{T}_{\mathcal{P}}\!\downarrow\!\alpha = (\mathbf{T}_{\mathcal{P}}\!\uparrow\!\alpha)^{\mathcal{P}}$. (By Prop. 2.7.1, 2.9.1 and 2.10.1). We still have to prove $\mathbf{T}_{\mathcal{P}}\!\uparrow\!\alpha \supseteq \mathbf{T}_{\mathcal{P}}\!\downarrow\!\alpha$.

Let $\mathbf{T}_{\mathcal{P}}\!\downarrow\!\alpha \vDash s \to t$, then (Prop. 4.8.3) $\nvdash_{\mathrm{ws}} s \not\to t$. Hence by completeness of $\mathsf{TSS}(\mathcal{P})$, $\vdash_{\mathrm{ws}} s \to t$. Now by Prop. 4.8.1, $\mathbf{T}_{\mathcal{P}}\!\uparrow\!\alpha \vDash s \to t$. This shows that $\mathbf{T}_{\mathcal{P}}\!\downarrow\!\alpha \subseteq \mathbf{T}_{\mathcal{P}}\!\uparrow\!\alpha$. $\boxtimes$

A Just supported is not enough

This appendix deals with an example that serves as extra explanation. In Sect. 4.1 we proved that the sound and complete rewrite sets for PRS $\mathcal{P}$ correspond to well supported models of $\mathsf{TSS}(\mathcal{P})$. The definition of well-supportedness is quite intricate, as it requires that for each transition L in a model $\mathcal{M}$, there exist negative premises N, such that $N \vdash_+ L$ and $\mathcal{M} \models N$. There exists a much simpler definition of supportedness:

Definition A.1. *[7, Def. 5] A transition relation $\mathcal{M}$ is* supported *if for every transition $L \in \mathcal{M}$, there is a rule instance $\frac{H}{L}$ such that $\mathcal{M} \models H$.*

Instead of the existence of a proof of L with true negative premises, now simply a rule with conclusion L is required, with true premises. However, note that when L appears among H, then the support for L is not very convincing. In this case the presence of L would be used to make sure that L is forced. Such a circularity can also be less visible. The circularity is avoided in the definition of *well*-supportedness (Def. 3.6). Indeed, we can find an example of a PRS that has no sound and complete rewrite set, but whose corresponding TSS has a least supported model. In the sequel, $\mathcal{P}$ refers to the following example.

Example A.2. [1, Example 2.12]

$$\begin{array}{|rcl|} \hline 1 & \to & A(1) \\ \big| A(0) & \to & 1 \\ \downarrow A(x) & \to & 0 \\ \hline \end{array}$$

$\boxtimes$

In [1] it is shown that this system has no meaning. The problem lies in the fact that the rewrite $A(1) \to 0$ is allowed if and only if $1 \not\twoheadrightarrow 0$. This however is precisely the case if $A(1) \not\to 0$. Hence no sound and complete rewrite set can exist.

Applying the translation of Def. 3.9, we get a TSS consisting of the fixed rules C1, C2, T1 and T2, together with:

$$\frac{}{1 \to A(1)}\ (\text{R1}) \qquad \frac{}{A(0) \to 1}\ (\text{R2}) \qquad \frac{x \not\twoheadrightarrow 0}{A(x) \to 0}\ (\text{R3})$$

Remember that models of $\mathsf{TSS}(\mathcal{P})$ are of the form (R, C, T), where R is the rewrite set, and C and T interpret the context- and transitive closure, respectively. As shown below, *supportedness* does not guarantee that T really equals $\longrightarrow\!\!\!\!\!\rightarrow_R$. Lemma 4.4 shows, that for *well*-supported models, this is guaranteed.

We claim that $\mathcal{M} := (R, C, T)$ as defined below is the least supported model of $\mathsf{TSS}(\mathcal{P})$. This is proved by 1, 2 and 3 below.

$$\begin{aligned} R &:= \{(1, A(1)),\ (A(0), 1)\} \\ C &:= \longrightarrow_R \\ T &:= \twoheadrightarrow_R \cup \{(x, 0) \mid x \text{ a closed term}\} \end{aligned}$$

We have no R3-rewrites in R. So in order to make R3 true, its premise must be false. This is done by ensuring that each term "reduces" to 0 in T. We have

1. $\mathcal{M}$ *is a model of* $\mathsf{TSS}(\mathcal{P})$. Clearly, R1, R2, C1, C2 and T1 hold in $\mathcal{M}$. R3 holds, because its premise is never true. As to T2, assume $x \longrightarrow_R y$ and yTz. Now either $y \twoheadrightarrow_R z$, in which case also $x \twoheadrightarrow_R z$, or $z = 0$. In both cases we have xTz. Hence T2 also holds in $\mathcal{M}$.

2. $\mathcal{M}$ *is supported.* Elements of R are supported by rules R1 or R2. Elements of C are supported by rules C1 or C2. The $\twoheadrightarrow_R$-elements of T are supported by T1 or T2. Finally, the $(x, 0)$ elements of T can be supported as follows. If $x = 0$, then T1 supports it. For $x = A^n(1)$, we find as support

$$\frac{A^n(1) \longrightarrow A^{n+1}(1) \quad A^{n+1}(1) \twoheadrightarrow 0}{A^n(1) \twoheadrightarrow 0}\ \text{(T2)}$$

   Both premises hold in $\mathcal{M}$. For $x = A^{n+1}(0)$ we find as support

$$\frac{A^{n+1}(0) \longrightarrow A^n(1) \quad A^n(1) \twoheadrightarrow 0}{A^{n+1}(0) \twoheadrightarrow 0}\ \text{(T2)}$$

   Again, both premises are true in $\mathcal{M}$.

3. $\mathcal{M}$ *is contained in any supported model* $\mathcal{M}' := (R', C', T')$. As $\mathcal{M}'$ is a model of $\mathsf{TSS}(\mathcal{P})$, surely $R \subseteq R'$ (because R1 and R2 hold); from C1 and C2 we derive $C = \longrightarrow_R \subseteq \longrightarrow_{\mathcal{R}'} \subseteq C'$; and by T1 and T2, we have $\twoheadrightarrow_R \subseteq C'^* \subseteq T'$.

   We still have to show that $(x, 0) \in T'$ for all closed x. For x is 0, this is to make T1 true. Assume towards a contradiction, that $T' \vDash A^n(1) \not\twoheadrightarrow 0$. R3 holds in $\mathcal{M}'$, so $R' \vDash A^{n+1}(1) \to 0$, hence by C1, T1, T2, $T' \vDash A^{n+1}(1) \twoheadrightarrow 0$. Using R1, C1, C2, we derive $C' \vDash A^n(1) \longrightarrow A^{n+1}(1)$. Now by T2, we get $T' \vDash A^n(1) \twoheadrightarrow 0$. Contradiction. Hence $T' \vDash A^n(1) \twoheadrightarrow 0$. But then, using R2, C1, C2, T2, we also $T' \vDash A^{n+1}(0) \twoheadrightarrow 0$ (via $A^n(1)$). Hence, $T \subseteq T'$.

1, 2 and 3 together yield that (R, C, T) is the least supported model of $\mathsf{TSS}(\mathcal{P})$.

This shows that Thm. 4.5 is not true if we replace “well supported” by “supported”. But we can prove the following theorem.

Theorem A.3. *Let $\mathcal{P}$ be a PRS. The following two statements are equivalent:*

1. *$\mathcal{P}$ has a unique sound and complete rewrite set.*
2. *$\mathsf{TSS}(\mathcal{P})$ has a least supported model of the form $(R, \longrightarrow_R, \twoheadrightarrow_R)$ for some R.*

Proof: Appropriately modify Prop. 4.3. As any well-supported model is also supported ([7, Prop. 3]), Prop. 4.2 can be modified too. The result follows. $\boxtimes$

References

[1] J. C. M. Baeten, J. A. Bergstra, J. W. Klop, and W. P. Weijland. Term-rewriting systems with rule priorities. *Theoretical Computer Science*, 67(2-3):283–302, October 1989.

[2] R.N. Bol and J.F. Groote. The meaning of negative premises in transition system specifications (extended abstract). In *Annual International Colloquium on Automata, Languages and Programming*, 1991.

[3] W.J. Fokkink and C. Verhoef. A conservative look at term deduction systems with variable binding. Logic Group Preprint Series 140, Utrecht University, 1995.

[4] J.R. Kennaway. The specificity rule for lazy pattern-matching in ambiguous term rewrite systems. In *Proc. 3rd European Symposium on Programming*, volume 432 of *LNCS*, pages 256–270. Springer-Verlag, 1990.

[5] A. Laville. Comparison of priority rules in pattern matching and term rewriting. *Journal of Symbolic Computation*, 11:321–347, 1991.

[6] Chilukuri K. Mohan. Priority rewriting: Semantics, confluence, and conditionals. In Nachum Dershowitz, editor, *Proc. 3rd Int. Conf. on Rewriting Techniques and Applications*, pages 278–291, Chapel Hill, NC, April 1989. LNCS 355, Springer-Verlag, Berlin.

[7] R. J. van Glabbeek. The meaning of negative premises in transition system specifications II. Technical Note STAN//CS-TN-95-16, Stanford University, Department of Computer Science, February 1995. Available via URL `http://theory.stanford.edu/people/rvg/`.