

Kripke semantics for provability logic GLP

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Abstract

A well-known polymodal provability logic **GLP** is complete w.r.t. the arithmetical semantics where modalities correspond to reflection principles of restricted logical complexity in arithmetic [9, 5, 8]. This system plays an important role in some recent applications of provability algebras in proof theory [2, 3]. However, an obstacle in the study of **GLP** is that it is incomplete w.r.t. any class of Kripke frames. In this paper we provide a complete Kripke semantics for **GLP**. First, we isolate a certain subsystem **J** of **GLP** that is sound and complete w.r.t. a nice class of finite frames. Second, appropriate models for **GLP** are defined as the limits of chains of finite expansions of models for **J**. The techniques involves unions of n -elementary chains and inverse limits of Kripke models. All the results are obtained by purely modal-logical methods formalizable in elementary arithmetic.

This paper is devoted to a modal-logical study of polymodal provability logic **GLP** introduced by Giorgi Japaridze [9, 10] as early as in 1986. This logic describes in the style of provability logic all the universally valid schemata for the reflection principles of restricted logical complexity in arithmetic. Recently, important applications of **GLP** have been found in proof theory and ordinal analysis of arithmetic, which stimulated further interest towards **GLP** (see ref. [2] and ref. [3] for a more recent survey).

The modal-logical study of **GLP** was initiated by Konstantin Ignatiev [7, 8] who simplified Japaridze's arithmetical completeness theorem and established Craig's interpolation and fixed-point properties for this logic. He also gave a normal form theorem and a universal Kripke model for the closed

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fragment of **GLP**. Some of these results have been adapted by George Boolos and incorporated into his popular book on provability logic [5], where one can find a very readable exposition of (a bimodal version of) **GLP**.

Nevertheless, some natural questions about purely modal-logical properties of **GLP** have been left open after Ignatiev's work. The main difficulty in the study of **GLP** comes from the fact that it is not complete w.r.t. any class of Kripke frames. Ignatiev failed to give an adequate Kripke-style semantics for **GLP**. Moreover, his methods partially relied on arithmetical semantics for **GLP**, on the one hand, and on the use of transfinite induction up to ϵ_0 , on the other, and thus were not formalizable in Peano arithmetic. Yet, formalizability turned out to be essential for some of the above-mentioned proof-theoretic applications of **GLP** (see ref. [1] for a detailed discussion of these aspects).

This has led the authors of ref. [1] to rethink the approach to **GLP** taken by Ignatiev and Boolos and to search for alternative treatments. In ref. [1] a normal form theorem for the closed fragment of **GLP** is established by finitary methods (formalizable in a weak subsystem of Peano arithmetic). These methods are based on bisimulation arguments that allow to isolate finite n -elementary substructures in a universal model, similar to Ignatiev's one, for the closed fragment of **GLP**.

In the present paper we develop this approach further and solve the main remaining open question on the modal logic **GLP**. We give a complete Kripke-style semantics for **GLP**, which allows to establish its modal-logical properties such as decidability and Craig interpolation by finitary methods. General Kripke models for **GLP** can be presented as the limits of n -elementary chains of finite models generated by a certain 'blow-up' operation. A universal model for the closed fragment of **GLP** — isomorphic to the one introduced in ref. [1] and somewhat deviating from Ignatiev's — can be obtained in this way from the simplest linear frames.

The paper is organized as follows. First, we introduce a subsystem **J** of **GLP** which is complete w.r.t. a natural class of finite Kripke frames and provides a sufficiently good approximation to **GLP**. We prove two completeness theorems for **J**: the one for a general kind of (finite) Kripke frames, called **J**-frames, and the one for a more restricted class of nicer looking frames, called *stratified frames*. Then we introduce the 'blow-up' operations that can be applied to any finite stratified frame and yield models of arbitrarily large fragments of **GLP**. This would already be sufficient for a proof of a weak completeness result for **GLP**. However, to obtain a stronger result we need to nicely glue such models together.

To do that in a reasonable way we present two general techniques, which

— to the best of our knowledge — have not been elaborated very deeply in the modal-logical literature: inverse limits of directed families of Kripke models (connected by p-morphisms), and unions of n -elementary chains of Kripke models. The main technical result of the paper occurs in Section 7 where it is shown that the blow-up operations preserve, in some sense, n -elementary extensions of stratified models. In Section 9 we prove our main results for **GLP**. Notably, the completeness theorem for the given semantics turns out to be technically easier than the soundness one.

1 A subsystem of GLP

GLP is a propositional modal logic formulated in a language with infinitely many modalities $[0]$, $[1]$, $[2]$, etc. **GLP** is given by the following axiom schemata and rules:

- Axioms:**
- (i) Boolean tautologies;
 - (ii) $[n](\varphi \rightarrow \psi) \rightarrow ([n]\varphi \rightarrow [n]\psi)$;
 - (iii) $[n]([n]\varphi \rightarrow \varphi) \rightarrow [n]\varphi$;
 - (iv) $[m]\varphi \rightarrow [n][m]\varphi$, for $m \leq n$.
 - (v) $\langle m \rangle \varphi \rightarrow [n]\langle m \rangle \varphi$, for $m < n$.
 - (vi) $[m]\varphi \rightarrow [n]\varphi$, for $m \leq n$;

Rules: modus ponens, $\varphi \vdash [n]\varphi$.

The system given by Axioms (i)–(v) was isolated by Ignatiev; we call it **I**. Ignatiev has shown that **I** is complete w.r.t. the class of (finite) Kripke frames $(\mathcal{W}; R_0, R_1, \dots)$ satisfying the following conditions:

- R_k is a converse well-founded, transitive ordering relation on \mathcal{W} , for each $k \geq 0$;
- $\forall x, y (xR_n y \Rightarrow \forall z (xR_m z \Leftrightarrow yR_m z))$ if $m < n$. (I)

We call such frames *Ignatiev frames*, or **I**-frames. Notice that there can be no more than one arrow between any two points in an Ignatiev frame. Otherwise, one obtains a contradiction with the irreflexivity of the smallest of the two relations.

Let **J** denote the system obtained from **I** by adding the axiom schema:

- (vii) $[m]\varphi \rightarrow [m][n]\varphi$, if $m \leq n$.

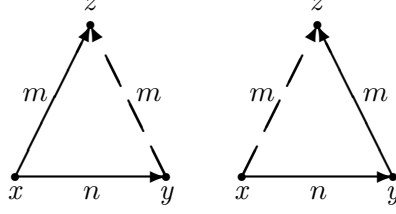


Figure 1: Frame condition (I). Dashed arrows represent the relations that must exist given the solid arrows.

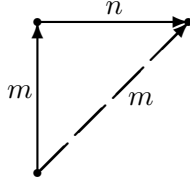


Figure 2: Frame condition (J).

Clearly, Axiom (vii) is provable in **GLP**:

$$\begin{aligned} \mathbf{GLP} \vdash [m]\varphi &\rightarrow [m][m]\varphi \\ &\rightarrow [m][n]\varphi, \quad \text{by (vi) and normality.} \end{aligned}$$

We will show that **J** is complete and enjoys finite model property w.r.t. a class of somewhat nicer frames than those for **I**.

We call a **J-frame** an Ignatiev frame satisfying

$$\bullet \forall x, y (xR_my \ \& \ yR_nz \Rightarrow xR_mz) \text{ if } m \leq n. \quad (\mathbf{J})$$

Theorem 1 ***J** is sound and complete w.r.t. (finite) **J**-frames.*

Proof. The soundness part is easy. For the completeness part, assume Δ is a set of formulas closed under subformulas, *modified negation*: $\sim\varphi := \psi$ if $\varphi = \neg\psi$, for some ψ ; $\sim\varphi := \neg\varphi$, otherwise, and the following operation:

$$[n]\varphi, [m]\psi \in \Delta \Rightarrow [m]\varphi \in \Delta.$$

We call such a set Δ *adequate*.

Let $\ell(\Delta) = \{n \in \omega : [n]\varphi \in \Delta \text{ for some } \varphi\}$. Clearly, every finite set of formulas Γ can be extended to a finite adequate set $\Delta \supseteq \Gamma$ such that $\ell(\Delta) = \ell(\Gamma)$.

Let us fix some finite adequate Δ . Below we shall assume that all the modalities range within $\ell(\Delta)$. We consider the following filtrated canonical model structure.

$$\mathcal{W} := \{x : x \text{ is a maximal } \mathbf{J}\text{-consistent set of formulas from } \Delta\}.$$

For any $x, y \in \mathcal{W}$ we let xR_ny if the following conditions hold:

1. For any $[n]\varphi \in \Delta$ such that $[n]\varphi \in x$,

$$\varphi \in y \text{ \& } \forall k \geq n (k \in \ell(\Delta) \Rightarrow [k]\varphi \in y);$$

2. For any $[m]\varphi \in \Delta$ such that $m < n$, $[m]\varphi \in x \Leftrightarrow [m]\varphi \in y$;
3. There is a $[n]\varphi \in \Delta$ such that $[n]\varphi \in y$ and $[n]\varphi \notin x$.

Lemma 1.1 *\mathcal{W} is a \mathbf{J} -frame.*

Proof. Condition 3 guarantees the irreflexivity of the relations R_n .

Assume xR_ny , xR_mz and $m < n$; we prove yR_mz . Indeed, if $[m]\varphi \in y$ then $[m]\varphi \in x$, since $m < n$. Hence $\varphi, [k]\varphi \in z$, for $k \geq m$. If $k < m$ then $[k]\varphi \in y \Leftrightarrow [k]\varphi \in x \Leftrightarrow [k]\varphi \in z$. Finally, we have $[m]\psi \in z$, $[m]\psi \notin x$, for some ψ . Hence, $[m]\psi \notin y$ because $m < n$.

Assume xR_ny , yR_mz and $m \leq n$; we prove xR_mz . Indeed, if $[m]\varphi \in x$ then $[m]\varphi \in y$, since $m \leq n$. Hence $\varphi, [k]\varphi \in z$, for $k \geq m$. If $k < m$ then $[k]\varphi \in x \Leftrightarrow [k]\varphi \in y \Leftrightarrow [k]\varphi \in z$. Finally, we have $[m]\psi \in z$, $[m]\psi \notin y$, for some ψ . Hence, $[m]\psi \notin x$ because $m \leq n$.

Assume xR_my , yR_nz and $m \leq n$; we prove xR_mz . If $m \leq k \leq n$, then $[m]\varphi \in x$ implies $[k]\varphi \in y$ and $[k]\varphi \in z$. If $k \geq n$, then $[m]\varphi \in x$ implies $[n]\varphi \in y$ and $\varphi, [k]\varphi \in z$. Finally, since xR_my , there is a ψ such that $[m]\psi \in y$, $[m]\psi \notin x$. Since $m < n$ we also have $[m]\psi \in z$, and we are done.

□

We define the evaluation of propositional variables on \mathcal{W} by letting $\mathcal{W}, x \Vdash p \iff p \in x$.

Lemma 1.2 *For any $\varphi \in \Delta$, $\mathcal{W}, x \Vdash \varphi \iff \varphi \in x$.*

Proof. This is completely standard by induction on the length of φ . We only treat the crucial case that $\varphi = [m]\varphi_0$. If $[m]\varphi_0 \in x$ and xR_my , then $\varphi_0 \in y$, by the definition of R_m on \mathcal{W} . Hence, by the induction hypothesis, $\mathcal{W}, y \Vdash \varphi_0$. Since this holds for any y , we have $\mathcal{W}, x \Vdash [m]\varphi_0$.

Assume $[m]\varphi_0 \notin x$. Since x is maximal **J**-consistent, $\neg[m]\varphi_0 \in x$. Let Φ be the union of the following sets of formulas:

1. $\Phi_1 := \{[n]\psi, \psi : [m]\psi \in x, n \geq m\};$
2. $\Phi_2 := \{[k]\psi : [k]\psi \in x, k < m\};$
3. $\Phi_3 := \{\neg[k]\psi : \neg[k]\psi \in x, k < m\};$
4. $\Phi_4 := \{[m]\varphi_0, \sim\varphi_0\}.$

We show that Φ is **J**-consistent. Assume otherwise, then, identifying sets Φ_i with the conjunctions of their elements,

$$\mathbf{J} \vdash \Phi_1 \wedge \Phi_2 \wedge \Phi_3 \rightarrow ([m]\varphi_0 \rightarrow \varphi_0).$$

Hence, by Löb's axiom,

$$\begin{aligned} \mathbf{J} \vdash [m]\Phi_1 \wedge [m]\Phi_2 \wedge [m]\Phi_3 &\rightarrow [m]([m]\varphi_0 \rightarrow \varphi_0) \\ &\rightarrow [m]\varphi_0. \end{aligned}$$

On the other hand, we notice that for $n \geq m$, by Axiom (vii),

$$\mathbf{J} \vdash [m]\psi \rightarrow [m]([n]\psi \wedge \psi),$$

for any formula ψ , whence $\mathbf{J} \vdash \Phi_1 \rightarrow [m]\Phi_1$. Secondly, if $k < m$, by Axioms (iv) and (v) we have, respectively,

$$\mathbf{J} \vdash [k]\psi \rightarrow [m][k]\psi, \tag{1}$$

$$\mathbf{J} \vdash \neg[k]\psi \rightarrow [m]\neg[k]\psi. \tag{2}$$

Hence, $\mathbf{J} \vdash \Phi_2 \rightarrow [m]\Phi_2$ and $\mathbf{J} \vdash \Phi_3 \rightarrow [m]\Phi_3$. Therefore,

$$\mathbf{J} \vdash \Phi_1 \wedge \Phi_2 \wedge \Phi_3 \rightarrow [m]\varphi_0,$$

which implies that x is **J**-inconsistent, quod non.

Thus, Φ is consistent and we can find a maximal consistent set $y \supseteq \Phi$. Then, $\mathcal{W}, y \not\Vdash \varphi_0$, by the induction hypothesis, and xR_my , by the definition of R_m . Hence, $\mathcal{W}, x \not\Vdash [m]\varphi_0$. \square

From the previous lemma we obtain a proof of Theorem 1 in a standard way. Assume $\mathbf{J} \not\models \varphi$. Consider a finite adequate set Δ containing φ and the corresponding model \mathcal{W} . Let x be any maximal \mathbf{J} -consistent set of formulas from Δ containing $\sim\varphi$. Then $\mathcal{W}, x \not\models \varphi$, by Lemma 1.2. \square

Remark. In a subsequent paper we shall establish Craig interpolation and fixed point properties for \mathbf{J} and for **GLP** using a modification of this proof.

Let us visualize the structure of a \mathbf{J} -model \mathcal{W} . Let \bar{R}_m denote the reflexive, transitive closure of the relation $R_m \cup R_{m+1} \cup \dots$, and let E_m denote the symmetric, transitive, reflexive closure of the same relation. E_m -equivalence classes will be called *m-planes*. We have the following simple properties:

- Each m -plane is partitioned into $m+1$ -planes, since E_{m+1} refines E_m .
- All points in an $m+1$ -plane are R_m -incomparable, in fact, R_n -incomparable for any $n \leq m$.

Assume $x_1 S_{m_1} x_2 S_{m_2} \dots S_{m_k} x_{k+1}$ with all $m_i > m$, where S_j denotes either R_j or the inverse relation R_j^{-1} . If $x_1 R_n x_{k+1}$, then using property (I) one successively obtains $x_2 R_n x_{k+1}$, $x_3 R_n x_{k+1}$, \dots , $x_{k+1} R_n x_{k+1}$, which contradicts the irreflexivity of R_n .

- There is an ordering relation R_m between $m+1$ -planes defined by $\alpha R_m \beta$ if $\exists x \in \alpha \exists y \in \beta x R_m y$. We have, by (I):

$$\alpha R_m \beta \iff \exists y \in \beta \forall x \in \alpha x R_m y.$$

- Assume α and β are $m+1$ -planes and $\alpha R_m \beta$. Let β_α denote the set $\{y \in \beta : \exists x \in \alpha x R_m y\}$. Then β_α is upwards closed w.r.t. \bar{R}_{m+1} , by property (J).

2 P-morphisms and limits

An m -plane of a \mathbf{J} -model \mathcal{A} can be considered as a model in the restricted signature R_m, R_{m+1} , etc. Kripke models in this signature will be called *m-models*. Most of the content of this section works for general Kripke models.

To every 0-model there corresponds a k -model obtained by renaming every R_i by R_{i+k} , for each i . We call this transformation *k-lifting*. The

opposite transformation is called *k-lowering*. All the notions defined below for 0-models can be obviously lifted to *k*-models. We shall often use this fact without mention.

Any $k + 1$ -model \mathcal{A} also gives rise to a *k*-model $\mathcal{B} = (\mathcal{A}; R_k)$ with R_k empty. We denote such a \mathcal{B} by $\{\mathcal{A}\}$. Notice that this operation is quite different from lowering.

Morphisms. An *embedding* of 0-models is an injective function $f : \mathcal{A} \rightarrow \mathcal{B}$ preserving the evaluation of variables and such that, for all $x, y \in \mathcal{A}$ and any k , $xR_k y$ if and only if $f(x)R_k f(y)$. Obviously, in this case \mathcal{A} can be identified with a *submodel* of \mathcal{B} , that is, a subset of \mathcal{B} together with all the inherited relations and the evaluation of variables. A submodel $\mathcal{A} \subseteq \mathcal{B}$ is called *upwards closed* if, for all $x \in \mathcal{A}$, $y \in \mathcal{B}$, and $k \geq 0$, $xR_k y$ implies $y \in \mathcal{A}$.

A *p-morphism* $f : \mathcal{A} \rightarrow \mathcal{B}$ is a function satisfying the following requirements for any k :

- (i) For all $a, b \in \mathcal{A}$, $aR_k b$ implies $f(a)R_k f(b)$.
- (ii) For all $a \in \mathcal{A}$ and $b \in \mathcal{B}$, if $f(a)R_k b$ then there is $a' \in \mathcal{A}$ such that $f(a') = b$.
- (iii) For any variable p , $\mathcal{A}, a \Vdash p$ if and only if $\mathcal{B}, f(a) \Vdash p$.

It is easy to check that p-morphisms are closed under composition and that a p-morphic image of a **J**-model is a **J**-model.

If a p-morphism $f : \mathcal{A} \rightarrow \mathcal{B}$ is injective, it happens to be an embedding. Such embeddings will be called *end-embeddings*. In this case \mathcal{A} can be identified with an upwards closed submodel of \mathcal{B} . The main property of p-morphisms is formulated in the following standard lemma (see [4, 6]).

Lemma 2.1 *Suppose $f : \mathcal{A} \rightarrow \mathcal{B}$ is a p-morphism. Then, for each $x \in \mathcal{A}$ and any formula φ ,*

$$\mathcal{A}, x \Vdash \varphi \iff \mathcal{B}, f(x) \Vdash \varphi.$$

Inverse limits. Assume that $(I, <)$ is a directed partial ordering, that is, an ordering satisfying

$$\forall \alpha, \beta \in I \exists \gamma \in I (\alpha \preceq \gamma \text{ and } \beta \preceq \gamma).$$

Let $(\mathcal{A}_\alpha)_{\alpha \in I}$ be a family of 0-models such that for each pair $\alpha, \beta \in I$ with $\alpha < \beta$ there is an p-morphism $f_{\alpha\beta} : \mathcal{A}_\beta \rightarrow \mathcal{A}_\alpha$. We require that these

p-morphisms satisfy the conditions $f_{\alpha\beta} \circ f_{\beta\gamma} = f_{\alpha\gamma}$, if $\alpha \prec \beta \prec \gamma$, and let $f_{\alpha\alpha} = \text{id}_{\mathcal{A}_\alpha}$.

The *inverse limit* $\lim_{\alpha \in I} \mathcal{A}_\alpha$ is the subset of the direct product $\prod_{\alpha \in I} \mathcal{A}_\alpha$ consisting of those elements $x = (x_\alpha)_{\alpha \in I}$ such that

$$\forall \alpha, \beta \in I (\alpha \prec \beta \Rightarrow x_\alpha = f_{\alpha\beta}(x_\beta)).$$

The relations R_k on $\lim_{\alpha \in I} \mathcal{A}_\alpha$ are defined by

$$x R_k y \iff \forall \alpha \in I x_\alpha R_k y_\alpha \text{ in } \mathcal{A}_\alpha.$$

We also define: $\lim_{\alpha \in I} \mathcal{A}_\alpha, x \Vdash p$ iff $\forall \alpha \in I \mathcal{A}_\alpha, x_\alpha \Vdash p$.

Let $f_\alpha : \lim_{\alpha \in I} \mathcal{A}_\alpha \rightarrow \mathcal{A}_\alpha$ denote the canonical mapping $x \mapsto x_\alpha$. From now on we shall only consider *countable* I .

Lemma 2.2 *For each $\alpha \in I$, f_α is a p-morphism.*

Proof. We check the three conditions of p-morphisms.

(i) If $x R_k y$, then clearly $f_\alpha(x) R_k f_\alpha(y)$.

(iii) If $x \in \lim_{\alpha \in I} \mathcal{A}_\alpha$ then, for all $\alpha, \beta \in I$, $\mathcal{A}_\alpha, x_\alpha \Vdash p$ iff $\mathcal{A}_\beta, x_\beta \Vdash p$, for any variable p . Indeed, if $\alpha, \beta \preceq \gamma$, then $\mathcal{A}_\alpha, x_\alpha \Vdash p$ iff $\mathcal{A}_\gamma, x_\gamma \Vdash p$, because $f_{\alpha\gamma}$ is a p-morphism and $x_\alpha = f_{\alpha\gamma}(x_\gamma)$, and similarly for x_β . Thus, f_α preserves the evaluation of variables.

(ii) Assume $x \in \lim_{\alpha \in I} \mathcal{A}_\alpha$ and $x_\alpha R_k w$ in \mathcal{A}_α . We need to construct a $y \in \lim_{\alpha \in I} \mathcal{A}_\alpha$ such that $y_\alpha = w$. Let us enumerate all $\beta \in I$ in a sequence $\beta_0 = \alpha, \beta_1, \beta_2, \dots$. For each i we construct a $y_i \in \mathcal{A}_{\beta_i}$ such that $x_{\beta_i} R_k y_i$ and $y_i = f_{\beta_i \beta_j}(y_j)$ whenever $\beta_i \prec \beta_j$. Suppose J is a finite subset of I such that y_j , for all $\beta_j \in J$, are already constructed. (Initially, we put $J = \{\beta_0\}$ and $y_0 := w$.) We inductively assume that J has a supremum β_s .

Let i be the least such that $\beta_i \notin J$. If $\beta_i \prec \beta_s$ let $y_i := f_{\beta_i \beta_s}(y_s)$. Otherwise, take the first β_m such that $\beta_i \prec \beta_m$ and $\beta_s \prec \beta_m$. Using Condition (ii) for the p-morphism $f_{\beta_s \beta_m}$ find a $y_m \in \mathcal{A}_{\beta_m}$ such that $x_{\beta_m} R_k y_m$ and $f_{\beta_s \beta_m}(y_m) = y_s$. Add β_i and β_m to J and let $y_i := f_{\beta_i \beta_m}(y_m)$. Thus, β_m will be the new supremum of J .

Obviously, the increasing sequence of finite sets J exhausts I . Hence, $(y_i)_{i \geq 0}$ defines an element y of $\lim_{\alpha \in I} \mathcal{A}_\alpha$ such that $f_\alpha(y) = y_0 = w$. It is also clear that $x R_k y$, since we have $f_\beta(x) R_k f_\beta(y)$ for all $\beta \in I$. \square

Remark. It should be noted that the category of Kripke models and p-morphisms is not closed under limits. Our inverse limit construction does not, in general, satisfy the universal property of limits. We can only state the following weaker lemma.

Lemma 2.3 *Suppose \mathcal{B} is a set such that for each $\alpha \in I$ there is a mapping $g_\alpha : \mathcal{B} \rightarrow \mathcal{A}_\alpha$ such that, for all $\alpha \prec \beta$, $f_{\alpha\beta} \circ g_\beta = g_\alpha$. Then there is a unique mapping $g : \mathcal{B} \rightarrow \lim_{\alpha \in I} \mathcal{A}_\alpha$ such that, for all $\alpha \in I$, $g_\alpha = f_\alpha \circ g$.*

Proof. We define $g(x) := (g_\alpha(x))_{\alpha \in I}$ and the uniqueness part is also clear. However, it is in general not the case that f is a p-morphism if \mathcal{B} is a Kripke model and all f_α are p-morphisms, because of the failure of Condition (ii) for f . \square

In some sense, it would be natural to deal with a modified category closed under limits. I believe this can be achieved by switching to topological Kripke models. However, in this paper we are interested in the underlying finite combinatorial structures and the minimal requirements to obtain an effective completeness result for **GLP**. So, we would like to postpone the development of a more general theory to a further paper. What we need from inverse limits is stated in Lemma 2.2 and the following obvious lemma.

Lemma 2.4 *Suppose \mathcal{A}_α is a **J**-model, for all $\alpha \in I$. Then $\lim_{\alpha \in I} \mathcal{A}_\alpha$ is a **J**-model.*

Proof. It is easy to check that R_k is a transitive irreflexive relation and to verify Conditions (I) and (J) for $\lim_{\alpha \in I} \mathcal{A}_\alpha$. Let us check (J), the other conditions are checked similarly.

Assume xR_my and yR_nz in $\lim_{\alpha \in I} \mathcal{A}_\alpha$, where $m \leq n$. Then, for each $\alpha \in I$, $x_\alpha R_m y_\alpha$ and $y_\alpha R_n z_\alpha$. Hence, $x_\alpha R_m z_\alpha$, for each α , that is, $xR_m z$.

Similarly, an infinite chain $x^1 R_k x^2 R_k x^3 R_k \dots$ induces an infinite chain $x_\alpha^1 R_k x_\alpha^2 R_k \dots$ in each \mathcal{A}_α . Hence, $\lim_{\alpha \in I} \mathcal{A}_\alpha$ is conversely well-founded. \square

3 Stratified models

In this section we improve upon Theorem 1. Let us call a *stratified frame* a **J**-frame \mathcal{W} satisfying the following additional condition:

$$\forall x, y, z (zR_mx \ \& \ yR_nx \Rightarrow zR_my) \quad \text{if } m < n. \quad (S)$$

Hence, in a stratified frame, for any $m+1$ -planes α, β such that $\alpha R_m \beta$, any point of β is R_m -accessible from any point of α ; in other words $\beta_\alpha = \beta$. Thus, the R_0 -ordering on a stratified frame is completely determined by the R_0 -ordering of its 1-planes, R_1 is determined by the R_1 -ordering of its

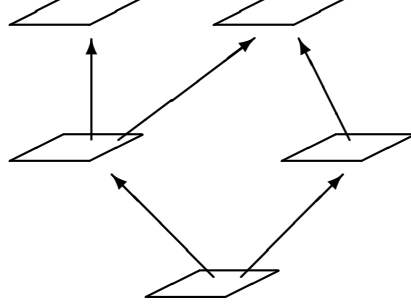


Figure 3: A stratified model viewed as a three-dimensional structure. The horizontal squares are the 1-planes; the arrows represent R_0 -relations between them (transitivity is assumed). Each 1-plane has a similar structure w.r.t. R_1 , R_2 , etc.

2-planes on each 1-plane, etc. Hence, we can think of stratified frames as *hereditary partial orderings* (see Fig. 3).¹

A stratified m -model is *trivial* if all of its relations are empty. To a finite stratified m -model \mathcal{A} we can associate a number $rk_m(\mathcal{A})$ called *rank* of \mathcal{A} . We define $rk_m(\mathcal{A}) = 0$, if \mathcal{A} is trivial; otherwise,

$$rk_m(\mathcal{A}) := \max_{\alpha \in \mathcal{A}} rk_{m+1}(\alpha) + 1,$$

where α runs through all $m + 1$ -planes of \mathcal{A} . Essentially, rk measures the maximal depth of nestings of planes in \mathcal{A} . We shall often use induction on the rank of a stratified model.

We shall prove that **J** is complete w.r.t. finite stratified models. This is a corollary of the following proposition.

Proof. To every finite **J**-model \mathcal{W} there is a finite stratified model \mathcal{W}^* and a surjective p-morphism $f : \mathcal{W}^* \rightarrow \mathcal{W}$. \square

Proof. We argue by induction on the rank of \mathcal{W} . Notice that a model \mathcal{W} is stratified iff every 1-plane of \mathcal{W} is stratified and condition (S) holds for $m = 0$. In the latter case we say that \mathcal{W} is 0-stratified. Given a model

¹A singleton is a hereditary partial ordering. Any hereditary partial ordering is a partial ordering whose elements are (previously constructed and lifted by 1) hereditary partial orderings. There is an obvious analogy between this notion and that of hereditary finite set.

\mathcal{W} we first construct a 0-stratified model \mathcal{W}_0 and a surjective p-morphism $f_0 : \mathcal{W}_0 \rightarrow \mathcal{W}$.

We define the following *resolution* operation on a finite **J**-model \mathcal{W} . Let α, β be 1-planes such that β is an immediate R_0 -successor of α . Add to \mathcal{W} a new 1-plane γ isomorphic to β_α (with all the inherited relations R_i and the same evaluation of variables). For any $x \in \gamma$ let x' denote the corresponding element of β . Define an ordering R'_0 on the extended model $\mathcal{W}' = \mathcal{W} \sqcup \gamma$ as follows:

1. $\forall x \in \gamma \forall y \in \mathcal{W} (xR'_0y \iff x'R_0y)$;
2. $\forall x \in \mathcal{W} \forall y \in \gamma (xR'_0y \iff x \in \alpha \vee \exists z \in \alpha xR_0z)$;
3. $\forall x \in \alpha \forall y \in \beta \neg xR'_0y$.

If none of these cases holds, $x, y \in \mathcal{W}$ and we define xR'_0y iff xR_0y .

It is easy to check that R'_0 is transitive and irreflexive and that \mathcal{W}' is a **J**-frame (since it also has a ‘plane structure’). We say that \mathcal{W}' is obtained from \mathcal{W} by resolving the 1-plain β over α .

Consider a function $g : \mathcal{W}' \rightarrow \mathcal{W}$ such that $g(x) := x'$, if $x \in \gamma$, and $g(x) := x$, otherwise. Define the evaluation of variables on \mathcal{W}' by

$$\mathcal{W}', x \Vdash p \iff \mathcal{W}, g(x) \Vdash p.$$

Lemma 3.1 *g is a surjective p-morphism.*

Proof. Conditions (i) and (iii) are obvious. We check Condition (ii). Assume $g(a)R_kb$. If $b \notin \beta$ we have $g^{-1}(b) = \{b\}$ and aR_kb in \mathcal{W}' . If $b \in \beta$ there are two cases: $k = 0$ or $k > 0$. In the first case $g(a) = a$. We also have aR_0b unless $a \in \alpha$. But then $b \in \beta_\alpha$ and hence there is a $c \in \gamma$ such that $g(c) = b$.

In the second case $g(a) \in \beta$. Either $a = g(a)$ and aR_kb , or $a \in \gamma$. In the latter case, since β_α is upwards closed w.r.t. \bar{R}_1 , $b \in \beta_\alpha$. Hence, there is a $c \in \gamma$ such that $g(c) = b$. \square

We call a pair of 1-planes $\langle \alpha, \beta \rangle$ *bad neighbors* if β is an immediate R_0 -successor of α and $\beta_\alpha \neq \beta$. We construct a 0-stratified model \mathcal{W}_0 and a surjective p-morphism $f_0 : \mathcal{W}_0 \rightarrow \mathcal{W}$ by induction on the number of bad neighbors in \mathcal{W} .

If this number is 0, then \mathcal{W} is 0-stratified. Otherwise, consider a pair of bad neighbors $\langle \alpha, \beta \rangle$ with an R_0 -maximal β . Let \mathcal{W}' be obtained by resolving β over α . In \mathcal{W}' , α and β are not connected, and the new plane γ does not add any bad neighbors: by Condition 2, α is the only immediate

predecessor of γ , but $\langle \alpha, \gamma \rangle$ are good neighbors. Since β was maximal, bad neighbors of the form $\langle \gamma, \delta \rangle$ are impossible. So, \mathcal{W}' has one pair of bad neighbors less than \mathcal{W} . By the induction hypothesis, there is a surjective p-morphism $f' : \mathcal{W}_0 \rightarrow \mathcal{W}'$. Composing it with the p-morphism $g : \mathcal{W}' \rightarrow \mathcal{W}$ yields the result.

Notice that $rk_0(\mathcal{W}_0) = rk_0(\mathcal{W})$. Let $(\alpha_i)_{i \in I}$ be the family of all the 1-planes in \mathcal{W}_0 . Since $rk_1(\alpha_i) < rk_0(\mathcal{W})$, by the induction hypothesis there is a stratified 1-model α_i^* and a surjective p-morphism $f_i : \alpha_i^* \rightarrow \alpha_i$, for each $i \in I$. \mathcal{W}^* will be obtained by replacing in \mathcal{W}_0 each α_i by α_i^* and retaining the R_0 -order between these 1-planes. We then define $f'' : \mathcal{W}^* \rightarrow \mathcal{W}_0$ by letting $f''(x) := f_i(x)$ if $x \in \alpha_i$. It is easy to see that f'' is a surjective p-morphism. Hence, the composition of f'' and f_0 yields the required p-morphism f . \square

As a corollary we obtain

Theorem 2 $\mathbf{J} \vdash \varphi \iff \mathcal{W} \models \varphi$, for all (finite) stratified models \mathcal{W} .

In what follows we will need a slight modification of this corollary. Let \mathcal{A} be a stratified model. The *root* of \mathcal{A} is the 1-plane α such that $\alpha R_0 \beta$ for any other 1-plane β in \mathcal{A} . \mathcal{A} is called *rooted* if such a root exists. \mathcal{A} is called *hereditarily rooted* if \mathcal{A} is rooted and each 1-plane α in \mathcal{A} is hereditarily rooted.

A hereditarily rooted model \mathcal{A} has a distinguished element, called its *hereditary root*, inductively defined as follows. If \mathcal{A} is trivial, then $\mathcal{A} = \{a\}$ and a is the hereditary root. Otherwise, the hereditary root of \mathcal{A} coincides with the hereditary root of α , where α is the root of \mathcal{A} .

Corollary 3.2 *If $\mathbf{J} \not\vdash \varphi$, then there is a hereditarily rooted stratified model \mathcal{A} such that $\mathcal{A}, a \not\models \varphi$, where a is the hereditary root of \mathcal{A} .*

Proof (sketch). The construction is a variant of a very standard one, so we only sketch a proof. We start with a finite stratified model \mathcal{W} where φ is false and construct a model \mathcal{W}' in which every m -plane is rooted and a surjective p-morphism $f : \mathcal{W}' \rightarrow \mathcal{W}$. This is done by induction on the rank of \mathcal{W} . An m -plane α with k R_m -minimal $m+1$ -planes β_1, \dots, β_k can be replaced, modulo a p-morphism, by a disjoint union of k rooted $m+1$ -planes isomorphic to the submodels of α of the form $\beta_i \cup \{x \in \alpha : \beta_i R_m x\}$. This can be done recursively as in the proof of Theorem 3.

Having constructed \mathcal{W}' we obtain a node $a \in \mathcal{W}'$ such that $\mathcal{W}', a \not\models \varphi$. Now let \mathcal{A} be the submodel of \mathcal{W}' generated by a , that is,

$$\mathcal{A} := \{a\} \cup \{x \in \mathcal{W}' : \exists k \ a R_k x\}.$$

It is easy to see that \mathcal{A} is an upwards closed submodel of \mathcal{W} , hence $\mathcal{A}, a \not\models \varphi$. Also, by induction on rank we show that every m -plane generated by a is hereditarily rooted and a is its hereditary root. All the other planes in \mathcal{A} are the same as in \mathcal{W} , hence they are hereditarily rooted, as well. \boxtimes

Next, we make a few observations about stratified models and p-morphisms. Notice that by Lemma 3 a p-morphic image of a stratified model need not be stratified. P-morphisms of stratified models respect the plane structure only in the following weak sense.

Lemma 3.3 *Suppose \mathcal{A}, \mathcal{B} are stratified models and $f : \mathcal{A} \rightarrow \mathcal{B}$ is a p-morphism. Then, for each m -plane α in \mathcal{A} there is an m -plane β in \mathcal{B} such that $f(\alpha) \subseteq \beta$ and $f \upharpoonright \alpha : \alpha \rightarrow \beta$ is a p-morphism of m -models.*

Proof. If $x_0 S x_1 S \cdots S x_n$ in \mathcal{B} , where S denotes R_k or R_k^{-1} for $k \geq m$, then using the monotonicity of f we successively prove $f(x_0) E_m f(x_i)$ in \mathcal{B} , for all $i \leq n$. Hence, $f(x_0)$ and $f(x_n)$ belong to the same m -plane. P-morphism properties of $f \upharpoonright \alpha$ are clear. \boxtimes

In the following, it will be technically convenient to deal with a special kind of p-morphisms and end-embeddings preserving all the 1-planes in stratified 0-models. We call a p-morphism of stratified 0-models $f : \mathcal{A} \rightarrow \mathcal{B}$ *special* if, for each 1-plane α of \mathcal{A} , $f(\alpha)$ is a 1-plane of \mathcal{B} and f is an isomorphism between 1-models α and $f(\alpha)$. A p-morphism f is called *full* if for each 1-plane α of \mathcal{A} , $f(\alpha)$ is a 1-plane of \mathcal{B} , $f \upharpoonright \alpha$ is full as a p-morphism of 1-models, and every 1-plane β of \mathcal{B} equals $f(\alpha)$, for some α .

Obviously, a full p-morphism is surjective and a surjective special p-morphism is full. It is also easy to check that an inverse limit of a family of stratified models is stratified.

4 Bisimulations

We shall use the standard notion of n -bisimilarity (see e.g. [4]). Let a Kripke model \mathcal{A} (in the language of **GLP**) be given. We define n -bisimilarity equivalence relations \sim_n on \mathcal{A} , for each $n \geq 0$, by induction on n .

Definition 4.1 *Let $x, x' \in \mathcal{A}$.*

- $x \sim_0 x'$ if x and x' force the same variables.
- $x \sim_{n+1} x'$ if

- (i) $x \sim_n x'$;
- (ii) $\forall k \geq 0 \forall y (xR_k y \Rightarrow \exists y' (x'R_k y' \& y \sim_n y'))$;
- (iii) $\forall k \geq 0 \forall y' (x'R_k y' \Rightarrow \exists y (xR_k y \& y' \sim_n y))$.

Let $dp(\varphi)$ denote the modality depth of φ , that is, the maximal number of nested modalities in φ . The following lemma is standard.

Lemma 4.2 *For any $x, y \in \mathcal{A}$, if $x \sim_n y$ then $x \Vdash \varphi$ iff $y \Vdash \varphi$, for every φ with $dp(\varphi) \leq n$.*

Proof. By an easy induction on n with a subsidiary induction on the length of φ . \square

P-morphisms and inverse limits preserve n -bisimilarity equivalence relations, for any n .

Lemma 4.3 *Let $f : \mathcal{A} \rightarrow \mathcal{B}$ be a p -morphism. Then $x \sim_n y$ in \mathcal{A} if and only if $f(x) \sim_n f(y)$ in \mathcal{B} .*

Proof. An easy induction on n . \square

Corollary 4.4 *Suppose $\mathcal{A} = \lim_{\alpha \in I} \mathcal{A}_\alpha$ and $\alpha \in I$. Then $x \sim_n y$ in \mathcal{A} iff $x_\alpha \sim_n y_\alpha$ in \mathcal{A}_α .*

We are interested in the question when the n -bisimilarity relation on a submodel $\mathcal{B} \subseteq \mathcal{A}$ coincides with that on \mathcal{A} (restricted to \mathcal{B}). In the following lemmata we list a few simple situations when this is the case.

Corollary 4.5 *If \mathcal{B} is an upwards closed submodel of \mathcal{A} , then $x \sim_n y$ in \mathcal{B} iff $x \sim_n y$ in \mathcal{A} .*

A slight generalization is as follows.

Lemma 4.6 *Let $\mathcal{B} \subseteq \mathcal{A}$ be a submodel such that for all $k \geq 0$*

$$\forall x, y \in \mathcal{B} \forall z \in \mathcal{A} \setminus \mathcal{B} (xR_k z \Rightarrow yR_k z).$$

Then $x \sim_n y$ in \mathcal{B} iff $x \sim_n y$ in \mathcal{A} .

Lemma 4.7 *Let \mathcal{A} be a stratified model and let α be one of its m -planes considered as an m -model. Then $x \sim_n y$ in α iff $x \sim_n y$ in \mathcal{A} .*

Proof. We apply induction on $m \geq 0$ with a subsidiary induction on n . Basis of induction is trivial, since one can consider \mathcal{A} itself as its unique 0-plane. To prove the statement for an $m + 1$ -plane α let β be the unique m -plane containing α . Assume $x \sim_n y$ in α and show by an easy induction on n that $x \sim_n y$ in β . Conclude that $x \sim_n y$ in \mathcal{A} by the induction hypothesis for m . \square

Now we formulate a property of stratified models which will allow us to build finite partial models of **GLP**. General models for **GLP** will be defined as the limits of such structures.

Definition 4.8 A model \mathcal{A} satisfies *n-similarity property* for R_m if

$$\forall x, y \in \mathcal{A} (xR_{m+1}y \Rightarrow \exists y' (xR_my' \ \& \ y \sim_n y')).$$

\mathcal{A} satisfies *n-similarity property* if it does so for each R_m , $m \geq 0$.

Lemma 4.9 Assume a stratified model \mathcal{A} satisfies *n-similarity property* for R_m . Then

$$\mathcal{A} \models [m]\varphi \rightarrow [m+1]\varphi,$$

for each φ with $dp(\varphi) \leq n$.

Proof. Assume $x \not\models [m+1]\varphi$, then there is a $y \in \mathcal{A}$ such that $xR_{m+1}y$ and $y \not\models \varphi$. Pick a y' such that xR_my' and $y' \sim_n y$. By Lemma 4.2, $y' \not\models \varphi$, hence $x \not\models [m]\varphi$. \square

We notice that if \mathcal{A} satisfies the *n-similarity property* for R_k then so does any upwards closed submodel $\mathcal{B} \subseteq \mathcal{A}$. Indeed, if $x \in \mathcal{B}$ and $xR_{k+1}y$, xR_ky' and $y \sim_n y'$ in \mathcal{A} , then both $y, y' \in \mathcal{B}$ and $y \sim_n y'$ holds in \mathcal{B} by Lemma 4.5.

5 Blowing up stratified models

A finite stratified model is never a model for **GLP**, unless all relations R_k with $k > 0$ are empty. Here we describe a ‘blow-up’ operation that transforms a finite stratified model into a model satisfying *m-similarity property*.

First, we are going to define two auxiliary operations on (stratified) models.

Definition 5.1 Let $(\mathcal{A}_i)_{i \in I}$ be a family of 0-models. $\coprod_{i \in I} \mathcal{A}_i$ denotes the *disjoint union* of all these models (all the relations and the evaluation of variables are inherited from \mathcal{A}_i , for $i \in I$).

Next we define the operation of *ordered sum* of (stratified) models.

Definition 5.2 Let (I, R) be a conversely well-founded partial ordering. Suppose we have a function associating with each $i \in I$ a 0-model \mathcal{A}_i . Let $\sum_{i \in I} \mathcal{A}_i$ denote the disjoint union of all the universes of these models $\bigsqcup_{i \in I} \mathcal{A}_i$ R_0 -ordered by the union of the orderings R_0 on each \mathcal{A}_i and the relations xR_0y such that $x \in \mathcal{A}_i$, $y \in \mathcal{A}_j$ and iRj in I . The relations R_k , for any $k > 0$, and the evaluation of variables are inherited from all the models \mathcal{A}_i . This makes $\sum_{i \in I} \mathcal{A}_i$ a stratified model, provided all the models \mathcal{A}_i are stratified.

$\sum_{i < m} \mathcal{A}_i$, for $m \leq \omega$, will denote the model $\sum_{i \in I} \mathcal{A}_i$, where (I, R) is the ordering $(\{0, \dots, m-1\}, >)$ or $(\omega, >)$ for $m = \omega$. (Thus, \mathcal{A}_0 occupies the highest position in this ordering.)

Obviously, each \mathcal{A}_i is embedded into $\sum_{i \in I} \mathcal{A}_i$. In general, this embedding is not an end-embedding. However, the embedding of \mathcal{A}_0 into $\sum_{i < m} \mathcal{A}_i$ is.

Notice that any stratified 0-model can be viewed as the ordered sum of all its 1-planes α :

$$\mathcal{A} = \sum_{\alpha \in \mathcal{A}} \{\alpha\}.$$

We will often use the following two simple facts.

- If $x, y \in \mathcal{A}_i$, then $x \sim_n y$ in \mathcal{A}_i iff $x \sim_n y$ in $\sum_{i \in I} \mathcal{A}_i$. This follows from Lemma 4.6.
- If, for each $i \in I$, there is a surjective p-morphism $f_i : \mathcal{B}_i \rightarrow \mathcal{A}_i$, then there is a (unique) surjective p-morphism $f : \sum_{i \in I} \mathcal{B}_i \rightarrow \sum_{i \in I} \mathcal{A}_i$ that coincides with f_i on each \mathcal{B}_i .

Notice that the statement is, in general, false for non-surjective p-morphisms. Actually, we have already used this fact in the proof of Theorem 3.

Next, we define the operation of *m-blowup* which can be applied to a finite rooted stratified 1-model \mathcal{A} and transforms it into a stratified 0-model $\mathcal{A}^{(m)}$. Informally speaking, $\mathcal{A}^{(m)}$ is obtained from \mathcal{A} by putting R_0 -above \mathcal{A} a few series of m copies of certain parts of \mathcal{A} , the series being linearly ordered by R_0 , and repeating this operation for each of these copies as far as it goes.

We consider \mathcal{A} as a finite strict partial ordering R_1 of its 2-planes. For every 2-plane α in \mathcal{A} let \mathcal{A}_α denote the 1-submodel generated by α , that is, the set $\{x \in \mathcal{A} : \alpha R_1 x \text{ or } x \in \alpha\}$ with the inherited orderings R_1, R_2 , etc. By induction on R_1 -depth of α we define 0-models $\mathcal{A}_\alpha^{(m)}$, where $m < \omega$.

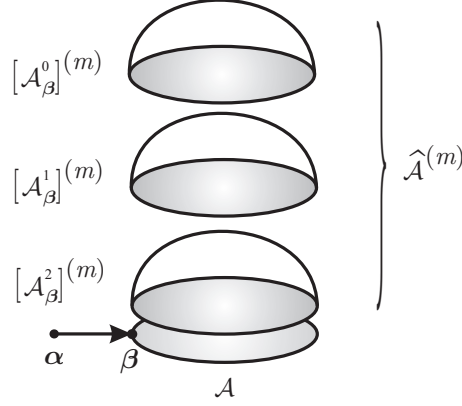


Figure 4: $\mathcal{A}^{(m)}$, m -blowup of a 1-schema \mathcal{A} (where $m = 3$, α is the root of \mathcal{A} and β its only immediate successor). The grey areas are the 1-planes isomorphic to \mathcal{A}_β . ‘Bubbles’ above them represent the parts isomorphic to $\mathcal{A}_\beta^{(m)}$.

Definition 5.3 If $\mathcal{A}_\alpha = \{\alpha\}$ is trivial let $\mathcal{A}_\alpha^{(m)} := \{\mathcal{A}_\alpha\}$, that is, $\mathcal{A}_\alpha^{(m)}$ is the same model considered as a 0-model with empty R_0 . Otherwise, by induction hypothesis $\mathcal{A}_\beta^{(m)}$ is defined for any β such that $\alpha R_1 \beta$. Let $\alpha_0, \dots, \alpha_{s-1}$ be all the immediate R_1 -successors of α . Define $\mathcal{B}_k := \sum_{i < m} \mathcal{A}_{\alpha_k}^{(m)}$ and $\hat{\mathcal{A}}_\alpha^{(m)} := \coprod_{k < s} \mathcal{B}_k$. Finally, $\mathcal{A}_\alpha^{(m)}$ is obtained by putting a copy of the 1-model \mathcal{A}_α R_0 -below $\hat{\mathcal{A}}_\alpha^{(m)}$ as a new root 1-plane. Thus, $\mathcal{A}_\alpha^{(m)}$ is essentially the ordered sum $\hat{\mathcal{A}}_\alpha^{(m)} + \{\mathcal{A}_\alpha\}$ and one can combine the definition of $\mathcal{A}_\alpha^{(m)}$ into a formula:

$$\mathcal{A}_\alpha^{(m)} = \coprod_{k < s} \sum_{i < m} \mathcal{A}_{\alpha_k}^{(m)} + \{\mathcal{A}_\alpha\}.$$

Notice that \mathcal{B}_k is a linearly ordered chain of m isomorphic copies of $\mathcal{A}_{\alpha_k}^{(m)}$. We will denote the i -th copy of $\mathcal{A}_\beta^{(m)}$ within $\sum_{i < m} \mathcal{A}_\beta^{(m)}$ by $[\mathcal{A}_\beta^i]^{(m)}$, for $i < m$, counting from above (see Fig. 4). Each $[\mathcal{A}_\beta^i]^{(m)}$ has a copy of \mathcal{A}_β as its root which is denoted \mathcal{A}_β^i .

The m -blowup operation is then lifted to k -models for $k > 1$ in an obvious way. Given a k -model \mathcal{A} we denote its $k + 1$ -planes α , β , etc. and write $\alpha \in \mathcal{A}$ to indicate that α is a $k + 1$ -plane in \mathcal{A} .

The following two lemmas state that the m -blowup operation behaves well w.r.t. p-morphisms.

Lemma 5.4 Suppose α, β are 2-planes in a 1-model \mathcal{A} and $\alpha R_1 \beta$. Then, for each m , there is a special end-embedding $f : \sum_{i < m} \mathcal{A}_\beta^{(m)} \rightarrow \hat{\mathcal{A}}_\alpha^{(m)}$.

Proof. Induction on the R_1 -depth of α . We prove the induction step. Let α_1 be an immediate successor of α such that $\alpha_1 R_1 \beta$ or $\alpha_1 = \beta$. In the first case, by the induction hypothesis there is a special end-embedding $g : \sum_{i < m} \mathcal{A}_\beta^{(m)} \rightarrow \hat{\mathcal{A}}_{\alpha_1}^{(m)}$. Then, by the construction of $\mathcal{A}_{\alpha_1}^{(m)}$, one obtains the following sequence of end-embeddings:

$$\sum_{i < m} \mathcal{A}_\beta^{(m)} \longrightarrow \hat{\mathcal{A}}_{\alpha_1}^{(m)} \longrightarrow \mathcal{A}_{\alpha_1}^{(m)} \longrightarrow \sum_{i < m} \mathcal{A}_{\alpha_1}^{(m)} \longrightarrow \hat{\mathcal{A}}_\alpha^{(m)}.$$

Clearly, all the embeddings are special. In the second case, one just takes the last of the above embeddings. \boxtimes

Lemma 5.5 Suppose \mathcal{A}, \mathcal{B} are rooted stratified 1-models and $f : \mathcal{B} \rightarrow \mathcal{A}$ is a full p -morphism. Then there is a full p -morphism $g : \mathcal{B}^{(m)} \rightarrow \mathcal{A}^{(m)}$.

Proof. We argue by induction on the R_1 -height of \mathcal{B} . Let β be the root of \mathcal{B} and let β_1, \dots, β_t denote all the immediate successors of β in the ordering of 2-planes. For each β_k let $\alpha_k := f(\beta_k)$. Since f is full, α_k is a 2-plane in \mathcal{A} and $f(\mathcal{B}_{\beta_k}) = \mathcal{A}_{\alpha_k}$. Then, by the induction hypothesis one obtains a full p -morphism $g_k : \mathcal{B}_{\beta_k}^{(m)} \rightarrow \mathcal{A}_{\alpha_k}^{(m)}$ which induces a full p -morphism

$$h_k : \sum_{i < m} \mathcal{B}_{\beta_k}^{(m)} \rightarrow \sum_{i < m} \mathcal{A}_{\alpha_k}^{(m)},$$

by the surjectivity of g_k .

Let α be the root of \mathcal{A} . Since $f(\beta) R_1 f(\beta_k) = \alpha_k$, clearly $\alpha R_1 \alpha_k$. By the previous lemma there is a special end-embedding $\bar{h}_k : \sum_{i < m} \mathcal{A}_{\alpha_k}^{(m)} \rightarrow \hat{\mathcal{A}}_\alpha^{(m)}$. This induces a mapping $h : \hat{\mathcal{B}}_\beta^{(m)} \rightarrow \hat{\mathcal{A}}_\alpha^{(m)}$ such that $h(x) = \bar{h}_k(h_k(x))$ if $x \in \sum_{i < m} \mathcal{B}_{\beta_k}^{(m)}$, for any $k < t$. It is easy to check that h is a full p -morphism. Hence, it can also be combined with f and induces a full p -morphism $g : \mathcal{B}^{(m)} \rightarrow \mathcal{A}^{(m)}$. \boxtimes

Definition 5.6 Let α be the root 2-plane of \mathcal{A} . We inductively define a natural *projection* function $\pi : \mathcal{A}^{(m)} \rightarrow \mathcal{A}$. By definition, π is identical on \mathcal{A} considered as the root 1-plane of $\mathcal{A}^{(m)}$. We define the restriction of π to $\hat{\mathcal{A}}^{(m)}$.

For each $k < s$ let $\mathcal{A}_k := \mathcal{A}_{\alpha_k}$. By induction hypothesis, there is a projection function $\pi_k^i : [\mathcal{A}_k^i]^{(m)} \rightarrow \mathcal{A}_k^i$, for each $k < s, i < m$. If $x \in [\mathcal{A}_k^i]^{(m)}$, then $\pi(x) := y$, where $y \in \mathcal{A}_k$ is the point corresponding to $\pi_k^i(x)$ in the isomorphic model \mathcal{A}_k^i .

Lemma 5.7 π restricted to any 1-plane in $\mathcal{A}^{(m)}$ is a special end-embedding of 1-models.

Proof. Induction on the height of \mathcal{A} . \square

For any $x \in \mathcal{A}$ let $F(x)$ denote the fiber of x within $\mathcal{A}^{(m)}$, that is, $F(x) := \pi^{-1}(x)$. We say that a point $x \in \mathcal{A}^{(m)}$ has level i , denoted $\ell(x) = i$, if $x \in [\mathcal{A}_k^i]^{(m)}$, for some $k < s$. If x belongs to the root \mathcal{A} we stipulate $\ell(x) = \infty$.

Lemma 5.8 Assume $k < m$ and $a \in \mathcal{A}$. All the points $x \in \mathcal{A}^{(m)}$ such that $x \in F(a)$ and $\ell(x) \geq k$ are k -bisimilar to a .

Proof. Induction on k . The statement holds for $k = 0$ because the evaluation of variables at all points of $F(a)$ is the same.

Consider the case $k + 1$. Let $x \in \mathcal{A}^{(m)}$ be a point such that $\ell(x) \geq k + 1$ and $x \in F(a)$. We show that $x \sim_{k+1} a$.

Indeed, by the induction hypothesis we have $x \sim_k a$, hence Condition (i) holds.

Suppose $xR_n y$, $n > 0$. Then we have $aR_n b$ where $b = \pi(y)$, and by the induction hypothesis $b \sim_k y$. If $xR_0 y$ then also $aR_0 y$, hence Condition (ii) holds.

Suppose $aR_n b$, $n > 0$. Then $b \in \mathcal{A}$, hence there is a $y \in F(b)$ which belongs to the same 1-plane as x . We have $xR_n y$ and $y \sim_k b$ by the induction hypothesis.

Suppose $aR_0 y$. Let $i = \ell(y)$ that is $y \in [\mathcal{A}_j^i]^{(m)}$, for some j . If $i \leq k$ then $xR_0 y$ and we are done. If $i > k$ let y_1 be the point corresponding to y in $[\mathcal{A}_j^k]^{(m)}$. Since $\ell(x) \geq k + 1$, we have $xR_0 y_1$. By the induction hypothesis $y_1 \sim_k y$, hence Condition (iii) holds. \square

Lemma 5.9 If $k < m$, $\mathcal{A}^{(m)}$ satisfies the k -similarity property for R_0 .

Proof. Suppose $x \in \mathcal{A}^{(m)}$ and $xR_1 y$. We argue by induction on the construction of $\mathcal{A}^{(m)}$. If $x \in [\mathcal{A}_k^i]^{(m)}$, for some $i < m$ and $k < s$, use the induction hypothesis and the fact that $[\mathcal{A}_k^i]^{(m)}$ as a submodel of $\mathcal{A}^{(m)}$ satisfies the conditions of Lemma 4.6. If $x \in \mathcal{A}$, then $y \in \mathcal{A}$ and there is a point $y' \in F(y)$ such that $\ell(y') \geq k$. Then $y' \sim_k y$. \square

Lemma 5.10 If $n > 0$ and \mathcal{A} satisfies the k -similarity property for R_n , then so does $\mathcal{A}^{(m)}$.

Proof. By Lemma 5.7 each 1-plane β in $\mathcal{A}^{(m)}$ is embeddable into \mathcal{A} by a special end-embedding. Hence, by Lemma 4.5 the 1-model β satisfies the k -similarity property for R_n . By Lemma 4.7 we conclude that all 1-planes of \mathcal{A} satisfy the k -similarity property for R_n also in \mathcal{A} . Since $n > 0$, it follows that \mathcal{A} itself satisfies the k -similarity property for R_n . \square

The previous lemmata lift to m -blowups of stratified k -models for any $k > 1$. However, we need to ensure the m -similarity property simultaneously for all R_n . Hence, we need a stronger notion of *global m -blowup* of a stratified model.

Definition 5.11 Given a hereditarily rooted finite stratified k -model \mathcal{A} we define a k -model $\mathfrak{B}_m(\mathcal{A})$ by induction on the rank of \mathcal{A} :

$$\mathfrak{B}_m(\mathcal{A}) := \sum_{\alpha \in \mathcal{A}} \mathfrak{B}_m(\alpha)^{(m)}.$$

Informally speaking, we apply the m -blowup operation to each $k+1$ -plane α of \mathcal{A} and order the resulting k -models by the ordering of $k+1$ -planes in \mathcal{A} .

Lemma 5.12 Let \mathcal{A} be a 0-model and $k < m$. $\mathfrak{B}_m(\mathcal{A})$ satisfies the k -similarity property for each R_n .

Proof. Induction on the rank of \mathcal{A} . Basis is trivial. For the induction step consider two cases.

CASE 1. $n = 0$. Each of the models $\mathfrak{B}_m(\alpha)^{(m)}$ satisfies the k -similarity property for R_0 by Lemma 5.9. Hence, by Lemma 4.6, so does the ordered sum $\sum_{\alpha \in \mathcal{A}} \mathfrak{B}_m(\alpha)^{(m)}$.

CASE 2. $n > 0$. By the induction hypothesis, each $\mathfrak{B}_m(\alpha)$ satisfies the k -similarity property for R_n . Therefore, by Lemma 5.10 so do all the models $\mathfrak{B}_m(\alpha)^{(m)}$. Lemma 4.6 again yields the result. \square

6 Elementary submodels and chains

Here we present an easy technique which can be seen as an analog in the realm of Kripke models of the well-known method of elementary chains in model theory. We begin with the following basic definition from [1].

Let \mathcal{A} be any Kripke model (in the language of **GLP**) and let $\mathcal{B} \subseteq \mathcal{A}$ be a submodel of \mathcal{A} .

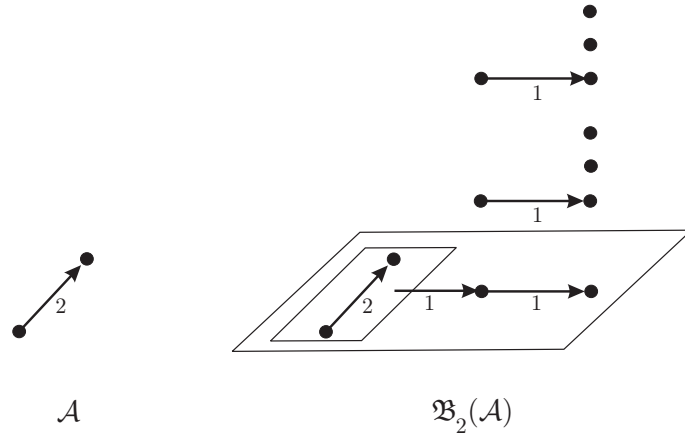


Figure 5: Global 2-blowup of a two-point stratified model \mathcal{A} . $\mathfrak{B}_2(\mathcal{A})$ can be viewed as a three-dimensional structure: horizontal layers are 1-planes, the upwards going R_0 -arrows between them are not shown. The outer parallelogram represents the root 1-plane of $\mathfrak{B}_2(\mathcal{A})$; it is isomorphic to the 1-model $\mathcal{A}^{(2)}$ (with \mathcal{A} considered as a 2-model). The inner parallelogram is its root 2-plane isomorphic to \mathcal{A} . Notice that $\mathfrak{B}_2(\mathcal{A})$ is hereditarily linear, that is, the eight 1-planes are linearly ordered by R_0 , the set of 2-planes in each of these 1-planes is linearly ordered by R_1 , etc.

Definition 6.1 For $n > 0$, we write $\mathcal{B} \prec_n \mathcal{A}$ if, for each $x \in \mathcal{B}$ and $y \in \mathcal{A}$ such that $xR_k y$, there is a $y' \in \mathcal{B}$ such that $y' \sim_{n-1} y$ in \mathcal{A} and $xR_k y'$. In this case, we say that \mathcal{B} is an n -elementary submodel of \mathcal{A} . We also stipulate $\mathcal{B} \prec_0 \mathcal{A}$.

Lemma 6.2 Suppose $\mathcal{B} \prec_n \mathcal{A}$. Then, $\mathcal{B}, x \models \varphi$ iff $\mathcal{A}, x \models \varphi$, for any $x \in \mathcal{B}$ and any φ with $dp(\varphi) \leq n$.

Proof. For $n = 0$ the statement is obvious. For $n > 0$, it follows by an easy induction on the length of φ .

Assume $\varphi = \langle k \rangle \psi$ and $\mathcal{A}, x \models \varphi$ where $x \in \mathcal{B}$. Then there is a $y \in \mathcal{A}$ such that $xR_k y$ and $\mathcal{A}, y \models \psi$. By the given condition, there is a $y' \in \mathcal{B}$ such that $xR_k y'$ and $\mathcal{A}, y' \models \psi$, since $y' \sim_{n-1} y$ and $dp(\psi) < n$. Hence, by the induction hypothesis, $\mathcal{B}, y' \models \psi$ and $\mathcal{B}, x \models \langle k \rangle \psi$. The other cases are quite obvious. \square

Lemma 6.3 If $\mathcal{A}_1 \prec_n \mathcal{A}_2$ then, for all $x, y \in \mathcal{A}_1$, $x \sim_n y$ in \mathcal{A}_1 iff $x \sim_n y$ in \mathcal{A}_2 .

Proof. Induction on n ; we only verify the induction step.

Assume $x \sim_{n+1} y$ in \mathcal{A}_1 . Consider any $x' \in \mathcal{A}_2$ such that $xR_k x'$. Since $\mathcal{A}_1 \prec_{n+1} \mathcal{A}_2$, there is a $x'' \in \mathcal{A}_1$ such that $xR_k x''$ and $x'' \sim_n x'$ in \mathcal{A}_2 . Using $x \sim_{n+1} y$ pick a $y' \in \mathcal{A}_1$ such that $yR_k y'$ and $y' \sim_n x''$ in \mathcal{A}_1 . By the induction hypothesis we also have $y' \sim_n x''$ in \mathcal{A}_2 . Hence, $x' \sim_n y'$ in \mathcal{A}_2 .

Assume $x, y \in \mathcal{A}_1$ and $x \sim_{n+1} y$ in \mathcal{A}_2 . Consider any $x' \in \mathcal{A}_1$ such that $xR_k x'$. Pick a $y' \in \mathcal{A}_2$ such that $yR_k y'$ and $y' \sim_n x'$. Since $\mathcal{A}_1 \prec_{n+1} \mathcal{A}_2$, there is a $y'' \in \mathcal{A}_1$ such that $yR_k y''$ and $y'' \sim_n y'$ in \mathcal{A}_2 . We have $y'' \sim_n y$ in \mathcal{A}_2 and hence, by the induction hypothesis, in \mathcal{A}_1 . \square

Lemma 6.4 (i) If $m < n$ and $\mathcal{A}_0 \prec_n \mathcal{A}_1$, then $\mathcal{A}_0 \prec_m \mathcal{A}_1$.

(ii) If $\mathcal{A}_0 \prec_n \mathcal{A}_1 \prec_n \mathcal{A}_2$, then $\mathcal{A}_1 \prec_n \mathcal{A}_2$.

Proof. Part (i) is obvious. To prove Part (ii) assume $x \in \mathcal{A}_0$, $y \in \mathcal{A}_2$ and $xR_k y$. Pick a $y' \in \mathcal{A}_1$ such that $xR_k y'$ and $y' \sim_{n-1} y$ in \mathcal{A}_2 . Further, pick a $y'' \in \mathcal{A}_0$ such that $xR_k y''$ and $y'' \sim_{n-1} y'$ in \mathcal{A}_1 . By Lemma 6.3 we have $y'' \sim_{n-1} y'$ in \mathcal{A}_2 , and hence $y'' \sim_{n-1} y$ in \mathcal{A}_2 . \square

Definition 6.5 Let an ordinal λ be given. An n -elementary chain of length λ is a sequence of models of the form

$$\mathcal{A}_0 \prec_n \mathcal{A}_1 \prec_n \cdots \prec_n \mathcal{A}_\alpha \prec_n \cdots, \quad \alpha < \lambda$$

such that $\mathcal{A}_\alpha \prec_n \mathcal{A}_\beta$ whenever $\alpha < \beta$. The *union* or the *limit* of the chain is the model with the universe $\mathcal{A}^* := \bigcup_{\alpha < \lambda} \mathcal{A}_\alpha$ whose relations R_k are the unions of the corresponding relations in all \mathcal{A}_α , for $\alpha < \lambda$. We also define, for any $x \in \mathcal{A}^*$ and any variable p , $\mathcal{A}^*, x \models p$ iff $\mathcal{A}_\alpha, x \models p$, for some $\alpha < \lambda$.

Lemma 6.6 *Assume $(\mathcal{A}_\alpha)_{\alpha < \lambda}$ is an n -elementary chain with the limit \mathcal{A}^* .*

(i) *For any $\alpha < \lambda$ and $x, y \in \mathcal{A}_\alpha$, $x \sim_n y$ in \mathcal{A}_α iff $x \sim_n y$ in \mathcal{A}^* .*

(ii) *$\mathcal{A}_\alpha \prec_n \mathcal{A}^*$ for all $\alpha < \lambda$.*

Proof. Part (i) is proved by induction on n ; we only verify the induction step.

Assume $x \sim_{n+1} y$ in \mathcal{A}_α . Consider any $x' \in \mathcal{A}^*$ such that $x R_k x'$. There is a $\beta < \lambda$ such that $x' \in \mathcal{A}_\beta$. We may assume $\alpha \leq \beta$. Since $\mathcal{A}_\alpha \prec_{n+1} \mathcal{A}_\beta$, there is a $x'' \in \mathcal{A}_\alpha$ such that $x R_k x''$ and $x'' \sim_n x'$ in \mathcal{A}_β . Using $x \sim_{n+1} y$ pick a $y' \in \mathcal{A}_\alpha$ such that $y R_k y'$ and $y' \sim_n x''$ in \mathcal{A}_α . By Lemma 6.3 we also have $y' \sim_n x''$ in \mathcal{A}_β . Hence, $x' \sim_n y'$ in \mathcal{A}_β and in \mathcal{A}^* , using the induction hypothesis.

Assume $x, y \in \mathcal{A}_\alpha$ and $x \sim_{n+1} y$ in \mathcal{A}^* . Consider any $x' \in \mathcal{A}_\alpha$ such that $x R_k x'$. Pick a $y' \in \mathcal{A}^*$ such that $y R_k y'$ and $y' \sim_n x'$. For some $\beta \geq \alpha$, $y' \in \mathcal{A}_\beta$. Since $\mathcal{A}_\alpha \prec_{n+1} \mathcal{A}_\beta$, there is a $y'' \in \mathcal{A}_\alpha$ such that $y R_k y''$ and $y'' \sim_n y'$ in \mathcal{A}_β . We have $y'' \sim_n y$ in \mathcal{A}_β and hence, by the induction hypothesis, in \mathcal{A}^* and in \mathcal{A}_α .

Part (ii) is proved using Part (i) as follows. Let $x \in \mathcal{A}_\alpha$, $y \in \mathcal{A}^*$ and $x R_k y$. For some $\beta \geq \alpha$, $y \in \mathcal{A}_\beta$. Since $\mathcal{A}_\alpha \prec_n \mathcal{A}_\beta$, there is a $y' \in \mathcal{A}_\alpha$ such that $y' \sim_{n-1} y$ in \mathcal{A}_β . By Part (i), $y' \sim_{n-1} y$ also holds in \mathcal{A}^* . \square

It is fairly easy to see that the definition of the limit of an n -elementary chain of models and the theorem above can be extended to more general limits of directed families of models. In the following we shall use, however, only very simple linear chains as above.

Corollary 6.7 *Suppose $(\mathcal{A}_\alpha)_{\alpha < \lambda}$ is a sequence of models such that*

$$\forall \alpha < \lambda \ \mathcal{A}_\alpha \prec_n \mathcal{A}_{\alpha+1} \text{ and } \mathcal{A}_\beta = \bigcup_{\alpha < \beta} \mathcal{A}_\alpha \text{ if } \beta \text{ is a limit ordinal.}$$

Then $(\mathcal{A}_\alpha)_{\alpha < \lambda}$ is an n -elementary chain.

Proof. Transfinite induction on α using Lemma 6.4 (ii) for successor α and Lemma 6.6 (ii) for limit α . \square

Corollary 6.8 Suppose $(\mathcal{A}_n)_{n < \omega}$ is a sequence of models such that

$$\mathcal{A}_0 \prec_0 \mathcal{A}_1 \prec_1 \mathcal{A}_2 \prec_2 \cdots$$

and let $\mathcal{A}^* := \bigcup_{n < \omega} \mathcal{A}_n$. Then $\forall n \mathcal{A}_n \prec_n \mathcal{A}^*$.

Proof. By Lemma 6.4

$$\mathcal{A}_n \prec_n \mathcal{A}_{n+1} \prec_n \mathcal{A}_{n+2} \prec_n \cdots$$

is an n -elementary chain and \mathcal{A}^* is its limit. Hence, the result follows from Lemma 6.6. \square

7 Models for GLP

Here we construct models for **GLP** as the limits of n -blowups of finite stratified models. Let \mathcal{A}, \mathcal{B} be stratified models.

Definition 7.1 A submodel $\mathcal{A} \subseteq \mathcal{B}$ is called *regular n -elementary* if the following conditions hold:

- $\forall x \in \mathcal{B} \setminus \mathcal{A} \exists y \in \mathcal{A} (xR_0y \text{ and } x \sim_{n-1} y)$;
- For any 1-plane α in \mathcal{B} , either $\alpha \cap \mathcal{A} = \emptyset$ or $\alpha \cap \mathcal{A}$ is a regular n -elementary submodel of α .

An embedding $f : \mathcal{A} \rightarrow \mathcal{B}$ is called regular n -elementary if so is the submodel $f(\mathcal{A}) \subseteq \mathcal{B}$.

Lemma 7.2 If $\mathcal{A} \subseteq \mathcal{B}$ is regular n -elementary, then $\mathcal{A} \prec_n \mathcal{B}$.

Proof. Assume $x \in \mathcal{A}$ and $xR_ky \in \mathcal{B} \setminus \mathcal{A}$. If $k = 0$ find a $y' \in \mathcal{A}$ such that $y' \sim_{n-1} y$ and yR_0y' by regularity. Then xR_0y' and y' is as required.

If $k > 0$ there is a 1-plane $\alpha \in \mathcal{B}$ such that $x, y \in \alpha$ and $\alpha \cap \mathcal{A} \neq \emptyset$. Then $\alpha \cap \mathcal{A}$ is a regular n -elementary submodel of α and by the induction hypothesis there is a $y' \in \alpha \cap \mathcal{A}$ such that xR_ky' and $y \sim_{n-1} y'$ in α . By Lemma 4.7, $y \sim_{n-1} y'$ also holds in \mathcal{B} . \square

Lemma 7.3 Let \mathcal{A} be a finite rooted 1-model. Then, for each $n > 0$, there is a regular n -elementary special embedding of $\mathcal{A}^{(n)}$ into $\mathcal{A}^{(n+1)}$.

Proof. We define a special embedding $f : \mathcal{A}^{(n)} \rightarrow \mathcal{A}^{(n+1)}$ by induction on the height of \mathcal{A} . Then we will show that the embedding is regular n -elementary.

If the height of \mathcal{A} is 0, both models coincide with $\{\mathcal{A}\}$ and the embedding is the identity mapping.

Otherwise, let $\mathcal{A}_k := \mathcal{A}_{\alpha_k}$ for $k < s$ be as in Definition 5.3. By the induction hypothesis for each k one has a special embedding $f_k : \mathcal{A}_k^{(n)} \rightarrow \mathcal{A}_k^{(n+1)}$. The embedding f of $\mathcal{A}^{(n)}$ into $\mathcal{A}^{(n+1)}$ acts as f_k by mapping the i -th copy $[\mathcal{A}_k^i]^{(n)}$ into $[\mathcal{A}_k^i]^{(n+1)}$, for each $i < n$. The root \mathcal{A} of $\mathcal{A}^{(n)}$ is mapped identically to the root of $\mathcal{A}^{(n+1)}$.

For a proof of regular n -elementarity, we first observe by an easy induction that f is indeed a special embedding (1-planes are mapped isomorphically). Hence, the second condition of n -regularity is automatically fulfilled.

Assume now that $x \in \mathcal{A}^{(n+1)} \setminus f(\mathcal{A}^{(n)})$.

CASE 1: $x \in [\mathcal{A}_k^i]^{(n+1)}$ for some $k < s$ and $i < n$. Then by the induction hypothesis and Lemma 4.6 we find a $y \in f([\mathcal{A}_k^i]^{(n)})$ such that xR_0y and $x \sim_{n-1} y$.

CASE 2: $x \in [\mathcal{A}_k^n]^{(n+1)}$ for some $k < s$. Take a $y \in \mathcal{A}_k^{n-1}$ such that $\pi(y) = \pi(x)$. By Lemma 5.8, $y \sim_{n-1} x$ and obviously xR_0y . \square

The following crucial lemma states that the blow-up operation almost preserves regular m -elementary embeddings. The word ‘almost’ refers to the fact that we can only insure the m -elementarity of the embedding $\mathcal{A}^{(n)} \rightarrow \mathcal{B}^{(n)}$ if \mathcal{A} is a *treelike* model.² In general, however, it is cumbersome to deal with treelike models, since $\mathcal{A}^{(n)}$ need not be a treelike model, even if \mathcal{A} was. Instead, we replace $\mathcal{A}^{(n)}$ by a somewhat larger model \mathcal{C} such that there is a p-morphism $\mathcal{C} \rightarrow \mathcal{A}^{(n)}$ and \mathcal{C} is m -elementarily embeddable into $\mathcal{B}^{(n)}$. This situation can be depicted by a diagram:

$$\mathcal{A}^{(n)} \leftarrow \mathcal{C} \xrightarrow[m]{\quad} \mathcal{B}^{(n)}$$

In the following, the two-head arrows will always denote full p-morphisms, and the arrows with a subscript m (regular) m -elementary embeddings.

An embedding $f : \mathcal{A} \rightarrow \mathcal{B}$ is called *root preserving*, if both \mathcal{A} and \mathcal{B} are rooted k -models and the root 1-plane of \mathcal{A} is mapped to the root 1-plane of \mathcal{B} .

Lemma 7.4 *Suppose \mathcal{B} is a rooted 1-model and $f : \mathcal{A} \rightarrow \mathcal{B}$ is a regular m -elementary embedding. Then, for each $n \geq m$, there is a rooted 0-model \mathcal{C} and regular m -elementary embedding $g : \mathcal{C} \rightarrow \mathcal{B}^{(n)}$ such that*

²In the sense that the ordering R_1 on the set of 2-planes of \mathcal{A} is treelike.

- (i) If f is root-preserving, then g is root-preserving and there is a full p -morphism $h : \mathcal{C} \twoheadrightarrow \mathcal{A}^{(n)}$.
- (ii) Otherwise, g is not root-preserving and there is a full p -morphism $h : \mathcal{C} \twoheadrightarrow \coprod_{\alpha \in \min \mathcal{A}} \sum_{i < n} \mathcal{A}_\alpha^{(n)}$, where $\min \mathcal{A}$ denotes the set of R_1 -minimal 2-planes in \mathcal{A} .

Proof. We construct the required embeddings by induction on the height of \mathcal{B} . The two statements are proved simultaneously. Without loss of generality we assume that $\mathcal{A} \subseteq \mathcal{B}$ and f is the identity mapping. Basis of induction is trivial.

For the induction step let β denote the root 2-plane of \mathcal{B} and β_1, \dots, β_t be all the immediate R_1 -successors of β . Let \mathcal{B}_k for $k = 1, \dots, t$ be the submodels of \mathcal{B} generated by β_k , respectively, and let $\mathcal{A}_k := \mathcal{A} \cap \mathcal{B}_k$. Since \mathcal{B}_k is upwards closed, by Lemma 4.5 $\mathcal{A}_k \prec_m \mathcal{B}_k$ is regular for each $k < t$. We consider the following cases.

CASE 1: f is not root-preserving. Then \mathcal{A} is the union of all \mathcal{A}_k for $k < t$. Consider any $k < t$.

SUBCASE 1.1: The embedding of \mathcal{A}_k into \mathcal{B}_k is not root-preserving. Take the uppermost copy \mathcal{B}_k^0 of \mathcal{B}_k in $\mathcal{B}^{(n)}$. By the induction hypothesis there is a diagram

$$\coprod_{\alpha \in \min \mathcal{A}_k} \sum_{i < n} \mathcal{A}_\alpha^{(n)} \xleftarrow{\quad} \mathcal{C}_k \xrightarrow[m]{g'_k} [\mathcal{B}_k^0]^{(n)},$$

where g'_k is a regular m -elementary embedding which is not root-preserving. Let g_k denote the composition of g'_k and the end-embedding of $[\mathcal{B}_k^0]^{(n)}$ into $\sum_{i < n} \mathcal{B}_k^{(n)}$. To show that g_k is regular m -elementary we prove the following auxiliary lemma.

Lemma 7.5 Assume $x \in \mathcal{B}_k^0$ and $y \in \mathcal{B}^{(n)}$ such that $\pi(x) = \pi(y)$ and $y R_0 x$. Then $y \sim_m x$.

Proof. We prove $y \sim_l x$ by induction on $l \leq m$. For $l = 0$ the claim is obvious. Consider the case $l + 1 \leq m$.

Suppose $y R_0 z$ in $\mathcal{B}^{(n)}$. Then $z \in [\mathcal{B}_k^i]^{(n)}$ for some $i < n$. If $i = 0$ then either $z \in \mathcal{B}_k^0$ or $z \in \widehat{\mathcal{B}_k^0}^{(n)}$. In the second case obviously $x R_0 z$ and we are done. In the first case notice that $z \notin g'_k(\mathcal{C}_k)$ because g'_k is not root-preserving. Since g'_k is regular m -elementary, there is a $z' \in g'_k(\mathcal{C}_k)$ such

that $z' \sim_{m-1} z$ and zR_0z' . Since x and z belong to the same 1-plane \mathcal{B}_k^0 we also have xR_0z' and, since $m \geq l+1$, $z' \sim_l z$ as required.

If $i > 0$ find a z' such that $z' \in \mathcal{B}_k^0$ and $\pi(z') = \pi(z)$. Then zR_0z' and by the induction hypothesis $z' \sim_l z$. Reason as before: using the regularity of g'_k find a $z'' \in g'_k(\mathcal{C}_k)$ such that $z'' \sim_{m-1} z'$ and $z'R_0z''$. Then xR_0z'' and $z \sim_l z''$ as required. \square

As an immediate corollary of this lemma we obtain for any $x \in [\mathcal{B}_k^i]^{(n)}$ with $i > 0$ an $x' \in \mathcal{B}_k^0$ such that $\pi(x) = \pi(x')$ and hence $x' \sim_m x$. Since g'_k is regular m -elementary we find an $x'' \in g'_k(\mathcal{C}_k)$ such that $xR_0x'R_0x''$ and $x' \sim_{m-1} x''$. Hence, g_k is regular m -elementary.

SUBCASE 1.2: \mathcal{A}_k is rooted and the embedding of \mathcal{A}_k into \mathcal{B}_k is root-preserving. Then, by the induction hypothesis, for each $i < n$, there is a diagram:

$$\mathcal{A}_k^{(n)} \xleftarrow{h_k} \mathcal{D}_k \xrightarrow{\frac{g_{ik}}{m}} [\mathcal{B}_k^i]^{(n)},$$

where g_{ik} is a root-preserving regular m -elementary embedding. Since h_k is a full p-morphism, this naturally lifts to a diagram:

$$\sum_{i < n} \mathcal{A}_k^{(n)} \xleftarrow{\quad} \sum_{i < n} \mathcal{D}_k \xrightarrow{\frac{g_k}{m}} \sum_{i < n} \mathcal{B}_k^{(n)},$$

By Lemma 4.6 for any i the relations \sim_{m-1} on $[\mathcal{B}_k^i]^{(n)}$ and on $\sum_{i < n} [\mathcal{B}_k^i]^{(n)}$ coincide. Hence, g_k is regular m -elementary and we let $\mathcal{C}_k := \sum_{i < n} \mathcal{D}_k$.

Thus, in both Cases 1.1 and 1.2, for each $k < t$, we obtain a diagram

$$\coprod_{\alpha \in \min \mathcal{A}_k} \sum_{i < n} \mathcal{A}_\alpha^{(n)} \xleftarrow{\quad} \mathcal{C}_k \xrightarrow{\frac{g_k}{m}} \sum_{i < n} \mathcal{B}_k^{(n)}.$$

(In Case 1.2, \mathcal{A}_k has a unique minimal element.) Since $\hat{\mathcal{B}}^{(n)}$ is isomorphic to a disjoint union $\coprod_{k < t} \sum_{i < n} \mathcal{B}_k^{(n)}$, we obtain

$$\coprod_{k < t} \coprod_{\alpha \in \min \mathcal{A}_k} \sum_{i < n} \mathcal{A}_\alpha^{(n)} \xleftarrow{\quad} \coprod_{k < t} \mathcal{C}_k \xrightarrow{\frac{g'}{m}} \hat{\mathcal{B}}^{(n)}.$$

Clearly, since $\mathcal{A} = \bigcup_{k < t} \mathcal{A}_k$, there is a full p-morphism

$$\coprod_{k < t} \coprod_{\alpha \in \min \mathcal{A}_k} \sum_{i < n} \mathcal{A}_\alpha^{(n)} \longrightarrow \coprod_{\alpha \in \min \mathcal{A}} \sum_{i < n} \mathcal{A}_\alpha^{(n)},$$

and there is a canonical end-embedding $\hat{\mathcal{B}}^{(n)} \rightarrow \mathcal{B}^{(n)}$. This yields a diagram

$$\coprod_{\alpha \in \min \mathcal{A}} \sum_{i < n} \mathcal{A}_\alpha^{(n)} \xleftarrow{\quad} \coprod_{k < t} \mathcal{C}_k \xrightarrow{\frac{g}{m}} \mathcal{B}^{(n)},$$

where we have to check that g is regular m -elementary.

To this end, it is sufficient to consider the points $x \in \mathcal{B}$ on the root 1-plane of $\mathcal{B}^{(n)}$. By the regularity of f one can find a $y \in \mathcal{A}$ such that $y \sim_{m-1} x$ in \mathcal{B} and xR_1y . Obviously, for some $k < t$, $y \in \mathcal{A}_k$. Consider a $y' \in \mathcal{B}_k^{n-1}$ such that $\pi(y') = y$. Then yR_0y' and, by Lemma 5.8, $y' \sim_{n-1} y$, whence $x \sim_{m-1} y'$. Since $y' \in \sum_{i < n} [\mathcal{B}_k^i]^{(n)}$, one can find a $y'' \in g_k(\mathcal{C}_k)$ such that $y'' \sim_{m-1} y'$. Then we have xR_0y'' and $x \sim_{m-1} y''$, as required.

CASE 2: \mathcal{A} is rooted and f is root-preserving.

SUBCASE 2.1: The embedding of \mathcal{A}_k into \mathcal{B}_k is not root-preserving. Then, by the induction hypothesis, there is a diagram

$$\coprod_{\alpha \in \min \mathcal{A}_k} \sum_{i < n} \mathcal{A}_\alpha^{(n)} \llcorner \mathcal{C}_k \xrightarrow{\frac{g'_k}{m}} [\mathcal{B}_k^0]^{(n)},$$

where g'_k is not root-preserving. By the same reasoning as in Subcase 1.1 we obtain that the composition of g'_k and the embedding of $[\mathcal{B}_k^0]^{(n)}$ into $\sum_{i < n} [\mathcal{B}_k^i]^{(n)}$ is regular m -elementary.

SUBCASE 2.2: \mathcal{A}_k is rooted and the embedding of \mathcal{A}_k into \mathcal{B}_k is root-preserving. Then, as in Subcase 1.2, we obtain

$$\sum_{i < n} \mathcal{A}_k^{(n)} \llcorner \mathcal{C}_k \xrightarrow{\frac{g_k}{m}} \sum_{i < n} \mathcal{B}_k^{(n)},$$

where g_k is root-preserving.

Thus, in each case we have

$$\coprod_{\alpha \in \min \mathcal{A}_k} \sum_{i < n} \mathcal{A}_\alpha^{(n)} \llcorner \mathcal{C}_k \xrightarrow{\frac{g_k}{m}} \sum_{i < n} \mathcal{B}_k^{(n)}.$$

As in Case 1, we now put together all the embeddings g_k , which defines a diagram

$$\coprod_{k < t} \coprod_{\alpha \in \min \mathcal{A}_k} \sum_{i < n} \mathcal{A}_\alpha^{(n)} \llcorner \coprod_{k < t} \mathcal{C}_k \xrightarrow{\frac{g'}{m}} \hat{\mathcal{B}}^{(n)}.$$

Since \mathcal{A} is rooted, there is a full p-morphism

$$\coprod_{k < t} \coprod_{\alpha \in \min \mathcal{A}_k} \sum_{i < n} \mathcal{A}_\alpha^{(n)} \longrightarrow \hat{\mathcal{A}}^{(n)},$$

so we obtain

$$\hat{\mathcal{A}}^{(n)} \llcorner \coprod_{k < t} \mathcal{C}_k \xrightarrow{\frac{g'}{m}} \hat{\mathcal{B}}^{(n)}.$$

We extend g' to $g : \mathcal{C} \rightarrow \mathcal{B}^{(n)}$, where $\mathcal{C} := \coprod_{k < t} \mathcal{C}_k + \{\mathcal{A}\}$, by specifying $g(x) := f(x)$ if $x \in \mathcal{A}$, and $g(x) := g'(x)$, otherwise. Similarly to Case 1, it is easy to check that g is regular m -elementary. Similarly, h' is naturally extended to a full p -morphism $h : \mathcal{C} \rightarrow \mathcal{A}^{(n)}$ by letting $h(x) := x$ if $x \in \mathcal{A}$, and $h(x) := h'(x)$, otherwise. Thus, we obtain

$$\mathcal{A}_\alpha^{(n)} \xleftarrow{h} \mathcal{C} \xrightarrow[m]{g} \mathcal{B}^{(n)},$$

as required. \square

From this lemma we now obtain its main corollary.

Lemma 7.6 *If \mathcal{A} is a hereditarily rooted finite stratified model, then for each n there is a hereditarily rooted model \mathcal{C} and a diagram*

$$\mathfrak{B}_n(\mathcal{A}) \xleftarrow{\quad} \mathcal{C} \xrightarrow[n]{g} \mathfrak{B}_{n+1}(\mathcal{A}),$$

such that g is a regular n -elementary root-preserving embedding.

Proof. Induction on the rank of \mathcal{A} .

Recall that $\mathfrak{B}_m(\mathcal{A})$ is inductively defined by

$$\mathfrak{B}_m(\mathcal{A}) := \sum_{\alpha \in \mathcal{A}} \mathfrak{B}_m(\alpha)^{(m)}.$$

By induction hypothesis, for each 1-plane $\alpha \in \mathcal{A}$, one has a diagram

$$\mathfrak{B}_n(\alpha) \xleftarrow{\quad} \mathcal{C}_\alpha \xrightarrow[n]{} \mathfrak{B}_{n+1}(\alpha).$$

By Lemma 7.4 one obtains

$$\mathcal{C}_\alpha^{(n)} \xleftarrow{\quad} \mathcal{D}_\alpha \xrightarrow[n]{} \mathfrak{B}_{n+1}(\alpha)^{(n)}.$$

This can be extended to the left using Lemma 5.5 and to the right using Lemma 7.3, so that one obtains

$$\mathfrak{B}_n(\alpha)^{(n)} \xleftarrow{\quad} \mathcal{C}_\alpha^{(n)} \xleftarrow{\quad} \mathcal{D}_\alpha \xrightarrow[n]{} \mathfrak{B}_{n+1}(\alpha)^{(n)} \xrightarrow[n]{} \mathfrak{B}_{n+1}(\alpha)^{(n+1)}.$$

Let $\mathcal{C} := \sum_{\alpha \in \mathcal{A}} \mathcal{D}_\alpha$. By putting together the constructed mappings one obtains

$$\sum_{\alpha \in \mathcal{A}} \mathfrak{B}_m(\alpha)^{(m)} \xleftarrow{\quad} \mathcal{C} \xrightarrow[n]{} \sum_{\alpha \in \mathcal{A}} \mathfrak{B}_{n+1}(\alpha)^{(n+1)},$$

as required. \square

Thus, given a hereditarily rooted stratified model \mathcal{A} we obtain an infinite diagram:

$$\begin{array}{ccccccc}
 \mathfrak{B}_1(\mathcal{A}) & & \mathfrak{B}_2(\mathcal{A}) & & \mathfrak{B}_3(\mathcal{A}) & & \mathfrak{B}_4(\mathcal{A}) & \cdots & (*) \\
 & \nwarrow & \uparrow 1 & \nwarrow & \uparrow 2 & \nwarrow & \uparrow 3 & \nwarrow & \\
 & & \mathcal{C}_{12} & & \mathcal{C}_{23} & & \mathcal{C}_{34} & & \cdots
 \end{array}$$

Now we note the following simple lemma.

Lemma 7.7 *$f : \mathcal{A} \rightarrow \mathcal{B}$ be a p -morphism and let $\mathcal{B}_0 \prec_n \mathcal{B}$ be an n -elementary submodel. Then $f^{-1}(\mathcal{B}_0) \prec_n \mathcal{A}$.*

Proof. Let $\mathcal{A}_0 := f^{-1}(\mathcal{B}_0)$ and assume $x \in \mathcal{A}_0$ and $xR_k y$. We have $f(x)R_k f(y)$, hence there is a $z \in \mathcal{B}_0$ such that $f(x)R_k z$ and $f(y) \sim_{n-1} z$. Since $f(x)R_k z$ we can find a $u \in \mathcal{A}$ such that $xR_k u$ and $f(u) = z$. By Lemma 4.3 we also have $u \sim_{n-1} y$. \square

In terms of diagrams this can be restated as follows.

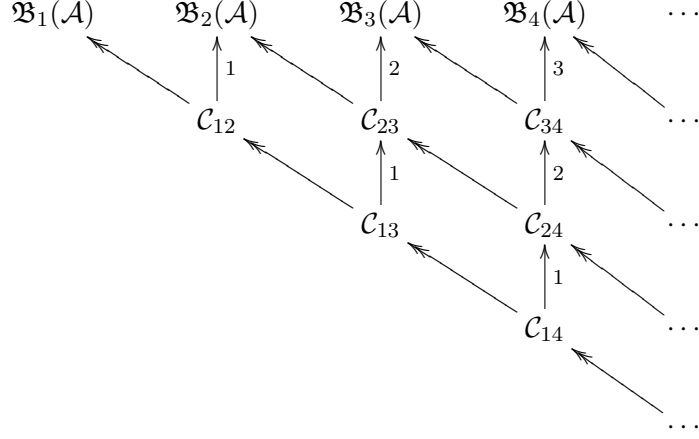
Corollary 7.8 *A diagram*

$$\begin{array}{ccc}
 \mathcal{B} & \leftarrow & \mathcal{A} \\
 \uparrow n & & \\
 \mathcal{C} & &
 \end{array}$$

can be completed to the following commutative diagram:

$$\begin{array}{ccc}
 \mathcal{B} & \leftarrow & \mathcal{A} \\
 \uparrow n & & \uparrow n \\
 \mathcal{C} & \leftarrow & \mathcal{D}
 \end{array}$$

Thus, the diagram (*) can be extended in the following way.



Denoting $\mathcal{C}_{ii} := \mathfrak{B}_i(\mathcal{A})$ we can consider ‘diagonal’ sequences of full p-morphisms of the form

$$\mathcal{C}_{nn} \longleftarrow \mathcal{C}_{n,n+1} \longleftarrow \mathcal{C}_{n,n+2} \longleftarrow \dots$$

Let $\mathcal{B}_n := \lim_{k \geq 0} \mathcal{C}_{n,n+k}$ be the inverse limit of this sequence of models. The inverse limit comes together with a canonical full p-morphism $f_{nk} : \mathcal{B}_n \rightarrow \mathcal{C}_{n,n+k}$.

Lemma 7.9 *For each n , there is an n -elementary embedding $e_n : \mathcal{B}_n \rightarrow \mathcal{B}_{n+1}$ such that the following diagram commutes, for all $k \geq 1$:*

$$\begin{array}{ccc} \mathcal{C}_{n+1,n+k} & \longleftarrow & \mathcal{B}_{n+1} \\ \uparrow n & & \uparrow n \\ \mathcal{C}_{n,n+k} & \longleftarrow & \mathcal{B}_n \end{array}$$

Proof. An element of \mathcal{B}_n is a sequence $\vec{x} = (x_i)_{i \geq n}$ such that $x_i \in \mathcal{C}_{ni}$ and $x_i = f_{ij}(x_j)$ whenever $i < j$. We map \vec{x} to the sequence $(y_i)_{i > n}$ by setting $y_i := e_{i,i+1}(x_i)$. It is easy to check that this defines an embedding $e_n : \mathcal{B}_n \rightarrow \mathcal{B}_{n+1}$. We show that it is n -elementary.

Assume $\vec{x} \in \mathcal{B}_n$, $\vec{y} \in \mathcal{B}_{n+1}$ and $e_n(\vec{x}) R_k \vec{y}$. Consider $y_{n+1} \in \mathcal{C}_{n+1,n+1}$. Obviously $x_{n+1} R_k y_{n+1}$. Since $e_{n,n+1}$ is n -elementary, there is a $z_{n+1} \in \mathcal{C}_{n,n+1}$ such that $x_{n+1} R_k z_{n+1}$ and $e_{n,n+1}(z_{n+1}) \sim_{n-1} y_{n+1}$.

Using the properties of p-morphisms, we can construct a sequence of elements $\vec{z} := (z_i)_{i \geq n}$ such that $z_i \in \mathcal{C}_{ni}$, $x_i R_k z_i$ and $f_{ij}(z_j) = z_i$, whenever $i < j$. (Let $z_n := f_{n,n+1}(z_{n+1})$ and, for $i > n+1$, z_i will be obtained

by successively applying the second condition of p-morphisms.) Then, by Lemma 5.5, for all $i > n$, $e_{ni}(z_i) \sim_{n-1} y_i$. By Corollary 4.4 this implies $e_n(\vec{z}) \sim_{n-1} \vec{y}$ in \mathcal{B}_{n+1} . \square

By this lemma we obtain a chain of n -elementary embeddings

$$\mathcal{B}_1 \xrightarrow{1} \mathcal{B}_2 \xrightarrow{2} \mathcal{B}_3 \xrightarrow{3} \cdots$$

Let $\mathfrak{B}_\omega(\mathcal{A})$ denote the union (or rather the direct limit) of this chain. We can consider each \mathcal{B}_n as submodel of $\mathfrak{B}_\omega(\mathcal{A})$. From Corollary 6.8 we thus obtain

Corollary 7.10 *For each $n < \omega$, $\mathfrak{B}_n(\mathcal{A}) \leftarrow \mathcal{B}_n \prec_n \mathfrak{B}_\omega(\mathcal{A})$.*

Corollary 7.11 *$\mathfrak{B}_n(\mathcal{A})$ and $\mathfrak{B}_\omega(\mathcal{A})$ satisfy the same modal formulas φ such that $dp(\varphi) \leq n$.*

Corollary 7.12 *$\mathfrak{B}_\omega(\mathcal{A})$ enjoys the m -similarity property, for each $m < \omega$.*

Proof. Assume $xR_{k+1}y$ in $\mathfrak{B}_\omega(\mathcal{A})$. Then, for some $i < \omega$, $x, y \in \mathcal{B}_i$. Select n larger than both i and m . Since $\mathcal{B}_n \twoheadrightarrow \mathfrak{B}_n(\mathcal{A})$ and $\mathfrak{B}_n(\mathcal{A})$ enjoys the m -similarity property, so does \mathcal{B}_n . Hence, we can find a $z \in \mathcal{B}_n$ such that $z \sim_m y$ and $xR_k z$. By Lemma 6.3, the same relations hold in $\mathfrak{B}_\omega(\mathcal{A})$. \square

Corollary 7.13 *$\mathfrak{B}_\omega(\mathcal{A})$ is a model of GLP.*

Proof. Let φ be an axiom of **J** and $dp(\varphi) = n$. Clearly, φ is satisfied in $\mathfrak{B}_n(\mathcal{A})$. Hence, it also holds in $\mathfrak{B}_\omega(\mathcal{A})$. The validity of the monotonicity schema follows from Corollary 7.12. \square

8 Hereditarily linear models and the closed fragment of GLP

A stratified model is called *hereditarily linear* if R_0 is a linear ordering of the set of 1-planes and each 1-plane is hereditarily linear. It is not difficult to see that the blow-up operations preserve hereditary linearity of models. This yields a considerable simplification in the limit construction described in the previous section. Namely, one obtains a simpler formulation of Lemma ?? and Corollary 7.10 as follows.

Lemma 8.1 *If \mathcal{A} is a hereditarily linear model, then there is an n -elementary root-preserving embedding $\mathfrak{B}_n(\mathcal{A}) \rightarrow_n \mathfrak{B}_{n+1}(\mathcal{A})$, for each n .*

Thus, $\mathfrak{B}_\omega(\mathcal{A})$ in this case can be considered just as a union of elementary chain

$$\mathfrak{B}_1(\mathcal{A}) \prec_1 \mathfrak{B}_2(\mathcal{A}) \prec_2 \dots$$

Let \mathcal{A}_m be the model consisting of just two nodes a, b such that aR_mb . We ignore the evaluation of variables and work in the closed fragment of **GLP**. It can be shown that the models $\mathfrak{B}_\omega(\mathcal{A}_m)$ are isomorphic to the upper parts of a universal model for the closed fragment of **GLP** studied in [1]. In fact,

$$\mathfrak{B}_\omega(\mathcal{A}_m) \simeq \mathcal{U}_{\omega_m},$$

where $\omega_0 = 1$ and $\omega_{n+1} = \omega^{\omega_n}$. A careful proof of this claim is somewhat lengthy, and it is more natural to set it up in a context where the definition of blow-up is generalized to infinite upwards well-founded models. Therefore, we leave a proof outside the present paper. Notice that in this case one obtains hereditarily linear models such that their R_0 order types (on the set of 1-planes) approximate the ordinal ϵ_0 from below. The situation is similar with blowing up general finite stratified models.

9 Completeness results for GLP

In this section \mathcal{A} will always denote a finite hereditary rooted stratified model. We also write $\mathcal{A} \Vdash \varphi$ to mean $\mathcal{A}, a \Vdash \varphi$, where a is the hereditary root of \mathcal{A} .

First, we formulate a partial completeness theorem for **GLP**. Let **GLP** _{m} denote the system **GLP**, where the monotonicity schema (vi) is restricted to formulas φ of modal depth $dp(\varphi) \leq m$ only.

Lemma 9.1 *If $\mathbf{GLP}_m \vdash \varphi$ then, for all \mathcal{A} , for all $n > m$, $\mathfrak{B}_n(\mathcal{A}) \models \varphi$.*

Proof. An easy induction on the length of proof. To check the validity of the restricted monotonicity schema use Corollary 5.12 and Lemma 4.9. \square

Lemma 9.2 *If $dp(\varphi) \leq m + 1$ and $\mathbf{GLP}_m \not\vdash \varphi$, then there is a model \mathcal{A} such that $\mathfrak{B}_p(\mathcal{A}) \not\models \varphi$, for any p .*

Proof. Let $M(\varphi) := \bigwedge_{i < s} ([m_i]\varphi_i \rightarrow [m_i + 1]\varphi_i)$, where $[m_i]\varphi_i$ for $i < s$ are all subformulas of φ of the form $[k]\psi$. Further, let $M^+(\varphi) := M(\varphi) \wedge \bigwedge_{i \leq n} [i]M(\varphi)$ where $n := \max_{i < s} m_i$.

Clearly, $\mathbf{GLP}_m \vdash M^+(\varphi)$, for $dp(\varphi_i) \leq dp(\varphi) - 1 \leq m$, for all i . Hence, if $\mathbf{GLP}_m \not\vdash \varphi$ then $\mathbf{J} \not\vdash M^+(\varphi) \rightarrow \varphi$. By Corollary 3.2 there is a hereditarily

rooted model \mathcal{A} such that $\mathcal{A} \not\models M^+(\varphi) \rightarrow \varphi$. In other words, $\mathcal{A}, a \not\models \varphi$ and $\mathcal{A}, a \models M^+(\varphi)$, where a is the hereditary root of \mathcal{A} . It is easy to see that, in this case, $\mathcal{A} \models M(\varphi)$ (each point of \mathcal{A} different from a is accessible from a by some R_i).

We inductively define a projection function $\pi^* : \mathfrak{B}_p(\mathcal{A}) \rightarrow \mathcal{A}$ as follows. If \mathcal{A} is trivial, π^* is the identity mapping. Otherwise, recall that

$$\mathfrak{B}_p(\mathcal{A}) = \sum_{\alpha \in \mathcal{A}} \mathfrak{B}_p(\alpha)^{(p)}.$$

Let π_α^* be the corresponding projection associated with $\mathfrak{B}_p(\alpha)$ and let $\pi_\alpha : \mathfrak{B}_p(\alpha)^{(p)} \rightarrow \mathfrak{B}_p(\alpha)$ be the natural projection function defined in 5.6. Then we let $\pi^*(x) := \pi_\alpha^*(\pi_\alpha(x))$, for any $x \in \mathfrak{B}_p(\alpha)^{(p)}$.

Lemma 9.3 *For each subformula ψ of φ ,*

$$\forall x \in \mathfrak{B}_p(\mathcal{A}) \ (\mathfrak{B}_p(\mathcal{A}), x \Vdash \psi \iff \mathcal{A}, \pi^*(x) \Vdash \psi).$$

Proof. We slightly generalize the situation and prove the following auxiliary lemma.

Lemma 9.4 *Let \mathcal{B} be a hereditarily rooted stratified model such that $\mathcal{B} \models M(\varphi)$. Let β be one of its k -planes and α_i , for $i \in I$, be all the $k+1$ -planes in β , so that $\beta = \sum_{i \in I} \{\alpha_i\}$. Let \mathcal{B}' be a model obtained by replacing β in \mathcal{B} by $\beta' := \sum_{i \in I} \alpha_i^{(p)}$. Let $\pi : \mathcal{B}' \rightarrow \mathcal{B}$ be a natural projection function acting as the standard projections π_i on each of the models $\alpha_i^{(p)}$ and identical on all points other than those in β' . Then*

$$\forall x \in \mathcal{B}' \ (\mathcal{B}', x \Vdash \psi \iff \mathcal{B}, \pi(x) \Vdash \psi).$$

Proof. Induction on the build-up of ψ . If ψ is a variable or is obtained by a boolean connective, the result is clear. We consider the case $\psi = [n]\theta$.

Suppose $x \Vdash [n]\theta$, $x \in \mathcal{B}'$. Then, for all $y \in \mathcal{B}'$ such that $xR_n y$, $y \Vdash \theta$. Assume $y' \in \mathcal{B}$ and $\pi(x)R_n y'$. We claim that there is a $y \in \mathcal{B}'$ such that $\pi(y) = y'$. If one of $\pi(x), y'$ is outside β the claim is easy. If both of them are in β then $n \geq k$. If $n = k$, $\pi(x) \in \alpha_i$ and $y' \in \alpha_j$ for different indices i, j . Then one can take any $y \in \pi_j^{-1}(y')$ and one obviously has $xR_n y$ in \mathcal{B}' . If $n > k$, the claim follows from the fact that each π_i restricted to any $k+1$ -plane in $\alpha_i^{(p)}$ is an end-embedding into α_i , by Lemma 5.7. Hence, we can find a suitable y within the $k+1$ -plane of $x \in \mathcal{B}'$.

By the induction hypothesis, since $y \Vdash \theta$, we have $y' \Vdash \theta$. This holds for all y' such that $\pi(x)R_n y'$, hence $\pi(x) \Vdash [n]\theta$.

Suppose $x \not\models [n]\theta$, $x \in \mathcal{B}'$. Then there is a $y \in \mathcal{B}'$ such that $xR_n y$ and $y \not\models \theta$. Again, we only consider the case when both $x, y \in \beta'$, for otherwise one easily obtains $\pi(x)R_n\pi(y)$ and $\pi(y) \not\models \theta$ by the induction hypothesis. If $x, y \in \beta'$ then $n \geq k$, since β' is a k -plane. In case $n > k$ one also obtains $\pi(x)R_n\pi(y)$ by Lemma 5.7.

Suppose now that $n = k$. Consider two subcases:

CASE 1: $x \in \alpha_i^{(p)}$ and $y \in \alpha_j^{(p)}$ for some $i, j \in I$ such that $i \neq j$. Then clearly $\pi(x)R_n\pi(y)$ in \mathcal{B} and we are done.

CASE 2: $x, y \in \alpha_i^{(p)}$. Since $xR_n y$ we have $y \in \hat{\alpha}_i$ and by the induction hypothesis $\mathcal{B}, y \not\models \theta$. Consider the hereditary root a of α_i . Since $y \in \hat{\alpha}_i$ and α_i is a $n + 1$ -plane in β , we have $aR_{n+1}\pi(y)$. Hence, $a \not\models [n + 1]\theta$. Since $a \Vdash M(\varphi)$ it follows that $a \not\models [n]\theta$. Then, there is a $z \in \mathcal{B}$ such that $aR_n z$ and $z \not\models \theta$. Since both a and $\pi(x)$ belong to the same $n + 1$ -plane α_i , we also have $\pi(x)R_n z$. Hence, $\pi(x) \not\models [n]\theta$. \square

Obviously, Lemma 9.4 can be applied successively to all the k -planes β of \mathcal{B} . Let us call the resulting model $\mathfrak{P}_k(\mathcal{B})$ and the associated projection function $\pi_k : \mathfrak{P}_k(\mathcal{B}) \rightarrow \mathcal{B}$. Then by Lemma 9.4, for each subformula ψ of φ ,

$$\mathfrak{P}_k(\mathcal{B}), x \Vdash \psi \iff \mathcal{B}, \pi_k(x) \Vdash \psi.$$

Now we prove Lemma 9.3. Define: $\mathcal{A}_0 := \mathcal{A}$, $\mathcal{A}_{i+1} := \mathfrak{P}_{r-i}(\mathcal{A}_i)$, where r is the rank of \mathcal{A} . Obviously, the evaluation of subformulas of φ is also preserved in all \mathcal{A}_i .

We claim: for each i , \mathcal{A}_i is obtained by replacing all $(r - i)$ -planes β of \mathcal{A} by $\mathfrak{B}_p(\beta)$. Indeed, if $i = 0$, β is trivial and $\mathfrak{B}_p(\beta) = \beta$. For the induction step, \mathcal{A}_{i+1} is obtained by replacing each $(r - i - 1)$ -plane $\beta = \sum_{\alpha \in \beta} \{\alpha\}$ in \mathcal{A}_i by $\sum_{\alpha \in \beta} \alpha^{(p)}$. By the induction hypothesis, \mathcal{A}_i is obtained from \mathcal{A} by replacing all $(r - i)$ -planes α in \mathcal{A} by $\mathfrak{B}_p(\alpha)$. Hence, $\beta = \sum_{\alpha \in \beta'} \{\mathfrak{B}_p(\alpha)\}$, for some $(r - i - 1)$ -plane β' in \mathcal{A} . Then \mathcal{A}_{i+1} is obtained by replacing each β' by $\sum_{\alpha \in \beta'} \mathfrak{B}_p(\alpha)^{(p)} = \mathfrak{B}_p(\beta')$, as required.

As an immediate corollary we obtain $\mathcal{A}_r = \mathfrak{B}_p(\mathcal{A})$. It is also easy to prove that $\pi^* = \pi_1 \circ \pi_2 \circ \dots \circ \pi_r$. This proves the lemma. \square

From Lemma 9.3 we obtain $\mathfrak{B}_p(\mathcal{A}) \not\models \varphi$, which proves the lemma. \square

Combining Lemmas 9.1 and 9.2 we obtain the following result.

Theorem 3 *Suppose $dp(\varphi) \leq m + 1$. Then $\mathbf{GLP}_m \vdash \varphi$ if and only if, for all \mathcal{A} , $\mathfrak{B}_{m+1}(\mathcal{A}) \models \varphi$.*

Now we prove our main theorem.

Theorem 4 *The following statements are equivalent:*

- (i) $\mathbf{GLP} \vdash \varphi$;
- (ii) $\mathbf{J} \vdash M^+(\varphi) \rightarrow \varphi$;
- (iii) For all \mathcal{A} , $\mathfrak{B}_\omega(\mathcal{A}) \models \varphi$;
- (iv) For all \mathcal{A} , there is an m such that $\mathfrak{B}_m(\mathcal{A}) \models \varphi$;
- (v) For all \mathcal{A} , $\mathfrak{B}_{m_0}(\mathcal{A}) \models \varphi$ where $m_0 = dp(\varphi)$.

Proof. (ii) \Rightarrow (i) and (v) \Rightarrow (iv) are obvious.

(iv) \Rightarrow (ii): If $\mathbf{J} \not\vdash M^+(\varphi) \rightarrow \varphi$, then there is a model \mathcal{A} such that $\mathcal{A} \not\models M^+(\varphi) \rightarrow \varphi$. As in the proof of Theorem 9.2, by Lemma 9.3 we obtain $\mathfrak{B}_m(\mathcal{A}) \not\models \varphi$, for each $m \geq 1$.

(iii) \Rightarrow (v) If $\mathfrak{B}_{m_0}(\mathcal{A}) \not\models \varphi$, where $m_0 = dp(\varphi)$, then by Corollary 7.11 $\mathfrak{B}_\omega(\mathcal{A}) \not\models \varphi$.

(i) \Rightarrow (iii): by Corollary 7.13, $\mathfrak{B}_\omega(\mathcal{A})$ is a model of \mathbf{GLP} . \square

We remark that in Statements (iii)–(v) the quantifier over all models \mathcal{A} can be bounded. The bound depends on the size of φ , which provides a decision procedure for \mathbf{GLP} .

Corollary 9.5 *\mathbf{GLP} is conservative over \mathbf{GLP}_m for formulas φ such that $dp(\varphi) \leq m + 1$.*

Proof. If $\mathbf{GLP}_m \not\vdash \varphi$ and $dp(\varphi) \leq m + 1$, by Lemma 9.2 there is a model \mathcal{A} such that $\mathfrak{B}_n(\mathcal{A}) \not\models \varphi$, for each n . By Part (iii) of the main theorem, this yields $\mathbf{GLP} \not\vdash \varphi$. \square

Open question. What is the optimal complexity of the decision procedure for \mathbf{GLP} ? Does \mathbf{GLP} belong to PSPACE?

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