Primitive Recursive Realizability and Basic Propositional Logic

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Abstract

Two notions of primitive recursive realizability for arithmetic sentences are considered. The first one is strictly primitive recursive realizability introduced by Z. Damnjanovic in 1994. We prove that intuitionistic predicate logic is not sound with this kind of realizability. Namely there exists an arithmetic sentence which is deducible in the intuitionistic predicate calculus but is not strictly primitive recursively realizable. Another variant of primitive recursive realizability was introduced by S. Salehi in 2000. This kind of realizability is defined for the formulas of Basic Arithmetic introduced by W. Ruitenburg in 1998. We prove that these two notions of primitive recursive realizability are essentially different. Namely there exists arithmetic sentence being also a sentence of Basic Arithmetic which is strictly primitive recursively realizable but is not realizable by Salehi. The negation of such a sentence is realizable by Salehi but is not strictly primitive recursively realizable. The relation between Basic Propositional Logic and strictly primitive recursive realizability is studied. We consider a sequent variant of Basic Propositional Calculus. Notions of strictly primitive recursive realizability for arithmetic and propositional sequents are defined. We prove that every sequent deducible in Basic Propositional Calculus is strictly primitive recursively realizable. An example of a sequent which is deducible in Intuitionistic Propositional Calculus but is not strictly primitive recursively realizable is proposed.

1 Introduction

Informal Intuitionistic Semantics

Realizability semantics have their origin in the informal intuitionistic semantics of mathematical sentences. From the intuitionistic point of view a sentence is true if it is proved. Thus the truth of a sentence is connected with its proof. In order to avoid any confusion with formal proofs we shall use the term "a justification" instead of "a proof". Such understanding of the meaning of sentences

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leads to an original interpretation of logical connectives and quantifiers stated by L. E. J. Brouwer, A. N. Kolmogorov, and A. Heyting. Namely for every true sentence A we can consider its justification as a text justifying A. Now if A and B are sentences, then

- a justification of a conjunction $A \,\&\, B$ is a text containing a justification of A and a justification of B;
- a justification of a disjunction $A \vee B$ is a text containing a justification of A or a justification of B and indicating what of them is justified;
- a justification of an implication $A \rightarrow B$ is a text describing a general effective method of obtaining a justification of B from every justification of A;
- a justification of a negative sentence $\neg A$ is a justification of the sentence $A \rightarrow \bot$, where $\bot$ is a certainly absurd sentence having no justification.

If $A(x)$ is a predicate with a parameter x over a domain M given in appropriate way, then

- a justification of a universal sentence $\forall x A(x)$ is a text describing a general effective method which allows to obtain a justification of $A(m)$ for every given $m \in M$;
- a justification of an existential sentence $\exists x A(x)$ is a text indicating a concrete $m \in M$ and containing a justification of the sentence $A(m)$.

Of course, this semantics is very informal and is not precise from the mathematical point of view. It can be made more precise if we define in a mathematical mode two key notions used in the above description of the informal semantics, namely the notions of a justification and a general effective operation. One way is to consider justifications as formal proofs in a formal system. This was done by S. N. Artemov [2] who introduced *the logic of proofs*. Operations over proofs are also defined in this approach. Logic of proofs is considered now as a final point in the development of intuitionistic semantics.

Recursive Realizability by Kleene

Another way of a specification of the informal intuitionistic semantics is to define a mathematically precise notion of a general effective operation. There is such notion in mathematics, namely the notion of algorithm. Thus it is enough to make justifications suitable for application to them of algorithms. This was done by S. C. Kleene [6] who introduced in 1945 the notion of *recursive realizability*. The main idea of Kleene was to consider natural numbers as the codes of justifications and partial recursive functions as general effective operations. Partial recursive functions are coded by natural numbers by means of the Gödel enumeration. A code of a justification of a sentence is called *a realization* of the sentence. The relation "a natural number e realizes a closed arithmetic formula A" (denoted as $e\,\mathbf{r}\,A$) is defined by induction on the number of logical connectives and quantifiers in A.

- If A is an atomic sentence $t_1 = t_2$, then $e\,\mathbf{r}\,A \rightleftharpoons [e = 0$ and A is true$]$.

Let A and B be arithmetic sentences. Then

- $e\,\mathbf{r}\,(A \,\&\, B) \rightleftharpoons [e$ is of the form $2^a \cdot 3^b$ and $a\,\mathbf{r}\,A$, $b\,\mathbf{r}\,B]$;
- $e\,\mathbf{r}\,(A \vee B) \rightleftharpoons [e$ is of the form $2^a \cdot 3^0$ and $a\,\mathbf{r}\,A$ or e is of the form $2^b \cdot 3^1$ and $b\,\mathbf{r}\,B]$;
- $e\,\mathbf{r}\,(A \to B) \rightleftharpoons [e$ is the Gödel number of a partial recursive function ϕ such that for every a if $a\,\mathbf{r}\,A$, then $\phi(a)\,\mathbf{r}\,B]$;
- $e\,\mathbf{r}\,\neg A \rightleftharpoons [e\,\mathbf{r}\,(A \to 0 = 1)]$.

Let $A(x)$ be an arithmetic formula with the only parameter x. Then

- $e\,\mathbf{r}\,\forall x A(x) \rightleftharpoons [e$ is the Gödel number of a general recursive function f such that $f(n)\,\mathbf{r}\,A(n)$ for every $n]$;
- $e\,\mathbf{r}\,\exists x A(x) \rightleftharpoons [e$ is of the form $2^n \cdot 3^a$ and $a\,\mathbf{r}\,A(n)]$.

An arithmetic sentence A is called *realizable* if there is e such that $e\,\mathbf{r}\,A$. An arithmetic formula $A(x_1, \dots, x_m)$ with the only parameters $x_1, \dots, x_m$ is called realizable if the sentence $\forall x_1 \dots \forall x_m A(x_1, \dots, x_m)$ is realizable.

Various notions of recursive realizability for predicate and propositional formulas can be defined. Let $\mathcal{A}$ be a predicate formula with predicate variables $P_1, \dots, P_n$; denote it by $\mathcal{A}(P_1, \dots, P_n)$. An arithmetic instance of $\mathcal{A}$ is every formula $\mathcal{A}(A_1, \dots, A_n)$ obtained from $\mathcal{A}$ by substituting predicates presented by arithmetic formulas $A_1, \dots, A_n$ for the predicate variables $P_1, \dots, P_n$. A closed predicate formula $\mathcal{A}$ is called

- *realizable* if every arithmetic instance of $\mathcal{A}$ is realizable;
- *effectively realizable* if there exists an algorithm allowing to find a realization of every closed arithmetic instance of $\mathcal{A}$;
- *uniformly realizable* if there exists a natural number realizing every closed arithmetic instance of $\mathcal{A}$.

Obviously, if a predicate formula is uniformly realizable, then it is effectively realizable and every effectively realizable predicate formula is realizable. These three notions are essentially different: there are realizable predicate formulas which are not effectively realizable and there are effectively realizable predicate formulas which are not uniformly realizable (see [11]). The problem of relations between these notions for propositional formulas is still open.

Some facts on recursive realizability have to be mentioned.

1. Every formula deducible in Intuitionistic Arithmetic HA is realizable (D. Nelson [8]).

2. There exists a propositional formula which is uniformly realizable but is not deducible in Intuitionistic Propositional Calculus IPC (G. F. Rose [14]).

3. The set of (the Gödel numbers of) realizable (effectively realizable, uniformly realizable) predicate formulas is not arithmetical (V. E. Plisko [10], [11]).

It follows from the first fact that intuitionistic logic is sound with recursive realizability, but the second and the third ones mean its incompleteness relative to this kind of semantics.

Primitive Recursive Realizability

Rather negative result on incompleteness of intuitionistic propositional logic relative to recursive realizability provoked considering variants of realizability using other classes of computable functions instead of the whole class of partial recursive functions. The class of primitive recursive functions is one of them. Unfortunately, there is a serious obstacle for defining a notion of realizability based on the primitive recursive functions because there is no primitive recursive function universal for the primitive recursive functions. Of course, we could consider a general recursive enumeration of the primitive recursive functions and formally define an analog of Kleene's realizability. But in this case applying an index of the function to an argument would not be a primitive recursive operation. Nevertheless such a definition of primitive recursive realizability was considered. As it follows from a private communication by F. L. Varpakhovskii, a primitive recursive variant of realizability was studied by him and other Moscow logicians in the early 1970th and it was found that IPC is not sound with this kind of realizability. This result was considered as absolutely negative and even was not published. Later the same approach to the primitive recursive realizability was used by S. Salehi [16] for an interpretation of Basic Arithmetic BA.

Another approach to the primitive recursive realizability was proposed by Z. Damnjanovic [4]. He used the ideas of Kleene's realizability and Kripke models. In order to avoid the obstacle mentioned above, Damnjanovic considers a hierarchy of the primitive recursive functions introduced by A. Grzegorczyk [5] and improved by P. Axt [3]. The class of primitive recursive functions is presented as a union of the classes $\mathbf{E}_n$ $(n = 0, 1, 2, \ldots)$ such that for every class $\mathbf{E}_n$ there exists a universal function in the next class $\mathbf{E}_{n+1}$ and one can use indexing of the functions in the class $\mathbf{E}_n$ relative to this universal function. In this case, application of an index of the function in a given class to an argument becomes a primitive recursive operation even though from the next class. This was a reason to call Damnjanovic's relizability *strictly primitive recursive realizability.*

We prove that Intuitionistic Predicate Calculus is not sound with the strictly primitive recursive realizability but Basic Propositional Calculus BPC is sound. We consider a sequent variant of BPC and define the notions of strictly primitive recursive realizability for the formulas and sequents of BA and for propositional sequents. It is proved that every sequent deducible in BPC is strictly primitive recursively realizable. An example of a sequent deducible in IPC which is not strictly primitive recursively realizable is proposed. The problem of soundness of Basic Predicate Calculus with strictly primitive recursive realizability is still open.

Predicate logics of primitive recursive realizabilities by Damnjanovic and by Salehi were studied by B. H. Park [9] and D. Viter [18]. It was proved that both logics are not arithmetical. Thus a question on coincidence of two these variants of primitive recursive realizability is rather actual. We prove that these variants are essentially different. Namely we propose a sentence in the common part of the ordinary language of arithmetic and the language of BA which is strictly primitive recursively realizable but is not realizable by Salehi. The negation of this sentence is realizable by Salehi but is not strictly primitive recursively realizable.

Many of the results presented in this paper are already published (see [12], [13]). The purpose of the paper is a self-contianed exposition of results and problems connected with primitive recursive realizabilities especially in view of some open problems in this area. We begin with a description of an indexing of the primitive recursive functions equally acceptable for both notions of primitive recursive realizability.

2 Indexing of the Primitive Recursive Functions

Primitive recursive functions are the functions obtained by substitution and recursion from the *basic functions*: the constant function $O(x) = 0$, the successor function $S(x) = x + 1$, the projection functions $I_i^m(x_1, ..., x_m) = x_i$, where $m = 1, 2, ...$, $1 \le i \le m$. The class of *elementary* (by Kalmar) functions is defined as the least class containing the constant $f(x) = 1$, the functions I_i^m, and the functions $f(x, y) = x + y$, $f(x, y) = x \div y$, where

$$x \div y = \begin{cases} 0 & \text{if} \quad x < y, \\ x - y & \text{if} \quad x \ge y, \end{cases}$$

and closed under the operations of substitution, summation

$$\varphi(\bar{x}, y) = \sum_{i=0}^{y} \psi(\bar{x}, i),$$

and multiplication

$$\varphi(\bar{x}, y) = \prod_{i=0}^{y} \psi(\bar{x}, i),$$

$\bar{x}$ being the list $x_1, \ldots, x_m$. If $a_0, \ldots, a_n$ are natural numbers, then $\langle a_0, \ldots, a_n \rangle$ denotes the number $p_0^{a_0} \cdot \ldots \cdot p_n^{a_n}$, where $p_0, \ldots, p_n$ are sequential prime numbers ($p_0 = 2$, $p_1 = 3$, $p_2 = 5, \ldots$). Note that the functions $\pi(i) = p_i$ and $f(x, y) = \langle x, y \rangle$ are elementary. For $a \ge 1$ and $i \ge 0$ let $[a]_i$ denote the exponent of p_i under decomposition of a into prime factors. Therefore, $[a]_i = a_i$ if $a = \langle a_0, \ldots, a_n \rangle$. For definiteness let $[0]_i = 0$ for every i. Note that the function $\mathsf{exp}(x, i) = [x]_i$ is elementary.

An $(m+1)$-ary function f is obtained by *bounded recursion* from an m-ary function g, an $(m+2)$-ary function h, and an $(m+1)$-ary function j if the following conditions are fulfilled:

$$f(0, x_1, \ldots, x_m) = g(x_1, \ldots, x_m),$$

$$f(y+1, x_1, \ldots, x_m) = h(y, f(y, x_1, \ldots, x_m), x_1, \ldots, x_m),$$
$$f(y, x_1, \ldots, x_m) \leq j(y, x_1, \ldots, x_m)$$

for every $x_1, \ldots, x_m, y$. Thus f is obtained by bounded recursion from g, h, j if f is obtained by primitive recursion from g, h and is bounded by j.

For given functions $\theta_1, \ldots, \theta_k$ let $\mathbf{E}[\theta_1, \ldots, \theta_k]$ denote the least class containing $\theta_1, \ldots, \theta_k$, the successor function S, all the constant functions, and all the projection functions I_i^m and closed under substitution and bounded recursion. Consider a sequence of functions

$$f_0(x, y) = y + 1,$$

$$f_1(x, y) = x + y,$$
$$f_2(x, y) = (x+1) \cdot (y+1),$$
$$f_{n+1}(y, 0) = f_n(y+1, y+1),$$
$$f_{n+1}(y, x+1) = f_{n+1}(f_{n+1}(x, y), x)$$

for $n \geq 2$. A. Grzegorczyk [5] introduced a hierarchy of the classes of functions $\mathcal{E}^n$, where $\mathcal{E}^n = \mathbf{E}[f_n]$. The class $\mathcal{E}^3$ contains all the elementary functions. It was shown by Grzegorczyk [5] that the union of the classes $\mathcal{E}^n$ is exactly the class of primitive recursive functions.

P. Axt [3] improved the description of the Grzegorczyk hierarchy in two directions. Obviously, in general, we can not know if there exists a function f obtained by the bounded recursion from the given functions g, h, j. Axt has shown that for $n \geq 4$ the usual bounded recursion in the definition of the classes $\mathcal{E}^n$ can be replaced by the following scheme applicable to every triple of the functions g, h, j of appropriate arities:

$$f(0, \bar{x}) = g(\bar{x}),$$

$$f(y+1, \bar{x}) = h(y, f(y, \bar{x}), \bar{x}) \cdot \mathsf{sg}(j(y, \bar{x}) \div f(y, \bar{x})) \cdot \mathsf{sg}(f(y, \bar{x})),$$

where $\mathsf{sg}(x) = \begin{cases} 0 & \text{if} \quad x = 0, \\ 1 & \text{if} \quad x > 0. \end{cases}$

The second improvement proposed by Axt was the construction of the Grzegorczyk classes by a general scheme of constructing hierarchies of the classes of functions described by Kleene [7]. Namely for every collection of functions $\Theta = \{\theta_1, \ldots, \theta_m\}$ we denote by $\mathbf{E}^4[\Theta]$ the least class including Θ, containing S, all the constant functions and the projection functions, the functions sg, $\div$, f_4 and closed under substitution and Axt's bounded recursion. A way of indexing the functions which are primitive recursive relative to $\theta_1, \ldots, \theta_m$ is proposed in [7]. This way can be adapted to an indexing of the class $\mathbf{E}^4[\Theta]$. The functions in $\mathbf{E}^4[\Theta]$ obtain indexes according to their definition from the initial functions by substitution and Axt's bounded recursion. We list below the possible defining schemes for a function φ and specify on the right its index.

( )	$\varphi(x_1,\dots,x_{k_i}) = \theta_i(x_1,\dots,x_{k_i})$	$\langle 0, k_i, i\rangle$
(I)	$\varphi(x) = x+1$	$\langle 1, 1\rangle$
(II)	$\varphi(x_1,\dots,x_n) = q$	$\langle 2, n, q\rangle$
(III)	$\varphi(x_1,\dots,x_n) = x_i$ (where $1 \le i \le n$)	$\langle 3, n, i\rangle$
(IV)	$\varphi(x) = \mathsf{sg}(x)$	$\langle 4, 1\rangle$
(V)	$\varphi(x,y) = x \div y$	$\langle 5, 2\rangle$
(VI)	$\varphi(x,y) = f_4(x,y)$	$\langle 6, 2\rangle$
(VII)	$\varphi(\bar{x}) = \psi(\chi_1(\bar{x}), ..., \chi_k(\bar{x}))$	$\langle 7, m, g, h_1, \dots, h_k\rangle$
(VIII)	$\begin{cases} \varphi(0,\bar{x}) = \psi(\bar{x}) \\ \varphi(y+1,\bar{x}) = \chi(y, f(y,\bar{x}), \bar{x})\times \\ \times\mathsf{sg}(\zeta(y,x) \div \varphi(y,\bar{x})) \cdot \mathsf{sg}(\varphi(y,\bar{x})) \end{cases}$	$\langle 8, m+1, g, h, j\rangle$

Here $\bar{x} = x_1,\dots,x_m$, $g, h_1,\dots,h_k, h, j$ are indexes of the functions ψ, $\chi_1,\dots,\chi_k$, χ, ζ.

Let $In^\Theta(b)$ mean that b is an index of some function in the described indexing of the class $\mathbf{E}^4[\Theta]$. It is shown in [3] that $In^\Theta(b)$ is an elementary predicate. If $In^\Theta(b)$, then ef_b^Θ denotes the $[b]_1$-ary function in $\mathbf{E}^4[\Theta]$ indexed by b. Following [3] we set

$$\mathsf{ef}^\Theta(b,a) = \begin{cases} \mathsf{ef}_b^\Theta([a]_0,\dots,[a]_{[b]_1 \div 1}) & \text{if} \quad In^\Theta(b), \\ 0 & \text{else.} \end{cases}$$

Therefore the function ef^Θ is universal for the class $\mathbf{E}^4[\Theta]$ and is not in this class. Obviously, every function in $\mathbf{E}^4[\Theta]$ has infinitely many indexes relative to the universal function ef^Θ. Now following Axt we define for every n a function of two variables e_n by letting

$$\mathsf{e}_0(b,a) = 0,$$

$$\mathsf{e}_{n+1}(b,a) = \mathsf{ef}^{\{\mathsf{e}_0,\dots,\mathsf{e}_n\}}(b,a).$$

Finally, the class $\mathbf{E}_n$ is defined as $\mathbf{E}^4[\mathsf{e}_0,\dots,\mathsf{e}_n]$. Axt proved (see [3, p. 58]) that $\mathbf{E}_n = \mathcal{E}^{n+4}$ for every $n \ge 0$.

Let $In(n,b)$ mean that b is an n-index, i.e. an index of a function in $\mathbf{E}_n$. It is shown in [3] that the predicate $In(n,b)$ is elementary. It follows immediately from the definition of the indexing of the classes $\mathbf{E}^4[\Theta]$ that every n-index of a function is also an m-index of the same function for every $m > n$, thus $In(n,b)$ and $m > n$ imply $In(m,b)$. Note that if an n-index b of a primitive recursive function $\varphi(\bar{x})$ is given, we can effectively find the value $\varphi(\bar{m})$ for every $\bar{m}$. Indeed, as it follows from the definition of indexing the class $\mathbf{E}_n$, computing the value $\varphi(\bar{x})$ is reduced to computing the values of a finite number of functions whose indexes are less than b. Thus the process of calculation must terminate.

We see that the function e_{n+1} is not in the class $\mathbf{E}_n$ but for every b the unary function $\psi_b^n(x) = \mathsf{e}_{n+1}(b, x)$ is in $\mathbf{E}_n$. Namely $\mathsf{e}_{n+1}(b, x)$ is the constant function $O(x) = 0$ if b is not an index of a function in $\mathbf{E}_n$ and $\mathsf{e}_{n+1}(b, x)$ is the function $\varphi([x]_0, \ldots, [x]_{m-1})$ if b is an index of an m-ary function $\varphi \in \mathbf{E}_n$. Note that an n-index of the function ψ_b^n can be found primitive recursively from b. Namely if b is an n-index of an m-ary function φ, let d_0 be a 0-index of the function exp and $k_i = \langle 7, 1, d_0, \langle 3, 1, 1\rangle, \langle 2, 1, i\rangle\rangle$. Obviously, k_i is a 0-index of the function $[x]_i$; then $\langle 7, 1, b, k_0, \ldots, k_{m-1}\rangle$ is an n-index of $\varphi([x]_0, \ldots, [x]_{m-1})$. Unfortunately, m depends on b. We avoid this obstacle by defining a function $\xi(b, i)$ as

$$\xi(b, 0) = \langle 7, 1, b, k_0\rangle,$$

$$\xi(b, i+1) = \xi(b, i) \cdot p_{i+4}^{k_{i+1}}.$$

Thus $\xi(b, i) = \langle 7, 1, b, k_0, \ldots, k_i\rangle$. Let $\varepsilon(b) = \xi(b, [b]_1 \dot{-} 1)$. We see that if b is an n-index of an m-ary function φ, then $\varepsilon(b)$ is an n-index of the unary function $\varphi([x]_0, \ldots, [x]_{m-1})$. Now let $\mathsf{in}(n, b)$ be an elementary function such that $\mathsf{in}(n, b) = 1$ if $In(n, b)$ and $\mathsf{in}(n, b) = 0$ else. We define a binary function α in the following way:

$$\alpha(n, b) = (1 \dot{-} \mathsf{in}(n, b)) \cdot \langle 2, 1, 0\rangle + \mathsf{in}(n, b) \cdot \varepsilon(b).$$

Clearly, $\alpha(n, b)$ is an n-index of ψ_b^n and α is a primitive recursive function. Let $\alpha_n(b) = \alpha(n, b)$.

Following [7] consider the function

$$\mathsf{sb}_n^m(z, y_1, \ldots, y_m) = \langle 7, n, z, \langle 2, n, y_1\rangle, \ldots, \langle 2, n, y_m\rangle, \langle 3, n, 1\rangle, \ldots, \langle 3, n, n\rangle\rangle.$$

Obviously, this function is elementary and when z is a k-index of a primitive recursive function $\varphi(y_1, \ldots, y_m, x_1, \ldots, x_n)$, then for each fixed $y_1, \ldots, y_m$ the number $\mathsf{sb}_n^m(z, y_1, \ldots, y_m)$ is a k-index of the function

$$\phi(x_1, \ldots, x_n) = \varphi(y_1, \ldots, y_m, x_1, \ldots, x_n).$$

If e is a k-index of an $(m+n)$-ary function $\varphi(y_1, \ldots, y_m, x_1, \ldots, x_n)$, let us write $\Lambda x_1, \ldots, x_n.\varphi(y_1, \ldots, y_m, x_1, \ldots, x_n)$ for $\mathsf{sb}_n^m(e, y_1, \ldots, y_m)$. Thus

$$\Lambda x_1, \ldots, x_n.\varphi(y_1, \ldots, y_m, x_1, \ldots, x_n)$$

is an elementary function of $y_1, \ldots, y_m$ and for every lists of natural numbers $\bar{n} = n_1, \ldots, n_m$, $\bar{k} = k_1, \ldots, k_n$ we have

$$\mathsf{e}_{k+1}(\Lambda x_1, \ldots, x_n.\varphi(\bar{n}, x_1, \ldots, x_n), \langle \bar{k}\rangle) = \varphi(\bar{n}, \bar{k}).$$

3 Strictly Primitive Recursive Realizability

Definition of Strictly Primitive Recursive Realizability

Damnjanovic [4] defined a relation "t realizes A at the level n" denoted as $t \Vdash_n A$, where t, n are natural numbers, A is a closed formula of the first-order language of arithmetic containing symbols for all the primitive recursive functions and using logical symbols $\&, \vee, \rightarrow, \forall, \exists$, the formula $\neg A$ being considered as an abbreviation for $A \rightarrow 0 = 1$. The relation $t \Vdash_n A$ is defined by induction on the number of logical symbols in A.

- If A is atomic, then $t \Vdash_n A \rightleftharpoons [t = 0$ and A is true$]$.
- $t \Vdash_n (B \,\&\, C) \rightleftharpoons [[t]_0 \Vdash_n B$ and $[t]_1 \Vdash_n C]$.
- $t \Vdash_n (B \vee C) \rightleftharpoons [[t]_0 = 0$ and $[t]_1 \Vdash_n B$ or $[t]_0 = 1$ and $[t]_1 \Vdash_n C]$.
- $t \Vdash_n (B \rightarrow C) \rightleftharpoons [In(n,t)$ and $(\forall j \geq n) In(j, \mathsf{e}_{j+1}(t, \langle j \rangle))$ and $(\forall j \geq n)\, \forall b\, (b \Vdash_j B \Rightarrow \mathsf{e}_{j+1}(\mathsf{e}_{j+1}(t, \langle j \rangle), \langle b \rangle) \Vdash_j C)]$.
- $t \Vdash_n \exists x B(x) \rightleftharpoons [[t]_1 \Vdash_n B([t]_0)]$.
- $t \Vdash_n \forall x B(x) \rightleftharpoons [In(n,t)$ and $\forall m\, \mathsf{e}_{n+1}(t, \langle m \rangle) \Vdash_n B(m)]$.

A closed arithmetic formula A is called *strictly primitive recursively realizable* if $t \Vdash_n A$ for some t and n. Note that it follows from $t \Vdash_n A$ that $t \Vdash_m A$ for every $m > n$. Let us say that A is *realizable at the level* n if $t \Vdash_n A$ for some t.

We see that the ideas of Kleene's realizability and Kripke models are used in the definition of strictly primitive recursive realizability. Similarity to the Kripke models is demonstrated in the clause for implication. Namely we can imagine a series of worlds indexed by the naturals such that computable functions in the n-th world are just the primitive recursive functions in the class $\mathbf{E}_n$. Thus a realization of the sentence $B \rightarrow C$ at the level n is (an n-index of) a computable function f admissible at this level such that for every $j \geq n$ the value $f(j)$ is a j-index of a computable function admissible at the level j which acts at the level j as Kleene's realization, namely it maps every realization of B at the level j into a realization of C at the same level.

Some facts are immediately implied by the definition of strictly primitive recursive realizability.

Proposition 3.1 *1) If a closed arithmetic formula A is not strictly primitive recursively realizable, then $a_0 \Vdash_0 \neg A$, where $a_0 = \langle 2, 1, \langle 2, 1, 0 \rangle \rangle$.*

2) For every closed arithmetic formula A if the formula $\neg A$ is strictly primitive recursively realizable, then A is not strictly primitive recursively realizable.

3) A closed arithmetic formula A is strictly primitive recursively realizable if and only if $\neg\neg A$ is strictly primitive recursively realizable.

4) An arithmetic formula A is strictly primitive recursively realizable if and only if $a_0 \Vdash_0 \neg\neg A$.

Proof. 1) Let a closed arithmetic formula A be not strictly primitive recursively realizable. We have to prove that $a_0 \Vdash_0 (A \rightarrow 0 = 1)$. Obviously, a_0 is a 0-index of the constant function f whose only value is $\langle 2, 1, 0 \rangle$. Thus $In(0, a_0)$ holds. We prove that for every $j \geq 0$ the number $f(j)$ is a j-index of a function g such that for every b

$$b \Vdash_j A \Rightarrow g(b) \Vdash_j 0 = 1. \tag{1}$$

Indeed, $f(j) = \langle 2, 1, 0 \rangle$ is a 0-index (therefore also a j-index) of a constant function $g(x) = 0$. Thus $In(j, f(j))$ holds. As A is not strictly primitive recursively realizable, the condition (1) is obviously fulfilled because the premise is false for every b.

2) Let the formula $\neg A$ be strictly primitive recursively realizable at a level n_1. Suppose that A is also strictly primitive recursively realizable at a level n_2. Then the formulas $\neg A$ and A are both realizable at the level $n = \max(n_1, n_2)$. Let $a \Vdash_n A$, $b \Vdash_n \neg A$, i.e. $b \Vdash_n (A \to 0 = 1)$. Then it follows from the definition of strictly primitive recursive realizability that $\mathsf{e}_{n+1}(\mathsf{e}_{n+1}(b, \langle n \rangle), \langle a \rangle) \Vdash_n 0 = 1$ and $0 = 1$ is realizable but this is impossible as it follows from the clause for atomic formulas in the definition of strictly primitive recursive realizability. Thus A is not strictly primitive recursively realizable if $\neg A$ is strictly primitive recursively realizable.

3) Let A be strictly primitive recursively realizable. Then it follows from 2) that $\neg A$ is not strictly primitive recursively realizable and it follows from 1) that $\neg\neg A$ is strictly primitive recursively realizable. Conversely, if $\neg\neg A$ is strictly primitive recursively realizable, then it follows from 2) that $\neg A$ is not strictly primitive recursively realizable. But if A is not strictly primitive recursively realizable, then it follows from 1) that $\neg A$ is strictly primitive recursively realizable. Thus A must be strictly primitive recursively realizable.

4) If $a_0 \Vdash_0 \neg\neg A$, then it follows from 3) that A is strictly primitive recursively realizable. Conversely, if A is strictly primitive recursively realizable, then it follows from 2) that $\neg A$ is not strictly primitive recursively realizable and it follows from 1) that $a_0 \Vdash_0 \neg\neg A$. □

Recursively Enumerable Predicates in the Semantics of Strictly Primitive Recursive Realizability

It is known that in the language of arithmetic containing symbols for the primitive recursive functions every recursively enumerable predicate $P(\bar{x})$ is expressed by a Σ_1-formula of the form $\exists y A(\bar{x}, y)$, where $A(\bar{x}, y)$ is an atomic formula. This means that for every finite sequence of natural numbers $\bar{m}$ $P(\bar{m})$ holds if and only if the formula $\exists y A(\bar{m}, y)$ is true.

Proposition 3.2 *Let a recursively enumerable predicate $P(\bar{x})$ be expressed by a Σ_1-formula $\exists y A(\bar{x}, y)$, where $A(\bar{x}, y)$ is an atomic formula. Then for every $\bar{m}$ $P(\bar{m})$ holds if and only if the formula $\neg\neg\exists y A(\bar{m}, y)$ is strictly primitive recursively realizable.*

Proof. Let $P(\bar{m})$. Then $\exists y A(\bar{m}, y)$ is true. This means that atomic formula $A(\bar{m}, n)$ is true for some n. Therefore, $0 \Vdash_0 A(\bar{m}, n)$ and $\langle n, 0 \rangle \Vdash_0 \exists y A(\bar{m}, y)$. Thus the formula $\exists y A(\bar{m}, y)$ is strictly primitive recursively realizable. It follows from Proposition 3.1 that the formula $\neg\neg\exists y A(\bar{m}, y)$ is strictly primitive recursively realizable. Conversely, if $\neg\neg\exists y A(\bar{m}, y)$ is strictly primitive recursively realizable, then by Proposition 3.1, $\exists y A(\bar{m}, y)$ is strictly primitive recursively realizable too, consequently, $t \Vdash_n \exists y A(\bar{m}, y)$ for some t, n. Then $[t]_1 \Vdash_n A(\bar{m}, [t]_0)$.

As $A(x,y)$ is atomic, the formula $A(\bar{m},[t]_0)$ is true. Therefore $\exists y A(\bar{m},y)$ is also true and $P(\bar{m})$ holds. $\Box$

Proposition 3.3 *Let a recursively enumerable predicate $P(\bar{x})$ be expressed by a Σ_1-formula $\exists y A(\bar{x},y)$, where $A(\bar{x},y)$ is an atomic formula. Then for every $\bar{m}$ the following conditions are equivalent:*
1) $P(\bar{m})$ holds;
2) the formula $\neg\neg\exists y A(\bar{m},y)$ is strictly primitive recursively realizable;
3) $a_0 \Vdash_0 \neg\neg\exists y A(\bar{m},y)$.

Proof. This is an immediate consequence of Propositions 1 and 2. $\Box$

Strictly Primitive Recursive Realizability and Intuitionistic Logic

Theorem 3.1 *There exists a closed arithmetic formula which is deducible in the intuitionistic predicate calculus but is not strictly primitive recursively realizable.*

Proof. For convenience we shall consider the first-order language of arithmetic and one of predicate logic with the propositional constant $\top$ ("truth") having in view that the predicate formula $\top$ is an axiom of the intuitionistic predicate calculus, the arithmetic formula $\top$ is true and notion of strictly primitive recursive realizability is defined in such a way that $\top$ means the same as $0=0$.

Consider a ternary predicate

$$e_{x+1}([y]_0,[y]_1)=z. \tag{2}$$

This predicate is decidable. Indeed, if naturals x,y,z are given and $In(x,[y]_0)$ is false, then (2) holds if and only if $z=0$. If $In(x,[y]_0)$ is true, we can apply the primitive recursive function with the index $[y]_0$ in the class $\mathbf{E}_x$ to the argument $[y]_1$ and check if the result is z.

As every decidable predicate is recursively enumerable, the predicate (2) is expressed by an arithmetic Σ_1-formula $\exists u A(x,y,z,u)$, the formula $A(x,y,z,u)$ being atomic. Let $B(x,y,z)$ be $\neg\neg\exists u A(x,y,z,u)$. Now we define a formula Φ as

$$\forall x(\forall y(\top \to \exists z B(x,y,z)) \to \forall y \exists z B(x,y,z)).$$

Evidently, Φ is deducible in the intuitionistic predicate calculus. We shall prove that Φ is not strictly primitive recursively realizable.

We have defined in Section 2 a primitive recursive function α. Let n_0 be a natural number such that $\alpha \in \mathbf{E}_{n_0}$. Then obviously, $\alpha_n \in \mathbf{E}_{n_0}$ for every n.

Lemma 3.1 *If $n \geq n_0$, then there exists a natural t such that*

$$t \Vdash_n \forall y(\top \to \exists z B(n,y,z)).$$

Proof. Let $n \geq n_0$ be fixed, $\delta(y) = \langle 7,1,\alpha_n([y]_0),\langle 2,1,[y]_1\rangle\rangle$. Obviously, for every b the number $\delta(b)$ is an n-index of the constant function $\varphi(x) = \psi^n_{[b]_0}([b]_1)$. Further, let c_0 be a 0-index of the function $\zeta(x) = 2^x \cdot 3^{a_0}$ and

$\beta(y) = \langle 7, 1, c_0, \delta(y)\rangle$. For every b the value $\beta(b)$ is an n-index of the constant function $\xi(x) = 2^{\psi^n_{[b]_0}([b]_1)} \cdot 3^{a_0}$. Finally, let $\gamma(y) = \langle 2, 1, \beta(y)\rangle$. Thus,

$$\gamma(y) = \langle 2, 1, \langle 7, 1, c_0, \langle 7, 1, \alpha_n([y]_0), \langle 2, 1, [y]_1\rangle\rangle\rangle\rangle.$$

For every b the value $\gamma(b)$ is an n-index of the constant function $\phi(x) = \beta(b)$.

We see that $\gamma \in \mathbf{E}_{n_0}$, hence, $\gamma \in \mathbf{E}_n$. Let t be an n-index of γ. Let us prove that

$$t \Vdash_n \forall y(\top \to \exists z B(n, y, z)). \tag{3}$$

By the definition of the strictly primitive recursive realizability, (3) means that $In(n, t)$ holds (this condition is shown above to be fulfilled) and

$$\mathsf{e}_{n+1}(t, \langle b\rangle) \Vdash_n (\top \to \exists z B(n, b, z))$$

for every b. As $\mathsf{e}_{n+1}(t, \langle b\rangle) = \gamma(b)$, we have to prove for every b that

$$\gamma(b) \Vdash_n (\top \to \exists z B(n, b, z)). \tag{4}$$

The condition (4) means that

1) $\gamma(b)$ is an n-index of a function $\phi \in \mathbf{E}_n$ such that

2) for any $j \geq n$ the number $\phi(j)$ is a j-index of a unary function $\xi \in \mathbf{E}_j$ such that for every a if $a \Vdash_j \top$, then $\xi(a) \Vdash_j \exists z B(n, b, z)$.

As it was remarked, $\gamma(b)$ is an n-index of the constant function $\phi(x) = \beta(b)$ in $\mathbf{E}_0$. Thus 1) is fulfilled. Moreover, for every j the value $\phi(j)$ is $\beta(b)$ being an n-index of the constant function $\xi(x) = 2^{\psi^n_{[b]_0}([b]_1)} \cdot 3^{a_0}$. Hence, $\xi \in \mathbf{E}_0$. Thus if $a \Vdash_j \top$, then $\xi(a) = 2^{\psi^n_{[b]_0}([b]_1)} \cdot 3^{a_0}$. Let $d = \psi^n_{[b]_0}([b]_1)$. We have to prove that $2^d \cdot 3^{a_0} \Vdash_j \exists z B(n, b, z)$, i.e. $a_0 \Vdash_j B(n, b, d)$. Note that $B(x, y, z)$ is of the form $\neg\neg\exists u A(x, y, z, u)$ with atomic $A(x, y, z, u)$. By Proposition 3.2, it is sufficient to prove that $\mathsf{e}_{n+1}([b]_0, [b]_1) = d$, but this is evident because $\mathsf{e}_{n+1}([b]_0, [b]_1) = \psi^n_{[b]_0}([b]_1))$. Thus 2) is also fulfilled. □

Lemma 3.2 *For every n there exists no natural t such that*

$$t \Vdash_n \forall y \exists z B(n, y, z).$$

Proof. Let $t \Vdash_n \forall y \exists z B(n, y, z)$. This means that t is an n-index of a function $f \in \mathbf{E}_n$ such that for every b $f(b) \Vdash_n \exists z B(n, b, z)$, i.e. $[f(b)]_1 \Vdash_n B(n, b, [f(b)]_0)$. Thus for every b the formula $B(n, b, [f(b)]_0)$ is strictly primitive recursively realizable. By Proposition 3.2, in this case, $\mathsf{e}_{n+1}([b]_0, [b]_1) = [f(b)]_0$ for every b. In particular,

$$\mathsf{e}_{n+1}(x, y) = [f(\langle x, y\rangle)]_0 \tag{5}$$

for every x, y. The expression on the right hand in (5) defines a function in the class $\mathbf{E}_n$, consequently, $\mathsf{e}_{n+1} \in \mathbf{E}_n$. This contradiction completes the proof of Lemma 3.2. □

To complete the proof of Theorem 3.1 let us assume that $t \Vdash_n \Phi$ for some t, n. We may suppose that $n \geq n_0$. Then t is an n-index of a function $g \in \mathbf{E}_n$ such that

$$g(m) \Vdash_n \forall y(\top \to \exists z B(m,y,z)) \to \forall y \exists z B(m,y,z)$$

for every m. In particular,

$$g(n) \Vdash_n \forall y(\top \to \exists z B(n,y,z)) \to \forall y \exists z B(n,y,z).$$

This implies that $\mathsf{e}_{n+1}(g(n), \langle n \rangle)$ is an n-index of a function h such that

$$h(a) \Vdash_n \forall y \exists z B(n,y,z) \tag{6}$$

if $a \Vdash_n \forall y(\top \to \exists z B(n,y,z))$. By Lemma 3.1, there exists such a. It yields (6) in a contradiction with Lemma 3.2. $\Box$

4 Primitive Recursive Realizability by Salehi

Definition of Primitive Recursive Realizability by Salehi

Another notion of primitive recursive realizability was introduced by S. Salehi (see [15], [16]) for the formulas of Basic Arithmetic. Basic Arithmetic BA is a formal system of arithmetic based on Basic Logic which is weaker than Intuitionistic Logic. The language of BA contains symbols for the primitive recursive functions and differs from the usual language of arithmetic by the mode of using the universal quantifier. Namely the quantifier $\forall$ is used only in the formulas of the form $\forall \bar{x}(A \to B)$, where $\bar{x}$ is a finite (possibly empty) list of variables, A and B being formulas. If the list $\bar{x}$ is empty, then $\forall \bar{x}(A \to B)$ is written merely as $(A \to B)$. Obviously, every formula of BA using universal quantifiers $\forall \bar{x}$ only with empty or one-element list of variables $\bar{x}$ is also a formula of the usual arithmetic language.

Salehi [16] defined a relation $t\mathbf{r}^{PR}A$, where t is a natural number, A is a closed formula in the language of BA. The definition is by induction on the number of logical symbols in A. Let $PR(b)$ mean that b is an index of a unary primitive recursive function, that is $\exists n In(n,b)$ and $[b]_1 = 1$. If $PR(b)$, let ψ_b be the function ψ_b^n, where n is such that $In(n,b)$.

- If A is atomic, then $t\mathbf{r}^{PR}A \rightleftharpoons [A \text{ is true}]$.
- $t\mathbf{r}^{PR}(B \,\&\, C) \rightleftharpoons [[t]_0\mathbf{r}^{PR}B \text{ and } [t]_1\mathbf{r}^{PR}C]$.
- $t\mathbf{r}^{PR}(B \vee C) \rightleftharpoons [[t]_0 = 0 \text{ and } [t]_1\mathbf{r}^{PR}B \text{ or } [t]_0 \neq 0 \text{ and } [t]_1\mathbf{r}^{PR}C]$.
- $t\mathbf{r}^{PR}\exists x B(x) \rightleftharpoons [[t]_1\mathbf{r}^{PR}B([t]_0)]$.
- $t\mathbf{r}^{PR}\forall \bar{x}(B(\bar{x}) \to C(\bar{x})) \rightleftharpoons$
$[PR(t) \text{ and } \forall b, \bar{m}(b\mathbf{r}^{PR}B(\bar{m}) \Rightarrow \psi_t(\langle b, \bar{m}\rangle)\mathbf{r}^{PR}C(\bar{m}))]$.

A closed arithmetic formula A is called *primitive recursively realizable* if $t\mathbf{r}^{PR}A$ for some t. Salehi [16] proved that every formula deducible in BA is primitive recursively realizable.

Some facts are immediately implied by the definition of primitive recursive realizability by Salehi.

Proposition 4.1 *1) If a closed arithmetic formula A is not primitive recursively realizable by Salehi, then $\langle 2,1,0\rangle\mathbf{r}^{PR}\neg A$.*

2) For every closed arithmetic formula A if the formula $\neg A$ is primitive recursively realizable by Salehi, then A is not primitive recursively realizable by Salehi.

3) A closed arithmetic formula A is primitive recursively realizable by Salehi if and only if $\neg\neg A$ is primitive recursively realizable by Salehi.

4) An arithmetic formula A is primitive recursively realizable by Salehi if and only if $\langle 2,1,0\rangle\mathbf{r}^{PR}\neg\neg A$.

Proof. 1) Let a closed arithmetic formula A be not primitive recursively realizable by Salehi. We have to prove that $\langle 2,1,0\rangle\mathbf{r}^{PR}(A \to 0 = 1)$. Obviously, $\langle 2,1,0\rangle$ is an index of the unary constant primitive recursive function f whose only value is 0. We prove that for every a

$$a\mathbf{r}^{PR}A \Rightarrow f(a)\mathbf{r}^{PR}0 = 1. \tag{7}$$

Indeed, as A is not primitive recursively realizable by Salehi, the condition (7) is obviously fulfilled because the premise is false for every a.

2) Let the formula $\neg A$ be primitive recursively realizable by Salehi. Suppose that A is also primitive recursively realizable by Salehi. Then the formulas $\neg A$ and A are both realizable. Let $a\mathbf{r}^{PR}A$, $b\mathbf{r}^{PR}\neg A$, i.e. $b\mathbf{r}^{PR}(A \to 0 = 1)$. Then it follows from the definition of primitive recursive realizability by Salehi that $\psi_b(a)\mathbf{r}^{PR}(0 = 1)$ and the formula $0 = 1$ is realizable but this is impossible as it follows from the clause for atomic formulas in the definition of primitive recursive realizability by Salehi. Thus A is not primitive recursively realizable by Salehi if $\neg A$ is primitive recursively realizable by Salehi.

3) Let A be primitive recursively realizable by Salehi. Then it follows from 2) that $\neg A$ is not primitive recursively realizable by Salehi and it follows from 1) that $\neg\neg A$ is primitive recursively realizable by Salehi. Conversely, if $\neg\neg A$ is primitive recursively realizable by Salehi, then it follows from 2) that $\neg A$ is not primitive recursively realizable by Salehi. But if A is not primitive recursively realizable by Salehi, then it follows from 1) that $\neg A$ is primitive recursively realizable by Salehi. Thus A must be primitive recursively realizable by Salehi.

4) If $\langle 2,1,0\rangle\mathbf{r}^{PR}\neg\neg A$, then it follows from 3) that A is primitive recursively realizable by Salehi. Conversely, if A is primitive recursively realizable by Salehi, then it follows from 2) that $\neg A$ is not primitive recursively realizable by Salehi and it follows from 1) that $\langle 2,1,0\rangle\mathbf{r}^{PR}\neg\neg A$. $\square$

Recursively Enumerable Predicates in the Semantics of Primitive Recursive Realizability by Salehi

Proposition 4.2 *Let a recursively enumerable predicate $P(\bar{x})$ be expressed by a Σ_1-formula $\exists y A(\bar{x}, y)$, where $A(\bar{x}, y)$ is an atomic formula. Then for every $\bar{m}$ $P(\bar{m})$ holds if and only if the formula $\neg\neg\exists y A(\bar{m}, y)$ is primitive recursively realizable by Salehi.*

Proof. Let $P(\bar{m})$. Then $\exists y A(\bar{m}, y)$ is true. This means that atomic formula $A(\bar{m}, n)$ is true for some n. Therefore, $0\mathbf{r}^{PR}A(\bar{m}, n)$ and $\langle n, 0\rangle\mathbf{r}^{PR}\exists y A(\bar{m}, y)$. Thus the formula $\exists y A(\bar{m}, y)$ is primitive recursively realizable by Salehi. It follows from Proposition 4.1 that the formula $\neg\neg\exists y A(\bar{m}, y)$ is primitive recursively realizable by Salehi. Conversely, if $\neg\neg\exists y A(\bar{m}, y)$ is primitive recursively realizable by Salehi, then by Proposition 4.1, $\exists y A(\bar{m}, y)$ is also primitive recursively realizable by Salehi, consequently, $a\mathbf{r}^{PR}\exists y A(\bar{m}, y)$ for some a. Then $[a]_1\mathbf{r}^{PR}A(\bar{m}, [a]_0)$. As $A(x, y)$ is atomic, the formula $A(\bar{m}, [a]_0)$ is true. Therefore $\exists y A(\bar{m}, y)$ is also true and $P(\bar{m})$ holds. $\square$

Proposition 4.3 *Let a recursively enumerable predicate $P(\bar{x})$ be expressed by a Σ_1-formula $\exists y A(\bar{x}, y)$, where $A(\bar{x}, y)$ is an atomic formula. Then the following conditions are equivalent for every $\bar{m}$:*
1) $P(\bar{m})$ holds;
2) the formula $\neg\neg\exists y A(\bar{m}, y)$ is primitive recursively realizable by Salehi;
3) $\langle 2, 1, 0\rangle\mathbf{r}^{PR}\neg\neg\exists y A(\bar{m}, y)$.

Proof. This is an immediate consequence of Propositions 4.1 and 4.2. $\square$

5 Relation Between Two Notions of Primitive Recursive Realizability

Theorem 5.1 *There exists a closed arithmetic formula which is strictly primitive recursively realizable but is not primitive recursively realizable by Salehi.*

Proof. Consider a ternary predicate

$$\mathsf{e}_{x+1}([y]_0, [y]_1) = z.$$

It is decidable and expressed by an arithmetic Σ_1-formula $\exists u A(x, y, z, u)$ with atomic $A(x, y, z, u)$. Let $B(x, y, z)$ be $\neg\neg\exists u A(x, y, z, u)$.

Consider a binary predicate

$$\mathsf{e}_x(x, \langle x\rangle) = y.$$

It is also decidable and expressed by an arithmetic Σ_1-formula $\exists v C(x, y, v)$ with atomic $C(x, y, v)$. Let $D(x, y)$ be $\neg\neg\exists v C(x, y, v)$.

Consider the formula

$$\forall x(\forall y(\top \to \exists z B(x, y, z)) \to \exists y D(x, y)). \tag{8}$$

Clearly, this arithmetic formula is a formula in the language of Basic Arithmetic too, thus both concepts of primitive recursive realizability are defined for it. The formula (8) is strictly primitive recursively realizable but is not primitive recursively realizable by Salehi.

Lemma 5.1 *The formula (8) is strictly primitive recursively realizable.*

Proof. Let p_0 be a 0-index of the elementary function $\langle x, y\rangle$, $\overline{\mathsf{sg}}(x) = 1 \dot{-} \mathsf{sg}(x)$, j_0 be a 0-index of the unary elementary function $\varphi(x) = \langle x\rangle$, $k_0(k) = \langle 7, 1, j_0, \langle 2, 1, k\rangle\rangle$ (thus $k_0(k)$ is a 0-index of the unary constant elementary function $\phi(x) = \langle k\rangle$), $q_0(k) = \langle 7, 1, \langle 0, 2, k\rangle, \langle 2, 1, k\rangle, k_0(k)\rangle$. Let $G(k, j)$ denote the number

$$\mathsf{sg}(k \dot{-} j) \cdot \langle 2, 1, 0\rangle + \overline{\mathsf{sg}}(k \dot{-} j) \cdot \langle 7, 1, p_0, q_0(k), \langle 2, 1, a_0\rangle\rangle,$$

a_0 being the same as in Proposition 3.2. Let $f(k) = \Lambda j.G(k, j)$. As it was remarked above, f is an elementary function, thus, $f \in \mathbf{E}_0$. We prove that t realizes the formula (8) at the level 0. This means that $In(0, t)$ holds (this condition is satisfied) and

$$\forall k\, f(k) \Vdash_0 \forall y(\top \to \exists z B(k, y, z)) \to \exists y D(k, y).$$

Let k be fixed. Let us prove that

$$f(k) \Vdash_0 \forall y(\top \to \exists z B(k, y, z)) \to \exists y D(k, y).$$

This means that $f(k)$ is a 0-index of a function $g \in \mathbf{E}_0$ and for every j

$$In(j, g(j)) \tag{9}$$

holds and $g(j)$ is a j-index of a function $h \in \mathbf{E}_j$ such that

$$\forall a[a \Vdash_j \forall y(\top \to \exists z B(k, y, z)) \Rightarrow h(a) \Vdash_j \exists y D(k, y)]. \tag{10}$$

It follows from the definition of f that $f(k)$ is an index of the function $g(j) = G(k, j)$. As $G(k, j)$ is elementary, g is elementary too and $g \in \mathbf{E}_0$. Let us prove that for every j the conditions (9) and (10) hold.

Let $j < k$. In this case, $g(j) = \langle 2, 1, 0\rangle$, i.e. $g(j)$ is a 0-index of the constant function $h(x) = 0$, thus $In(0, g(j))$ holds, therefore $In(j, g(j))$ holds too and the condition (9) is satisfied. Let us prove that (10) holds. The reason is trivial: there is no natural a such that

$$a \Vdash_j \forall y(\top \to \exists z B(k, y, z)). \tag{11}$$

Indeed, if (9) holds for some a, then a is a j-index of a function $\psi \in \mathbf{E}_j$ such that

$$\forall m\, \psi(m) \Vdash_j (\top \to \exists z B(k, m, z)).$$

Let m be fixed and $\psi(m) = d$. Thus

$$d \Vdash_j (\top \to \exists z B(k, m, z)).$$

This means that $In(j,d)$ holds and for every $i \geq j$ the number $\mathsf{e}_{j+1}(d, \langle i \rangle)$ is an i-index of a function $\chi \in \mathbf{E}_i$ such that

$$\forall b\, [b \Vdash_i \top \Rightarrow \chi(b) \Vdash_i \exists z B(k,m,z)].$$

In particular, for $i = j$ we have $0 \Vdash_j \top$, $In(j, \mathsf{e}_{j+1}(d, \langle j \rangle)$, and

$$\mathsf{e}_{j+1}(\mathsf{e}_{j+1}(d, \langle j \rangle), \langle 0 \rangle) \Vdash_j \exists z B(k,m,z).$$

Then

$$[\mathsf{e}_{j+1}(\mathsf{e}_{j+1}(d, \langle j \rangle), \langle 0 \rangle)]_1 \Vdash_j B(k,m, [\mathsf{e}_{j+1}(\mathsf{e}_{j+1}(d, \langle j \rangle), \langle 0 \rangle)]_0).$$

Therefore the formula

$$B(k,m,[\mathsf{e}_{j+1}(\mathsf{e}_{j+1}(\psi(m), \langle j \rangle), \langle 0 \rangle)]_0) \tag{12}$$

is strictly primitive recursively realizable. As the formula $B(x,y,z)$ is of the form $\neg\neg\exists u A(x,y,z,u)$ with atomic $A(x,y,z,u)$, it follows from (12) and Proposition 3.3 that

$$\mathsf{e}_{k+1}([m]_0, [m]_1) = [\mathsf{e}_{j+1}(\mathsf{e}_{j+1}(\psi(m), \langle j \rangle), \langle 0 \rangle)]_0.$$

Then for every natural u, v

$$\mathsf{e}_{k+1}(u,v) = [\mathsf{e}_{j+1}(\mathsf{e}_{j+1}(\psi(\langle u,v \rangle), \langle j \rangle), \langle 0 \rangle)]_0. \tag{13}$$

The expression on the right hand in (13) defines a primitive recursive function in the class $\mathbf{E}_{j+1}$. Thus it follows from (13) that e_{k+1} is also in the class $\mathbf{E}_{j+1}$, but this is impossible bacause $j < k$. Thus the assumption that (11) holds for some a leads to a contradiction. The case $j < k$ is completely considered and in this case (10) is proved.

Now let us consider the case $j \geq k$. In this case,

$$g(j) = \langle 7, 1, p_0, q_0(k), \langle 2, 1, a_0 \rangle \rangle =$$

$$= \langle 7, 1, p_0, \langle 7, 1, \langle 0, 2, k \rangle, \langle 2, 1, k \rangle, k_0(k) \rangle, \langle 2, 1, a_0 \rangle \rangle =$$

$$= \langle 7, 1, p_0, \langle 7, 1, \langle 0, 2, k \rangle, \langle 2, 1, k \rangle, \langle 7, 1, j_0, \langle 2, 1, k \rangle \rangle \rangle, \langle 2, 1, a_0 \rangle \rangle.$$

We see that $g(j)$ is an index of a unary function h obtained from the function $\langle x, y \rangle$ by substituting the function with index

$$q_0 = \langle 7, 1, \langle 0, 2, k \rangle, \langle 2, 1, k \rangle, \langle 7, 1, j_0, \langle 2, 1, k \rangle \rangle \rangle$$

for x and the constant function with the only value a_0 for y. The number q_0 is an index of the function obtained from the function $\mathsf{e}_k(x,y)$ by substituting the constant function with the only value k for x and the constant function with the only value $\langle k \rangle$ for y. Obviously, the function with the index q_0 is in the class $\mathbf{E}_k$, therefore $In(k, g(j))$ holds. Consequently, $In(j, g(j))$ holds too because $j \geq k$. Thus the condition (9) is satisfied. Let us prove that (10) holds. Let a be such that (11) holds. We prove that $h(a) \Vdash_j \exists y D(k, y))$. As it was just shown, $h(a) = \langle \mathsf{e}(k, \langle k \rangle), a_0 \rangle$. Thus we have to prove that

$$\langle \mathsf{e}(k, \langle k \rangle), a_0 \rangle \Vdash_j \exists y D(k, y)). \tag{14}$$

As $D(x, y)$ is of the form $\neg\neg\exists v C(x, y, v)$ with atomic $C(x, y, v)$, it follows from Proposition 3.3 that $a_0 \Vdash_0 D(k, \mathsf{e}(k, \langle k \rangle))$. Thus,

$$\langle \mathsf{e}(k, \langle k \rangle), a_0 \rangle \Vdash_0 \exists y D(k, y)),$$

therefore (14) is also proved. $\Box$

Lemma 5.2 *The formula (8) is not primitive recursively realizable by Salehi.*

Proof. Let us assume the contrary, i.e. that the formula (8) is primitive recursively realizable by Salehi. This means that there exists a primitive recursive function f such that

$$\forall a, k[a\mathbf{r}^{PR}\forall y(\top \to \exists z B(k, y, z)) \Rightarrow f(a, k)\mathbf{r}^{PR}\exists y D(k, y)]. \tag{15}$$

We prove that there exists a unary elementary function μ such that for every k

$$\mu(k)\mathbf{r}^{PR}\forall y(\top \to \exists z B(k, y, z)). \tag{16}$$

Namely let $\nu(k) = \langle 7, 2, p_0, \langle 7, 2, \langle 0, 2, k+1 \rangle, m_1, m_2 \rangle, \langle 2, 2, \langle 2, 1, 0 \rangle \rangle \rangle$, where p_0 is the same as in Lemma 5.1, $m_1 = \langle 7, 2, l_0, \langle 3, 2, 1 \rangle \rangle$, $m_2 = \langle 7, 2, r_0, \langle 3, 2, 1 \rangle \rangle$, l_0 and r_0 being 0-indexes of the functions $[x]_0$ and $[x]_1$ respectively. We see that $\nu(k)$ is an index of a binary function $\xi(y, a)$ obtained from the function $\langle x, y \rangle$ by substituting the function $\mathsf{e}_{k+1}([y]_0, [y]_1)$ for x and the binary constant function with the only value $\langle 2, 1, 0 \rangle$ for y. Thus $\chi(y, a) = \langle \mathsf{e}_{k+1}([y]_0, [y]_1), \langle 2, 1, 0 \rangle \rangle$ for every y, a. Let $\mu(k) = \langle 7, 1, \nu(k), l_0, r_0 \rangle$. Obviously, $\psi_{\mu(k)}(\langle y, a \rangle = \chi(y, a)$ for every y, a. It follows from Proposition 4.3 that for every m

$$\langle 2, 1, 0 \rangle \mathbf{r}^{PR} B(k, m, \mathsf{e}_{k+1}([m]_0, [m]_1)).$$

Therefore, $\chi(m, 0)\mathbf{r}^{PR}\exists z B(k, m, z)$ for every m. But this means that (16) is satisfied.

It follows from (16) and (15) that $\forall k\, f(k, \mu(k))\mathbf{r}^{PR}\exists y D(k, y)$. But then

$$\forall k\, [f(k, \mu(k))]_1 \mathbf{r}^{PR}\, D(k, [f(k, \mu(k))]_0).$$

Thus for every k the formula $D(k, [f(k, \mu(k))]_0)$ is primitive recursively realizable by Salehi. As $D(x, y)$ is of the form $\neg\neg\exists v C(x, y, v)$ with atomic $C(x, y, v)$, it follows from Proposition 4.3 that for every k

$$\mathsf{e}_k(k, \langle k \rangle) = [f(k, \mu(k))]_0.$$

This means that the function $g(x) = \mathsf{e}_x(x, \langle x\rangle)$ is primitive recursive. But it is impossible. Indeed, if $g \in \mathbf{E}_n$, then the function $h(x) = g(x) + 1$ is also in the class $\mathbf{E}_n$ and there exists m such that $\forall x\, h(x) = \mathsf{e}_{n+1}(m, \langle x\rangle)$. As every function in the class $\mathbf{E}_n$ has infinitely many n-indexes, we can suppose that $m > n + 1$. Then m is also an $(m-1)$-index of the function h. Thus, $\mathsf{e}_m(m, \langle x\rangle) = h(x)$ for every x. Now putting $x = m$ we obtaine a contradiction:

$$\mathsf{e}_m(m, \langle m\rangle) = h(m) = g(m) + 1 = \mathsf{e}_m(m, \langle m\rangle) + 1.$$

□

Lemmas 5.1 and 5.2 give together Theorem 5.1.

Theorem 5.2 *There exists a closed arithmetic formula which is primitive recursively realizable by Salehi but is not strictly primitive recursively realizable.*

Proof. Let Φ be the negation of the formula (8). By Theorem 5.1, the formula (8) is not primitive recursively realizable by Salehi; then by Proposition 4.1, Φ is primitive recursively realizable by Salehi. By Theorem 5.1, the formula (8) is strictly primitive recursively realizable; then by Proposition 3.1, Φ is not strictly primitive recursively realizable. □

6 Strictly Primitive Recursive Realizability and Basic Logic

Basic Propositional Calculus BPC was first introduced by A. Visser [17]. BPC is related to the modal logic K4 just as IPC is related to S4 by the Gödel translation. As it follows from the Salehi result mentioned above, his variant of primitive recursive realizability can be considered as a kind of constructive interpretation of BPC. We consider the question on soundness of BPC with strictly primitive recursive realizability. We shall use a sequent variant of BPC as it is presented in [1]. A propositional (an arithmetic) *sequent* is an expression of the form $A \Rightarrow B$, A and B being propositional (resp. arithmetic) formulas in the language containing propositional constants $\top$ and $\bot$.

Let $A(\bar{x})$ and $B(\bar{x})$ be arithmetic formulas with only free variables in the list $\bar{x} = x_1, \ldots, x_m$. A sequent $A(\bar{x}) \Rightarrow B(\bar{x})$ is called strictly primitive recursively realizable at the level n if and only if there exists an $(m+1)$-ary function $f \in \mathbf{E}_n$ such that for every $a, \bar{c}$ and every $j \geq n$ if $a \Vdash_j A(\bar{c})$, then $f(a, \bar{c}) \Vdash_j B(\bar{c})$. A sequent is realizable if there exists n such that this sequent is realizable at the level n. A propositional sequent $A(P_1, \ldots, P_n) \Rightarrow B(P_1, \ldots, P_n)$ is strictly primitive recursively realizable (at the level n) if for every arithmetic formulas $C_1, \ldots, C_n$ the sequent $A(C_1, \ldots, C_n) \Rightarrow B(C_1, \ldots, C_n)$ is strictly primitive recursively realizable (at the level n).

Axioms and rules for BPC:

1. $A \Rightarrow A$ 2. $A \Rightarrow \top$ 3. $\bot \Rightarrow A$

4. $A \,\&\, (B \vee C) \Rightarrow (A \,\&\, B) \vee (A \,\&\, C)$

5. $\dfrac{A \Rightarrow B \quad B \Rightarrow C}{A \Rightarrow C}$

6. (a) $\dfrac{A \Rightarrow B \quad A \Rightarrow C}{A \Rightarrow B \,\&\, C}$ (b) $\dfrac{A \Rightarrow B \,\&\, C}{A \Rightarrow B}$ (c) $\dfrac{A \Rightarrow B \,\&\, C}{A \Rightarrow C}$

7. (a) $\dfrac{A \Rightarrow C \quad B \Rightarrow C}{A \vee B \Rightarrow C}$ (b) $\dfrac{A \vee B \Rightarrow C}{A \Rightarrow C}$ (c) $\dfrac{A \vee B \Rightarrow C}{B \Rightarrow C}$

8. $\dfrac{A \,\&\, B \Rightarrow C}{A \Rightarrow B \rightarrow C}$

9. $(A \rightarrow B) \,\&\, (B \rightarrow C) \Rightarrow A \rightarrow C$

10. $(A \rightarrow B) \,\&\, (A \rightarrow C) \Rightarrow A \rightarrow B \,\&\, C$

11. $(A \rightarrow B) \,\&\, (B \rightarrow C) \Rightarrow A \vee B \rightarrow C$

Theorem 6.1 *Every sequent deducible in* BPC *is strictly primitive recursively realizable.*

Proof. We prove that every axiom is strictly primitive recursively realizable and every rule of inference preserves realizability. Let $A(\bar{x})$, $B(\bar{x})$, $C(\bar{x})$ be arithmetic formulas with parameters in the list $\bar{x}$.

Axiom 1. $A(\bar{x}) \Rightarrow A(\bar{x})$.

This sequent is realizable at every level. For the corresponding function f we can put $f(a, \bar{x}) = a$.

Axiom 2. $A(\bar{x}) \Rightarrow \top$.

This sequent is realizable at every level. We can put $f(a, \bar{x}) = 0$.

Axiom 3. $\bot \Rightarrow A(\bar{x})$.

This sequent is realizable at every level. We can put $f(a, \bar{x}) = a$.

Axiom 4. $A(\bar{x}) \,\&\, (B(\bar{x}) \vee C(\bar{x})) \Rightarrow (A(\bar{x}) \,\&\, B(\bar{x})) \vee (A(\bar{x}) \,\&\, C(\bar{x}))$.

This sequent is realizable at every level. We can put

$$f(a, \bar{x}) = 2^{[[a]_1]_0} \cdot 3^{2^{[a]_0} \cdot 3^{[[a]_1]_1}}.$$

Rule 5. $\dfrac{A(\bar{x}) \Rightarrow B(\bar{x}) \quad B(\bar{x}) \Rightarrow C(\bar{x})}{A(\bar{x}) \Rightarrow C(\bar{x})}$.

Let the sequents in the premise be both realizable at a level n. Thus there are functions $f, g \in \mathbf{E}_n$ such that for every $a, b, \bar{c}$ and every $j \geq n$ if $a \Vdash_j A(\bar{c})$, then $f(a, \bar{c}) \Vdash_j B(\bar{c})$ and if $b \Vdash_j B(\bar{c})$, then $g(b, \bar{c}) \Vdash_j C(\bar{c})$. Obviously, if $j \geq n$ and $a \Vdash_j A(\bar{c})$, then $g(f(a, \bar{c}), \bar{c}) \Vdash_j C(\bar{c})$. As the function $h(a, \bar{x}) = g(f(a, \bar{x}), \bar{x})$ is in the class $\mathbf{E}_n$, the sequent $A(\bar{x}) \Rightarrow C(\bar{x})$ is realizable at the level n.

Rule 6.

(a) $\dfrac{A(\bar{x}) \Rightarrow B(\bar{x}) \quad A(\bar{x}) \Rightarrow C(\bar{x})}{A(\bar{x}) \Rightarrow B(\bar{x}) \,\&\, C(\bar{x})}$.

Let the sequents in the premise be both realizable at a level n. Thus there are functions $f, g \in \mathbf{E}_n$ such that for every $a, \bar{c}$ and every $j \geq n$ if $a \Vdash_j A(\bar{c})$, then $f(a, \bar{c}) \Vdash_j B(\bar{c})$ and $g(a, \bar{c}) \Vdash_j C(\bar{c})$. Obviously, if $j \geq n$ and $a \Vdash_j A(\bar{c})$, then $2^{f(a,\bar{c})} \cdot 3^{g(a,\bar{c})} \Vdash_j B(\bar{c}) \,\&\, C(\bar{c})$. As the function $h(a, \bar{x}) = 2^{f(a,\bar{x})} \cdot 3^{g(a,\bar{x})}$ is in the class $\mathbf{E}_n$, the sequent $A(\bar{x}) \Rightarrow B(\bar{x}) \,\&\, C(\bar{x})$ is realizable at the level n.

(b) $\dfrac{A(\bar{x}) \Rightarrow B(\bar{x}) \,\&\, C(\bar{x})}{A(\bar{x}) \Rightarrow B(\bar{x})}$.

Let the sequent in the premise be realizable at a level n. Thus there is a function $f \in \mathbf{E}_n$ such that for every $a, \bar{c}$ and every $j \geq n$ if $a \Vdash_j A(\bar{c})$, then $f(a, \bar{c}) \Vdash_j B(\bar{c}) \,\&\, C(\bar{x})$. Obviously, if $j \geq n$ and $a \Vdash_j A(\bar{c})$, then

$$[f(a, \bar{c})]_0 \Vdash_j B(\bar{c}).$$

As the function $h(a, \bar{x}) = [f(a, \bar{x})]_0$ is in the class $\mathbf{E}_n$, the sequent $A(\bar{x}) \Rightarrow B(\bar{x})$ is realizable at the level n.

(c) $\dfrac{A(\bar{x}) \Rightarrow B(\bar{x}) \,\&\, C(\bar{x})}{A(\bar{x}) \Rightarrow C(\bar{x})}$.

Let the sequent in the premise be realizable at a level n. Thus there is a function $f \in \mathbf{E}_n$ such that for every $a, \bar{c}$ and every $j \geq n$ if $a \Vdash_j A(\bar{c})$, then $f(a, \bar{c}) \Vdash_j B(\bar{c}) \,\&\, C(\bar{x})$. Obviously, if $j \geq n$ and $a \Vdash_j A(\bar{c})$, then

$$[f(a, \bar{c})]_1 \Vdash_j C(\bar{c}).$$

As the function $h(a, \bar{x}) = [f(a, \bar{x})]_1$ is in the class $\mathbf{E}_n$, the sequent $A(\bar{x}) \Rightarrow C(\bar{x})$ is realizable at the level n.

Rule 7.

(a) $\dfrac{A(\bar{x}) \Rightarrow C(\bar{x}) \, B(\bar{x}) \Rightarrow C(\bar{x})}{A(\bar{x}) \vee B(\bar{x}) \Rightarrow C(\bar{x})}$.

Let the sequents in the premise be both realizable at a level n. Thus there are functions $f, g \in \mathbf{E}_n$ such that for every $a, b, \bar{c}$ and every $j \geq n$ if $a \Vdash_j A(\bar{c})$, then $f(a, \bar{c}) \Vdash_j C(\bar{c})$ and if $b \Vdash_j B(\bar{c})$, then $g(b, \bar{c}) \Vdash_j C(\bar{c})$. Obviously, if $j \geq n$ and $a \Vdash_j A(\bar{c}) \vee B(\bar{c})$, then

$$([a]_0 \cdot g([a]_1, \bar{c}) + (1 \div [a]_0) \cdot f([a]_1, \bar{c}) \Vdash_j C(\bar{c}).$$

As the function $h(a, \bar{x}) = [a]_0 \cdot g([a]_1, \bar{x}) + (1 \div [a]_0) \cdot f([a]_1, \bar{x})$ is in the class $\mathbf{E}_n$, the sequent $A(\bar{x}) \vee B(\bar{x}) \Rightarrow C(\bar{x})$ is realizable at the level n.

(b) $\dfrac{A(\bar{x}) \vee B(\bar{x}) \Rightarrow C(\bar{x})}{A(\bar{x}) \Rightarrow C(\bar{x})}$.

Let the sequent in the premise be realizable at a level n. Thus there is a function $f \in \mathbf{E}_n$ such that for every $a, \bar{c}$ and every $j \geq n$ if $a \Vdash_j A(\bar{c}) \vee B(\bar{c})$, then $f(a, \bar{c}) \Vdash_j C(\bar{c})$. Obviously, if $j \geq n$ and $a \Vdash_j A(\bar{c})$, then

$$f(2^0 \cdot 3^a, \bar{c}) \Vdash_j C(\bar{c}).$$

As the function $h(a, \bar{x}) = f(2^0 \cdot 3^a, \bar{x})$ is in $\mathbf{E}_n$, the sequent $A(\bar{x}) \Rightarrow C(\bar{x})$ is realizable at the level n.

(c) $\dfrac{A(\bar{x}) \vee B(\bar{x}) \Rightarrow C(\bar{x})}{B(\bar{x}) \Rightarrow C(\bar{x})}$.

Let the sequent in the premise be realizable at a level n. Thus there is a function $f \in \mathbf{E}_n$ such that for every $a, \bar{c}$ and every $j \geq n$ if $a \Vdash_j A(\bar{c}) \vee B(\bar{c})$, then $f(a, \bar{c}) \Vdash_j C(\bar{c})$. Obviously, if $j \geq n$ and $a \Vdash_j B(\bar{c})$, then

$$f(2^1 \cdot 3^a, \bar{c}) \Vdash_j C(\bar{c}).$$

As the function $h(a, \bar{x}) = f(2^1 \cdot 3^a, \bar{x})$ is in $\mathbf{E}_n$, the sequent $B(\bar{x}) \Rightarrow C(\bar{x})$ is realizable at the level n.

Rule 8.
$\dfrac{A(\bar{x}) \,\&\, B(\bar{x}) \Rightarrow C(\bar{x})}{A(\bar{x}) \Rightarrow B(\bar{x}) \rightarrow C(\bar{x})}$.

Let the sequent in the premise be realizable at a level n. Thus there is a function $f \in \mathbf{E}_n$ such that for every $a, \bar{c}$ and every $j \geq n$ if $a \Vdash_j A(\bar{c}) \,\&\, B(\bar{c})$, then $f(a, \bar{c}) \Vdash_n C(\bar{c})$. Put $g(a, \bar{x}) = \Lambda k.\Lambda y.f(2^a \cdot 3^y, \bar{x})$. Let us prove that for every $a, \bar{c}$ and every $j \geq n$, if $a \Vdash_j A(\bar{c})$, then $g(a, \bar{c}) \Vdash_j (B(\bar{c}) \rightarrow C(\bar{c}))$. Let $j \geq n$ and $a \Vdash_j A(\bar{c})$. We have to prove that $g(a, \bar{c})$ is a j-index of a function h such that for every $k \geq j$ the value $h(k)$ is a k-index of a function that maps every realization at the level k of the formula $B(\bar{c})$ into a realization at the same level of the formula $C(\bar{c})$. We see that for every $k \geq j$, $h(k) = \Lambda y.f(2^a \cdot 3^y, \bar{x})$. We have to prove that $h(k)$ is a k-index of a function p such that for every b, if $b \Vdash_k B(\bar{c})$, then $p(b) \Vdash_k C(\bar{c})$. Obviously, $h(k)$ is a 0-index of an elementary function p such that $p(b) = f(2^a \cdot 3^b)$ for every b. Let $b \Vdash_k B(\bar{c})$. As $a \Vdash_j A(\bar{c})$ and $k \geq j$, we have $a \Vdash_k A(\bar{c})$ and $2^a \cdot 3^b \Vdash_k A(\bar{c}) \,\&\, B(\bar{c})$. As $k \geq n$, by the main property of the function f we have $f(2^a \cdot 3^b) \Vdash_k C(\bar{c})$. This means that $p(b) \Vdash_k C(\bar{c})$.

Axiom 9. $(A(\bar{x}) \to B(\bar{x}))\,\&\,(B(\bar{x}) \to C(\bar{x})) \Rightarrow A(\bar{x}) \to C(\bar{x}))$.

We prove that this sequent is realizable at every level n. Let

$$\alpha(x,y) = \langle 7, 1, y, x\rangle.$$

Obviously, α is an elementary function and if x and y are n-indexes of unary primitive recursive functions χ and ϕ respectively, then $\alpha(x,y)$ is an n-index of the function $\varphi(t) = \phi(\chi(t))$. Let a be a 0-index of α. Let d be a 0-index of the function $\psi_d(x,y) = \langle 7, 1, \mathsf{a}, x, y\rangle$.

Lemma 6.1 *For every natural numbers a, b, n and closed formulas A, B, C if $a \Vdash_n (A \to B)$ and $b \Vdash_n (B \to C)$, then $\psi_d(a,b) \Vdash_n (A \to C)$.*

Proof. Let $a \Vdash_n (A \to B)$ and $b \Vdash_n (B \to C)$. Denote $\psi_d(a,b)$ by h and prove that $h \Vdash_n A \to C$, i.e. that for every $j \geq n$ the number $\psi_h(j)$ is a j-index of a function ϕ such that if $x \Vdash_j A$, then $\phi(x) \Vdash_j C$. Let $j \geq n$ be given. Note that h is a j-index of a function obtained by substituting the functions ψ_a and ψ_b into α, thus $\psi_h(j) = \alpha(\psi_a(j), \psi_b(j))$ and for every x, $\phi(x) = \psi_b(\psi_a(x))$. In particular, if $x \Vdash_n A$, then $\psi_a(x) \Vdash_n B$ and $\phi(x) \Vdash_n C$. $\Box$

Let $f(a, \bar{x}) = \langle 7, 1, \mathsf{a}, (a)_0, (a)_1\rangle$, i.e. $f(a,\bar{x}) = \psi_d((a)_0, (a)_1)$, where d is the same as in lemma 3.1. We prove that for every $a, \bar{c}$ and every $j \geq n$, if $a \Vdash_j (A(\bar{c}) \to B(\bar{c}))\,\&\,(B(\bar{c}) \to C(\bar{c})$, then $f(a,\bar{c}) \Vdash_j (A(\bar{c}) \to B(\bar{c}))$. Let $j \geq n$ and $a \Vdash_j (A(\bar{c}) \to B(\bar{c}))\,\&\,(B(\bar{c}) \to C(\bar{c})$. Then $(a)_0 \Vdash_j (A(\bar{c}) \to B(\bar{c}))$ and $(a)_1 \Vdash_j (B(\bar{c}) \to C(\bar{c}))$ and by lemma 6.1 $f(a,\bar{c}) \Vdash_j (A(\bar{c}) \to C(\bar{c}))$.

Axiom 10. $(A(\bar{x}) \to B(\bar{x}))\,\&\,(A(\bar{x}) \to C(\bar{x})) \Rightarrow A(\bar{x}) \to B(\bar{x})\,\&\,C(\bar{x}))$.

This sequent is realizable at every level n. Let $\beta(x,y) = 2^x \cdot 3^y$. β is an elementary function, let b be its 0-index. We define $\gamma(x,y) = \langle 7, 1, \mathsf{b}, x, y\rangle$. Obviously, γ is an elementary function; let c be its 0-index. Now let d be a 0-index of the elementary function $\psi_d(x,y) = \langle 7, 1, \mathsf{c}, x, y\rangle$.

Lemma 6.2 *For every natural numbers a, b, n and closed formulas A, B, C, if $a \Vdash_n (A \to B)$ and $b \Vdash_n (A \to C)$, then $\psi_d(a,b) \Vdash_n (A \to B\,\&\,C)$.*

Proof. Let $a \Vdash_n (A \to B)$ and $b \Vdash_n (A \to C)$. This means that for every $j \geq n$, the number $\psi_a(j)$ is a j-index of a function ϕ_0 such that if $x \Vdash_j A$, then $\phi_0(x) \vdash_n B$ and $\psi_b(j)$ is a j-index of a function ϕ_1 such that if $x \Vdash_j A$, then $\phi_1(x) \vdash_n C$. Denote $\psi_d(a,b)$ by h and prove that $h \Vdash_n A \to B\,\&\,C$, i.e. that for every $j \geq n$, the number $\psi_h(j)$ is a j-index of a function ϕ such that if $x \Vdash_j A$, then $\phi(x) \Vdash_j B\,\&\,C$. Let $j \geq n$ be given. Note that h is a j-index of a function obtained by substituting the functions ψ_a and ψ_b into γ, thus $\psi_h(j) = \gamma(\psi_a(j), \psi_b(j)) = \langle 7, 1, \mathsf{b}, \psi_a(j), \psi_b(j)\rangle$ is an index of a function obtained by substituting ϕ_0 and ϕ_1 into β and for every x, $\phi(x) = 2^{\phi_0(x)} \cdot 3^{\phi_1(x)}$. In particular, if $x \Vdash_j A$, then $\phi_0(x) \vdash_j B$, $\phi_1(x) \vdash_j C$, therefore $\phi(x) \Vdash_j B \,\&\, C$. $\square$

Let $f(a, \bar{x}) = \langle 7, 1, \mathsf{a}, (a)_0, (a)_1 \rangle$, i.e. $f(a, \bar{x}) = \psi_d((a)_0, (a)_1)$, where d is the same as in lemma 3.2. We prove that for every $a, \bar{c}$ and every $j \geq n$, if $a \Vdash_j (A(\bar{c}) \to B(\bar{c})) \,\&\, (A(\bar{c}) \to C(\bar{c})$, then $f(a, \bar{c}) \Vdash_j (A(\bar{c}) \to B(\bar{c}) \,\&\, C(\bar{c}))$. Let $j \geq n$ and $a \Vdash_j (A(\bar{c}) \to B(\bar{c})) \,\&\, (A(\bar{c}) \to C(\bar{c})$. Then $(a)_0 \Vdash_j (A(\bar{c}) \to B(\bar{c}))$ and $(a)_1 \Vdash_j (A(\bar{c}) \to C(\bar{c}))$ and by lemma 6.2 $f(a, \bar{c}) \Vdash_j (A(\bar{c}) \to B(\bar{c}) \,\&\, C(\bar{c}))$.

Axiom 11. $(A(\bar{x}) \to C(\bar{x})) \,\&\, (B(\bar{x}) \to C(\bar{x})) \Rightarrow A(\bar{x}) \vee B(\bar{x}) \to C(\bar{x})$.

We prove that this sequent is realizable at every level n. Let

- s be a 0-index of the elementary function $\sigma(x, y) = x + y$,
- p be a 0-index of the elementary function $\pi(x, y) = x \cdot y$,
- l be a 0-index of the elementary function $\lambda(x) = (x)_0$,
- r be a 0-index of the elementary function $\rho(x) = (x)_1$,
- d be a 0-index of the elementary function $\delta(x) = \overline{\mathrm{sg}}((x)_0)$.

Let

$$\mathsf{s}_0(y) = \langle 7, 1, \mathsf{p}, \mathsf{l}, \langle 7, 1, y, \mathsf{r} \rangle \rangle,$$
$$\mathsf{s}_1(x) = \langle 7, 1, \mathsf{p}, \mathsf{d}, \langle 7, 1, x, \mathsf{r} \rangle \rangle,$$
$$\gamma(x, y) = \langle 7, 1, \mathsf{s}, \mathsf{s}_0(y), \mathsf{s}_1(x) \rangle.$$

Obviously, γ is an elementary function; let c be its 0-index. Let d be a 0-index of the elementary function $\psi_d(x, y) = \langle 7, 1, \mathsf{c}, x, y \rangle$.

Lemma 6.3 *For every natural numbers a, b, n and closed formulas A, B, C, if $a \Vdash_n (A \to C)$ and $b \Vdash_n (B \to C)$, then $\psi_d(a, b) \Vdash_n (A \vee B \to C)$.*

Proof. Let $a \Vdash_n (A \to C)$ and $b \Vdash_n (B \to C)$. This means that for every $j \geq n$, the number $\psi_a(j)$ is a j-index of a function ϕ_0 such that

$$x \Vdash_j A \Rightarrow \phi_0(x) \vdash_n C$$

and $\psi_b(j)$ is a j-index of a function ϕ_1 such that

$$x \Vdash_j B \Rightarrow \phi_1(x) \vdash_n C.$$

Denote $\psi_d(a, b)$ by h and prove that $h \Vdash_n (A \vee B \to C)$, i.e. that for every $j \geq n$, the number $\psi_h(j)$ is a j-index of a function ϕ such that

$$x \Vdash_j (A \vee B) \Rightarrow \phi(x) \Vdash_j C.$$

Let $j \geq n$ be given. Note that h is a j-index of a function obtained by substituting the functions ψ_a and ψ_b into γ, thus

$$\psi_h(j) = \gamma(\psi_a(j), \psi_b(j)) = \langle 7, 1, \mathsf{s}, \mathsf{s}_0(\psi_b(j)), \mathsf{s}_1(\psi_a(j))\rangle$$

is an index of the function

$$\phi(x) = \psi_{\mathsf{s}_0(\psi_b(j))}(x) + \psi_{\mathsf{s}_1(\psi_a(j))}(x).$$

Note that

$$\mathsf{s}_0(\psi_b(j)) = \langle 7, 1, \mathsf{p}, \mathsf{l}, \langle 7, 1, \psi_b(j), \mathsf{r}\rangle\rangle$$

is a j-index of the function $\varphi_0(x) = (x)_0 \cdot \phi_1((x)_1)$ and

$$\mathsf{s}_1(\psi_a(j)) = \langle 7, 1, \mathsf{p}, \mathsf{d}, \langle 7, 1, \psi_a(j), \mathsf{r}\rangle\rangle$$

is a j-index of the function $\varphi_1(x) = \overline{\mathrm{sg}}((x)_0) \cdot \phi_0((x)_1)$. Thus for every x,

$$\phi(x) = (x)_0 \cdot \phi_1((x)_1) + \overline{\mathrm{sg}}((x)_0) \cdot \phi_0((x)_1).$$

Now let $x \Vdash_j (B \vee C)$. If $(x)_\top$, then $(x)_1 \Vdash_j A$ and $\phi_0((x)_1) \Vdash_j C$. But in this case $\phi(x) = \phi_0((x)_1)$, thus $\phi(x) \Vdash_j C$. If $(x)_0 = 1$, then $(x)_1 \Vdash_j B$ and $\phi_1((x)_1) \Vdash_j C$. But in this case $\phi(x) = \phi_1((x)_1)$, thus $\phi(x) \Vdash_j C$. □

Let $f(a, \bar{c}) = \langle 7, 1, \mathsf{a}, (a)_0, (a)_1\rangle$, i.e. $f(a, \bar{c}) = \psi_d((a)_0, (a)_1)$, where d is the same as in Lemma 6.3. We prove that for every $a, \bar{c}$ and every $j \geq n$, if $a \Vdash_j (A(\bar{c}) \to C(\bar{c}))\,\&\,(B(\bar{c}) \to C(\bar{c})$, then $f(a, \bar{c}) \Vdash_j (A(\bar{c}) \vee B(\bar{c}) \to C(\bar{c}))$. Let $j \geq n$ and $a \Vdash_j (A(\bar{c}) \to C(\bar{c}))\,\&\,(B(\bar{c}) \to C(\bar{c})$. Then $(a)_0 \Vdash_j (A(\bar{c}) \to C(\bar{c}))$ and $(a)_1 \Vdash_j (B(\bar{c}) \to C(\bar{c}))$ and by Lemma 6.3 $f(a, \bar{c}) \Vdash_j (A(\bar{c}) \vee B(\bar{c}) \to C(\bar{c}))$. □

Theorem 6.2 *There exists a sequent deducible in* IPC *which is not strictly primitive recursively realizable.*

Proof. Consider the sequent $\top \to P \Rightarrow P$. It is known that this sequent is deducible in IPC but not in BPC. In the proof of Theorem 3.1 we considered the ternary predicate

$$e_{x+1}([y]_0, [y]_1) = z$$

and the formula $B(x, y, z)$ of a special kind expressing this predicate. We prove that the sequent $\top \to \exists z B(x, y, z) \Rightarrow \exists z B(x, y, z)$ is not realizable. Let it be realizable at a level n. This means that there is a ternary function $f \in \mathbf{E}_n$ such that for every a, k, l and every $j \geq n$, if $a \Vdash_j (\top \to \exists z B(k, l, z))$, then $f(a, k, l) \Vdash_j \exists z B(k, l, z)$. Consider the function $g(a, l) = f(a, n, l)$. Obviously, $g \in \mathbf{E}_n$ and for every a, l and every $j \geq n$, if $a \Vdash_j (\top \to \exists z B(n, l, z))$, then $g(a, l) \Vdash_j \exists z B(n, l, z)$. In particular, if $a \Vdash_n (\top \to \exists z B(n, l, z))$, then $g(a, l) \Vdash_n \exists z B(n, l, z)$. Lemma 3.1 states that the formula $\forall y(\top \to \exists z B(n, y, z))$ is realizable at the level n. This means that there is a unary function $h \in \mathbf{E}_n$ such that for every l, $h(l) \Vdash_n (\top \to \exists z B(n, l, z))$. Thus $g(h(l), l) \Vdash_n \exists z B(n, l, z)$ for every l. This means that the formula $\forall y \exists z B(x, y, z)$ is realizable at the level n. But it is impossible by Lemma 3.2. □

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