Closed Fragments of Provability Logics of Constructive Theories

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Abstract

In this paper we give a new proof of the characterization of the closed fragment of the provability logic of Heyting's Arithmetic. We also provide a characterization of the closed fragment of the provability logic of Heyting's Arithmetic plus Markov's Principle and Heyting's Arithmetic plus Primitive Recursive Markov's Principle.

Key words: Provability Logic, Constructive Arithmetic

MSC2000 codes: 03F30, 03F45, 03F50

Dedicated to Craig Smoryński on the occasion of his 60th birthday.

Contents

1 Introduction

Friedman's 35th problem was to give a decision procedure for the closed fragment of the provability logic of Peano Arithmetic, PA. See [Fri75]. It was indepently solved by van Benthem (see: [vB74]), Boolos (see: [Boo76]) and Bernardi & Montagna (unpublished). The story of this result is told in [BS91]. The solution of Friedman's question would seem to be the end of the story of closed fragments: the characterization given is simple and definitive. Moreover, as we now know, it is amazingly stable. The same characterization works for all Σ^0_1-sound recursively enumerable extensions of Buss' theory S^1_2.[1] Very roughly speaking, one could say that it works in all theories in which one can arithmetize employing the usual recursive presentations of syntax. Thus, it almost works for all theories for which the question makes sense at all.

However, the situation changes dramatically, when we go on to consider the provability logics of *constructive* theories. New obstacles arise to the proof. We proceed to the next level in difficulty. Different theories will have markedly different closed fragments. In my preprint [Vis85], I gave characterizations and corresponding decision procedures for the closed fragments of HA and $\mathsf{HA}^\star$.[2] The characterizations are reasonably simple. However, in the case of HA, the verification of the characterization involves a complicated algorithm. I was not happy with its presentation and tried to improve it in [Vis94]. Finally the result was published in [Vis02]. Unfortunately, this final presentation still employs the original unperspicuous algorithm.

In the present paper, I will give a new proof of my old result by formalizing a Smoryński style Kripke model argument. It is somewhat surprising that the possibility of such a formalization was so long overlooked. One reason that it was so easy to overlook, is that one embeds classical methods inside a purely constructive argument. The new proof, with minor adaptations, also yields a characterization of the closed fragments of the provability logics of $\mathsf{HA} + \mathsf{MP}$ and $\mathsf{HA} + \mathsf{MP}_{\mathsf{PR}}$.

Why are these results interesting? We can view the question from two perspectives. The first perspective starts with the arithmetical theories. These theories are natural theories. This study throws light on a salient property of these theories. The second perspective is the perspective of the closed fragments themselves. The description of the closed fragment is a direct generalization of the second incompleteness theorem. In the context of the results for constructive theories, the classical case appears in a new light. We understand it as a special case of a more general pattern. Thus, enlarging the scope of our enquiry to the constructive case tells us something essential about closed fragments. Finally, apart from any specific perspective, one can say that a beautiful underlying mathematical structure is revealed.

[1] The case where we do not have Σ^0_1-soundness just gives rise to a trivial variation.

[2] For information about $\mathsf{HA}^\star$ see [Vis82], [dJV96], [Vis02] and [Vis06].

Prerequisites

We presuppose a background in provability logic. See e.g. [Boo93] or [Smo85]. We also presuppose basic knowledge of constructive arithmetic. See e.g [Tro73] or [TvD88]. The chapter [Smo73] of [Tro73] will be especially relevant.

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2 The Framework

The results of the paper fit a reasonably simple framework. In this section we describe this framework.

The question we are studying is to characterize the closed fragments of the provability logics of certain arithmetical theories. This introduces two kinds of theories and their connection. We have closed constructive provability logics and arithmetical theories. The connection between the kinds is formed by translations of the language of modal logic without propositional variables into the language of arithmetic. In our framework we will employ a third kind of theories, to wit theories of degrees of falsity. The theories of degrees of falsity are connected to closed provability logics via a translation of their language into the language of modal logic without propositional variables. We proceed to introduce these three kinds of theories and the relevant translations.

2.1 Degrees of Falsity

The intended arithmetical meaning of the degrees of falsity is: iterated inconsistency statements for the given arithmetical theory T.

Let $\omega^+ := \omega \cup \{\infty\}$. We let α, β, ..., range over the degrees of falsity. We use m, n, ..., for the finite degrees of falsity, i.e., the natural numbers. We equip ω^+ with the usual ordering and define $\infty + 1 := \infty$. Note that the successor function remains injective under this extension.

The language $\mathcal{D}$ of theories of degrees of falsity is the language given by:

- $\phi ::= \alpha \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid (\phi \to \phi)$.

The theory Basic is given by intuitionistic propositional logic with 0 in the role of $\perp$ and ∞ in the role of $\top$, plus the principles $\vdash \alpha \to \beta$, for $\alpha \leq \beta$. A theory of degrees of falsity is any extension of Basic in $\mathcal{D}$. We will use Δ, Γ, ..., to range over arbitrary theories of degrees of falsity. We identify $\neg\,\phi$ with $(\phi \to 0)$.

An extension of Γ of Basic is called *p-sound* iff it does not imply any sentence of the form $\alpha \to \beta$, for $\beta < \alpha$. We will be mainly interested in p-sound theories. Here are some further important notions.

- A theory Γ is *decent* if, for every ϕ and for every n larger than all finite m occurring in ϕ, we have $\Gamma \vdash n \to \phi$ implies $\Gamma \vdash \phi$.

- $\alpha_\Gamma(\phi) := \mathsf{max}\{\alpha \mid \Gamma \vdash \alpha \to \phi\}$. If there is no such maximum, $\alpha_\Gamma(\phi)$ is undefined.

The decency of Γ is a rather natural sufficient condition for the totality of α_Γ.

Theorem 2.1 *If Γ is decent, then, for any ϕ, we have that $\alpha_\Gamma(\phi)$ is defined. Moreover, $\alpha_\Gamma(\phi)$ is either ∞ or a finite k occurring in ϕ.*

Proof

Suppose Γ is decent. Let n be the maximal finite number occurring in ϕ. In case $\Gamma \vdash (n+1) \to \phi$, we have, by decency, that $\alpha_\Gamma(\phi) = \infty$.

Otherwise, there is a maximal $k \leq n$ such that $\Gamma \vdash k \to \phi$. So $\alpha_\Gamma(\phi) = k$. We show that k occurs in ϕ. Let $\widetilde{\phi}$ be the result of replacing every occurrence of any $m \geq k$ in ϕ by ∞. We have $\Gamma \vdash k \to (\phi \leftrightarrow \widetilde{\phi})$. It follows that $\Gamma \vdash k \to \widetilde{\phi}$, and, hence, that $\Gamma \vdash \widetilde{\phi}$. Let $\widetilde{k}$ be the smallest $s \geq k$ occurring in ϕ. Clearly, $\Gamma \vdash \widetilde{k} \to (\phi \leftrightarrow \widetilde{\phi})$, and hence $\Gamma \vdash \widetilde{k} \to \phi$. So, by the maximality of k, we have that $\widetilde{k} = k$, and, hence, that k occurs in ϕ. ❑

Here are some salient extensions of Basic.

- $\mathsf{Stronglöb} := \mathsf{Basic} + \{((\alpha \to \beta) \to \beta) \mid \beta < \alpha\}$,
- $\mathsf{Stable} := \mathsf{Basic} + \{\neg\neg\,\alpha \to \alpha \mid \alpha \in \omega^+\}$,
- $\mathsf{Classical} := \mathsf{Basic} + \{\alpha \vee \neg\,\alpha \mid \alpha \in \omega^+\}$.

Each of these theories is p-sound and decent. (In this paper we will prove the p-soundness and decency of Basic and Stable.)

2.2 Closed Constructive Provability Logics

We turn to closed constructive provability logics. The modal language $\mathcal{L}^0_\Box$ of closed logics is given as follows.

- $\phi ::= \bot \mid \top \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid (\phi \to \phi) \mid \Box\phi$.

The theory $i\mathsf{GL}^0$, intuitionistic Löb's Logic on zero variables, is the $\mathcal{L}^0_\Box$-theory axiomatized by intuitionistic propositional logic plus the following axioms and rules.

L1. $\vdash (\Box\phi \wedge \Box(\phi \to \psi)) \to \Box\psi$,

L2. $\vdash \Box\phi \to \Box\Box\phi$,

L3. $\vdash \Box(\Box\phi \to \phi) \to \Box\phi$,

L4. $\vdash \phi \;\; \Rightarrow \;\; \vdash \Box\phi$.

A closed provability logic is any extension of $i\mathsf{GL}^0$ in $\mathcal{L}^0_\Box$ that is closed under $\mathsf{L4}$. We let Λ, Θ, ..., range over closed provability logics. We define *the modal degrees of falsity* as follows.

- $\Box^0\bot := \bot$,
- $\Box^{n+1}\bot := \Box\Box^n\bot$,
- $\Box^\infty\bot := \top$.

We translate $\mathcal{D}$ into $\mathcal{L}^0_\Box$ via the translation emb_0 given by the following clauses.

- $\mathsf{emb}_0(\alpha) := \Box^\alpha\bot$,
- emb_0 commutes with the propositional connectives.

Consider any closed logic Λ. We define:

- $\mathsf{TDF}_\Lambda := \{\phi\in\mathcal{D} \mid \Lambda \vdash \mathsf{emb}_0(\phi)\}$.
  The theory TDF_Λ is *the theory of degrees of falsity of* Λ.
- For ϕ, ψ in $\mathcal{D}$: $\phi \;\mathrel{\vdash\!\!\sim}_\Lambda\; \psi \;\;:\Leftrightarrow\;\; \Lambda \vdash \Box\,\mathsf{emb}_0(\phi) \to \Box\,\mathsf{emb}_0(\psi)$.
  The relation $\mathrel{\vdash\!\!\sim}_\Lambda$ is the relation of *provably deductive consequence* w.r.t. Λ. We write $\approx_\Lambda$ for the induced equivalence relation.

Note that TDF is monotonic w.r.t. theory extension.

Since emb_0 *is a fixed embedding we will, when no confusion is possible, omit it.* Thus, e.g., we will treat TDF_Λ as if it is a *fragment* of Λ. We will write α_Λ for $\alpha_{\mathsf{TDF}_\Lambda}$. We will say that Λ is p-sound if TDF_Λ is p-sound and that Λ is decent if TDF_Λ is decent.

We prove a number of basic theorems. Consider any closed provability logic Λ. Clearly, $\vdash_{\mathsf{TDF}_\Lambda}$ and $\mathrel{\vdash\!\!\sim}_\Lambda$ are preorders on the language of $\mathcal{D}$ of theories of degrees of falsity. Moreover, $\mathrel{\vdash\!\!\sim}_\Lambda$ extends $\vdash_{\mathsf{TDF}_\Lambda}$. I.o.w., the identity mapping supports a functor π_λ from the preorder category $\vdash_{\mathsf{TDF}_\Lambda}$ to the preorder category $\mathrel{\vdash\!\!\sim}_\Lambda$.

Theorem 2.2 *Consider any p-sound closed provability logic Λ. Suppose π_Λ has a left adjoint Φ_Λ whose range consists of degrees of falsity. Then we have:*

1. *Φ_Λ is a functor, i.e. Φ_Λ is monotonic.*
2. *$\Phi_\Lambda(\phi) \vdash_{\mathsf{TDF}_\Lambda} \phi$.*
3. *$\phi \;\mathrel{\vdash\!\!\sim}_\Lambda\; \Phi_\Lambda(\phi)$.*
4. *$\Phi_\Lambda(\phi) \approx_\Lambda \phi$.*
5. *$\Phi_\Lambda(\beta) = \beta$.*
6. *The function α_Λ is total and $\alpha_\Lambda(\phi) = \Phi_\Lambda(\phi)$.*

Proof

Items (1), (2) and (3) are well known facts about adjunctions, noting that we have suppressed π_λ, since it is supported by the idenity mapping. See [Mac71]. Item (4) follows from (2) and (3).

We prove (5), By (2), we have $\Phi_\Lambda(\beta) \vdash_{\mathsf{TDF}_\Lambda} \beta$. So, by p-soundness, we have: $\Phi_\Lambda(\beta) \leq \beta$. Moreover, we have: $\beta \;\mathrel{\vdash\!\sim}_\Lambda \Phi_\Lambda(\beta)$. So $\Lambda \vdash \Box\beta \to \Box\,\Phi_T(\beta)$. I.o.w., $\Lambda \vdash (\beta+1) \to (\Phi_\Lambda(\beta)+1)$. By p-soundness, we have $\beta + 1 \leq \Phi_\Lambda(\beta)+1$, so $\beta \leq \Phi_\Lambda(\beta)$.

We prove (6). It is sufficient to show that Φ_Λ satisfies the conditions for α_Λ. First note that $\Phi_\Lambda(\phi) \vdash_{\mathsf{TDF}_\Lambda} \phi$. Suppose $\beta \vdash_{\mathsf{TDF}_\Lambda} \phi$. It follows that $\beta \;\mathrel{\vdash\!\sim}_\Lambda \phi$, and, hence, $\Phi_\Lambda(\beta) \vdash_{\mathsf{TDF}_\Lambda} \Phi_\Lambda(\phi)$. Since $\beta = \Phi_\Lambda(\beta)$ and Λ is p-sound, it follows that $\beta \leq \Phi_\Lambda(\phi)$. ❑

We state a sufficient condition for the existence of a left adjoint of π_Λ.

Theorem 2.3 *Suppose Λ is a p-sound closed provability logic. Suppose further that α_Λ is total and, for all ϕ in $\mathcal{D}$, we have $\phi \;\mathrel{\vdash\!\sim}_\Lambda \alpha_\Lambda(\phi)$. Then, α_Λ is the left adjoint of π_Λ.*

Proof

Let Λ be a closed provability logic. Suppose that α_Λ is total and $\phi \;\mathrel{\vdash\!\sim}_\Lambda \alpha_\Lambda(\phi)$.

Suppose $\alpha_\Lambda(\phi) \vdash_{\mathsf{TDF}_\Lambda} \psi$. Then, $\phi \;\mathrel{\vdash\!\sim}_\Lambda \alpha_\Lambda(\phi)$ and $\alpha_\Lambda(\phi) \;\mathrel{\vdash\!\sim}_\Lambda \psi$. We may conclude that $\phi \;\mathrel{\vdash\!\sim}_\Lambda \psi$.

Conversely, suppose $\phi \;\mathrel{\vdash\!\sim}_\Lambda \psi$. Since, $\alpha_\Lambda(\phi) \vdash_{\mathsf{TDF}_\Lambda} \phi$, we have $\alpha_\Lambda(\phi) \;\mathrel{\vdash\!\sim}_\Lambda \phi$. So, we find: $\alpha_\Lambda(\phi) \;\mathrel{\vdash\!\sim}_\Lambda \phi$ and $\phi \;\mathrel{\vdash\!\sim}_\Lambda \psi$ and $\psi \;\mathrel{\vdash\!\sim}_\Lambda \alpha_\Lambda(\psi)$. We may conclude: $\alpha_\Lambda(\phi) \;\mathrel{\vdash\!\sim}_\Lambda \alpha_\Lambda(\psi)$. By the p-soundness, we find $\alpha_\Lambda(\phi) \leq \alpha_\Lambda(\psi)$. We may conclude $\alpha_\Lambda(\phi) \vdash_{\mathsf{TDF}_\Lambda} \alpha_\Lambda(\psi)$ and $\alpha_\Lambda(\psi) \vdash_{\mathsf{TDF}_\Lambda} \psi$. Hence, $\alpha_\Lambda(\phi) \vdash_{\mathsf{TDF}_\Lambda} \psi$. ❑

Remark 2.4 Note that the proof did not use all the data of the theorem. We did not use the maximality of α_T but only the fact that $\alpha_\Lambda(\phi) \vdash_{\mathsf{TDF}_\Lambda} \phi$. ❑

Let Λ be a p-sound closed provability logic for which α_Λ is a left adjoint of π_Λ. We define the function $\mathsf{nf}_\Lambda : \mathcal{L}^0_\Box \to \mathcal{D}$.

- $\mathsf{nf}_\Lambda(\alpha) := \alpha$,
- nf_Λ commutes with the propositional connectives,
- $\mathsf{nf}_\Lambda(\Box\phi) := \alpha_\Lambda(\mathsf{nf}_\Lambda(\phi)) + 1$.

We have the following theorem.

Theorem 2.5 *Let Λ be a p-sound closed provability logic for which α_Λ is a left adjoint of π_Λ. We have $\Lambda \vdash \phi \leftrightarrow \mathsf{nf}_\Lambda(\phi)$.*

Proof

The proof is a simple induction on ϕ, using Theorem 2.2. ❑

Theorem 2.5 yields a characterization of a Λ that satisfies its conditions in terms of nf_Λ, since we have $\Lambda \vdash \phi$ iff $\mathsf{nf}_\Lambda(\Box\phi) = \infty$. In its turn nf_Λ is completely determined by α_Λ, which is fixed by TDF_Λ. This result will be the central in our characterization of the closed fragments of HA, $\mathsf{HA} + \mathsf{MP}$ and $\mathsf{HA} + \mathsf{MP}_{\mathsf{PR}}$.

Remark 2.6 Note that we can view the result as follows. Suppose Λ satisfies the conditions of Theorem 2.5. We can now *enrich* the language of Λ with new constants α —now not considered as defined— and we can add axioms $\vdash \alpha \leftrightarrow \Box^\alpha\bot$ to Λ. We extend the necessitation rule to the system with the new axioms. Let's call the resulting system Λ^+. Now Theorem 2.5 tells us that we have *box-elimination* in Λ^+. ❑

Consider any theory of degrees of falsity Γ for which α_Γ is defined. We can construct a closed provability logic from Γ in the following way. We first extend the language of Γ with the modal operator $\Box$. For any ϕ in $\mathcal{D}$, we add an axiom $\vdash \Box\phi \leftrightarrow (\alpha_\Gamma(\phi) + 1)$. Next, we replace 0 and ∞ by $\bot$ and $\top$. Finally, we take the reduct of the resulting theory to the language without $1, 2, \ldots$ We call the theory thus obtained AL_Γ, the associated logic of Γ. It is easy to see that AL_Γ is a closed provability logic.

Example 2.7 The operation AL is not monotonic w.r.t. theory extension. E.g., we have $\mathsf{AL}_{\mathsf{Basic}} \vdash \Box\neg\neg\Box\bot \to \Box\Box\bot$, but $\mathsf{AL}_{\mathsf{Stable}} \nvdash \Box\neg\neg\Box\bot \to \Box\Box\bot$. ❑

Here are the expected consequences of our definition.

Theorem 2.8 *Suppose that Γ is p-sound and that α_Γ is total. We have:*

1. $\mathsf{TDF}_{\mathsf{AL}_\Gamma} = \Gamma$,
2. α_Γ *is left adjoint to* π_{AL_Γ},
3. $\Lambda = \mathsf{AL}_\Gamma$ *iff* α_Γ *is left adjoint to* π_Λ,

Proof

Item (1) is immediate since AL_Γ is just a 'definitional extension' of Γ. Item (2) follows from Theorem 2.3. Finally, (3) follows from (2) and Theorem 2.2. ❑

Remark 2.9 The box of AL_Γ is in a sense the minimal box of a closed provability logic which has Γ as its theory of degrees of falsity. Suppose α_Γ is total. Let Λ be any closed provability logic such that $\mathsf{TDF}_\Lambda = \Gamma$. We have, for ϕ in $\mathcal{D}$:

$$\begin{aligned} \Lambda \vdash (\alpha_\Gamma(\phi) + 1) &\to \Box\,\alpha_\Gamma(\phi) \\ &\to \Box\,\phi \end{aligned}$$

❑

Open Question 2.10 Let Γ be p-sound and suppose that α_Γ is total. Consider all logics Λ with $\mathsf{TDF}_\Lambda = \Gamma$ ordered by theory extension. It is easy to see that $i\mathsf{GL} + \{\mathsf{emb}_0(\phi) \mid \Gamma \vdash \phi\}$ is the minimal element of this structure. Moreover, since every formula of AL_Γ is equivalent to $\mathsf{emb}_0(\phi)$ for some ϕ in $\mathcal{D}$, we have, for every strict extension Θ of AL_Γ, that TDF_Θ is a strict extension of Γ. Ergo, AL_Γ is a maximum of our structure. It would be interesting to have more information about this structure. E.g., are there other maxima? ❑

Open Question 2.11 Can we axiomatize $\mathsf{AL}_{\mathsf{Basic}}$ and $\mathsf{AL}_{\mathsf{Stable}}$? I conjecture that such an axiomatization will demand extended schemes involving finite disjunctions and conjunctions of variable length. ❑

2.3 Arithmetical Theories

Let T be any constructive ce theory with a designated interpretation K of $i\mathsf{S}^1_2$, the intuitionistic version of Buss' $i\mathsf{S}^1_2$. We write $\Box_T A$ for the formalization of: *A is provable in T*. This formalization is supposed to be executed 'inside' the interpretation K. We interpret the formulas of $\mathcal{L}^0_\Box$ in T, via an interpretation emb^T_1 that commutes with the propositional connectives and sends $\Box\phi$ to $\Box_T\,\mathsf{emb}^T_1(\phi)$. We define the closed fragment of T as follows.

- $\mathsf{CF}_T := \{\phi \mid T \vdash \mathsf{emb}^T_1(\phi)\}$.

Note that CF need *not* be monotonic w.r.t. extension of theories. We define an interpretation of emb^T_2 of $\mathcal{D}$ into the language of T by:

- $\mathsf{emb}^T_2 := \mathsf{emb}^T_1 \circ \mathsf{emb}_0$.

The degrees of falsity of T are the formulas $\Box^\alpha_T\bot := \mathsf{emb}^T_2(\alpha)$. We write TDF_T for: $\mathsf{TDF}_{\mathsf{CF}_T}$, and $\mathrel{|\!\sim}_T$ for: $\mathrel{|\!\sim}_{\mathsf{CF}_T}$.

We will study the closed fragments of three salient theories: HA, $\mathsf{HA} + \mathsf{MP}_{\mathsf{PR}}$, and $\mathsf{HA}+\mathsf{MP}$. To prepare our results we will consider a specific class of theories containing the target theories. Let's say that T is a *ha-theory* if T is a theory in the language of arithmetic that is HA-verifiably a Π^0_2-conservative extension of HA. Note that this notion is *intensional*: the arithmetical formula defining the axiom set of a given theory should be part of the data specifying the theory. This formula is needed to make sense of the question of a theory being a ha-theory. We can easily provide examples of pairs of theories that are extensionally the same of which one is a ha-theory and of which the other is not. For the theories we are considering like $\mathsf{HA} + \mathsf{MP}$, we assume that they are defined by a natural formula.

The ha-theories include HA, $\mathsf{HA}^\star$, $\mathsf{HA}+\mathsf{MP}$, $\mathsf{HA}+\mathsf{MP}_{\mathsf{PR}}$, $\mathsf{HA}+\mathsf{ECT}_0$, $\mathsf{MA} := \mathsf{HA} + \mathsf{ECT}_0 + \mathsf{MP}$ and PA. Note that ha-theories are Π^0_2-sound. Note that it follows that the closed fragment of any ha-theory is p-sound.

For all ha-theories U, V, and W, we have $U \vdash \Box^\alpha_V\bot \leftrightarrow \Box^\alpha_W\bot$, since any ha-theory is a HA-verifiably a Π^0_2-conservative extension of HA. So, modulo U-provable equivalence, the degrees of falsity of any of our theories are the same as the corresponding degrees of falsity of any other of our theories. This justifies us in suppressing the embedding emb_3^T in the context of any ha-theory. We will treat TDF_T simply as a fragment of T. We use α_T for α_{TDF_T}, etc.

For the theories HA, $\mathsf{HA}^\star$, $\mathsf{HA}+\mathsf{MP}$, $\mathsf{HA}+\mathsf{MP}_{\mathsf{PR}}$ and PA, all the information about the closed fragment is contained in the theory of the degrees of falsity. In fact, for each target theory T of this list, we have $\mathsf{CF}_T = \mathsf{AL}_{\mathsf{TDF}_T}$. Here are the theories of the degrees of falsity of our target theories.

1. $\mathsf{TDF}_{\mathsf{HA}} = \mathsf{Basic}$,
2. $\mathsf{TDF}_{\mathsf{HA}^\star} = \mathsf{Basic} + \{((\alpha \to \beta) \to \beta) \mid \beta < \alpha\} =: \mathsf{Stronglöb}$,
3. $\mathsf{TDF}_{\mathsf{HA+MP}} = \mathsf{DF}_{\mathsf{HA+MP_{PR}}} = \mathsf{Basic} + \{\neg\neg\,\alpha \to \alpha \mid \alpha \in \omega^+\} =: \mathsf{Stable}$,
4. $\mathsf{TDF}_{\mathsf{PA}} := \mathsf{Basic} + \{\alpha \vee \neg\,\alpha \mid \alpha \in \omega^+\} =: \mathsf{Classical}$.

We will provide a new proof for (1) (see also [Vis02]), and we will prove (3) for the first time in this paper. For (2), see [Vis02]. Finally, (3) is an immediate consequence of the classical answer to Friedman's problem.

We will show that, for the theories HA, $\mathsf{HA}+\mathsf{MP}$, $\mathsf{HA}+\mathsf{MP}_{\mathsf{PR}}$, the mapping α_T is the left adjoint of π_T. Moreover, α_T is computable. This implies that nf_T is computable. Since we have, for ϕ in $\mathcal{L}^0_\Box$, that $\mathsf{CF}_T \vdash \phi$ iff $\mathsf{nf}_T(\Box\phi) = \infty$, we obtain a decision procedure for CF_T.

For the proofs of the analogous facts for $\mathsf{HA}^\star$, see [Vis02]. For PA, see the classical literature.

3 Small Beer

In this section we collect some small facts in the environment of our main result. This section can very well be skipped.

3.1 Structural Facts about Theories of Degrees of Falsity

A first observation is that α_Γ, if defined, is monotonic in Γ.

One can show by a Kripke model argument that the theories $\mathsf{Stronglöb}$ and $\mathsf{Classical}$ are maximal among p-sound theories.

Theorem 3.1 $\mathsf{Classical}$ *is the unique maximal p-sound extension of* Stable.

Proof

Let Γ be a p-sound extension of Stable. Suppose $\Gamma \vdash \phi$. We have:

$$\mathsf{Classical} \vdash \phi \leftrightarrow \bigwedge(\alpha \to \beta),$$

for some selection of pairs α, β. Since we have the principles of $\mathsf{Classical}$ inside double negation, we find: $\Gamma \vdash \neg\neg \bigwedge(\alpha \to \beta)$. Ergo, $\Gamma \vdash \bigwedge(\alpha \to \neg\neg\beta)$, and, hence, $\Gamma \vdash \bigwedge(\alpha \to \beta)$, since Γ extends Stable. So, we find, by p-soundness, $\alpha \leq \beta$, for each pair α, β in our conjunction. It follows that $\mathsf{Classical} \vdash \phi$. ❑

We may conclude that $\mathsf{TDF}_{\mathsf{MA}} \subseteq \mathsf{Classical}$, thus obtaining an upper bound. One can show that $\mathsf{TDF}_{\mathsf{MA}}$ is *strictly* between Stable and $\mathsf{Classical}$. Further study of $\mathsf{TDF}_{\mathsf{MA}}$ —evidently the most exciting theory of degrees of falsity— will have to wait for another paper.

Open Question 3.2 1. Are there more examples of maximal p-sound theories?

2. Is Stable the mimimal theory that has $\mathsf{Classical}$ as unique maximal p-sound extension?

3. Are there interesting subtheories of $\mathsf{Stronglöb}$ that have $\mathsf{Stronglöb}$ as unique maximal p-sound extension?

❑

3.2 Π^0_2-sentences of Theories of Degrees of Falsity

Suppose $\mathsf{HA} \vdash \phi \leftrightarrow P$, where P is Π^0_2 and where ϕ is (a standard interpretation of a sentence) in $\mathcal{D}$. By reasoning in Basic, we have $\mathsf{HA} \vdash \neg\neg\phi \leftrightarrow \bigwedge(\alpha \to \neg\neg\beta)$, where α, β ranges over a finite set of such pairs with $\beta < \alpha$. We find that: $\mathsf{HA} \vdash P \to (\alpha \to \neg\neg\beta)$. Hence by the Friedman translation w.r.t. β:

$$\mathsf{HA} \vdash P^\beta \to ((\alpha \vee \beta) \to (((\beta \vee \beta) \to \beta) \to \beta)).$$

Since P is Π^0_2, we have $\mathsf{HA} \vdash P \to P^\beta$. Ergo, $\mathsf{HA} \vdash P \to (\alpha \to \beta)$.

Conversely, we have $\mathsf{HA} \vdash \bigwedge(\alpha \to \beta) \to \neg\neg P$. Say P is $\forall x\, Sx$, where Sx is Σ^0_1. By Friedman translating w.r.t. Sa, we get:

$$\mathsf{HA} \vdash \bigwedge((\alpha \vee Sa) \to (\beta \vee Sa)) \to ((\forall x\, (Sx \vee Sa) \to Sa) \to Sa).$$

Hence $\mathsf{HA} \vdash \bigwedge(\alpha \to \beta) \to Sa$, and, so, $\mathsf{HA} \vdash \bigwedge(\alpha \to \beta) \to P$. Combining we get: $\mathsf{HA} \vdash \bigwedge(\alpha \to \beta) \leftrightarrow P$.

Note that our argument works for every ha-theory for which Friedman's translation works with Σ^0_1-formulas as 'superscript formulas'. For ha-theories T that are extensions of $\mathsf{HA} + \mathsf{MP}_{\mathsf{PR}}$ we can reason more directly.

Let T be a ha-theory that extends $\mathsf{HA} + \mathsf{MP}_{\mathsf{PR}}$. Clearly for any Π^0_2-formula Q, we have $T \vdash \neg\neg\, Q \leftrightarrow Q$. Suppose $T \vdash P \leftrightarrow \phi$, for P in Π^0_2 and ϕ in $\mathcal{L}_0$. We have $T \vdash \neg\neg\, P \leftrightarrow \neg\neg \bigwedge(\alpha \to \beta)$, by elementary reasoning. Since both P and $\bigwedge(\alpha \to \beta)$ are Π^0_2, we find: $T \vdash P \leftrightarrow \bigwedge(\alpha \to \beta)$.

3.3 TDF as a Mapping of Theories

We note that TDF_T is monotonic in T.

Here is another simple insight. Let T be a ha-theory. Then, $\mathsf{HA} + \mathsf{TDF}_T$ is a ha-theory and $\mathsf{TDF}_{\mathsf{HA}+\mathsf{TDF}_T} = \mathsf{TDF}_T$. It is important to note that to understand the insight properly we have to think *intensionally*. E.g., the formula defining the theory $\mathsf{HA}+\mathsf{TDF}_T$ goes into the verification of the fact that it is a subtheory of T. We assume that this formula is an obvious formula involving the formula defining the axiom set of T.

Open Question 3.3 If we run through ha-theories which theories of degrees of falsity are assumed. I.o.w, what is the range of the mapping TDF? E.g., is every ce r-sound theory of degrees of falsity extensionally equal to the TDF_T of a ha-theory T? ❑

We provide a modest partial result.

Theorem 3.4 *Let* Γ *be an intensionally given ce extension of* Stable *and suppose that:*

$$\mathsf{HA} \vdash \forall \alpha, \beta \, (\Box_{\mathsf{HA}+\mathsf{MP}_{\mathsf{PR}}+\Gamma}(\alpha \to \beta) \to \Box_{\mathsf{HA}+\mathsf{MP}_{\mathsf{PR}}}(\alpha \to \beta) \,).$$

Then, $\mathsf{HA}+\mathsf{MP}_{\mathsf{PR}}+\Gamma$ *is a ha-theory and* $\mathsf{TDF}_{\mathsf{HA}+\mathsf{MP}_{\mathsf{PR}}+\Gamma} = \Gamma$.

Proof

We assume the hypothesis of the theorem. Reason in HA. Suppose P is Π^0_2 and that $\Box_{\mathsf{HA}+\mathsf{MP}_{\mathsf{PR}}+\Gamma} P$. Then, for some finite conjunction γ of formulas from Γ, $\Box_{\mathsf{HA}+\mathsf{MP}_{\mathsf{PR}}}(\gamma \to P)$. It follows that $\Box_{\mathsf{HA}+\mathsf{MP}_{\mathsf{PR}}}(\neg\neg\, \gamma \to P)$. Moreover, for some finite set of pairs α, β, we have $\Box_{\mathsf{HA}+\mathsf{MP}_{\mathsf{PR}}}(\neg\neg\gamma \leftrightarrow \bigwedge(\alpha \to \beta))$. So we have (i) $\Box_{\mathsf{HA}+\mathsf{MP}_{\mathsf{PR}}+\Gamma} \bigwedge(\alpha \to \beta)$ and (ii) $\Box_{\mathsf{HA}+\mathsf{MP}_{\mathsf{PR}}}(\bigwedge(\alpha \to \beta) \to P)$. By our hypothesis, we have from (i): $\Box_{\mathsf{HA}+\mathsf{MP}_{\mathsf{PR}}} \bigwedge(\alpha \to \beta)$. So, by (ii), we find $\Box_{\mathsf{HA}+\mathsf{MP}_{\mathsf{PR}}} P$. Hence, $\Box_{\mathsf{HA}} P$. We may conclude that $\mathsf{HA}+\mathsf{MP}_{\mathsf{PR}}+\Gamma$ is a ha-theory.

Suppose $\mathsf{HA}+\mathsf{MP}_{\mathsf{PR}}+\Gamma \vdash \phi$. Then, since Γ extends Stable, by the results of the next section: $\Gamma \vdash \phi$. So, $\Gamma = \mathsf{TDF}_{\mathsf{HA}+\mathsf{MP}_{\mathsf{PR}}+\Gamma}$. ❑

4 The Main Result

We first provide Smoryński style Kripke arguments to characterize $\mathsf{TDF}_{\mathsf{HA}}$, $\mathsf{TDF}_{\mathsf{HA}+\mathsf{MP}_{\mathsf{PR}}}$ and $\mathsf{TDF}_{\mathsf{HA}+\mathsf{MP}}$. Then, by formalizing these arguments, we prove the existence of a left adjoint of the embedding of $\vdash_T$ into $\mathrel{|\!\sim}_T$, considered as preorderings on $\mathcal{D}$, for T one of these three theories.

4.1 Kripke Model Arguments

We show that Basic provides a complete axiomatization of $\mathsf{TDF}_{\mathsf{HA}}$. A Kripke model for Basic is a Kripke model in the usual sense with $\alpha \in \omega^+$ as atoms for the atomic forcing. Here we use 0 in the role of $\bot$ and ∞ in the role of $\top$. We demand: $k \Vdash n$ and $m \geq n$ implies $k \Vdash m$. A Kripke model for Basic is *finite* iff it has a finite number of nodes and, for some number N, we have that N is forced in all nodes.

Consider any Kripke model $\mathcal{K}$ for Basic such that $\mathcal{K} \nVdash \phi$. Let N be larger than all finite atoms occurring in ϕ. We transform our model to a model $\mathcal{K}'$ by resetting the atomic forcing to: $k \Vdash' n$ iff $k \Vdash n$ or $n \geq N$. It is easy to see that $\mathcal{K}'$ is again a model of Basic and $\mathcal{K}' \nVdash \phi$. We collapse $\mathcal{K}'$ by dividing out the maximal bisimulation, thus obtaining a finite model $\mathcal{J}$ of Basic with $\mathcal{J} \nVdash \phi$. We have shown:

Theorem 4.1 *The class of all finite Kripke models for* Basic *is complete for* Basic*.*

It follows immediately that Basic is decent. We have:

Theorem 4.2 *For all ϕ in $\mathcal{D}$, $\alpha_{\mathsf{Basic}}(\phi)$ is defined. Moreover $\alpha_{\mathsf{Basic}}(\phi)$ is computable by a multi-exponential algorithm.*

Proof

The fact that α_{Basic} is total, follows from decency.

Let N be the smallest finite degree of falsity not occurring in ϕ. We construct a finite Henkin model for ordinary propositional logic for the formula $\psi := (N \wedge \bigwedge\{(k \rightarrow \ell) \mid k < \ell < N\})$, where we take as the nodes of the model saturated subsets of the subformulas of ϕ and ψ plus ∞. One may show that, for $k \leq N$, we have $\mathsf{Basic} \vdash k \rightarrow \phi$ iff $\mathcal{H} \Vdash k \rightarrow \phi$. It follows that we can determine $\alpha_{\mathsf{Basic}}(\phi)$ by running through $\mathcal{H}$. ❑

We define, for finite Kripke models of Basic:

- ν_k is the largest n such that $k \nVdash n$.

Note that in a finite Kripke model the atomic forcing is completely determined by the ν_k. Consider any $\mathcal{D}$-formula ϕ. Suppose $\mathsf{Basic} \nvdash \phi$. We show that $\mathsf{HA} \nvdash \phi$. Consider a finite Kripke model for Basic that does not force ϕ at the root. We unravel this model to a finite Kripke tree, say $\mathcal{T}$. We build a model $\mathcal{T}^\star$ of HA on the frame of $\mathcal{T}$. We associate to every node k a model $\mathcal{M}_k$ of $\mathsf{PA} + \neg\,\nu_k + (\nu_k + 1)$ as follows.

To the root $\mathfrak{b}$ we assign an arbitrarily chosen model of $\mathsf{PA} + \neg\,\nu_{\mathfrak{b}} + (\nu_{\mathfrak{b}} + 1)$. (Such a model exists by Löb's Theorem and the Completeness Theorem.)

Suppose we have already assigned $\mathcal{M}_k$ to k. We will have insured that $\mathcal{M}_k$ satisfies $\mathsf{PA} + \neg\,\nu_k + (\nu_k + 1)$. Let ℓ be a direct successor of k. In case $\nu_\ell = \nu_k$, we put $\mathcal{M}_\ell := \mathcal{M}_k$. In case $\nu_\ell < \nu_k$, we proceed as follows. Note that $\neg\,\nu_k$ is PA-provably equivalent to $\mathsf{con}(\mathsf{PA} + \neg\,(\nu_k - 1))$. Since $\nu_\ell \leq \nu_k - 1$, we have:

$$\mathcal{M}_k \models \mathsf{PA} + \mathsf{con}(\mathsf{PA} + \neg\nu_\ell).$$

By Löb's Theorem, it follows that:

$$\mathcal{M}_k \models \mathsf{PA} + \mathsf{con}(\mathsf{PA} + \neg\nu_\ell + (\nu_\ell + 1)).$$

By the Feferman-Henkin construction, we can find an interpretation

$$\mathcal{H}_{\nu_\ell} : (\mathsf{PA} + \mathsf{con}(\mathsf{PA} + \neg\nu_\ell + (\nu_\ell + 1))) \rhd (\mathsf{PA} + \neg\nu_\ell + (\nu_\ell + 1)).$$

We take $\mathcal{M}_\ell := \mathsf{MOD}(\mathcal{H}_{\nu_\ell})(\mathcal{M}_k)$, i.e., the internal model of $\mathcal{M}_k$ given by $\mathcal{H}_{\nu_\ell}$. Clearly: $\mathcal{M}_\ell \models \mathsf{PA} + \neg\nu_\ell + (\nu_\ell + 1)$.

Since the cone above each node is internally definable in the model associated to that node, we find, by Smoryński's reasoning, that $\mathcal{T}^\star \Vdash \mathsf{HA}$. See [Smo73]. Moreover, since, for Σ^0_1-sentences S, we have $k \Vdash S \Leftrightarrow \mathcal{M}_k \models S$, we find: $k \Vdash n \Leftrightarrow \mathcal{M}_k \models n$. By our construction, it follows that $\mathcal{T}$ is equal to $\mathcal{T}^\star$ if we restrict $\mathcal{T}^\star$ to the degrees of falsity (modulo emb_3). We may conclude that $\mathcal{T}^\star \nVdash \phi$.

We can easily adapt our argument to establish a similar result for $\mathsf{HA}+\mathsf{MP}$ and $\mathsf{HA}+\mathsf{MP}_{\mathsf{PR}}$. The proof of the finite model property for Stable is word for word the same. It follows that Stable is decent.

Let's say that a finite Kripke model for Basic is *upwards reflecting* if, for every node k there is a top-node $k' \geq k$. such that $k \vdash \neg\,\nu_k$, i.o.w., such that the atomic forcing of k' is the same as the atomic forcing of k. By a simple argument one can show that a finite Kripke model for Basic is a finite Kripke model for Stable iff it is upwards reflecting. We may conclude:

Theorem 4.3 *We have completeness for* Stable *in the class of finite upwards reflecting Kripke models for* Basic.

Also the argument for the multi-exponential computability of α_{Stable} is the same.

Theorem 4.4 *We have:* α_{Stable} *is total and computable by a multi-exponential algorithm.*

We clearly have that Stable is a subtheory of $\mathsf{TDF}_{\mathsf{HA}+\mathsf{MP}_{\mathsf{PR}}}$ and $\mathsf{TDF}_{\mathsf{HA}+\mathsf{MP}}$. Suppose $\mathsf{Stable} \nvdash \phi$. We show that $\mathsf{HA} + \mathsf{MP} \nvdash \phi$. Let $\mathcal{T}$ be an upwards reflecting finite Kripke tree such that the root of $\mathcal{T}$ does not force ϕ. Consider the model $\mathcal{T}^\star$ constructed above. Note that, by the construction, it has the property that above each node k there is a top node k' such that $\mathcal{M}_{k'} = \mathcal{M}_k$.

We claim that $\mathcal{T}^\star$ forces Markov's Principle. Suppose $k \Vdash \forall x\,(A_0x \vee \neg\,A_0x)$ and $k \Vdash \neg\neg\,\exists x\ A_0x$. Let k' be a top node above k such that $\mathcal{M}_{k'} = \mathcal{M}_k$. Clearly, $k' \Vdash \exists x\ A_0x$. Let a be a domain element such that $k' \Vdash A_0a$. We find that $k \nVdash \neg A_0a$, and so, by decidability, $k \Vdash A_0a$. Thus, $k \Vdash \exists x\ A_0x$.

It follows that Stable provides a complete axiomatization for $\mathsf{TDF}_{\mathsf{HA}+\mathsf{MP}}$ and $\mathsf{TDF}_{\mathsf{HA}+\mathsf{MP}_{\mathsf{PR}}}$.

4.2 Formalization of Kripke Models

Our next step is to verify the completeness side of the two above proofs inside HA. The result gets the form: $\mathsf{HA} \vdash \Box_T \phi \to \Box_T \alpha_T(\phi)$, where T is one of our three target theories. By Theorem 2.3, this is precisely what we need to show that α_T is left adjoint to π_T. Thus, we obtain the desired characterization.

To realize our program, we will have to eliminate the reference to models and replace it fully by talk about interpretations. In fact we will view $k \Vdash A$ as an arithmetical formula B. Let $\mathcal{T}$ be any finite tree model of Basic which forces some finite N. Let the ordering of $\mathcal{T}$ be $\preceq$. We write $\prec_1$ for: direct successor in the $\preceq$-relation. We define a translation as follows.

- Suppose $k \prec_1 \ell$. Then, $\mathcal{H}_{k\ell} := \mathsf{ID}$, iff $\nu_k = \nu_\ell$ and $\mathcal{H}_{k\ell} := \mathcal{H}_{\nu_\ell}$, if $\nu_\ell < \nu_k$. Here $\mathcal{H}_{\nu_\ell}$ is as defined in Subsection 4.1. We write $F_{k\ell}$ for (a formula representing) the Dedekind embedding of ID into $\mathcal{H}_{k\ell}$. I.o.w., $F_{k\ell}$ is an arithmetical formula that represents the initial embedding of the numbers of the given extension of PA into the numbers of the interpretation $\mathcal{H}_{\nu_\ell}$.
- Suppose $k = \ell_0 \prec_1 \ell_1 \prec_1 \ldots \prec_1 \ell_{n-1} = k'$. Then:

  $\mathcal{H}_{kk'} := \mathcal{H}_{\ell_0\ell_1} \circ \ldots \circ \mathcal{H}_{\ell_{n-2}\ell_{n-1}}$.

  We write $\delta_{kk'}$ for $\delta_{\mathcal{H}_{kk'}}$.
- It is convenient to give our definitions by restricting the given Kripke model to the cone above a given node k. Thus, our definitions are for 'the model as it looks according to k'.

  Suppose $k \preceq k' = \ell_0 \prec_1 \ell_1 \prec_1 \ldots \prec_1 \ell_{n-1} = k''$. Then:

  $F_{kk'k''} := F^{\mathcal{H}_{k\ell_{n-2}}}_{\ell_{n-2}\ell_{n-1}} \circ \ldots \circ F^{\mathcal{H}_{k\ell_0}}_{\ell_0\ell_1}$.[3]

  Note that $F_{kk'k''}$ is the Dedekind embedding of $\mathcal{H}_{kk'}$ into $\mathcal{H}_{kk''}$.
- We define a mapping $(\cdot) \Vdash_{(\cdot)} (\cdot)$ from pairs of nodes and arithmetical sentences to arithmetical sentences.
  - ▼ $(k' \Vdash_k P\vec{x}) :\leftrightarrow (P\vec{x})^{\mathcal{H}_{kk'}}$,
  - ▼ $k' \Vdash_k (\cdot)$ commutes with $\bot$, $\top$, conjunction and disjunction.
  - ▼ $(k' \Vdash_k (A\vec{x} \to B\vec{x})) :\leftrightarrow$
    $\bigwedge_{k'' \succeq k'} \forall \vec{y}\, ((\vec{x} F_{kk'k''} \vec{y} \wedge k'' \Vdash_k A\vec{y}) \to k'' \Vdash_k B\vec{y})$.
  - ▼ $(k' \Vdash_k \forall u\, Au\vec{x}) :\leftrightarrow \bigwedge_{k'' \succeq k'} \forall u{:}\delta_{kk''}\, \forall \vec{y}\, (\vec{x} F_{kk'k''} \vec{y} \to k' \Vdash_k Au\vec{y})$.
  - ▼ $(k' \Vdash_k \exists u\, Au\vec{x}) :\leftrightarrow \exists u{:}\delta_{kk'}\, k' \Vdash_k Au\vec{x}$.

Let $T_k := \mathsf{PA} + \neg\nu_k + (\nu_k + 1)$.

Fact 4.5 *Suppose $k \preceq k' \preceq k'' \preceq \ell$. We have in T_k:*

i. $\mathcal{H}_{kk''} = \mathcal{H}_{kk'} \circ \mathcal{H}_{k'k''}$.

[3]Note the reversal of order here. This is because the MOD-functor is contravariant.

ii. $\delta_{kk''} = \delta_{k'k''}^{\mathcal{H}_{kk'}}$.

iii. $F_{kk'\ell} = F_{kk''\ell} \circ F_{kk'k''}$.

iv. $F_{kk''\ell} = F_{k'k''\ell}^{\mathcal{H}_{kk'}}$.

Proof

We treat (iv). Suppose $k'' = m_0 \prec_1 m_1 \prec_1 \ldots \prec_1 m_{n-1} = \ell$. We have in T_k:

$$\begin{array}{rcl} F_{k'k''\ell}^{\mathcal{H}_{kk'}} & = & (F_{m_{n-2}m_{n-1}}^{\mathcal{H}_{k'm_{n-2}}} \circ \ldots \circ F_{m_0m_1}^{\mathcal{H}_{k'm_0}})^{\mathcal{H}_{kk'}} \\ & = & (F_{m_{n-2}m_{n-1}}^{\mathcal{H}_{k'm_{n-2}}})^{\mathcal{H}_{kk'}} \circ \ldots \circ (F_{m_0m_1}^{\mathcal{H}_{k'm_0}})^{\mathcal{H}_{kk'}} \\ & = & F_{m_{n-2}m_{n-1}}^{\mathcal{H}_{kk'} \circ \mathcal{H}_{k'm_{n-2}}} \circ \ldots \circ F_{m_0m_1}^{\mathcal{H}_{kk'} \circ \mathcal{H}_{k'm_0}} \\ & = & F_{m_{n-2}m_{n-1}}^{\mathcal{H}_{km_{n-2}}} \circ \ldots \circ F_{m_0m_1}^{\mathcal{H}_{km_0}} \\ & = & F_{kk''\ell} \end{array}$$

❑

Fact 4.6 *Suppose $k \preceq k' \preceq k''$. We have:* $T_k \vdash (k'' \Vdash_k A) \leftrightarrow (k'' \Vdash_{k'} A)^{\mathcal{H}_{kk'}}$.

Proof

Suppose $k \preceq k' \preceq k''$. The proof is by induction on A.

For the base case, we reason in T_k. We have: $k'' \Vdash P\vec{x}$ iff $(P\vec{x})^{\mathcal{H}_{kk''}}$. But $\mathcal{H}_{kk''} = \mathcal{H}_{kk'} \circ \mathcal{H}_{k'k''}$. So, $(P\vec{x})^{\mathcal{H}_{kk''}}$ iff $((P\vec{x})^{\mathcal{H}_{k'k''}})^{\mathcal{H}_{kk'}}$ iff $(k'' \Vdash_{k'} P\vec{x})^{\mathcal{H}_{kk'}}$.

The cases of $\bot$, $\top$, conjunction, disjunction and existential quantification are easy.

We treat the case of implication. The case of the universal quantifier is similar. Reason in T_k. We have:

$$\begin{array}{rcl} k'' \Vdash_k (B\vec{x} \to C\vec{x}) & \leftrightarrow & \bigwedge_{\ell \succeq k''} \forall \vec{y}\, ((\vec{x}F_{kk''\ell}\vec{y} \wedge \ell \Vdash_k B\vec{y}) \to \ell \Vdash_k C\vec{y}) \\ & \leftrightarrow & \bigwedge_{\ell \succeq k''} \forall \vec{y} : \delta_{kk'}\ ((\vec{x}F_{k'k''\ell}^{\mathcal{H}_{kk'}}\vec{y} \wedge (\ell \Vdash_{k'} B\vec{y})^{\mathcal{H}_{kk'}}) \\ & & \qquad\qquad \to (\ell \Vdash_{k'} C\vec{y})^{\mathcal{H}_{kk'}}) \\ & \leftrightarrow & (\bigwedge_{\ell \succeq k''} \forall \vec{y}\, ((\vec{x}F_{kk''\ell}\vec{y} \wedge \ell \Vdash_k B\vec{y}) \to \ell \Vdash_k C\vec{y}))^{\mathcal{H}_{kk'}} \\ & \leftrightarrow & (k'' \Vdash_{k'} (B\vec{x} \to C\vec{x}))^{\mathcal{H}_{kk'}} \end{array}$$

❑

Fact 4.7 Persistence: *Suppose $k \preceq k' \preceq k''$. We have:*

$$T_k \vdash ((\vec{x}F_{kk'k''}\vec{y} \wedge k' \Vdash_k A\vec{x}) \to k'' \Vdash_k A\vec{y}).$$

Proof

The proof is by a simple induction on A. The atomic case uses the fact that the Dedekind embedding is an embedding of structures. ❑

Fact 4.8 *We have, verifiably in* HA*, that, for any finite* $\Gamma\vec{x}$*, if* $\Gamma\vec{x} \vdash_{\mathsf{IQC}} A\vec{x}$*, then:*

$$T_k \vdash \bigwedge_{k' \succeq k} \forall \vec{x} : \delta_{k'} \; ((\bigwedge_{B\vec{x} \in \Gamma\vec{x}} k' \Vdash_k B\vec{x}) \to k' \Vdash_k A\vec{x}).$$

Proof

The proof is by induction on the number of steps in the IQC-proof. ❑

Fact 4.9 *Let a finite tree for* Basic *be given. We have, verifiably in* HA*, that, for any* $k' \succeq k$ *and for any axiom* C *of* HA*, we have* $T_k \vdash (k' \Vdash_k C)$*.*

Proof

We treat, for example, the axiom $\vdash \forall x \; \mathsf{S}x \neq 0$. Reason in T_k. We have to show that $\bigwedge_{k' \succeq k} \forall x : \delta_{k'} \; \neg(k' \Vdash_k Sx = 0)$. In other words, we want to show that:

$$\bigwedge_{k' \succeq k} \forall x{:}\delta_{k'} \; \neg(Sx = 0)^{\mathcal{H}_{kk'}}.$$

But this is trivial, since $\mathcal{H}_{k,k'}$ is an interpretation of PA.

We turn to the case of induction. We reason in HA. Suppose we have verified $T_{k'} \vdash (k'' \Vdash_{k'} C)$, for all $k'' \succeq k' \succ k$ and for all instances of induction C. Fix k and k'. Consider any $Dx\vec{y}$. We define:

- $\mathsf{prog}_x(Dx\vec{y}) :\leftrightarrow D0\vec{y} \wedge \forall z \; (Dz\vec{y} \to D\mathsf{S}z\vec{y})$.
- $\mathsf{ind}_x(Dx\vec{y}) :\leftrightarrow (\mathsf{prog}_x(Dx\vec{y}) \to \forall x \; Dx\vec{y})$.

We want to prove: $T_k \vdash \vec{y} : \delta_{kk'} \to (k' \Vdash_k \mathsf{ind}_x(Dx\vec{y}))$. We have, for $k'' \succ k$,

$$T_{k''} \vdash (k'' \Vdash_{k''} \mathsf{ind}_x(Dx\vec{y})).$$

Since, $\mathcal{H}_{kk''}$ is an interpretation of $T_{k''}$ in T_k, it follows that:

$$(\dagger) \;\; T_k \vdash \vec{y} : \delta_{kk''} \to (k'' \Vdash_{k''} \mathsf{ind}_x(Dx\vec{y}))^{\mathcal{H}_{kk''}}.$$

We reason in T_k. Suppose that $k'' \succeq k'$ and $\vec{y} : \delta_{kk''}$ and $k'' \Vdash_k \mathsf{prog}_x(Dx\vec{y})$.

Case 1: Suppose first that $k'' \succ k$. It follows that $(k'' \Vdash_{k''} \mathsf{prog}_x(Dx\vec{y}))^{\mathcal{H}_{kk''}}$. By ($\dagger$), we have $(k'' \Vdash_{k''} \forall x \; Dx\vec{y})^{\mathcal{H}_{kk''}}$. Hence, $k'' \Vdash_k \forall x \; Dx\vec{y}$.

Case 2: Suppose $k'' = k$. It follows that $k' = k$. We have to show $k \Vdash_k \forall x\ Dx\vec{y}$. Note that by persistence, we have, for any $\ell \succ k$, and any $\vec{z}$ such that $\vec{y}F_{kk\ell}\vec{z}$, that $\ell \Vdash_k \mathsf{prog}_x(Dx\vec{z})$. Hence, by Case 1, we find $\ell \Vdash_k \forall x\ Dx\vec{z}$. So it is sufficient to show that $\forall x\ k \Vdash_k Dx\vec{y}$. From $k \Vdash_k \mathsf{prog}_x(Dx\vec{y})$, we can derive: $\mathsf{prog}_x(k \Vdash_k Dx\vec{y})$. Hence the desired result follows by the induction scheme of the classical theory T_k. ❑

Fact 4.10 *Suppose $S\vec{x}$ is a Σ^0_1-formula. Then*

$$T_k \vdash \vec{x} : \delta_{kk'} \to (S^{\mathcal{H}_{kk'}}\vec{x} \leftrightarrow (k' \Vdash_k S\vec{y})).$$

This fact is verifiable in HA.

Proof

The main part of the proof is by induction on the complexity of Δ_0-formulas A. The addition of the initial block of existential quantifiers is easy. We use the fact that we already know that HA is forced in our finite model, that HA proves the decidability of Δ_0-formulas and the fact that, for decidable A,

$$\mathsf{HA} \vdash \neg \forall x{<}t\ Ax \leftrightarrow \exists x{<}t\ \neg Ax.$$

We treat the case of the bounded universal quantifier. Suppose A is Δ_0 and satisfies the induction hypothesis. We reason in T_k. Suppose $\vec{y} : \delta_{kk'}$ and x is not among the $\vec{y}$.

$$\begin{array}{rcl}
k' \Vdash_k \forall x{<}t\vec{y}\ Ax\vec{y} & \to & \forall x{:}\delta_{kk'}\ ((x < t\vec{y})^{\mathcal{H}_{kk'}} \to k' \Vdash_k Ax\vec{y}) \\
& \to & \forall x{:}\delta_{kk'}\ ((x < t\vec{y})^{\mathcal{H}_{kk'}} \to A^{\mathcal{H}_{kk'}}x\vec{y}) \\
& \to & (\forall x{<}t\vec{y}\ Ax\vec{y})^{\mathcal{H}_{kk'}}
\end{array}$$

$$\begin{array}{rcl}
k' \nVdash_k \forall x{<}t\vec{y}\ Ax\vec{y} & \to & k' \Vdash_k \neg \forall x{<}t\vec{y}\ Ax\vec{y} \\
& \to & k' \Vdash_k \exists x{<}t\vec{y}\, \neg Ax\vec{y} \\
& \to & \exists x{:}\delta_{kk'}\ ((x < t\vec{y})^{\mathcal{H}_{kk'}} \wedge k' \Vdash_k \neg Ax\vec{y}) \\
& \to & \exists x{:}\delta_{kk'}\ ((x < t\vec{y})^{\mathcal{H}_{kk'}} \wedge k' \nVdash_k Ax\vec{y}) \\
& \to & \exists x{:}\delta_{kk'}\ ((x < t\vec{y})^{\mathcal{H}_{kk'}} \wedge \neg A^{\mathcal{H}_{kk'}}x\vec{y}) \\
& \to & (\exists x < t\vec{y}\, \neg Ax\vec{y})^{\mathcal{H}_{kk'}} \\
& \to & \neg (\forall x < t\vec{y}\ Ax\vec{y})^{\mathcal{H}_{kk'}}
\end{array}$$

❑

Remark 4.11 There is an alternative proof of Theorem 4.10 following an idea of Smoryński which avoids most of the inductive cases: one uses the fact that Matijacevič's theorem can be verified in HA. This proof has the advantage of being simpler. It has the disadvantage of using a major theorem. Moreover, it does not downwards generalize —as far as we know— to e.g. $iI\Delta_0$. ❑

Fact 4.12 *Let a finite Kripke tree for* Basic *be given. We have, for* $k \preceq k'$*, verifiably in* T_k*, that* $k' \nVdash_k \nu_{k'}$ *and* $k' \Vdash_k (\nu_{k'} + 1)$*.*

Proof

Reason in T_k. We have already seen that $k' \Vdash_k \mathsf{HA}$. We also have $k' \Vdash_k \nu_k'$ iff $\nu_{k'}^{\mathcal{H}_{kk'}}$. Since, $\mathcal{H}_{kk'}$ interprets $T_{k'}$, we have $\neg \nu_{k'}^{\mathcal{H}_{kk'}}$. Ergo, $k' \nVdash_k \nu_k'$. The other case is similar. ❑

Fact 4.13 *Let a finite Kripke tree for* Basic *be given. Suppose* $k \preceq k'$*. Suppose* $k' \Vdash \phi$*. We have* $T_k \vdash (k' \Vdash_k \phi)$*. Similarly, if* $k' \nVdash \phi$*, then,* $T_k \vdash (k' \nVdash_k \phi)$*.*

Proof

The proof is by induction on ϕ using Fact 4.12. ❑

Theorem 4.14 *Consider any* ϕ *in* $\mathcal{L}_0$*. We have:* $\mathsf{HA} \vdash \Box_{\mathsf{HA}}\phi \to \Box_{\mathsf{HA}}\alpha_{\mathsf{Basic}}(\phi)$*.*

Proof

In case $\alpha_{\mathsf{Basic}}(\phi) = \infty$, this is trivial. We assume that $\alpha_{\mathsf{Basic}}(\phi)$ is finite. By Kripke completeness, we can find a finite Kripke tree for Basic with root $\mathfrak{b}$, such that and $\mathfrak{b} \nVdash \phi$ and $\nu_{\mathfrak{b}} = \alpha_{\mathsf{Basic}}(\phi)$. Reason in HA. Suppose $\Box_{\mathsf{HA}}\phi$. It follows that $\Box_{T_{\mathfrak{b}}}(\mathfrak{b} \Vdash_{\mathfrak{b}} \phi)$. On the other hand, since $\mathfrak{b} \nVdash \phi$, we have: $\Box_{T_{\mathfrak{b}}}(\mathfrak{b} \nVdash_{\mathfrak{b}} \phi)$. Ergo, $\Box_{T_{\mathfrak{b}}}\bot$. Hence, $\Box_{\mathsf{PA}}((\nu_{\mathfrak{b}} + 1) \to \nu_{\mathfrak{b}})$. By Löb's Theorem, in combination with the verifiable Π^0_2-conservativity of PA over HA, we find that $\Box_{\mathsf{PA}}\nu_{\mathfrak{b}}$ and, thus, $\Box_{\mathsf{HA}}\nu_{\mathfrak{b}}$. In other words, $\Box_{\mathsf{HA}}\alpha_{\mathsf{Basic}}(\phi)$. ❑

To adapt our proof to $\mathsf{HA} + \mathsf{MP}_{\mathsf{PR}}$ and $\mathsf{HA} + \mathsf{MP}$. The only extra step is to verify in that, for a finite upwards reflecting tree $\mathcal{T}$, we have, in HA, that $k' \Vdash_k (\mathsf{HA} + \mathsf{MP})$. The argument needed to do that closely follows the argument at the end of Subsection 4.1. So we have:

Theorem 4.15 *Consider any* ϕ *in* $\mathcal{L}_0$*. Let* U *be either* $\mathsf{HA} + \mathsf{MP}_{\mathsf{PR}}$ *or* $\mathsf{HA} + \mathsf{MP}$*. We have:* $\mathsf{HA} \vdash \Box_U\phi \to \Box_U\alpha_{\mathsf{Stable}}(\phi)$*.*

We have proved the conditions of Theorem 2.3 for the pairs $\mathsf{CF}_{\mathsf{HA}}$, Basic and $\mathsf{CF}_{\mathsf{HA}+\mathsf{MP}_{\mathsf{PR}}}$, Stable and $\mathsf{CF}_{\mathsf{HA}+\mathsf{MP}}$, Stable. This completes our characterization of the closed fragments of HA, $\mathsf{HA} + \mathsf{MP}_{\mathsf{PR}}$, and $\mathsf{HA} + \mathsf{MP}$. We have:

- $\mathsf{CF}_{\mathsf{HA}} = \mathsf{AL}_{\mathsf{Basic}}$,
- $\mathsf{CF}_{\mathsf{HA+MP}} = \mathsf{CF}_{\mathsf{HA+MP_{PR}}} = \mathsf{AL}_{\mathsf{Stable}}$.

Example 4.16 We provide an example of a theory that does not conform to our pattern. Define $T := \mathsf{HA} + \{(n \to (1 \vee \neg 1)) \mid n \in \omega\}$. Since T is between HA and PA, we find that T is a ha-theory.

Suppose that $T \vdash \Box_T(1 \vee \neg 1)$. It follows by the Π^0_2-soundness of T that $T \vdash 1 \vee \neg 1$. By a simple Kripke model argument we can show that T has the disjunction property. So $T \vdash 1$ or $T \vdash \neg 1$. Quod non.

Suppose that $T \vdash \Box_T(1 \vee \neg 1) \leftrightarrow n$. It follows that $T \vdash (n+1) \to n$, and hence $T \vdash n$. Quod non.

We may conclude that, for no β, we have $T \vdash \Box_T(1 \vee \neg 1) \leftrightarrow \beta$. Note that TDF_T is not decent, so the question remains for a decent example that does not conform to the pattern. ❑

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