Growing Commas
A Study of Sequentiality and Concatenation

Albert Visser

Department of Philosophy

Utrecht University

Heidelberglaan 8, 3584 CS Utrecht

The Netherlands

`albert.visser@phil.uu.nl`

August 27, 2007

Abstract

In his paper [Grz05], Andrzej Grzegorczyk introduces a theory of concatenation TC. We show that TC does not define pairing. We determine a reasonable extension of TC that is sequential, i.e., has a good sequence coding.

Contents

1 Introduction

The supervenience of structured objects on strings of symbols is one of the central facts of human life. It underlies writing and speech. The possibility of this supervenience is based on mathematical facts.

We employ weak systems for strings and concatenation, to study the fine-structure of reasoning about this supervenience. The focus of this paper is the question: *how do finite sets emerge out of strings?*

We study a theory of concatenation TC introduced by Andrzej Grzegorczyk. We can think of this theory as having two classes of 'standard models', to wit: the finite strings of at least two letters and the decorated linear order types for classes of of letters or colors of at least two elements.

We introduce an extension of this theory that is sequential, i.e., in which a theory of finite sets is definable that is sufficient for coding. We will need a principle that says that, for any string x, all substrings of $\mathtt{a}$'s of x, are strictly majorizable by some string of $\mathtt{a}$'s y. This extension is valid in our 'standard models'. We show that the extension is interpretable in TC. As a corollary we obtain that Robinson's Arithmetic Q is interpretable in TC.

The trick we use to simulate sets is a modification of Quine's trick. Our commas are strings of $\mathtt{a}$'s flanked by $\mathtt{b}$'s. To distinguish the commas from the elements we make the commas larger and larger.

We show that TC itself is not sequential: it does not even have a pairing function.

> In TC we do not have a good notion of *occurrence*. We would like to define an occurrence of v in w as a pair (u, v), where $w = u * v * z$, for some z. However, since TC does not exclude that u is a proper initial segment of itself, we cannot pin down a uniquely determined place in w in this way. A secondary target of this paper is to understand how to reason in the absence of a good notion of occurrence. We will see that it is possible to simulate some of the usual reasoning involving occurrences.

2 Theories of Concatenation

In his paper [Grz05], Andrzej Grzegorczyk introduces a theory of concatenation TC. Grzegorczyk's theory is in essence an earlier theory due to Tarski plus axioms guaranteeing the existence of at least two letters or atoms. We will call Tarski's theory TC_0.

The theory TC has a binary function symbol $*$ for concatenation and two constants $\mathtt{a}$ and $\mathtt{b}$. The theory is axiomatized as follows.

TC1. $\vdash (x * y) * z = x * (y * z)$

$\mathsf{TC}2.\ \vdash x * y = u * v \rightarrow ((x = u \wedge y = v) \vee$
$\exists w\ ((x * w = u \wedge y = w * v) \vee (x = u * w \wedge y * w = v)))$

$\mathsf{TC}3.\ \vdash x * y \neq \mathtt{a}$

$\mathsf{TC}4.\ \vdash x * y \neq \mathtt{b}$

$\mathsf{TC}5.\ \vdash \mathtt{a} \neq \mathtt{b}$

Grzegorczyk calls axiom $\mathsf{TC}2$: the editor axiom. Tarski's theory TC_0 has only concatenation in its signature, and is axiomatized by $\mathsf{TC}1$ and $\mathsf{TC}2$.

Andrzej Grzegorczyk and Konrad Zdanowski have shown that TC is essentially undecidable. See their forthcoming paper *Undecidability and Concatenation*, which will appear in the Mostowski Volume. This result can be strengthened by showing that Robinson's Arithmetic Q is mutually interpretable with TC. See below. Note that TC_0 is undecidable —since it has an extension that parametrically interprets TC— but that TC_0 is not essentially undecidable: it is satisfied by a one-point model. It also has an extension that is a definitional extension of the theory of pure identity.

The theories TC and TC_0 are theories for concatenation without the empty string, i.o.w., without the unit element ε. We find it more convenient to work in a theory with unit. Our variant TC^ε of TC with empty string added looks as follows.

$\mathsf{TC}^\varepsilon 1.\ \vdash \varepsilon * x = x \wedge x * \varepsilon = x$

$\mathsf{TC}^\varepsilon 2.\ \vdash (x * y) * z = x * (y * z)$

$\mathsf{TC}^\varepsilon 3.\ \vdash x * y = u * v \rightarrow \exists w\ ((x * w = u \wedge y = w * v) \vee (x = u * w \wedge y * w = v))$

$\mathsf{TC}^\varepsilon 4.\ \vdash \mathtt{a} \neq \varepsilon$

$\mathsf{TC}^\varepsilon 5.\ \vdash x * y = \mathtt{a} \rightarrow (x = \varepsilon \vee y = \varepsilon)$

$\mathsf{TC}^\varepsilon 6.\ \vdash \mathtt{b} \neq \varepsilon$

$\mathsf{TC}^\varepsilon 7.\ \vdash x * y = \mathtt{b} \rightarrow (x = \varepsilon \vee y = \varepsilon)$

$\mathsf{TC}^\varepsilon 8.\ \vdash \mathtt{a} \neq \mathtt{b}$

The theories TC and TC^ε are bi-interpretable. See Appendix A.[1] We will also consider the theory $\mathsf{TC}_0^\varepsilon$, axiomatized by $\mathsf{TC}^\varepsilon 1, 2, 3$.

There is a somewhat different salient theory of concatenation which is in many respects a direct analogue of Robinson's Arithmetic Q. We call this theory: $\mathsf{Q}^{\mathsf{bin}}$. The axioms of this theory are as follows.

[1]Almost all desirable properties of theories are preserved modulo bi-interpretability — e.g. finite axiomatizability, κ-categoricity, sequentiality. Moreover, bi-interpretability is a bisimulation w.r.t. theory extension: if U is bi-interpretable with V and $U \subseteq U'$, then there is a $V' \supseteq V$ which is bi-interpretable with V.

$\mathsf{Q}^{\mathsf{bin}}1.$ $\vdash \mathsf{S}_\mathsf{a}x \neq \varepsilon$,

$\mathsf{Q}^{\mathsf{bin}}2.$ $\vdash \mathsf{S}_\mathsf{b}x \neq \varepsilon$,

$\mathsf{Q}^{\mathsf{bin}}3.$ $\vdash \mathsf{S}_\mathsf{a}x \neq \mathsf{S}_\mathsf{b}y$,

$\mathsf{Q}^{\mathsf{bin}}4.$ $\vdash \mathsf{S}_\mathsf{a}x = \mathsf{S}_\mathsf{a}y \to x = y$,

$\mathsf{Q}^{\mathsf{bin}}5.$ $\vdash \mathsf{S}_\mathsf{b}x = \mathsf{S}_\mathsf{b}y \to x = y$,

$\mathsf{Q}^{\mathsf{bin}}6.$ $\vdash x * \varepsilon = x$,

$\mathsf{Q}^{\mathsf{bin}}7.$ $\vdash x * \mathsf{S}_\mathsf{a}y = \mathsf{S}_\mathsf{a}(x * y)$,

$\mathsf{Q}^{\mathsf{bin}}8.$ $\vdash x * \mathsf{S}_\mathsf{b}y = \mathsf{S}_\mathsf{b}(x * y)$,

$\mathsf{Q}^{\mathsf{bin}}9.$ $\vdash x = \varepsilon \vee \exists y\ (x = \mathsf{S}_\mathsf{a}y \vee x = \mathsf{S}_\mathsf{b}y)$.

In Appendix B, we show that $\mathsf{Q}^{\mathsf{bin}}$ and TC^ε are mutually interpretable. Rachel Sterken proves in her master's thesis —which will soon be available on internet— that $\mathsf{Q}^{\mathsf{bin}}$ is mutually interpretable with Q. It follows that TC^ε is mutually interpretable with Q.

3 Basics of TC^ε

In this section we provide some basic facts concerning TC^ε. We define:

- $\mathsf{atom}(x) :\leftrightarrow x \neq \varepsilon \wedge \forall y, z\ (y * z = x \to (x = \varepsilon \vee y = \varepsilon))$.
- $x \subseteq y :\leftrightarrow \exists u, v\ y = u * x * v$.
- $x \subset y :\leftrightarrow \exists u, v\ (y = u * x * v \wedge (u \neq \varepsilon \vee v \neq \varepsilon))$.
- $x \subset^+ y :\leftrightarrow x \subseteq y \wedge \neg\, y \subseteq x$.
- $x \subseteq_{\mathsf{ini}} y :\leftrightarrow \exists v\ x * v = y$.
- $x \subseteq_{\mathsf{end}} y :\leftrightarrow \exists u\ u * x = y$.
- $y : \mathsf{N}_x :\leftrightarrow \forall z {\subseteq} y\ (x \subseteq z \vee z = \varepsilon)$.
  We will call a y in N_x an x-string.
- If $\prec$ is one of our pre-orderings, then we define:
  $y \prec_x z :\leftrightarrow \forall u : \mathsf{N}_x\ (u \subseteq y \to \exists v {\subseteq} z\ u \prec v)$.

Fact 3.1 *The theory TC^ε proves the following facts.*

1. *(a)* $(\mathsf{atom}(x) \wedge \mathsf{atom}(y) \wedge u * x = v * y) \to (u = v \wedge x = y)$.
   (b) $(\mathsf{atom}(x) \wedge \mathsf{atom}(y) \wedge x * u = y * v) \to (u = v \wedge x = y)$.
2. *Suppose* $\mathsf{atom}(x)$ *and* $u * v = x * w$*, Then, either* $u = \varepsilon$ *or, there is a* u_0*, such that* $u = x * u_0$ *and* $u_0 * v = w$*. Similarly for:* $u * v = w * x$*.*

Proof

We reason in TC^ε Suppose x is an atom.

Ad 1: We treat (a). Suppose $\mathsf{atom}(y)$ and $u * x = v * y$. Then, by the editor axiom, we have a w, such that (i) $u * w = v$ and $x = w * y$, or (ii) $u = v * w$ and $w * x = y$. In case (i), we have $w = \varepsilon$ and, hence $u = v$ and $x = y$. A contradiction. Case (ii) is similar. Item (b) is similar.

Ad 2: Suppose $u * v = x * w$. By the editor axiom, there is a z such that (i) $(u * z = x$ and $v = z * w)$ or (ii) $(u = x * z$ and $z * v = w)$.

In case (i), either (i1) $u = \varepsilon$ —and we are done— or (i2) $z = \varepsilon$ and $u = x$. In case (i2), we have $u = x$ and $v = w$. So, we can take $u_0 := \varepsilon$. In case (ii), we can take $u_0 := z$. ❑

Fact 3.2 *We have in TC^ε, the following facts.*

1. *The relation $\subseteq_{\mathsf{ini}}$ is a weak partial preordering with minimal element ε. The atoms in our sense are also atoms of this pre-ordering.*[2] *Our preordering is linear when restricted to the initial substrings of an element x.*
2. *The relation $\subseteq_{\mathsf{end}}$ is a weak partial preordering with minimal element ε. The atoms in our sense are also atoms of this weak pre-ordering. Our preordering is linear when restricted to the final substrings of x.*
3. *The relation $\subseteq$ is a partial preordering on the substrings of x with minimal element ε. The atoms in our sense are precisely the atoms of the preordering.*
4. $x \subseteq y * z \to (x \subseteq y \vee x \subseteq z \vee \exists x_0, x_1\ (x = x_0 * x_1 \wedge x_0 \subseteq_{\mathsf{end}} y \wedge x_1 \subseteq_{\mathsf{ini}} z))$,
5. *The relation $\subset$ is a partial preordering. The relation $\subset^+$ is a strong ordering. We have: $x \subset^+ y \to x \subset y$.*[3]

Proof

We reason in TC^ε. We only treat 4. Suppose $x \subseteq y * z$. So, for some u, v, we have $u * x * v = y * z$. By the editor axiom, there is a w, such that (a) $u * w = y$ and $x * v = w * z$, or (b) $u = y * w$ and $w * x * v = z$. In case (b), we have $x \subseteq z$ and we are done. We treat case (a). By the editor axiom, we have an r, such that (a1) $x * r = w$ and $v = r * z$, or (a2) $x = w * r$ and $r * v = z$. In case (a1) we have $x \subseteq w \subseteq y$, so $x \subseteq y$, and we are done. In case (a2), we take $x_0 := w$ and $x_1 := r$. ❑

[2]It is not difficult to produce a model to show that we cannot prove that the atoms in our sense are the only atoms of the preordering.

[3]It is easy to produce a countermodel to show that the converse does not generally hold.

Here is a definition.

- Let $I(x)$ be a formula. We treat $\{x \mid I(x)\}$ as a virtual class. Par abus de langage, we write I for $\{x \mid I(x)\}$. We take $\mathsf{DC}(I)(x) :\leftrightarrow \forall y{\subseteq}x\, I(y)$.

Fact 3.3 *In $\mathsf{TC}^{\varepsilon}$, we have the following. Suppose (the virtual class) I is closed under concatenation. Let $J := \mathsf{DC}(I)$. Then, J is closed under concatenation and downward closed under $\subseteq$.*

Proof

Reason in $\mathsf{TC}^{\varepsilon}$. Suppose I is closed under concatenation. Let $J := \mathsf{DC}(I)$. Clearly, J is downwards closed under $\subseteq$. Suppose x_0 and x_1 are in J. To show $x_0 * x_1$ is in J. Suppose $y \subseteq x_0 * x_1$. By Fact 3.2(4), we have that either (a) $y \subseteq x_0$, or (b) $y \subseteq x_1$, or (c), for some y_0, y_1, $y = y_0 * y_1$ and $y_0 \subseteq_{\mathsf{end}} x_0$ and $y_1 \subseteq_{\mathsf{ini}} x_1$. In cases (a) and (b), we immediately have that y is in I. In case (c), we find that y_0 is in I and y_1 is in I. Hence, by the closure of I under concatenation, y is in I. ❑

Fact 3.4 *We have in $\mathsf{TC}^{\varepsilon}$ the following facts.*

1. *N_x is closed under ε and concatenation. Moreover, it is downwards closed under taking substrings.*
2. *N_x is non trivial, i.e., not equal to $\{\varepsilon\}$, iff it contains x. If x is an atom, then N_x is non-trivial.*

Proof

We only treat the case that N_x is closed under concatenation. Let:

$$I_x(y) :\leftrightarrow y = \varepsilon \vee x \subseteq y.$$

Clearly, I_x is closed under concatenation and $\mathsf{N}_x = \mathsf{DC}(I_x)$. The desired result now follows from Fact 3.3. ❑

Our final fact, follows an idea of Pavel Pudlák. Consider any model of $\mathsf{TC}^{\varepsilon}_0$. Fix an element w. We call a sequence $(w_0, \ldots, w_k)$ a *partition* of w if we have that $w_0 * \cdots * w_k = w$. The partitions of w form a category with the following morphisms. $f : (u_0, \ldots, u_n) \to (w_0, \ldots, w_k)$ iff f is a surjective and weakly monotonic function from $n{+}1$ to $k{+}1$, such that, for any $i \leq k$, $w_i = u_s{*}\cdots{*}u_\ell$, where $\{j \mid f(j) = i\} = \{j \mid s \leq j \leq \ell\}$. We write $(u_0, \ldots, u_n) \leq (w_0, \ldots, w_k)$ for: $\exists f\; f : (u_0, \ldots, u_n) \to (w_0, \ldots, w_k)$. In this case we say that $(u_0, \ldots, u_n)$ is *a refinement* of $(w_0, \ldots, w_k)$.

Fact 3.5 *We work in any model of $\mathsf{TC}^{\varepsilon}_0$. Consider a w in the model. Then, any two partitions of w have a common refinement.*

Proof

Fix any model of $\mathsf{TC}^{\varepsilon}_0$. We first prove that, for all w, all pairs of partitions $(u_0, \ldots, u_n)$ and $(w_0, \ldots, w_k)$ of w have a common refinement, by induction of $n + k$.

If either n or k is 0, this is trivial. Suppose $(u_0, \ldots, u_{n+1})$ and $(w_0, \ldots, w_{n+1})$ are partitions of w. We have, by the editor axiom, that there is a v such that (a) $u_0 {*} \cdots {*} u_n {*} v = w_0 {*} \cdots {*} w_k$ and $u_{n+1} = v {*} w_{k+1}$, or (b) $u_0 {*} \cdots {*} u_n = w_0 {*} \cdots {*} w_k {*} v$ and $v * u_{n+1} = w_{k+1}$. By symmetry, we only need to treat case (a). By the induction hypothesis, there is a common refinement $(x_0, \ldots x_m)$ of $(u_0, \ldots, u_n, v)$ and $(w_0, \ldots, w_n)$. Let this be witnessed by f, resp. g. It is easily seen that $(x_0, \ldots x_m, w_{k+1})$ is the desired refinement with witnessing functions f' and g', where $f' := f[m + 1 : n + 1]$, $g' := g[m + 1 : k + 1]$. ❑

Note that the length of the common refinement produced by our proof is the sum of the lengths of our original partitions minus one.

> We will use refinements to simulate the presence of occurrences. Instead of working with occurrences in an absolute sense, we will treat them as places in a sufficiently fine refinement.

4 $\mathsf{TC}^{\varepsilon}$ and Sequentiality

In this section, we introduce the notion of sequentiality and give an extension of $\mathsf{TC}^{\varepsilon}$ that is extensional.

4.1 What is Sequentiality?

Adjunctive Set Theory AS is the theory in the language with $\in$ and $=$, which is axiomatized as follows.

$\mathsf{AS}1$. $\vdash \exists x\, \forall y\; y \notin x$ (empty set axiom)

$\mathsf{AS}2$. $\vdash \forall u, v\; \exists x\, \forall y\; (y \in x \leftrightarrow (y \in u \vee y = v))$ (adjunction axiom)

A theory is *sequential* iff it directly interprets adjunctive set theory AS. Direct interpretability means: interpretability without relativization of quantifiers, that sends identity to identity. Said differently, a theory is sequential if we can define a predicate $\in$ provably satisfying the axioms of AS.

Remark 4.1 The notion of sequential theory was introduced by Pavel Pudlák in his paper [Pud83]. Pudlák uses his notion for the study of the degrees of local multi-dimensional parametric interpretability. He proves that sequential theories are prime in this degree structure. In [Pud85], sequential theories provide the right level of generality for theorems about consistency statements.

The notion of sequential theory was independently invented by Friedman who called it *adequate theory*. See Smoryński's survey [Smo85].[4] Friedman uses the notion to provide the Friedman characterization of interpretability among finitely axiomatized sequential theories. (See also [Vis90] and [Vis92].) Moreover, he shows that ordinary interpretability and faithful interpretability among finitely axiomatized sequential theories coincide. (See also [Vis93] and [Vis05].)

Adjunctive Set Theory is mutually interpretable with Q. For the interpetability of AS in Q, see e.g. [Nel86] or [HP91]. Here is the story of the interpretability of Q in AS in a nutshell.

1. In [ST50], Wanda Smielew and Alfred Tarski announce the interpretability of Q in AS plus extensionality. See also [TMR53], p34.
2. A new proof of the Smielew-Tarski result is given by George Collins and Joseph Halpern in [CH70].
3. Franco Montagna and Antonella Mancini, in [MM94], give an improvement of the Smielew-Tarski result. They prove that Q can be interpreted in an extension of AS in which we stipulate the functionality of empty set and adjunction of singletons.
4. In appendix III of [MPS90], Jan Mycielski, Pavel Pudlák and Alan Stern provide the ingredients of the interpretation of Q in AS.

In a forthcoming paper we will provide another proof of the interpretability of Q in AS.

For further work concerning sequential theories, see, e.g., [Pud85], [Smo85], [MPS90], [HP91], [Vis93], [Vis98], [JV00], [Vis05], [Vis07]. ❑

4.2 Sequentiality from Concatenation

We want to define a notion of set from strings that are at least binary. How are we going to define $\{w_0, \ldots, w_{n-1}\}$? A first idea would be to take simply $w_0 * \cdots * w_{n-1}$. Of course, there is, in general, no way to retrieve precisely the w_i from this object. We need some kind of separator or comma. If we had a fresh letter, we would be done. However, our rules dictate that we must be able to make sets of all possible strings —so we have no extra letter available. One idea to create commas would be to employ a tally length function.

[4] An important difference is that in the definition, as given by Smoryński, Elementary Arithmetic EA (aka $I\Delta_0 + \mathsf{EXP}$) is stipulated to be interpretable in adequate theories. This demand is evidently much too strong.

Consider the following list of properties: $\Lambda_{\mathsf{a}}\varepsilon := \varepsilon$, $\Lambda_{\mathsf{a}}x := \mathsf{a}$, if x is an atom, $\Lambda_{\mathsf{a}}(x * y) := \Lambda_{\mathsf{a}}x * \Lambda_{\mathsf{a}}y$. A function that satisfies these properties is called a *tally length function.*[a]

In the model of finite strings, the tally length function is uniquely determined and has a very efficient computation on the two tape Turing machine. In the model of decorated linear order types, we also have a tally length function. (I don't know whether it is necessarily unique.) Thus, it seems to me a very reasonable function to add as a primitive.[b]

If we extend our language with a tally length function, there are several possibilities to define sets. E.g., we could create room for a comma by replacing the w_i by the result of doubling each atom in w_i. Define:

- $u \equiv_{\mathsf{a}} v :\leftrightarrow \Lambda_{\mathsf{a}}u = \Lambda_{\mathsf{a}}v$,
- $\mathsf{dubb}(w, \tilde{w}) :\leftrightarrow \forall u, x, v\ ((w = u * x * v \wedge \mathsf{atom}(x)) \rightarrow$ $\exists \tilde{u}, \tilde{v}\ (\tilde{u} \equiv_{\mathsf{a}} u * u \wedge \tilde{v} \equiv_{\mathsf{a}} v * v \wedge \tilde{w} = \tilde{u} * x * x * \tilde{v}))$,

Now we can represent $\{w_0, \ldots, w_{n-1}\}$ by:

$$\tilde{w}_0 * \mathsf{a} * \mathsf{b} * \tilde{w}_1 * \cdots * \mathsf{a} * \mathsf{b} * \tilde{w}_{n-1},$$

where $\mathsf{dubb}(w_i, \tilde{w}_i)$, for each i. A second idea is to represent the set $\{w_0, \ldots, w_{n-1}\}$ by:

$$w_0 * \mathsf{a} * w_1 * \cdots \mathsf{a} * w_{n-1} * w_0 * \mathsf{b} * w_1 * \cdots * \mathsf{b} * w_{n-1}.$$

To retrieve the w_i, we clearly need a relation like $\equiv_{\mathsf{a}}$. A third idea is to represent $\{w_0, \ldots, w_{n-1}\}$ by:

$$\Lambda_{\mathsf{a}}w_0 * \mathsf{a} * \mathsf{b} * \Lambda_{\mathsf{a}}w_1 * \cdots * \mathsf{a} * \mathsf{b} * \Lambda_{\mathsf{a}}w_{n-1} * \mathsf{a} * \mathsf{b} * \mathsf{b} * \\ w_0 * \mathsf{a} * \mathsf{b} * w_1 * \cdots * \mathsf{a} * \mathsf{b} * w_{n-1}$$

To make any of these ideas work we will need additional axioms over $\mathsf{TC}^{\varepsilon}$ plus the tally axioms.

[a] It is easy to produce a model of $\mathsf{TC}^{\varepsilon}$ that does not admit a tally length function. See Appendix C. One can also produce a model that admits different tally length functions and in which the range of these functions is *not* N_{a}. See Subsection 5.2.

[b] Of course, we can define a tally length function in a sufficiently strong extension of $\mathsf{TC}^{\varepsilon}$. However, that is *after* we coded sequences. We are now precisely considering it as a tool to define sequences.

In our treatment we will not use a tally length function. For one thing, it is nicer, of course, to avoid expanding the signature. More seriously, it seems to me that each of the ideas involving the tally length function involve the notion of occurrence of a substring, which we do not have in $\mathsf{TC}^{\varepsilon}$. Can we avoid, this presupposition? We remind the reader of Quine's way of representing sets. See [Qui46]. He represents $\{w_0, \ldots, w_{n-1}\}$ by:

$$w_0 * \mathsf{b} * u * \mathsf{b} * w_1 * \cdots * \mathsf{b} * u * \mathsf{b} * w_{n-1}.$$

Here u is an a-string strictly longer that all a-strings that are substrings of the w_i. This idea works perfectly in the context of a sufficiently strong theory, but it has the following disadvantage. If we want to adjoin an element to a set, we may have to update all commas in the given representation. This is a complex operation. We employ a variant of Quine's idea. We represent $\{w_0, \ldots, w_{n-1}\}$ by:

$$\mathsf{b} * u_0 * \mathsf{b} * w_0 * \mathsf{b} * u_1 * \mathsf{b} * w_1 * \cdots \mathsf{b} \; * u_{n-1} * \mathsf{b} * w_{n-1},$$

where the u_i are a-strings and $u_i \subseteq u_j$, if $i \leq j$. We demand that $w_i \subset^+_{\mathsf{a}} u_i$. This idea is derived from some lecture notes by Visser, de Moor and Walsteijn of 1986, to wit [VdMM86].

Here is the formal realization. We define:

- (u', u) is a *comma*, if u is an a-string and $u' \subseteq_{\mathsf{a}} u$.
- $x \in y$ if (i) there are commas (u', u) and (v', v), such that $(u', \mathsf{b}, u, \mathsf{b}, x)$ is a partition of v' and (v', b, v, u) is a partition of y, for some u, and $x \subset^+_{\mathsf{a}} u$, or (ii) there is a comma (u', u), such that $(u', \mathsf{b}, u, \mathsf{b}, x)$ is a partition of y and $x \subset^+_{\mathsf{a}} u$.
- $\emptyset := \varepsilon$.
- $\mathsf{adj}(x, y, z)$ iff, for some c, (x, c) is a comma and $(x, \mathsf{b}, c, \mathsf{b}, y)$ is a partition of z, and $y \subset^+_{\mathsf{a}} c$).

  $\mathsf{adj}(x, y, z)$ stands for adjunction, i.e. '$x \cup \{y\} = z$', without commitment to either the existence or the uniqueness of z.

4.3 Correctness of the Definitions

In this subsection, we show that the correctness of the joint definitions of $\in$ and adj, in $\mathsf{TC}^{\varepsilon}$.[5]

Theorem 4.2 *We have:* $\mathsf{TC}^{\varepsilon} \vdash \mathsf{adj}(x, y, z) \to \forall w \, (w \in z \leftrightarrow (w \in x \vee w = y))$.

[5]The fact that correctness can be verified in a weaker theory than the one we need for existence, by itself, does not give us much information. After all, we could define an alternative adjunction by:

- $\mathsf{adj}^{\star}(x, y, z) :\leftrightarrow \mathsf{adj}(x, y, z) \wedge \forall w \, (w \in z \leftrightarrow (w \in x \vee w = y))$.

For *this* definition correctness would be trivial and the whole burden of verification would be shifted to the existence clause. Still I like the fact that for this particular definition we only need $\mathsf{TC}^{\varepsilon}$ to see that adjunction, if defined, delivers the ordered goods.

Proof

We reason in any model of $\mathsf{TC}^{\varepsilon}$. Suppose $\mathsf{adj}(x,y,z)$. So, $\sigma := (x,\mathtt{b},c,\mathtt{b},y)$, is a partiation of z, where (x,c) is a comma and $y \subset^{+}_{\mathtt{a}} c$.

Suppose $w \in x$. If w is in x by the first disjunct of the definition of $\in$, then, trivially, w is in z. Suppose w is in x by the second disjunct. So, there is a comma (d',d) such that $(d',\mathtt{b},d,\mathtt{b},w)$ is a partition of x and $w \subset^{+}_{\mathtt{a}} d$. Since also (x,c) is a comma, we find that $w \in z$, by first clause of the definition of z..

Clearly $y \in z$ by the second clause of the definition of $\in$.

Suppose that $w \in z$. First, we consider the case that this is true by the first clause of the definition of $\in$. So, we have commas (d',d), (e',e) such that: $(d',\mathtt{b},d,\mathtt{b},w)$ is a partition of e' and $\tau := (d',\mathtt{b},d,\mathtt{b},w,\mathtt{b},e,u)$ is a partition of z, for some u, and $w \subset^{+}_{\mathtt{a}} d$.

Let $\zeta := (z_0,\ldots,z_m)$ be a common partition of $\sigma = (x,\mathtt{b},c,\mathtt{b},y)$ and $\tau = (d',\mathtt{b},d,\mathtt{b},w,\mathtt{b},e,u)$. Let f and g be the witnessing morphisms.

It will be pleasant to have a name for, e.g., $\mathtt{b}$-as-occurring-at-place-3-in-σ. We will call this item $(\sigma,3)$. Similarly, for other strings-as-occuring-at-places-in-a-partition.

It is easily seen that there is a unique i such that $z_i = \mathtt{b}$ and $f(i) = 3$. (There may by other i' with $f(i') = 3$, but, for such i', we must have $z_{i'} = \varepsilon$.) This i is the place of $(\sigma,3)$ relative to the context (ζ,f,g). Note that the number i is just dependent on ζ and f. However, par abus de langage, we will write it as 3_σ.[6] Similarly, we can define 5_τ as the unique j, such that $g(j) = 5$ and $z_j = \mathtt{b}$. We distinguish a number of cases.

Case 1: Suppose $3_\sigma \leq 5_\tau$. It follows that $e \subseteq y$ and $c \subseteq e'$. Hence, $e \subset_{\mathtt{a}} y \subset^{+}_{\mathtt{a}} c \subseteq_{\mathtt{a}} e$. It follows that $e \subset^{+} e$. Quod impossibile.

Case 2: Suppose that $1_\sigma < 5_\tau < 3_\sigma$. Since 2_σ is an occurrence of the $\mathtt{a}$-string c, we get a contradiction.

Case 3: Suppose $5_\tau = 1_\sigma$. In this case, we have: $(d',\mathtt{b},d,\mathtt{b},w)$ is a partition of x, and (d',d) is a comma and $w \subset_{\mathtt{a}} d$. So $w \in x$, by the second clause of the definition of $\in$.

Case 4: Suppose $5_\tau < 1_\sigma$. We have $f(5_\tau) = 0$. Clearly, every j, such that $g(j) = 5$, must be $< 1_\sigma$. But then, the j such that $g(j) = 6$ must also be $< 0_\sigma$, since τ_6 is the $\mathtt{a}$-string e. It follows that, for some v, $(d',\mathtt{b},d,w,\mathtt{b},e,v)$ is a partition of x. Moreover (d',d) and (e',e) are commas and $w \subset^{+}_{\mathtt{a}} d$. So $w \in x$, by the first clause of the definition of $\in$.

[6] So, note that it is possible that $3_\sigma \neq 3_\tau$, even if $\sigma = \tau$.

Next we suppose that $x \in z$ by the second clause of the definition of $\in$. So, we have, for some comma (d', d), that $\nu := (d', \mathsf{b}, d, \mathsf{b}, w)$ is a partition of z and $w \subset_{\mathsf{a}} d$. Let $\zeta = (z_0, \ldots, z_m)$ be a common refinement of $\sigma = (x, \mathsf{b}, c, \mathsf{b}, y)$ and $\nu = (d', \mathsf{b}, d, \mathsf{b}, w)$. Let f and g be witnessing functions.

Suppose $3_\sigma < 3_\tau$. In this case we may show, reasoning as in case 4 above, that $d \subseteq y$ and $c \subseteq d'$. So, $c \subseteq_{\mathsf{a}} d' \subseteq_{\mathsf{a}} d \subseteq_{\mathsf{a}} y \subset^+_{\mathsf{a}} c$. It follows that $c \subset^+ c$. A contradiction. Similarly, we may refute the supposition that $3_\sigma > 3_\tau$. We may conclude that $3_\sigma = 3_\tau$ and, thus, that $w = y$. ❑

4.4 Existence of Adjuncts

What we need to get existence is obviously something like a suitable collection principle. We define a-collection as follows:

a-coll $\vdash \forall x\, \exists y : \mathsf{N}_{\mathsf{a}}\; x \subseteq_{\mathsf{a}} y$.

Note however that this not enough. We need our a-string y strictly above the a-strings of x. So we need strong a-collection.

a-coll$^+$ $\vdash \forall x\, \exists y : \mathsf{N}_{\mathsf{a}}\; x \subset^+_{\mathsf{a}} y$.

It is immediate that $\mathsf{TC}^\varepsilon +$ a-coll$^+$ proves the existence clause of our adjunction axiom. Since the empty set axiom is trivial, we find: $\mathsf{TC}^\varepsilon +$ a-coll$^+$ is sequential.

We will now show that TC^ε interprets $\mathsf{TC}^\varepsilon +$ a-coll$^+$. We will do this by first interpreting the strictness axiom (defined below) and then interpreting a-coll. Here is the strictness axiom.

strict $\vdash \forall u\; u \not\subset u$.

It is easily seen that strictness plus a-coll implies a-coll$^+$: if we have $x \subseteq_{\mathsf{a}} y$, then we have $x \subset^+_{\mathsf{a}} y * \mathsf{a}$. Note that we interpret more than necessary. In the decorated order types strictness fails, but we do have strong a-collection.

Theorem 4.3 *The theory* TC^ε *interprets* $\mathsf{TC}^\varepsilon +$ strict *on an initial segment.*

Proof

Consider $I(u) :\leftrightarrow u \not\subset u$. We show that I is closed under concatenation. Suppose u_0 and u_1 are in I. Suppose, for some v_0, v_1, we have $v_0 * u_0 * u_1 * v_1 = u_0 * u_1$. By the editor axiom, there is a w such that (1) $v_0 * u_0 * w = u_0$ and $u_1 * v_1 = w * u_1$, or (2) $v_0 * u_0 = u_0 * w$ and $w * u_1 * v_1 = u_1$. Suppose we are in case (1). Since u_0 is in I, we find that $v_0 = w = \varepsilon$. It follows that $u_1 * v_1 = u_1$, and, hence, since u_1 is in I, that $v_1 = \varepsilon$.

By Fact 3.3, $J := \mathsf{DC}(I)$ is closed under concatenation and downwards closed under substrings. Also clearly, J contains ε, a and b. Noting that $\forall u\; u \not\subset u$ is universal, we find that relativization to J interprets $\mathsf{TC}^\varepsilon +$ strict. ❑

Theorem 4.4 *We can interpret* $\mathsf{TC}^\varepsilon +$ a-coll *in* TC^ε *on an initial segment.*

Proof

We work in TC^ε. We first form a predicate $\mathsf{N}^\star_\mathsf{a}(x)$, such that, (i) $\mathsf{N}^\star_\mathsf{a}$ is a subclass of N_a, (ii) $\mathsf{N}^\star_\mathsf{a}$ is closed under ε, a and concatenation, (iii) $\mathsf{N}^\star_\mathsf{a}$ is downwards closed under $\subseteq$, and (iv) $\mathsf{N}^\star_\mathsf{a}$ satisfies: for any x and y in $\mathsf{N}^\star_\mathsf{a}$, we have $x * y = y * x$.

Let $I(x) :\leftrightarrow \mathsf{a} * x = x * \mathsf{a}$. Let $J(x) :\leftrightarrow \forall y{:}I\ x * y = y * x$. Since, a is in I, we find that J is a subclass of I. It is easily seen that J is closed under concatenation. It follows that $K := \mathsf{DC}(J)$ is closed under concatenation and is downwards closed under taking substrings. Clearly, K contains ε and a. We take $\mathsf{N}^\star_\mathsf{a} := \mathsf{N}_\mathsf{a} \cap J$.

Suppose that, for $i = 0, 1$, we have $x_i \subseteq_\mathsf{a} y_i$ and $y_i : \mathsf{N}^\star_\mathsf{a}$. We show that: $x_0 * x_1 \subseteq_\mathsf{a} y_0 * y_1$. Let $z : \mathsf{N}_\mathsf{a}$ be a substring of $x_0 * x_1$. We want to show that z is a substring of $y_0 * y_1$. We have either (1) $z \subseteq x_0$, or (2) $z \subseteq x_1$, or (3), for some z_0, z_1, $z = z_0 * z_1$, $z_0 \subseteq_{\mathsf{end}} x_0$ and $z_1 \subseteq_{\mathsf{ini}} x_1$. In cases (1) and (2), we are immediately done. We treat case (3). We find that $z_0 \subseteq y_0$ and $z_1 \subseteq y_1$. We have, for certain v_{ij}, $y_i = v_{i0} * z_i * v_{i1}$. Since $v_{ij} \subseteq y_i$ and $z_i \subseteq y_i$, we have that v_{ij} and z_i are in $\mathsf{N}^\star_\mathsf{a}$. Hence:

$$y_0 * y_1 = v_{00} * z_0 * v_{01} * v_{10} * z_1 * v_{11} = v_{00} * v_{01} * z_0 * z_1 * v_{10} * v_{11}.$$

So $z \subseteq y_0 * y_1$.

Let $L(x) :\leftrightarrow \exists y{:}\mathsf{N}^\star_\mathsf{a}\ x \subseteq_\mathsf{a} y$. Clearly L is downwards closed under substrings. By the above, L is closed under concatenation. It is easily seen that ε, a and b are in L. Finally, trivially, $\mathsf{N}^\star_\mathsf{a}$ is contained in L. Thus, restriction to L interprets $\mathsf{TC}^\varepsilon + \mathsf{a\text{-}coll}$. ❑

Theorem 4.5 TC^ε *interprets* $\mathsf{TC}^\varepsilon + \mathsf{a\text{-}coll}^+$.

Proof

First interpret $\mathsf{TC}^\varepsilon + \mathsf{strict}$ in TC^ε. Then relativize to the class L of the previous theorem to obtain an interpretation of $\mathsf{TC}^\varepsilon + \mathsf{a\text{-}coll}$. Since strict can be written in purely universal form as: $\vdash \forall u, v, w\ (u = w * u * v \to (u = \varepsilon \wedge v = \varepsilon))$, we find that we will inherit strict in our interpretation. Finally, strict plus $\mathsf{a\text{-}coll}$ implies $\mathsf{a\text{-}coll}^+$. ❑

> We did not explore the tally length representations in the context of TC^ε and its extensions. It is very well possible that we can make a tally length representation work TC^ε plus axioms that are incomparable to strong a-collection. If that is true, the tally representations and the growing commas representation would both have relative advantages and disadvantages.

5 $\mathsf{TC}^{\varepsilon}$ does not have Pairing

We prove that $\mathsf{TC}^{\varepsilon}$ does not prove pairing by producing a model with 'too many automorphisms'. We first define what it is for a theory to have pairing. Let PAIR be the following theory.

PAIR 1. $\vdash (\mathsf{pair}(x,y,z) \wedge \mathsf{pair}(x',y',z)) \to (x = x' \wedge y = y')$

PAIR 2. $\vdash \forall x, y\, \exists z\ \mathsf{pair}(x,y,z)$

A theory has pairing if it directly interprets PAIR. I.o.w., if there we can define a predicate pair in the language of the theory that provably satisfies the axioms of PAIR. We first prove a lazy version of our result. Then we raise our standards and prove the strongest version I could think of.

5.1 A Model Construction

A *concatenation structure* is a model of $\mathsf{TC}_0^{\varepsilon}$.[7] There are many concatenation structures. E.g. all groups are concatenation structures. Also, if $(X, \leq)$ is a linear ordering with a minimal element, then (X, max) is a concatenation structure. The simplest example is any structure with $x * y = y$, if $y \neq \varepsilon$, and $x * y = x$, if $y = \varepsilon$.

We show that concatenation structures are closed under the operation $\circledast$ that is defined as follows. Consider concatenation structures $\mathcal{X}$ and $\mathcal{U}$. We will use $x, y, z, \ldots$ for the elements of $\mathcal{X}$ and $u, v, w, \ldots$ for the elements of $\mathcal{U}$. We employ p, q, r ambiguously for both. We use α, β, γ, for the elements of $\mathcal{X} \circledast \mathcal{U}$.

The elements of $\mathcal{X} \circledast \mathcal{U}$ are the elements x of $\mathcal{X}$, and triples (y, u, z), where y and z are in $\mathcal{X}$ and u is in $\mathcal{U}$. We assume that the triples (y, u, z) are disjoint from the x's. We write $*$ for concatenation if $\mathcal{X}$ and $\mathcal{U}$ and $\star$ for the new concatenation. We define:

$\star$	x'	(y', u', z')
x	$x * x'$	$(x * y', u', z')$
(y, u, z)	$(y, u, z * x')$	$(y, u * u', z')$

We prove that $\mathcal{X} \circledast \mathcal{U}$ is indeed a concatenation structure. Clearly, the unit of $\mathcal{X}$ functions as the new unit of $\mathcal{X} \circledast \mathcal{U}$. Here is the verification of associativity.

α	β	γ	$(\alpha \star \beta) \star \gamma)$	$\alpha \star (\beta \star \gamma)$
x	x'	x''	$(x * x') * x''$	$x * (x' * x'')$
x	x'	(y'', u'', z'')	$(x * (x' * y''), u'', z'')$	$((x * x') * y'', u'', z'')$
x	(y', u', z')	x''	$(x * y', u', z' * x'')$	$(x * y', u', z' * x'')$
x	(y', u', z')	(y'', u'', z'')	$(x * y', u' * u'', z'')$	$(x * y', u' * u'', z'')$
(y, u, z)	x'	x''	$(y, u, (z * x') * x'')$	$(y, u, z * (x' * x''))$
(y, u, z)	x'	(y'', u'', z'')	$(y, u * u'', z'')$	$(y, u * u'', z'')$
(y, u, z)	(y', u', z')	x''	$(y, u * u', z' * x'')$	$(y, u * u', z' * x'')$
(y, u, z)	(y', u', z')	(y'', u'', z'')	$(y, (u * u') * u'', z'')$	$(y, u * (u' * u''), z'')$

[7] Warning: The presence of the unit is essential to make the construction work.

Next we verify the editor axiom. Suppose $\alpha \star \beta = \gamma \star \delta$. We are looking for a witness of the editor axiom, say θ. We run though the possible cases.

case 1. All four elements involved are in $\mathcal{X}$. We are done by the editor axiom of $\mathcal{X}$.

case 2. Three elements are in $\mathcal{X}$. This is impossible.

case 3. Two elements are in $\mathcal{X}$ and they are both on the same side of the identity. This is impossible.

case 4. Two elements are in $\mathcal{X}$ and they are either α and γ, or β and δ. We have e.g.: $(y, u, z) \star x = (y', u', z') \star x'$. So, $(y, u, z * x) = (y', u', z' * x')$. Let p be provided by the editor axiom for $\mathcal{X}$, such that, e.g., $z * p = z'$ and $x = p * x'$. We take $\theta := p$. We have: $(y, u, z) \star p = (y, u, z') = (y', u', z')$ and $x = p \star x'$

case 5. Two elements are in $\mathcal{X}$ and they are either α and δ, or β and γ. We have e.g.: $(y, u, z) \star x = x' \star (y', u', z')$. So, $(y, u, z * x) = (x' * y', u', z')$. We take $\theta := (y', u, z)$. We have $(y, u, z) = x' \star (y', u, z)$ and $(y', u, z) \star x = (y', u', z')$.

case 6. One element is in $\mathcal{X}$. We have e.g.:

$$x \star (y, u, z) = (y', u', z') \star (y'', u'', z''),$$

i.e., $(x * y, u, z) = (y', u' * u'', z'')$. We take $\eta := (y, u', z')$. We have $x \star (y, u', z') = (x * y, u', z') = (y', u', z')$ and $(y, u', z') \star (y'', u'', z'') = (y, u' * u'', z'') = (y, u, z)$.

case 7. No elements are in $\mathcal{X}$. We have:

$$(y, u, z) \star (y', u', z') = (y'', u'', z'') \star (y''', u''', z''').$$

So, $(y, u * u', z') = (y'', u'' * u''', z''')$. Let p be provided by the editor axiom for $\mathcal{U}$, such that, e.g., $u * p = u''$ and $u' = p * u'''$. Take $\theta := (y', p, z'')$. We have: $(y, u, z) \star (y', p, z'') = (y, u * p, z'') = (y'', u'', z'')$ and $(y', p, z) \star (y''', u''', z''') = (y', p * u''', z''') = (y', u', z')$.

An important property of the construction $\circledast$ is that automorphisms of $\mathcal{X}$ and $\mathcal{U}$ can be lifted in the obvious way to automorphisms of $\mathcal{X} \circledast \mathcal{U}$.

5.2 No Pairing

We show that PAIR is not directly interpretable in $\mathsf{TC}^{\varepsilon}$. We consider the case of one-dimensional interpretations with no parameters. In the next subsection, we will consider parameters, multi-dimensionality and more.

Suppose there is a predicate pair in $\mathsf{TC}^{\varepsilon}$ satisfying the axioms of PAIR. Let $\mathcal{A}_2$ be the monoid on generators a and b. Let D be a domain with at least four elements. We extend D to a concatenation structure $D^{\dagger}$, by stipulating that

$d * e := e$ and by adding a unit to the structure. Note that any permutation of D is an automorphism of $D^\dagger$.

Consider $\mathcal{B} := \mathcal{A}_2 \circledast D^\dagger$. We identify the elements of $\mathcal{A}_2$ with their counterparts in the construction of $\mathcal{B}$ and the elements of $D^\dagger$ with the triples $(\varepsilon, d, \varepsilon)$. Note that ε of $\mathcal{A}$ maps to the unit of $\mathcal{B}$, and that a and b map to atoms of $\mathcal{B}$. So, $\mathcal{B}$ is a model of TC^ε (modulo expansion of the signature). (The unit of $D^\dagger$ does not map to the unit of $\mathcal{B}$, and the atoms of $D^\dagger$ do not map to atoms of $\mathcal{B}$.)

Let d and e be different elements of D. Suppose $\mathsf{pair}(d, e, \alpha)$. Suppose first that α is not a triple. In this case there is an automorphism of $\mathcal{B}$, mapping d to d', e to e and α to α, where d' is in $D \setminus \{d, e\}$. We get $\mathsf{pair}(d', e, \alpha)$. A contradiction.

Suppose $\alpha = (x, e', y)$, where $e' \in D \cup \{\varepsilon\}$. Clearly, one of d, e is not identical to e'. Suppose it is e.g. d. Let $d' \in D \setminus \{d, e, e'\}$. There is an automorphism of $\mathcal{B}$, mapping d to d', e to e and α to α. We get $\mathsf{pair}(d', e, \alpha)$. A contradiction.

We may conclude that TC^ε does not have pairing.

> Let Λ^0_{a} be the usual tally length function on binary strings. Let ϕ be an automorphism of $D^\dagger$. Define $\Lambda^\phi_{\mathsf{a}}\alpha := \Lambda^0_{\mathsf{a}}\alpha$, if α is in $\mathcal{A}_2$ and $\Lambda^\phi_{\mathsf{a}}\alpha := (\Lambda^0_{\mathsf{a}}x, \phi u, \Lambda^0_{\mathsf{a}}y)$, if α is (x, u, y), where x and y are in $\mathcal{A}_2$ and u is in $D^\dagger$.
> It is easy to see that we have indeed defined a tally length function, since triples can never be atoms.
> It follows that even if we add a tally length function we still cannot define pairing. Note that no (x, u, y) is in N_{a}. So, the range of $\Lambda^\phi_{\mathsf{a}}$ is not contained in N_{a}.
> If D has at least two elements, our model allows more than one tally length function. We also see that tally length functions need not be idempotent.

5.3 The Pro Version of No Pairing

We adapt the proof of the previous subsection to prove a stronger result. We borrow some ideas from the proof of Lemma 6.5 of a forthcoming paper by Harvey Friedman, called *The Inevitability of Logical Strength: Strict Reverse Mathematics*.

Let me explain what we are going to prove. A first step is to widen our concept of interpretation: we will consider multi-dimensional interpretations with parameters.[8]

[8] There is one further possible widening of our class of interpretations: we could consider piecewise interpretations, where the new domain is assembled out of possibly overlapping pieces. However, since TC^ε provides an infinity of closed terms that are pairwise provably different, one can show that piecewise interpretations, for the case at hand, can always be replaced by multi-dimensional ones.

A second step is that we widen the notion of direct interpretability. In the category of interpretations where we identify two interpretations whenever the interpreting theory proves that they are the same, direct interpretability can be defined as follows. Let ID be the pure theory of identity. Let $\iota_U : \mathsf{ID} \to U$, be the interpretation of ID in U, obtained by just reducing the signature. I.o.w., ι_U is the unique direct interpretation of ID in U. It is easy to see that $K : U \to V$ is direct iff $\iota_V = K \circ \iota_U$. We now take this characterization and re-interpret it in the category of interpretations where we count two interpretations as the same when the associated mappings of models are the same modulo isomorphism. We do not demand that these isomorphisms are definable in the interpreting theory. The notion that we obtain in this way is: cardinality preserving interpretation. An interpretation $K : U \to V$ is *cardinality preserving* if for any model $\mathcal{M}$ of V, the internal model $K(\mathcal{M})$ defined by K has the same cardinality as $\mathcal{M}$.

We will show that there is no cardinality preserving multi-dimensional interpretation with parameters of PAIR in TC^ε. It follows that TC^ε is not weakly bi-interpretable with a theory that has pairing.

We again work in the the model $\mathcal{A}_2 \circledast D^\dagger$ of the previous subsection. However, now we demand that D is uncountable. We will call the unit of $D^\dagger$, $\varepsilon^\dagger$ to distinguish it from the unit ε of $\mathcal{A}_2$.

Suppose we have an n-dimensional interpretation K of PAIR in TC^ε. Let P be a given finite set of parameters in the model. Let the domain of K be Δ (in the parameters). We may assume that P is a subset of (the embedded elements of) D plus $\varepsilon^\dagger$, since all other elements are definable from elements of D plus $\varepsilon^\dagger$.[9] Let the domain of K be Δ (given the parameters). Let E be the identity of K (given the parameters).

A *form* is an n-tuple $(t_0, \ldots, t_{n-1})$, where each t is either (i) an a,b-string, or (ii) a term of the form udu' where u and u' are a,b-strings and d is a parameter, or (iii) a term of the form $u\varepsilon^\dagger u'$ where u and u' are a,b-strings, or (iv) a term of the form uXu', where u and u' are a,b-strings and X is a variable. We identify forms modulo permutations of variables. An f-assigment σ is an injective mapping from variables to $D \setminus P$. We define σF, for F a form, in he obvious way. We define $\sigma[F]$ as the set of values of the variables of F.

Clearly, any n-tuple from the domain of $\mathcal{B}$ can be obtained as σF, for some σ and F. Conversely, every such element A uniquely determines σ and F, such that $A = \sigma F$.

We call a permutation of D *permissible* if it leaves each parameter in place. Two n-tuples have the same form iff they are mapped to each other by an admissible permutation. It follows that either all elements of a given form are in Δ, or none is. We have the following simple lemma.

Lemma 5.1 *Suppose that σF is in Δ and $(\sigma F)E(\tau F)$, where $\sigma[F]$ and $\tau[F]$ are disjoint. Then, there is only one instantiation of F, modulo E.*

[9] In fact, the a,b-strings are definable in the model. It follows that $\varepsilon^\dagger$ is definable.

Proof

Suppose that σF is in Δ and $(\sigma F)E(\tau F)$, where $\sigma[F]$ and $\tau[F]$ are disjoint. Let σ' and τ' be any other pair with $\sigma'[F]$ and $\tau'[F]$ disjoint. We can find a permissible permutation ϕ such that $\phi \circ \sigma = \sigma'$ and $\phi \circ \tau = \tau'$. It follows that $(\sigma' F)E(\tau' G)$. Let ν and ρ be any f-assigments. Let θ be an f-assignment where $\theta[F]$ is disjoint from $\nu[F]$ and $\rho[F]$ on the variables of F. We find: $(\nu F)E(\theta F)E(\rho F)$, and, hence $(\nu F)E(\rho F)$. So F contains only one element modulo E. ❑

We show that any F has at most one instance in Δ modulo E. Suppose F has an instance in Δ. If F has no variables we are immediately done. So suppose F has at least one variable.

Let $A, B, \ldots$ range over Δ. We define a non-functional sequence coding as follows (for standard n):

- $\mathsf{seq}_2(A_0, A_1, B) :\leftrightarrow \mathsf{pair}(A_0, A_1, B)$,
- $\mathsf{seq}_{k+3}(A_0, \ldots, A_{k+2}, B) :\leftrightarrow \exists C$
  $(\mathsf{seq}_{k+2}(A_0, \ldots, A_{k+1}, C) \wedge \mathsf{pair}(C, A_{k+2}, B))$.

Let $\sigma_0, \ldots, \sigma_n$ be a sequence of $n+1$ f-assignments such that the $\sigma_i[F]$ are pairwise disjoint. Suppose that $\mathsf{seq}_{n+1}(\sigma_0 F, \ldots, \sigma_n F, B)$, for some B. Say, B is of form τG, for some form G. The number of variables of G is smaller or equal to n. So, $\tau[G]$ is smaller or equal to n. It follows that $\tau[G]$ is disjoint from one of the $\sigma_i[F]$, say for i_0. Let ϕ be any admissible permutation that leaves the elements of $\tau[G]$ and of the $\sigma_i[F]$, for $i \neq i_0$ in place, but moves the elements of $\sigma_{i_0}[F]$ to a set disjoint from the $\sigma_{i_0}[F]$. We find that: $\mathsf{seq}_{n+1}(\phi\sigma_0 F, \ldots, \phi\sigma_n F, \phi B)$. It follows that:

$$\mathsf{seq}_{n+1}(\sigma_0 F, \ldots, \sigma_{i_0 - 1} F, \phi\sigma_{i_0} F, \sigma_{i_0+1} F, \ldots, \sigma_n F, B).$$

So, $(\sigma_{i_0} F)E(\phi\sigma_{i_0} F)$. It follows, by the lemma, that F contains, modulo E, only one object in Δ.

Since there are only countable many forms, it follows that Δ is countable (modulo E). So K is *not* cardinality preserving.

Note that our result still holds when we add a tally length function.

References

[CH70] G.F. Collins and J.D. Halpern. On the interpretability of Arithmetic in Set Theory. *The Notre Dame Journal of Formal Logic*, 11:477–483, 1970.

[Grz05] Andrzej Grzegorczyk. Undecidability without arithmetization. *Studia Logica*, 79:163–230, 2005.

[HP91] P. Hájek and P. Pudlák. *Metamathematics of First-Order Arithmetic*. Perspectives in Mathematical Logic. Springer, Berlin, 1991.

[JV00] J.J. Joosten and A. Visser. The interpretability logic of *all* reasonable arithmetical theories. *Erkenntnis*, 53(1–2):3–26, 2000.

[MM94] F. Montagna and A. Mancini. A minimal predicative set theory. *The Notre Dame Journal of Formal Logic*, 35:186–203, 1994.

[MPS90] J. Mycielski, P. Pudlák, and A.S. Stern. *A lattice of chapters of mathematics (interpretations between theorems)*, volume 426 of *Memoirs of the American Mathematical Society*. AMS, Providence, Rhode Island, 1990.

[Nel86] E. Nelson. *Predicative arithmetic*. Princeton University Press, Princeton, 1986.

[Pet90] P.P. Petkov, editor. *Mathematical logic, Proceedings of the Heyting 1988 summer school in Varna, Bulgaria*. Plenum Press, Boston, 1990.

[Pud83] P. Pudlák. Some prime elements in the lattice of interpretability types. *Transactions of the American Mathematical Society*, 280:255–275, 1983.

[Pud85] P. Pudlák. Cuts, consistency statements and interpretations. *The Journal of Symbolic Logic*, 50:423–441, 1985.

[Qui46] W.V. Quine. Concatenation as a Basis for Arithmetic. *The Journal of Symbolic Logic*, 11(4):105–114, 1946.

[Smo85] C. Smoryński. Nonstandard models and related developments. In L.A. Harrington, M.D. Morley, A. Šĕdrov, and S.G. Simpson, editors, *Harvey Friedman's Research on the Foundations of Mathematics*, pages 179–229. North Holland, Amsterdam, 1985.

[ST50] W. Smielew and A. Tarski. Mutual Interpretability of some essentially undecidable theories. In *Proceedings of the International Congress of Mathematicians*, volume 1, page 734. Cambridge, 1950.

[TMR53] A. Tarski, A. Mostowski, and R.M. Robinson. *Undecidable theories*. North–Holland, Amsterdam, 1953.

[VdMM86] A. Visser, O. de Moor, and Walsteijn M.J. How to prove the first and the second incompleteness theorem using concatenation. Technical Report RUU-CS-86-15, Department of Computer Science, Utrecht University, Budapestlaan 6, 3584 CD Utrecht, 1986.

[Vis90] A. Visser. Interpretability logic. In *[Pet90]*, pages 175–209, 1990.

[Vis92] A. Visser. An inside view of exp. *The Journal of Symbolic Logic*, 57:131–165, 1992.

[Vis93] A. Visser. The unprovability of small inconsistency. *Archive for Mathematical Logic*, 32:275–298, 1993.

[Vis98] A. Visser. An Overview of Interpretability Logic. In M. Kracht, M. de Rijke, H. Wansing, and M. Zakharyaschev, editors, *Advances in Modal Logic, vol 1*, CSLI Lecture Notes, no. 87, pages 307–359. Center for the Study of Language and Information, Stanford, 1998.

[Vis05] A. Visser. Faith & Falsity: a study of faithful interpretations and false Σ^0_1-sentences. *Annals of Pure and Applied Logic*, 131:103–131, 2005.

[Vis06] A. Visser. Categories of Theories and Interpretations. In Ali Enayat, Iraj Kalantari, and Mojtaba Moniri, editors, *Logic in Tehran. Proceedings of the workshop and conference on Logic, Algebra and Arithmetic, held October 18–22, 2003*, volume 26 of *Lecture Notes in Logic*, pages 284–341. ASL, A.K. Peters, Ltd., Wellesley, Mass., 2006.

[Vis07] A. Visser. Pairs, sets and sequences in first order theories. Logic Group Preprint Series 251, Department of Philosophy, Utrecht University, Heidelberglaan 8, 3584 CS Utrecht, http://www.phil.uu.nl/preprints/lgps/, 2007.

A Comparing TC and TC^ε

We show that TC is *bi-interpretable* with a corresponding theory TC^ε.[10] This means that there are interpretations $K : \mathsf{TC}^\varepsilon \to \mathsf{TC}$ and $M : \mathsf{TC} \to \mathsf{TC}^\varepsilon$ so that $K \circ M : \mathsf{TC} \to \mathsf{TC}$ is isomorphic to the interpretation $\mathsf{id}_{\mathsf{TC}}$ via a definable isomorphism F, and $M \circ K : \mathsf{TC}^\varepsilon \to \mathsf{TC}^\varepsilon$ is isomorphic to the interpretation $\mathsf{id}_{\mathsf{TC}^\varepsilon}$ via a definable isomorphism G.[11]

We can take K and M one-dimensional interpretations without parameters, We specify K, M, F, and G. We use C for the relational formulation of concatenation and E as an alternative way of writing identity.

- $\delta_K(x) :\leftrightarrow x = \mathtt{a} \vee \exists x_0\; x = \mathtt{b} * x_0$,
- $x \mathsf{E}_K y :\leftrightarrow x = y$,
- $\mathsf{C}_K(x,y,z) :\leftrightarrow (x = \mathtt{a} \wedge y = z) \vee (y = \mathtt{a} \wedge x = z) \vee$
  $\exists x_0, y_0\; (x = \mathtt{b} * x_0 \wedge y = \mathtt{b} * y_0 \wedge z = \mathtt{b} * x_0 * y_0)$,

[10] Gregorczyck and Zdanowski prove that TC interprets TC^ε in their forthcoming paper *Undecidability and Concatenation.* Our argument is a variation of their argument.

[11] See [Vis06] for detailed definitions.

- $\varepsilon_K := \mathsf{a}$,
- $\mathsf{a}_K = \mathsf{b} * \mathsf{a}$,
- $\mathsf{b}_K = \mathsf{b} * \mathsf{b}$,
- $\delta_M(x) :\leftrightarrow x \neq \varepsilon$,
- $x \mathsf{E}_M y :\leftrightarrow x = y$,
- $x *_M y := x * y$,
- $xFy :\leftrightarrow x = \mathsf{b} * y$,
- $xGy :\leftrightarrow (x = \mathsf{a} \wedge y = \varepsilon) \vee x = \mathsf{b} * y$.

The verification that our definitions work is routine. We note the important fact that the presence of atoms in TC^ε implies that $x * y = \varepsilon \to (x = \varepsilon \vee y = \varepsilon)$.

The author thinks that he can prove the even stronger theorem, to wit that TC and TC^ε are definitionally equivalent, but the proof still has to be written up.

B Comparing TC^ε and $\mathsf{Q}^{\mathsf{bin}}$

We first show how to interpret $\mathsf{Q}^{\mathsf{bin}}$ in TC^ε. We work in TC^ε. Define $I(x) :\leftrightarrow \forall y \subseteq_{\mathsf{ini}} x\, (y = \varepsilon \vee \exists z\, (z = y*\mathsf{a} \vee y = z*\mathsf{b})$. It is easy to see that I is closed under ε, a and b, and that it is downwards closed under $\subseteq_{\mathsf{ini}}$. We show that it is closed under concatenation. Suppose that x_0 and x_1 are in I and that $y \subseteq_{\mathsf{ini}} x_0 * x_1$, say $y * w = x_0 * x - 1$. We want to show: $(\dagger_y)$ $y = \varepsilon$ or $\exists z\, (z = y * \mathsf{a} \vee y = z * \mathsf{b})$

By the editor axiom, there is a u, such that (1) $y * u = x_0$ and $w = u * x_1$, or (2) $y = x_0 * u$ and $u * w = x_1$. In the first case, $y \subseteq_{\mathsf{ini}} x_0$ and, hence we have $\dagger_y$. In the second case, $u \subseteq_{\mathsf{ini}} x_1$. So, we have $\dagger_u$. If $u = \varepsilon$, we find $y = x_0$. So, $y \subseteq_{\mathsf{ini}} x_0$ and $\dagger_y$. Otherwise, for some z, (2.1) $u = z * \mathsf{a}$ or (2.2) $u = z * \mathsf{b}$. In case (1.1) it follows that $y = x_0 * u = x_0 * (z * \mathsf{a}) = (x_0 * z) * \mathsf{a}$. Case (1.2) is similar. We may conclude $\dagger_y$.

Our interpretation $K : \mathsf{Q}^{\mathsf{bin}} \to \mathsf{TC}$ is just relativization to I, where we set $\mathsf{S}_{\mathsf{a}} x := x * \mathsf{a}$ and $\mathsf{S}_{\mathsf{b}} x := x * \mathsf{b}$.

We provide the reverse interpretation $M : \mathsf{TC} \to \mathsf{Q}^{\mathsf{bin}}$. We work in $\mathsf{Q}^{\mathsf{bin}}$. Let $I(x) :\leftrightarrow \forall y, z\, (y * z) * x = y * (z * x)$. It is easy to see that I is closed under ε, S_{a}, S_{b}, concatenation and under the predecessor functions corresponding to S_{a} and S_{b}.

So relativization to I will give us an interpretation of $\mathsf{Q}^{\mathsf{bin}}$ plus the associativity of concatenation. We proceed to work in this theory. Let:

$$J(x) :\leftrightarrow \forall y, u, v\, (y*x = u*v \to \exists z\, ((y*z = u \wedge x = z*v) \vee (y = u*z \vee z*x = v))).$$

We define $\mathsf{a} := \mathsf{S_a}\varepsilon$ and $\mathsf{b} := \mathsf{S_b}\varepsilon$. Clearly, J is closed under ε. Suppose $x = \mathsf{a}$, $y * \mathsf{a} = u * v$. If $v = \varepsilon$, we can take $z := \mathsf{a}$. We have: $y * z = u$ and $x = z * v$. If $v = \mathsf{S_a} v_0$, we can take $z := v_0$. We have:

$$\mathsf{S_a}(u * z) = u * v_0 * \mathsf{a} = u * v = y * \mathsf{a} = \mathsf{S_a} y.$$

So, $y = u * z$. Moreover, $z * x = v_0 * \mathsf{a} = v$. It is easily seen that the case that $v = \mathsf{S_b} v_0$, leads to a contradiction. We may conclude that a is in J. By similar reasoning, we find that b is in J.

We now show that J is closed under concatenation. Suppose x_0 and x_1 are in J and $y * x_0 * x_1 = u * v$. For some z_0, we have (1) $y * x_0 * z_0 = u$ and $x_1 = z_0 * v$, or (2) $y * x_0 = u * z_0$ and $z_0 * x_1 = v$.

In case (1) we can take the desired $z := x_0 * z_0$. We have: $y * z = y * x_0 * z_0 = u$ and $z * v = x_0 * z_0 * v = x_0 * x_1$.

In case (2), we have a z_1, such that (2.1) $y * z_1 = u$ and $x_0 = z_1 * z_0$, or (2.2) $y = u * z_1$ and $z_1 * x_0 = z_0$. In case (2.1), we can take $z := z_1$. We have: $y * z = y * z_1 = u$ and $z * v = z_1 * v = z_1 * z_0 * x_1 = x_0 * x_1$. In case (2.2), we can take $z := z_1$. We have $u * z = u * z_1 = y$ and $z * x_0 * x_1 = z_1 * x_0 * x_1 = z_0 * x_1 = v$.

Finally, we define $J^\star(x) :\leftrightarrow \forall y \subseteq x\ J^\star(y)$. It is easily seen that $J^\star$ is closed under ε, a, b, and downwards closed under taking substrings. We show that $J^\star$ is closed under concatenation. Suppose x_0 and x_1 are in $J^\star$ and $y \subseteq x_0 * x_1$. We have, for some w_0, w_1, that $x_0 * x_1 = w_0 * y * w_1$. Since x_1 in in $J^\star$ and, a fortiori, in J, there is a z_0, such that (1) $x_0 * z_0 = w_0$ and $x_1 = z_0 * y * w_1$, or (2) $x_0 = w_0 * z_0$ and $z_0 * x_1 = y * w_1$. In case (1), we have that $y \subseteq x_1$, hence $y \in J$.

In case (2), we use again that x_1 is in J. We can find a z_1, such that (2.1) $z_0 * z_1 = y$ and $x_1 = z_1 * w_1$, or (2.2) $z_0 = y * z_1$ and $z_1 * x_1 = w_1$. In case (2.1), we note that, $z_0 \subseteq x_0$ and $z_1 \subseteq x_1$. Hence, z_0 and z_1 are both in J. Thus $y = z_0 * z_1$ is also in J. In case (2.2), we find that $y \subseteq z_0 \subseteq x_0$, so $y \subseteq x_0$ and y is in J.

Since, we can rewrite the editor axiom with substring-bounded quantifiers, it is easily see that relativization to $J^\star$ interprets TC^ε in $\mathsf{Q}^{\mathsf{bin}}$ plus the associativity of concatenation.

C A Model without a Tally

Consider a non-standard model $\mathcal{N}$ of true arithmetic. We consider the binary strings on its non-standard numbers. Since these strings can be coded in the model, we obtain a non-standard model $\mathcal{M}$ of the true theory of finite binary strings. Let's say that the strings of this model are the n-strings. Now consider the class A of n-strings with only standardly finitely many a's. This class is closed under empty string, atoms, concatenation and is downward closed under substrings. Consider the submodel $\mathcal{A}$ of $\mathcal{M}$ determined by A. It is clear that$\mathcal{A}$ satisfies TC^ε and does not have a tally function.

We call a formula $\Delta_0^{\subseteq}$ is it only contains substring bounded quantifiers. A formula is $\Pi_1^{\subseteq}$ if it is given by a $\Delta_0^{\subseteq}$-formula preceded by universal quantifiers.

We see that $\mathcal{A}$ satisfies all $\Pi_1^{\subseteq}$-sentences true in $\mathcal{M}$ and, hence, true in the standard model of binary strings. Thus, there is a model of $\mathsf{TC}^{\varepsilon}$ plus all $\Pi_1^{\subseteq}$-sentences which are true in the standard model of binary strings, in which there is no tally length function.

We note that the model $\mathcal{M}$ cannot be realized in decorated linear order types. Consider a non-standard a-string σ. Let α be the decorated linear order type associated with σ and let β be the decorated linear order type associated with a. Since we can iterate the predecessor operation ω times on σ, we see there is an decorated linear order type α_0 such that $\alpha = \alpha_0 * (\beta \cdot \breve{\omega})$. Since the predecessor of σ will be associated with the same decorated linear order type, we have a contradiction.