Regular Ultrafilters and Finite Square Principles

Juliette Kennedy*
Department of Mathematics and Statistics
University of Helsinki
Helsinki, Finland

Saharon Shelah †
Institute of Mathematics
Hebrew University
Jerusalem, Israel

Jouko Väänänen‡
Intitute for Logic, Language and Computation
University of Amsterdam
Amsterdam, Netherlands

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Abstract

We show that many singular cardinals λ above a strongly compact cardinal have regular ultrafilters D that violate the finite square principle $\Box^{fin}_{\lambda,D}$ introduced in [2]. For such ultrafilters D and cardinals λ there are models of size λ for which M^{λ}/D is not λ^{++}-universal

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and models M and N of size λ for which M^λ/D and N^λ/D are non-isomorphic. The existence of such ultrafilters and models was raised in [1].

1 Introduction

In [2] and [3] the equivalence of a finite square principle $\square^{fin}_{\lambda,D}$ with various model theoretic properties of structures of size λ and regular filters was established.

The model theoretic properties were the following: Firstly, if D is an ultrafilter, then $\square^{fin}_{\lambda,D}$ is equivalent to M^λ/D being λ^{++}-universal for each model M in a vocabulary of size $\leq \lambda$. Secondly, if $2^\lambda = \lambda^+$, then $\square^{fin}_{\lambda,D}$ is equivalent to M^λ/D and N^λ/D being isomorphic for any elementarily equivalent models M and N of size λ in a vocabulary of size $\leq \lambda$. The existence of such ultrafilters and models is related to Open Problems 18 and 19 in [1].

It was indicated in [3] how regular filters with $\neg\square^{fin}_{\lambda,D}$ can be obtained. In this paper we construct regular ultrafilters with $\neg\square^{fin}_{\lambda,D}$ for singular λ above a strongly compact. The importance of this lies in the above equivalence of $\square^{fin}_{\lambda,D}$with model theoretic questions. When we have failure of $\square^{fin}_{\lambda,D}$ for an ultrafilter, we get the failure of λ^{++}-universality of M^λ/D for some M, as well as failure of isomorphism of some regular ultrapowers M^λ/D and N^λ/D.

A filter D on a set I is said to be *regular* if there is a family $E \subseteq D$, such that $|E| = |I|$ and $\bigcap F = \emptyset$ for all infinite $F \subseteq E$. We then say that E is a *regular* family in D. Regular filters have given rise to hard set theoretical problems but at the same time they are very useful in model theory.

2 An equivalent condition for $\square^{fin}_{\lambda,D}$

The following finite square principle was introduced in [2]:

$\square^{fin}_{\lambda,D}$: There exist finite sets C^ξ_α and integers n_ξ for each $\alpha < \lambda^+$ and $\xi < \lambda$ such that for each ξ, α

- **(i)** $C^\xi_\alpha \subseteq \alpha + 1$
- **(ii)** If $B \subset \lambda^+$ is a finite set of ordinals and $\alpha < \lambda^+$ is such that $B \subseteq \alpha + 1$, then $\{\xi : B \subseteq C^\xi_\alpha\} \in D$

(iii) $\beta \in C^\xi_\alpha$ implies $C^\xi_\beta = C^\xi_\alpha \cap (\beta+1)$

(iv) $|C^\xi_\alpha| < n_\xi$

By results in [2] and [4], $\lambda^{<\lambda} = \lambda$ implies $\square^{fin}_{\lambda,D}$, and on the other hand, for singular strong limit λ and any regular filter D generated by λ sets, $\square^{fin}_{\lambda,D}$ implies $\square^*_\lambda$.

Regularity is the ultimate denial of countable completeness of the filter: not only is *some* infinite intersection of filter-elements empty, but *every* infinite intersection of elements of the subset E, which is as big as the domain of the filter itself, is empty. We now introduce an even stronger denial. Suppose we have a filter on a set of size λ. The condition (1) below states the existence of longer and longer regular sequences $\{X_\alpha : \alpha < \beta\}$, $\beta < \lambda^+$, which moreover cohere. It turns out that the existence of such a sequence is equivalent with $\square^{fin}_{\lambda,D}$:

Theorem 1 *Suppose D is a filter on λ. Then the following conditions are equivalent:*

(1) There are sets $\{B_{\alpha,\beta} : \alpha < \beta < \lambda^+\}$ in D such that

(1.1) If $\alpha < \beta < \gamma$, then $B_{\alpha,\gamma} \cap B_{\beta,\gamma} = B_{\alpha,\beta} \cap B_{\beta,\gamma}$

(1.2) If $\alpha_n < \alpha_{n+1} < \beta$ for $n < \omega$, then $\bigcap_n B_{\alpha_n,\beta} = \emptyset$.

(2) $\square^{fin}_{\lambda,D}$

Proof. Let us first assume (1) and derive (2). Let $D^\xi_\beta = \{\alpha < \beta : \xi \in B_{\alpha,\beta}\}$. By condition (1.2) above, the sets D^ξ_β are finite for each β. Let β_γ, $\gamma < \lambda^+$, be a strictly increasing sequence for which $|D^\xi_{\beta_\gamma}|$ is constant m_ξ. Let $C^\xi_\beta = D^\xi_{\beta_\gamma} \cup \{\beta_\gamma\}$ if $\beta = \beta_\gamma$, and $C^\xi_\beta = \{\beta\}$ otherwise. Let $n_\xi = m_\xi + 1$. We show that (i)-(iv) of $\square^{fin}_{\lambda,D}$ hold. Clause (i) holds by construction. To prove (ii), assume $X \subseteq \alpha + 1$ is finite. Note that

$$\bigcap_{\beta \in X} B_{\beta,\alpha} \subseteq \{\xi : X \subseteq C^\xi_\alpha\}.$$

Since D is a filter, we get $\{\xi : X \subseteq C^\xi_\alpha\} \in D$. To prove (iv), assume $\alpha \in C^\xi_\gamma$. If $\delta \in C^\xi_\alpha$ and $\delta < \alpha$, then

$$\xi \in B_{\delta,\alpha} \cap B_{\alpha,\gamma} = B_{\delta,\gamma} \cap B_{\alpha,\gamma},$$

whence $\delta \in C^\xi_\gamma$. Conversely, if $\delta \in C^\xi_\gamma$ and $\delta < \alpha$,

$$\xi \in B_{\delta,\gamma} \cap B_{\alpha,\gamma} = B_{\delta,\alpha} \cap B_{\alpha,\gamma},$$

whence $\delta \in C^\xi_\alpha$. Finally, (iv) is true by construction.

Let us then assume (2) and prove (1). Let $B_{\alpha,\beta} = \{\xi < \lambda : \alpha \in C^\xi_\beta\}$. By (ii) of $\square^{fin}_{\lambda,D}$, $B_{\alpha,\beta} \in D$. To prove (1.1), let $\alpha < \beta < \gamma$. If $\xi \in B_{\alpha,\gamma} \cap B_{\beta,\gamma}$, then $\alpha \in C^\xi_\gamma$ and $\beta \in C^\xi_\gamma$. By (iii), $C^\xi_\beta = C^\xi_\gamma \cap (\beta+1)$. Thus we may conclude $\alpha \in C^\xi_\beta$ and $\beta \in C^\xi_\gamma$, i.e. $\xi \in B_{\alpha,\beta} \cap B_{\beta,\gamma}$. Conversely, if $\xi \in B_{\alpha,\beta} \cap B_{\beta,\gamma}$, then $\alpha \in C^\xi_\beta$ and $\beta \in C^\xi_\gamma$. Again by (iii), $C^\xi_\beta = C^\xi_\gamma \cap (\beta+1)$. Thus $\alpha \in C^\xi_\gamma$ and $\beta \in C^\xi_\gamma$, i.e. finally, $\xi \in B_{\alpha,\gamma} \cap B_{\beta,\gamma}$.

To prove (1.2), assume $\alpha_n < \alpha_{n+1} < \beta$ for $n < \omega$, but $\bigcap_n B_{\alpha_n,\beta} \neq \emptyset$, say $\xi \in \bigcap_n B_{\alpha_n,\beta}$. Then each α_n is in C^ξ_β, which is impossible because the latter is finite. $\square$

In Theorem 4 we will construct a regular ultrafilter which does not have the strong regularity property of Theorem 1.

3 A partition property

Let $P_2(\lambda,\kappa)$ denote the following property of λ and κ:

> Suppose $c : [\lambda]^2 \to E$, where E is a filter on κ. Then there is an $i < \kappa$ such that for all $\chi < \lambda$ there is an increasing sequence ζ_β, $\beta \leq \chi$, of ordinals $< \lambda$ such that for all $\beta < \chi$ we have $i \in c(\{\zeta_\beta, \zeta_\chi\})$.

If λ is weakly compact, then $P_2(\lambda,\kappa)$ holds for all $\kappa < \lambda$. What is interesting about $P_2(\lambda,\kappa)$ is that it can hold also for successor cardinals λ:

Proposition 2 *Suppose $\kappa < \lambda$ and there is a κ^+-complete uniform ultrafilter on λ^+. Then $P_2(\lambda^+,\kappa)$.*

Proof. Fix $\zeta < \lambda^+$. Then $[\zeta+1,\lambda^+) \in \mathcal{E}$. Obviously,

$$[\zeta+1,\lambda^+) = \bigcup_{A\in E} b_A(\zeta) \in \mathcal{E},$$

where

$$b_A(\zeta) = \{\xi : \zeta < \xi < \lambda^+ \text{ and } c(\{\zeta,\xi\}) = A\},$$

and the union is disjoint. Since $\mathcal{E}$ is κ^+-complete, there is $A(\zeta) \in E$ such that $b_A(\zeta) \in \mathcal{E}$. Since λ^+ is regular, there is a stationary $Y \subseteq \lambda^+$ such that $A \restriction Y$ is constant, which we denote A^*. Let $i \in A^*$. The claim follows now by a pressing down argument. $\square$

Corollary 3 *Suppose $\kappa < \theta \leq \lambda$ where θ is strongly compact. Then $P_2(\lambda, \kappa)$ holds.*

Proof. Let F be the θ-complete filter $\{A \subseteq \lambda^+ : |\lambda^+ \setminus A| < \lambda^+\}$. By strong compactness there is a θ-complete uniform ultrafilter $\mathcal{E}$ on λ^+ extending F. Of course, $\mathcal{E}$ is now κ^+-complete and uniform. $\square$

4 Main result

If D_ξ is a filter on μ_ξ for each $\xi < \kappa$, $\lambda = \sup_\xi \mu_\xi$, and E is a filter on κ, we define

$$\Sigma_E D_\xi = \{A \subseteq \lambda : \{\xi : A \cap \mu_\xi \in D_\xi\} \in E\}.$$

It is easy to see that $\Sigma_E D_\xi$ is always a filter on λ, and moreover an ultrafilter, if E and each D_ξ are.

Theorem 4 *Let us assume*

(a) $P_2(\lambda^+, \kappa)$.

(b) $\lambda = \sup\{\lambda_\xi : \xi < \kappa\} = \sup\{\mu_\xi : \xi < \kappa\}$, *where* $\lambda_\xi < \mu_\xi < \kappa$.

(c) D_ξ *is a* λ_ξ*-regular ultrafilter on* μ_ξ *such that* $\mu_\xi \setminus \bigcup_{\zeta<\xi} \lambda_\zeta \in D_\xi$.

(e) E *is a regular ultrafilter on* κ.

Then $D = \Sigma_E D_\xi$ *is a regular ultrafilter on* λ *with* $\neg\square^{fin}_{\lambda,D}$.

Proof. It is easy to see that D is regular. We assume now D satisfies condition (2) of Theorem 1, and derive a contradiction. Let $B_{\alpha,\beta}$, $\alpha < \beta < \lambda^+$, be as in Theorem 1. Since $B_{\alpha,\beta} \in D$,

$$a(\alpha, \beta) =_{df} \{\xi : B_{\alpha,\beta} \cap \mu_\xi \in D_\xi\} \in E.$$

Claim 1: If $\zeta_1 < \zeta_2 < \zeta_3$, then $a(\zeta_1, \zeta_3) \cap a(\zeta_2, \zeta_3) \subseteq a(\zeta_1, \zeta_2)$.

To prove the Claim, assume $\zeta_1 < \zeta_2 < \zeta_3$ and $\xi \in a(\zeta_1, \zeta_3) \cap a(\zeta_2, \zeta_3)$. Thus $B_{\zeta_1,\zeta_3} \cap \mu_\xi \in D_\xi$ and $B_{\zeta_2,\zeta_3} \cap \mu_\xi \in D_\xi$. Hence $B_{\zeta_1,\zeta_3} \cap B_{\zeta_2,\zeta_3} \cap \mu_\xi \in D_\xi$. Now we use the fact that $B_{\zeta_1,\zeta_3} \cap B_{\zeta_2,\zeta_3} = B_{\zeta_1,\zeta_2} \cap B_{\zeta_2,\zeta_3}$ to conclude that $B_{\zeta_1,\zeta_2} \cap \mu_\xi \in D_\xi$, and thereby $\xi \in a(\zeta_1, \zeta_2)$. The Claim is proved.

Let $c(\{\zeta, \xi\}) = a(\zeta, \xi) \in E$. By $P_2(\lambda^+, \kappa)$ there are an $i < \kappa$ and an increasing sequence ζ_β, $\beta \leq \chi$, $\chi = \mu_i^+$, of ordinals $< \lambda^+$ such that for all $\beta < \chi$ we have $i \in c(\{\zeta_\beta, \zeta_\chi\})$. Let $Y = \{\zeta_\beta : \beta < \chi\}$.

Claim 2: If $\zeta_1 < \zeta_2$ in Y, then $B_{\zeta_1,\zeta_2} \cap \mu_i \in D_i$, i.e. $i \in a(\zeta_1, \zeta_2)$.

To prove the Claim, assume $\zeta_1 < \zeta_2$ in Y. Then $\zeta_\chi > \zeta_1, \zeta_2$. Thus $i \in a(\zeta_1, \zeta_\chi) \cap a(\zeta_2, \zeta_\chi)$. By Claim 1, $a(\zeta_1, \zeta_\chi) \cap a(\zeta_2, \zeta_\chi) \subseteq a(\zeta_1, \zeta_2)$, whence $i \in a(\zeta_1, \zeta_2)$ i.e. $B_{\zeta_1,\zeta_2} \cap \mu_i \in D_i$. The Claim is proved.

Let $\xi \in Y$ such that $|Y \cap \xi| > \mu_i$, and

$$Z_\alpha = \{\zeta \in Y \cap \xi : \alpha \in B_{\zeta,\xi} \cap \mu_i\}.$$

Claim 3: $Y \cap \xi = \bigcup\{Z_\alpha : \alpha < \mu_i\}$.

To prove this, assume $\zeta \in Y \cap \xi$. By Claim 2, $i \in a(\zeta, \xi)$. As $B_{\zeta,\xi} \cap \mu_i \in D_i$, we may pick $\alpha \in B_{\zeta,\xi} \cap \mu_i$. Now $\zeta \in Z_\alpha$. Claim 3 is proved.

As $|Y \cap \xi| > \mu_i$, there is α such that Z_α is infinite. Let $\alpha_0 < \alpha_1 < ...$ be an infinite increasing sequence in Z_α. Then $\alpha \in \bigcap_n B_{\alpha_n,\xi}$. This is a contradiction, as $\bigcap_n B_{\alpha_n,\xi} = \emptyset$. $\square$

Corollary 5 *Suppose θ is strongly compact. Then every cardinal $\lambda > \theta$ of cofinality $< \theta$ has a regular filter D such that $\square^{fin}_{\lambda,D}$ fails.*

5 Model theory

The background of $\square^{fin}_{\lambda,D}$ is the following question, asked by Chang and Keisler as Conjecture 18 in [1]:

> Let M and N be structures of cardinality $\leq \lambda$ in a language of size $\leq \lambda$ and let D be a regular ultrafilter over λ. If $M \equiv N$, then $M^\lambda/D \cong N^\lambda/D$.

The question is a natural one as most of the model theory regarding ultrapowers is centered on the regular ultrafilters. It is reasonable to assume GCH in this question, although it is not part of the question.

Another open problem that motivated the formulation of $\square^{fin}_{\lambda,D}$ is Conjecture 19 of [1]:

> If D is a regular ultrafilter over λ, then for all infinite M, M^λ/D is λ^{++}-universal.

The original motivation for the study of $\square^{fin}_{\lambda,D}$ was its equivalence with the above conjectures:

Theorem 6 ([3]) *Assume D is a regular ultrafilter on λ. Then the following conditions are equivalent:*

- (i) $\square^{fin}_{\lambda,D}$.
- (ii) *If M and N are elementarily equivalent models of a language of cardinality $\leq \lambda$, then the second player has a winning strategy in the Ehrenfeucht-Fraïssé game of length λ^+ on M^λ/D and N^l/D.*
- (v) *If M is a structure in a language of cardinality $\leq \lambda$, then M^λ/D is λ^{++}-universal.*

By means of Theorem 4 we can get the relative consistency of the failure of the above conjectures:

Corollary 7 *Suppose $P_2(\lambda^+, \kappa)$. Then λ has a regular ultrafilter D such that for some structure M in a language of cardinality $\leq \lambda$ the reduced product M^λ/D is not λ^{++}-universal.*

Corollary 8 *Suppose* $P_2(\lambda^+, \kappa)$. *Then* λ *has a regular ultrafilter* D *such that for some elementarily equivalent structures* M *and* N *of cardinality* λ *in a language of cardinality* $\leq \lambda$, *the reduced products* M^λ/D *and* N^λ/D *are non-isomorphic.*

In the above corollaries the vocabularies of the structures M and N can taken to be finite.

References

[1] C.C. Chang and J. Keisler, Model Theory, North-Holland.

[2] J. Kennedy and S. Shelah. On regular reduced products. *J. Symbolic Logic*, 67(3):1169–1177, 2002.

[3] J. Kennedy and S. Shelah. More on regular reduced products. *J. Symbolic Logic*, 69(4):1261–1266, 2004.

[4] S. Shelah, "Gap 1" two-cardinal principles and the omitting types theorem for $L(Q)$. Israel Journal of Mathematics vol 65 no. 2, 1989, 133–152.