Bisimulations, Model Descriptions and Propositional Quantifiers

Albert Visser

Department of Philosophy, Utrecht University
Heidelberglaan 8, 3584 CS Utrecht
email: `Albert.Visser@phil.ruu.nl`

Department of Mathematics, University of Amsterdam
Plantage Muidergracht, 1018 TV Amsterdam

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Abstract

In this paper we give perspicuous proofs of the existence of model descriptions for finite Kripke models and of Uniform Interpolation for the theories IPC, K, GL and $\mathsf{S4Grz}$, using bounded bisimulations.

1 Introduction

Bisimulation and bounded bisimulation can be used to 'visualize' the proofs of two classes of well known results in intuitionistic and modal propositional logic. The first class consists of results guaranteeing the existence of model descriptions for certain classes of finite Kripke models. The other class consists of interpolation and uniform interpolation results. The aim of this paper is to present proofs for results in these two classes of theorems as clearly and perspicuously as possible.

A model description in modal or intuitionistic logic of a Kripke model $\mathbb{K}$ with designated node k is a formula A such that every other node m in any other Kripke model $\mathbb{M}$ bisimulates with k if and only if it satisfies A. We show the existence of model descriptions for transitive models both in the modal and the intuitionistic (persistent) case, without explicitely programming the formulas.[1]

Ordinary interpolation for a given theory T says that if $\mathsf{T} \vdash A \to B$, then there is a formula $I(A, B)$ in the language containing only the shared propositional variables, say $\vec{q}$, such that $\mathsf{T} \vdash A \to I$ and $\mathsf{T} \vdash I \to B$. Uniform interpolation is a strengthening of ordinary interpolation in which the data in terms of which the interpolant is to be specified are weaker: the interpolant can be found from either A and $\vec{q}$ or from $\vec{q}$ and B. Thus, if uniform interpolation holds, there is, for every A and $\vec{q}$, a 'post-interpolant' $I(A, \vec{q})$ such that, for all B such that $\mathsf{T} \vdash A \to B$ and such that the shared propositional variables of A and B are among $\vec{q}$, we have $\mathsf{T} \vdash A \to I(A, \vec{q})$ and $\mathsf{T} \vdash I(A, \vec{q}) \to B$. Similarly there is a pre-interpolant. As we will see, uniform interpolation is equivalent to the possibility of interpreting certain propositional quantifiers in T. Yet another way of viewing the phenomenon is as the existence of *quantifier elimination* for certain quantifiers. In this paper I prove Uniform Interpolation for IPC (Intuitionistic Propositional Calculus), for K, for GL (Löb's Logic) and for $\mathsf{S4Grz}$.[2]

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[1] No originality is claimed for the result. It was proved earlier by Silvio Ghilardi.

[2] Uniform Interpolation for IPC was first proved by Pitts using proof theoretical methods. It was proved by the present method by Ghilardi and Zawadowski and, independently but later, by the author. Uniform Interpolation for GL was first proved by Shavrukov. It was proved by the present method by the author. To give the due credit it should

Uniform interpolation for S4Grz is rather surpising, since it fails for the closely related theory S4. We reproduce a version of the proof of Ghilardi and Zawadowski that Uniform Interpolation fails for S4 at the end of this paper.

2 Models

We start with introducing the notion of Kripke model and specifying some notations. A (Kripke) model is a structure $\mathbb{K} = \langle K, \prec, \models, \mathcal{P}\rangle$. Here:

- K is a non-empty set of nodes
- $\prec$ is a binary relation on K
- $\mathcal{P}$ is a (possibly empty) set of propositional variables[3]
- $\models$ is a relation between K and $\mathcal{P}$

We can, alternatively, view a model $\mathbb{K}$ as a function that assigns to a fixed set of pairwise disjoint labels $\{\underline{K}, \underline{\prec}, \underline{\models}, \underline{\mathcal{P}}\}$ the appropriate objects. In this style we will write e.g. $\mathcal{P}_{\mathbb{K}}$ for $\mathbb{K}(\underline{\mathcal{P}})$. We will say that $\mathbb{K}$ is a $\mathcal{P}$-model if $\mathcal{P}_{\mathbb{K}} = \mathcal{P}$. Similarly for $K, \models$-model, etcetera. Similar conventions will be employed for other kinds of models. Define: $\mathsf{PV}_{\mathbb{K}}(k) := \{p{\in}\mathcal{P}_{\mathbb{K}} \mid k \models_{\mathbb{K}} p\}$. Note that $\models$ and PV are interdefinable. $\vec{p}, \vec{q}, \vec{r}$ will range over finite sets of propositional variables. A model $\mathbb{K}$ is finite if both $K_{\mathbb{K}}$ and $\mathcal{P}_{\mathbb{K}}$ are finite. We will call the class of models Mod.

It is often pleasant to think in terms of *a node in a model*. It is worthwile to make this notion explicit. A *pointed* model is a structure $\mathbb{K} = \langle \mathbb{K}_0, k\rangle$, where $\mathbb{K}_0$ is a model, and k is a node of $\mathbb{K}_0$. A pointed model $\langle \mathbb{K}, b\rangle$ is called *rooted* if for all $k \in K$: $b \prec^* k$[4] b is called the *root*. We can confuse a class of models with its disjoint union, taking as new nodes the pointed models corresponding to the models of the class. We define, e.g., $\langle \mathbb{K}, k\rangle \prec \langle \mathbb{K}', k'\rangle :\Leftrightarrow \mathbb{K} = \mathbb{K}'$ and $k \prec_{\mathbb{K}} k'$. Thus, we can confuse a pointed model $\langle \mathbb{K}, k\rangle$ with a 'free floating' node k. Note that the disjoint union of all models is not strictly speaking a model in our sense. The set of popositional variables that is declared to be present need not be constant in different 'nodes'. It is essential for our purposes for this to be so, since we want to study transitions between nodes in different models that do not leave the set of variables present constant. The totality of pointed models will be called Pmod and the totality of rooted models Rmod.

Suppose $\mathbb{K}$ is a —possibly pointed— $\mathcal{P}$-model. Then $\mathbb{K}[\mathcal{Q}]$ is the $\mathcal{P} \cap \mathcal{Q}$-model obtained by restricting $\models_{\mathbb{K}}$ to $\mathcal{P} \cap \mathcal{Q}$. For any $k \in K$, $\mathbb{K}[k]$ is the rooted model $\langle K', k, \prec', \models', \mathcal{P}\rangle$, where $K' := \uparrow k := \{k'{\in}K \mid k \prec^* k'\}$ and where $\prec'$ and $\models'$ are the restrictions of $\prec$ respectively $\models$ to K'. (We will often simply write $\prec$ and $\models$ for $\prec'$ and $\models'$.) In case we are using the convention of confusing a node k with its pointed model, $\langle \mathbb{K}, k\rangle$, we will, e.g., write $k[\mathcal{Q}]$ for $\langle \mathbb{K}[\mathcal{Q}], k\rangle$.

We will consider several properties of models. $\mathbb{K}$ will be said to be *transitive* if $\prec_{\mathbb{K}}$ is transitive, etcetera. $\mathbb{K}$ is *persistent* if $\mathsf{PV}_{\mathbb{K}}$ is monotonic w.r.t. $\prec_{\mathbb{K}}$ and $\subseteq$.

It will be convenient to extend the natural numbers ω with an extra element ∞. Let ω^∞ be $\omega \cup \{\infty\}$. We let $\alpha, \beta, \ldots$ range over ω^∞. ω^∞ is equipped with the obvous ordering $\leq$. We extend addition by: $\infty + \alpha = \alpha + \infty = \infty$. We extend cut-off substraction in our structure by: $\infty - n = \infty$. We will avoid the question of what $\infty - \infty$ is.

Transitive models are going to play a special role in this paper so we will need some some special notions concerned with transitive models. Consider any *transitive* model $\mathbb{K}$. Define:

- $k \prec^+ k' :\Leftrightarrow k \prec k'$ and not $k' \prec k$

be pointed out that the method here is very similar to the one used by Ghilardi and Zawadowski and, independently, the author, to prove ther result for IPC. The result for S4Grz is, as far as I know, new in this paper.

[3]We take the set of propositional variables as 'internal' to the models (and the languages), because we want to think about model extensions, which involve changing the set of variables of the model.

[4]$\prec^*$ is the transitive reflexive closure of $\prec$.

$$\begin{tikzcd}
k' \arrow[r, dash, dashed, "\mathcal{Z}_\alpha"] & m' \\
k \arrow[u, "\prec_{\mathbb{K}}"] \arrow[r, dash, "\mathcal{Z}_{\alpha+1}"'] & m \arrow[u, dashed, "\prec_{\mathbb{M}}"']
\end{tikzcd}$$

Figure 1: *The zig$_{\alpha+1}$-property*

- $k \approx k' :\Leftrightarrow k = k'$ or ($k \prec k'$ and $k \prec k'$). So $\approx$ means *being in the same cluster.*
- $d_{\mathbb{K}}(k) := \sup(\{(d_{\mathbb{K}}(k')+1) \in \omega^\infty \mid k' \succ^+ k\})$
- If $\mathbb{K}$ is pointed with designated node k, we put: $d(\mathbb{K}) := d_{\mathbb{K}}(k)$

Note that if $k \prec^+ k'$, then $d_{\mathbb{K}}(k') \leq d_{\mathbb{K}}(k) - 1$. k is a top node if it is a top node w.r.t. $\prec^+$. Note that k is a top node precisely if $d_{\mathbb{K}}(k) = 0$.

3 Layered Bisimulation

In this section we introduce bisimulation and bounded bisimulation. To avoid formulating most definitions and theorems twice —once for bounded and once for ordinary bisimulation— we make use of a portmanteau notion: *layered bisimulation*.[5]

Consider $\mathcal{P}$-models $\mathbb{K}$ and $\mathbb{M}$. We write $K := K_{\mathbb{K}}$ and $M := K_{\mathbb{M}}$. A *layered bisimulation* or *ℓ-bisimulation* $\mathcal{Z}$ between $\mathbb{K}$ and $\mathbb{M}$ is a ternary relation between K, ω^∞ and M, satisfying the conditions specified below. We will consider $\mathcal{Z}$ also as an ω^∞-indexed set of binary relations between K and M writing $k\mathcal{Z}_\alpha m$ for $\langle k, \alpha, m\rangle \in \mathcal{Z}$. We often write $k\mathcal{Z}m$ for $k\mathcal{Z}_\infty m$. We give the conditions:

1. $k\mathcal{Z}_\alpha m \Rightarrow \mathsf{PV}_{\mathbb{K}}(k) = \mathsf{PV}_{\mathbb{M}}(m)$
2. $k' \succ_{\mathbb{K}} k\mathcal{Z}_{\alpha+1}m \Rightarrow$ there is an m' with $k'\mathcal{Z}_\alpha m' \succ_{\mathbb{M}} m$; i.o.w. $\succ_{\mathbb{K}} \circ \mathcal{Z}_{\alpha+1} \subseteq \mathcal{Z}_\alpha \circ \succ_{\mathbb{M}}$.
3. $k\mathcal{Z}_{\alpha+1}m \prec_{\mathbb{M}} m' \Rightarrow$ there is a k' with $k \prec_{\mathbb{K}} k'\mathcal{Z}_\alpha m'$; i.o.w. $\mathcal{Z}_{\alpha+1} \circ \prec_{\mathbb{M}} \subseteq \prec_{\mathbb{K}} \circ \mathcal{Z}_\alpha$

Note that we allow ℓ-bisimulations to be undefined on some nodes. They may even be empty. Note also that ℓ-bisimulations occur only between models for the same set of variables. We call (2) the *zig$_{\alpha+1}$-property* (see figure 1) and (3) the *zag$_{\alpha+1}$-property*. If $\alpha = \infty$ we simply speak of the zig- and the zag-property. A binary relation $\mathcal{Z}$ between $\mathbb{K}$ and $\mathbb{M}$ is a *bisimulation* between $\mathbb{K}$ and $\mathbb{M}$ iff $\{\langle k, \infty, m\rangle \mid k\mathcal{Z}m\}$ is an ℓ-bisimulation. We will simply confuse bisimulations $\mathcal{Z}$ with the corresponding ℓ-bisimulations. An ℓ-bisimulation $\mathcal{Z}$ is a *bounded bisimulation* if for some natural number n: $k\mathcal{Z}_\alpha m \Rightarrow \alpha \leq n$.

Let $\mathsf{ID}_{\mathbb{K}} := \{\langle k, \alpha, k\rangle \mid k \in K,\ \alpha \in \omega^\infty\}$. Suppose $\mathcal{Z}$ is an ℓ-bisimulation between $\mathbb{K}$ and $\mathbb{M}$ and that $\mathcal{U}$ is an ℓ-bisimulation between $\mathbb{M}$ and $\mathbb{N}$. We define $\mathcal{Z} \circ \mathcal{U}$ by: $(\mathcal{Z} \circ \mathcal{U})_\alpha := \mathcal{Z}_\alpha \circ \mathcal{U}_\alpha$, and $\widehat{\mathcal{Z}}$ by $(\widehat{\mathcal{Z}})_\alpha := \widehat{(\mathcal{Z}_\alpha)}$, where $\widehat{(.)}$ is the usual inverse on binary relations. $\mathcal{Z}^\alpha$ is the relation given by: $\mathcal{Z}^\alpha_\beta := \mathcal{Z}_{\alpha+\beta}$. We say that $\mathcal{Z}$ is *downward closed* if for all $\alpha \prec \beta$: $\mathcal{Z}_\beta \subseteq \mathcal{Z}_\alpha$, The *downward closure* $\mathcal{Z}{\downarrow}$ of $\mathcal{Z}$ is the smallest downwards closed relation extending $\mathcal{Z}$. In the following theorem we collect the necessary elementary facts.

[5] Bisimulation is used both in computer science and modal logic. See e.g. the papers in [18] for an impression. In model theory bisimulation and bounded bisimimulation appears in the guise of Ehrenfeucht games and back-and-forth equivalence. See e.g. [8].

Theorem 3.1 *1.* $\mathsf{ID}_{\mathbb{K}}$ *is an ℓ-bisimulation.*

2. $\mathcal{Z} \circ \mathcal{U}$ *is an ℓ-bisimulation between $\mathbb{K}$ and $\mathbb{N}$.*

3. $\widehat{\mathcal{Z}}$ *is an ℓ-bisimulation between $\mathbb{M}$ and $\mathbb{K}$.*

4. $\mathcal{Z}^{\alpha}$ *is an ℓ-bisimulation.*

5. *The downward closure of $\mathcal{Z}$ is an ℓ-bisimulation.*

6. *Suppose Z is a set of ℓ-bisimulations between $\mathbb{K}$ and $\mathbb{M}$. Then $\bigcup \mathsf{Z}$ is again an ℓ-bisimulation between $\mathbb{K}$ and $\mathbb{M}$. It follows that there is always a maximal ℓ-bisimulation, $\simeq^{\mathbb{K},\mathbb{M}}$ between two models. (1)-(5) imply that for any α:*

   - $\mathsf{ID}_{\mathbb{K}} \subseteq \simeq^{\mathbb{K},\mathbb{M}}$
   - $\simeq^{\mathbb{K},\mathbb{M}} \circ \simeq^{\mathbb{M},\mathbb{N}} \subseteq \simeq^{\mathbb{K},\mathbb{N}}$
   - $\simeq^{\mathbb{K},\mathbb{M}}$ *is downward closed.*

   Note that, by the above, each of the $\simeq^{\mathbb{K},\mathbb{M}}_{\alpha}$ is an equivalence relation.

7. *Consider $k \in K$ and $m \in M$. Let $\mathcal{Z}[k,m]$ be the restriction of $\mathcal{Z}$ to $\uparrow k \times \uparrow m$. Then $\mathcal{Z}[k,m]$ is an ℓ-bisimulation between $\mathbb{K}[k]$ and $\mathbb{M}[m]$.*

8. *Consider two transitive models $\mathbb{K}$ and $\mathbb{M}$. Consider the relation $\mathcal{W}$, given by:*

$$k\mathcal{W}_{\alpha}m :\Leftrightarrow \textit{for some } k', m' :\ k \approx k'\mathcal{Z}_{\alpha}m' \approx m \textit{ and } k \simeq_0 m.$$

   We have: $\mathcal{W}$ is an ℓ-bisimulation. It follows, e.g., taking $\mathbb{M} := \mathbb{K}$ and $\mathcal{Z} := \mathsf{ID}_{\mathcal{K}}$, that $\approx \cap \simeq_0$ is an ℓ-bisimulation on $\mathbb{K}$.

We will often drop the superscript of $\simeq^{\mathbb{K},\mathbb{M}}$ In case $\alpha = \infty$, we will drop the subscript of $\simeq^{\mathbb{K},\mathbb{M}}_{\alpha}$ (if no confusion is possible). We will say that k and m (considered as pointed models) *n-simulate* if $k \simeq_n m$ and that k and m *bisimulate* if $k \simeq m$. $\mathcal{Z}_{\alpha}$ is *full* if it is both total and surjective as a relation between K and M. We will say that $\mathbb{K}$ and $\mathbb{M}$ *α-bisimulate (bisimulate)*, or, $\mathbb{K} \cong_{\alpha} \mathbb{M}$ ($\mathbb{K} \cong \mathbb{M}$) if there is a full α-bisimulation (bisimulation) between them. $\mathcal{Z} : \mathbb{K} \cong_{\alpha} \mathbb{M}$ means that $\mathcal{Z}$ is a full α-bisimulation witnessing that $\mathbb{K} \cong_{\alpha} \mathbb{M}$. Note that for *rooted* models $\mathbb{K}$ and $\mathbb{M}$, we have: $\mathbb{K} \cong_{\alpha} \mathbb{M} \Leftrightarrow b_{\mathbb{K}} \simeq b_{\mathbb{M}}$.[6]

We can collapse $\mathbb{K}$ to an *α-irreducible (irreducible)* $\mathcal{P}$-model, $\mathbb{K}^{\alpha} := \mathsf{Coll}_{\alpha}(\mathbb{K})$, by dividing $\simeq_{\alpha}$ ($\simeq_{\infty}$) out. The construction is as follows:

- $[k]_{\alpha}$ is the $\simeq_{\alpha}$ equivalence class of k.
- $K^{\alpha} := \{[k]_{\alpha} \mid k \in K\}$; we let κ, κ' range over K^{α}.
- $\kappa \models p$ iff $\exists k{\in}\kappa\ \ k \models p$
- $\kappa \prec \kappa'$ iff $\exists k{\in}\kappa\, \exists k'{\in}\kappa'\ \ k \prec k'$

We collect the simple facts about the collapse in a theorem.

Theorem 3.2 *1. The mapping $\phi_{\mathbb{K}} : k \mapsto [k]_{\alpha}$ is a full, functional α-bisimulation (a* surjective α-p-morphism*) from $\mathbb{K}$ to $\mathbb{K}^{\alpha}$; note that ϕ considered as a relation is precisely the relation $\in$.*

2. *K^{α} is α-irreducible, i.e., if $\kappa \simeq_{\alpha} \kappa'$, then $\kappa = \kappa'$.*

3. *Suppose $\mathbb{K} \cong_{\alpha} \mathbb{M}$, then $\simeq_{\alpha}$ is a bijection between $\mathbb{K}^{\alpha}$ and $\mathbb{M}^{\alpha}$. It is an isomorphism of models if $\alpha = \infty$.*

[6]Note the difference between "$\mathbb{K} \cong_{\alpha} \mathbb{M}$" and "$\mathbb{K} \simeq_{\alpha} \mathbb{M}$". The first statement relates ordinary models and the second one pointed models, saying that their points α-bisimulate.

Proof

1. It is easy to see that ϕ is full, functional and that it preserves atoms. Moreover ϕ has the full zig$_\infty$-property: it is monotonic. Define, for $\gamma \leq \alpha$: $\mathcal{Z}_\gamma := \simeq_\gamma^{\mathbb{K},\mathbb{K}} \circ\phi$. In other words: $k\mathcal{Z}_\gamma\kappa$ iff for some k', $k \simeq_\gamma k'$ and $k' \in \kappa$. We show that $\mathcal{Z}$ is an ℓ-bisimulation between $\mathbb{K}$ and $\mathbb{K}^\alpha$. We check the zag direction. Suppose $\delta + 1 \leq \alpha$ and $k_0 \simeq_{\delta+1} k_1 \in \kappa_2 \prec \kappa_3$. This means that there are k_2, k_3 such that: $k_0 \simeq_{\delta+1} k_1 \simeq_\alpha k_2 \prec k_3 \in \kappa_3$. Since $\simeq$ is downward closed and closed under composition, it follows that: $k_0 \simeq_{\delta+1} k_2 \prec k_3 \in \kappa_3$. Hence, by the zag$_{\delta+1}$-property for $\simeq$, we find a k_4, with: $k_0 \prec k_4 \simeq_\delta k_3 \in \kappa_3$.

$$\begin{tikzcd}
k_4 \arrow[rr, dash, dashed, "\simeq_\delta"] & & k_3 \arrow[r, dash, "\in"] & \kappa_3 \\
k_0 \arrow[u, dashed, "\prec_{\mathbb{K}}"] \arrow[r, dash, "\simeq_{\delta+1}"] & k_1 \arrow[r, dash, "\simeq_\alpha"] & k_2 \arrow[u, "\prec_{\mathbb{K}}"'] \arrow[r, dash, "\in"] & \kappa_2 \arrow[u, "\prec_{\mathbb{K}^\alpha}"']
\end{tikzcd}$$

Clearly $\mathcal{Z}_\alpha$ is precisely ϕ, so we are done.

2. Suppose $\kappa \simeq_\alpha \kappa'$, then for $k \in \kappa$ and $k' \in \kappa'$, we have

$$k \simeq_\alpha^{\mathbb{K},\mathbb{K}^\alpha} \kappa \simeq_\alpha^{\mathbb{K}^\alpha,\mathbb{K}^\alpha} \kappa' \simeq_\alpha^{\mathbb{K}^\alpha,\mathbb{K}} k'.$$

Hence: $k \simeq_\alpha^{\mathbb{K},\mathbb{K}} k'$, and, thus, $\kappa = \kappa'$.

3. Suppose $\mathbb{K} \cong_\alpha \mathbb{M}$. Consider κ in $\mathbb{K}^\alpha$. Pick a k in κ. There is an m in $\mathbb{M}$ with $k \simeq_\alpha m$, so: $\kappa \simeq_\alpha k \simeq_\alpha m \simeq_\alpha [m]_\alpha$ and hence $\kappa \simeq_\alpha [m]_\alpha$. Thus $\simeq_\alpha$ is total between $\mathbb{K}^\alpha$ and $\mathbb{M}^\alpha$. Similarly for the surjectiveness. Suppose $\kappa \simeq_\alpha \mu$ and $\kappa \simeq_\alpha \mu'$. It follows that $\mu \simeq_\alpha \mu'$, and hence $\mu = \mu'$. $\simeq_\infty$ is structure preserving, since it is bijective and has the zig- and the zag-property.

❑

The following theorem tells us that the number of $\simeq_n$-equivalence classes on a model has a fixed finite bound that only depends on n.

Theorem 3.3 *Define $F(N, 0) := 2^N$, $F(N, n+1) := 2^{F(N,n)+N}$. Suppose $|\mathcal{P}| = \mathcal{N}$, then the number of possible $\simeq_n$ equivalence classes is smaller or equal to $F(N, n)$.*

Proof

By a simple induction on n, noting that the $n+1$-equivalence class of a node k is fully determined by the atoms forced in k and the n-equivalence classes of the nodes 'seen' by k. ❑

4 Changing variables

In this paper we are particularly interested in things like extending or even changing the forcing of the propositional variables on nodes. We introduce the relevant notions. In this section $k, k', m, m' \ldots$ will be pointed models.

- $k \simeq_{\alpha,\mathcal{Q}} m :\Leftrightarrow \mathcal{P}_k \cap \mathcal{Q} = \mathcal{P}_m \cap \mathcal{Q}$ and $k[\mathcal{Q}] \simeq_\alpha m[\mathcal{Q}]$. So, roughly, this means that k and m α-bisimulate w.r.t. the variables in $\mathcal{Q}$.

- $k \simeq_{\alpha,[\mathcal{Q}]} m :\Leftrightarrow k \simeq_{\alpha,\mathcal{Q}^c} m$ and $\mathcal{Q} \subseteq \mathcal{P}_m$. So, roughly, this means that k differs from m *modulo* $\simeq_\alpha$ only at $\mathcal{Q}$ and m is at least a $\mathcal{Q}$-node.
- $k \sqsubseteq_{\alpha,\mathcal{Q}} m :\Leftrightarrow k \simeq_{\alpha,\mathcal{P}_k} m$ and $\mathcal{Q} \cap \mathcal{P}_k = \emptyset$ and $\mathcal{Q} \cup \mathcal{P}_k = \mathcal{P}_m$. We will say that m is a $\mathcal{Q},\alpha$-*bisimulation extension* of k. In case $\alpha = \infty$, we will speak of a $\mathcal{Q}$-*bisimulation extension*.

5 Persistent models and bisimulation orderings

Upwards persistent models play an important role in this paper. We give an illustrative example of such a model and the application of the notions of the previous section to the model.

Example 5.1 We specify the model $\mathbb{I}$, which is the Henkin model of the one variable fragment of the Intuitionistic Propositional Calculus (IPC). It is also the one point compactification of the 1-characterizing model (in the sense of [19]) for IPC.

- $K_{\mathbb{I}} := \omega^\infty$
- $b_{\mathbb{I}} := \infty$
- $\alpha \prec \beta :\Leftrightarrow \beta + 1 < \alpha$ or $\alpha = \beta$
- $\mathcal{P}_{\mathbb{I}} := \{p\}$
- $\alpha \models p :\Leftrightarrow \alpha = 0$

All facts proved here about this model follow directly from known results in combination with later results of this paper. Since our purposes are illustrative, we prove the basic facts from scratch. In figure 2 we give a picture of $\mathbb{I}$. It is easily verified that $\prec_{\mathbb{I}}$ is a weak partial ordering. Moreover $\mathbb{I}$ is persistent. Below we collect some simple facts about $\mathbb{I}$. ❑

Theorem 5.2 *$\mathbb{I}$ is irreducible, i.e. any bisimulation $\mathcal{Z} : \mathbb{I} \simeq \mathbb{I}$ is the identity on ω^∞.*

Proof

Consider any $\mathcal{Z} : \mathbb{I} \simeq \mathbb{I}$. We may assume that $\mathcal{Z}$ is symmetrical. Suppose that for some α and β : $\alpha\mathcal{Z}\beta$ and $\alpha \neq \beta$. Let α be $<$-minimal such that there is a β with $\alpha\mathcal{Z}\beta$ and $\alpha \neq \beta$. Clearly $0 < \alpha < \infty$. Since $\alpha < \beta$, it follows that $(\alpha - 1) + 1 < \beta$ and hence $\beta \prec (\alpha - 1)$. Since $\alpha\mathcal{Z}\beta$, there must be a $\gamma \succ \alpha$ with $\gamma\mathcal{Z}(\alpha - 1)$. By the $<$-minimality of α, we find that $\gamma = (\alpha - 1)$, and hence $(\alpha - 1) \succ \alpha$. Quod non. ❑

Define $\alpha E_\gamma \beta :\Leftrightarrow \alpha = \beta$ or $(\alpha > \gamma$ and $\beta > \gamma$.

Theorem 5.3 *E is an ℓ-simulation between $\mathbb{I}$ and $\mathbb{I}$.*

Proof

We verify the zig-property. Suppose $\delta \succ \alpha E_{\gamma+1} \beta$. In case $\delta \succ \beta$, we have: $\delta E_\gamma \delta$ and $\alpha E_\gamma \beta \prec \delta$. So we are done. Suppose not $\delta \succ \beta$. It follows that $\beta \leq \delta + 1$. Moreover we have $\alpha \neq \beta$ and hence $\alpha > \gamma + 1$ and $\beta > \gamma + 1$. It follows that $\beta > \gamma$ and $\delta > \gamma$. We may conclude that: $\delta E_\gamma \beta$ and $\alpha E_{\gamma+1} \beta \prec \beta$. Again we are done. ❑

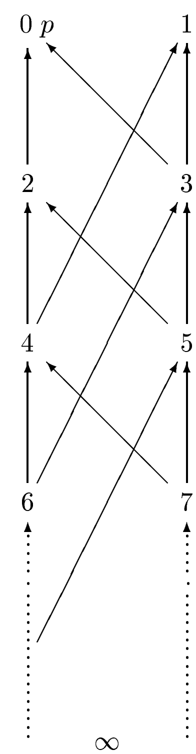


Figure 2: *The Henkin model of IPC(p)*

Define the 'shift right' function $\mathfrak{r}$ on ω^∞ by: $\mathfrak{r}(\alpha) :=$ the largest β such that $2.\beta \leq \alpha$. So $\mathfrak{r}(\infty) = \infty$.

Theorem 5.4 $d_{\mathbb{I}}(\alpha) = \mathfrak{r}(\alpha)$.

We state an evident fact about persistence.

Theorem 5.5 *Consider two models $\mathbb{K}$ and $\mathbb{M}$. Suppose that $\mathbb{K}$ is persistent and that $\mathbb{K} \cong \mathbb{M}$. Then $\mathbb{M}$ is persistent.*

If we are studying persistence it is often more natural to think in terms of certain orderings related to layered bisimulation, than in terms of layered bisimulation itself. We can think of these orderings as a kind of extension of the ordering in the model. For the rest of this section we think about persistent pointed $\mathcal{P}$-models. We let $k, k', m, m' \ldots$ range over such models.

- $k \preceq_0 m :\Leftrightarrow \mathsf{PV}(k) \subseteq \mathsf{PV}(m)$
- $k \preceq_{\alpha+1} m :\Leftrightarrow \mathsf{PV}(k) \subseteq \mathsf{PV}(m),\ \forall m' \succ m\, \exists k' \succ k'\ k' \simeq_\alpha m'$

In case $\alpha = \infty$, we will drop the subscript.

Theorem 5.6 *1. $\preceq_\alpha$ is a partial preordering on pointed, persistent $\mathcal{P}$-models.*

2. $k \prec k' \Rightarrow k \preceq_\alpha k'$.

3. $\alpha \leq \beta \Rightarrow \preceq_\beta \subseteq \preceq_\alpha$.

4. $k \simeq_\alpha m \Leftrightarrow k \preceq_\alpha m$ and $m \preceq_\alpha k$.

5. $k \preceq_\infty m \Leftrightarrow$ for some $k' \succeq k\ k' \simeq m$.

Proof

We prove (4). For $\alpha = 0$ this is easy. Suppose $\alpha > 0$. "$\Rightarrow$" Easy. "$\Leftarrow$" Suppose $k \preceq_\alpha m$ and $m \preceq_\alpha k$. We show that $\mathcal{U} := \simeq \cup \{\langle k, \alpha, m \rangle\}$ is an ℓ-bisimulation, and, hence, that $k \simeq_\alpha m$. Clearly $\mathsf{PV}(k) = \mathsf{PV}(m)$. The zig-property for $\mathcal{U}$ follows from the fact that $m \preceq_\alpha k$. The zag-property for $\mathcal{U}$ follows from the fact that $k \preceq_\alpha m$. ❑

6 The big jump: from finite to infinite

In some cases one can conclude that two nodes bisimulate from the fact that they n-bisimulate. This information can be usueful, since —as we shall see— in some cases it yields a cheap proof of the existence of formulas describing a model and formulas 'omitting' a model. We will be particularly interested in cases where the relevant n can be found by looking at one of the relevant models alone. We will restrict ourselves in this section to transitive models. Consider two transitive $\mathcal{P}$-models $\mathbb{K}$ and $\mathbb{M}$. Let $\mathcal{Z}$ be an ℓ-bisimulation between $\mathbb{K}$ and $\mathbb{M}$. We may assume that $\mathcal{Z}$ is downwards closed. We may also assume that $\mathcal{Z}$ is cluster-preserving i.e., $(\approx \circ \mathcal{Z} \circ \approx) \cap \simeq_0 \subseteq \mathcal{Z}$. If not we replace $\mathcal{Z}$ by $(\approx \circ \mathcal{Z} \circ \approx) \cap \simeq_0$. This is again an ℓ-bisimulation by theorem 3.1. Moreover, we have, by the transitivity of $\approx$:

$$\begin{aligned} (\approx \circ ((\approx \circ \mathcal{Z} \circ \approx) \cap \simeq_0) \circ \approx) \cap \simeq_0 &\subseteq (\approx \circ \approx \circ \mathcal{Z} \circ \approx \circ \approx) \cap \simeq_0 \\ &\subseteq (\approx \circ \mathcal{Z} \circ \approx) \cap \simeq_0 \end{aligned}$$

Theorem 6.1 *Define:*

- $\delta(k, m) := d_{\mathbb{K}}(k) + d_{\mathbb{M}}(m)$
- $k \mathcal{U} m :\Leftrightarrow k \mathcal{Z}_{\delta(k,m)+1} m$

Then $\mathcal{U}$ is a bisimulation between $\mathbb{K}$ and $\mathbb{M}$.

Proof

We prove that $\mathcal{U}$ has the zig-property. Suppose $k' \succ k \mathcal{Z}_{\delta(k,m)+1} m$. Hence, for some m': $k' \mathcal{Z}_{\delta(k,m)} m' \succ m$. It follows that $\mathsf{PV}(k') = \mathsf{PV}(m')$. In case $k' \approx k$ and $m' \approx m$, we find: $d_{\mathbb{K}}(k') = d_{\mathbb{K}}(k)$, $d_{\mathbb{M}}(m') = d_{\mathbb{M}}(m)$ and by cluster-preservation: $k' \mathcal{Z}_{\delta(k,m)+1} m'$. Hence, $k' \mathcal{Z}_{\delta(k',m')+1} m' \succ m$. In case not both $k' \approx k$ and $m' \approx m$, we find that $\delta(k',m') + 1 \leq \delta(k,m)$ and hence, by downwards closure: $k' \mathcal{Z}_{\delta(k',m')+1} m' \succ m$. ❑

In the transitive, reflexive, antisymmetric (i.e. wpo) case there is a marginal improvement.

Theorem 6.2 *Suppose $\mathbb{K}$ and $\mathbb{M}$ are transitive, reflexive and antisymmetric. Let $\mathcal{Z}$ and δ be as before. Define: $k \mathcal{U} m :\Leftrightarrow k \mathcal{Z}_{\delta(k,m)} m$. Then $\mathcal{U}$ is a bisimulation between $\mathbb{K}$ and $\mathbb{M}$.*

Proof

We prove that $\mathcal{U}$ has the zig-property. Suppose $k' \succ k \mathcal{U} m$. We are looking for m' with $k' \mathcal{U} m' \succ k$. In case $k' = k$ we may take $m' := m$. In case $k' \succ^+ k$, we have: $\delta(k,m) > 0$ and hence for some n: $k' \mathcal{Z}_{\delta(k,m)-1} n \succ k$. Since $\delta(k',n) \leq \delta(k,m) - 1$, we find: $k' \mathcal{Z}_{\delta(k',n)} n \succ k$. So we may take: $m' := n$. ❑

Example 6.3 Consider the model $\mathbb{I}$ of example 5.1. Clearly $\mathbb{I}$ satisfies the conditions for theorem 6.2. Note that for $i \in \omega$: $\delta(2i, 2i+1) = \mathfrak{r}(2i) + \mathfrak{r}(2i+1) = 2i$. Since $2i$ and $2i+1$ do not bisimulate, for no ℓ-simulation $\mathcal{Z}$ between $\mathbb{I}$ and $\mathbb{I}$: $(2i)\mathcal{Z}_{2i}(2i+1)$. On the other hand for $i > 0$, we have: $(2i)E_{2i-1}(2i+1)$. So theorem 6.2 is optimal. ❑

In the transitive, irreflexive (i.e. spo) case there is a substantial improvement:

Theorem 6.4 *Suppose $\mathbb{K}$ and $\mathbb{M}$ are transitive and irreflexive. Let $\mathcal{Z}$ be as before. Define:*

- $\mu(k,m) := \min(d_{\mathbb{K}}(k), d_{\mathbb{M}}(m))$
- $k \mathcal{U} m :\Leftrightarrow k \mathcal{Z}_{\mu(k,m)+1} m$

Then $\mathcal{U}$ is a bisimulation between $\mathbb{K}$ and $\mathbb{M}$.

Proof

We prove that $\mathcal{U}$ has the zig-property. Suppose $k' \succ k \mathcal{U} m$. There is an m' with: $k' \mathcal{Z}_{\mu(k,m)} m' \succ m$. Note that $d_{\mathbb{K}}(k') \leq d_{\mathbb{K}}(k) - 1$ and $d_{\mathbb{M}}(m') \leq d_{\mathbb{K}}(m) - 1$. Hence: $\mu(k,m) \geq \mu(k',m') + 1$. So we find: $k' \mathcal{Z}_{\mu(k',m')+1} m' \succ m$. ❑

Theorem 6.4 shows that in the irreflexive case n need only depend on the depth of our node in, say, $\mathbb{K}$. Surprisingly this phenomenon extends beyond this trivial case to the general transitive case. We first give the general argument. Then we give a sharper version for the case where $\mathbb{K}$ and $\mathbb{M}$ are partially ordered. The estimate of the result was sharpened from $2.d(k)+3$ to $2.d(k)+2$ by Giovanna d. Agostino.

Theorem 6.5 *Suppose $\mathbb{K}$ and $\mathbb{M}$ are transitive models. Define $k \mathcal{U} m :\Leftrightarrow k \simeq_{2.d(k)+2} m$. Then $\mathcal{U}$ is a bisimulation between $\mathbb{K}$ and $\mathbb{M}$.*

Proof

Suppose $k\mathcal{U}m$. Both in applying the zig- and the zag-property, we will arrive at a constellation: $k \prec k' \simeq_{2d(k)+1} m' \succ m$. Both the zig- and the zag-property applied to $k' \simeq_{2d(k)+1} m'$, yield a constellation: $k' \prec k'' \simeq_{2d(k)} m'' \succ m'$. Thus, we have (by transitivity in $\mathbb{M}$): $k \simeq_{2d(k)+2} m \prec m''$.

$$\begin{tikzcd}
k'' \arrow[r, dash, "2d(k)"] & m'' \arrow[r, dash, dashed, "2d(k)+1"] & k^* \\
k' \arrow[u, "\prec"] \arrow[r, dash, "2d(k)+1"] & m' \arrow[u, "\prec"'] & \\
k \arrow[u, "\prec"] \arrow[r, dash, "2d(k)+2"] & m \arrow[u, "\prec"'] \arrow[r, dash, "2d(k)+2"] & k \arrow[uu, dashed, "\prec"']
\end{tikzcd}$$

By the zig-property, we can find an k^* such that: $k \prec k^* \simeq_{2d(k)+1} m''$. It follows that:

$$k'' \simeq_{2d(k)} m'' \simeq_{2d(k)+1} k^*$$

By the downwards closure and transitivity of $\simeq_\alpha$, we find: $k'' \simeq_{2d(k)} k^*$. We distinguish two cases. (i) k'', k^* belong to the same cluster in $\mathbb{K}$. In this case we find: k'' and k^* bisimulate, since they force the same atoms (by theorem 3.1). (ii) k'' and k^* do not belong to the same cluster. In this case either $d(k'') < d(k)$ or $d(k^*) < d(k)$. It follows that $d(k'')+d(k^*)+1 \leq 2d(k)$. Applying theorem 6.1 (with $\mathbb{K}$ both in the role of $\mathbb{K}$ and of $\mathbb{M}$), we find: $k'' \simeq k^*$. So, in both cases, we have, $k'' \simeq k^*$. We find, $k'' \simeq k^* \simeq_{2d(k)+1} m''$, and, hence, $k'' \simeq_{2d(k)+1} m''$. We may conclude that k' and m' satisfy both the zig- and the zag-property for $\simeq_{2d(k)+2}$ and, so, $k' \simeq_{2d(k)+2} m'$, and, a fortiori, $k' \simeq_{2d(k')+2} m'$, i.e., $k'\mathcal{U}m'$. ❑

In the following theorem we sharpen the result for the case of partial orderings.

Theorem 6.6 *Suppose $\mathbb{K}$ and $\mathbb{M}$ are transitive, reflexive and antisymmetric. I.o.w. suppose that their accessibility relations are (weak) partial orderings. We will use $\preceq$ instead of $\prec$ to stress this fact. We use $\prec$ for the corresponding strict ordering. Let $\mathcal{Z}$ be a downward closed ℓ-bisimulation between $\mathbb{K}$ and $\mathbb{M}$. Define: $k\mathcal{U}m :\Leftrightarrow \exists p\ k\mathcal{Z}_{2.d(k)+1}p \preceq m$ and $k\mathcal{Z}_{2.d(k)}m$ Then $\mathcal{U}$ is a bisimulation between $\mathbb{K}$ and $\mathbb{M}$.*

Note that by downward closure: $k\mathcal{Z}_{2.d(k)+1}m \Rightarrow k\mathcal{U}m$.

Proof

Suppose $k\mathcal{U}m$ with witness p. We first prove that $\mathcal{U}$ has the zig-property. Suppose $k' \succeq k\mathcal{U}m$. We want an $m' \succeq m$ and a p' with p' witnesses $k'\mathcal{U}m'$. In case $k' = k$, take $p' := p$ and $m' := m$. Trivially p' witnesses $k'\mathcal{U}m'$. If $k' \succ k$ we find that $d_{\mathbb{K}}(k) > 0$. From $k\mathcal{Z}_{2.d(k)}m$, we get an $m' \succ m$ with $k'\mathcal{Z}_{2.d(k)-1}m'$. We have: $2.d_{\mathbb{K}}(k')+1 \leq 2.d_{\mathbb{K}}(k) - 1$, and hence $k'\mathcal{Z}_{2.d(k')+1}m'$. Moreover:

$k'\mathcal{Z}_{2.d(k')}n \succ m'$. Ergo we may take $p' := m'$.

$$\begin{tikzcd}
 & k'=k \arrow[r, dash, dashed, "\mathcal{Z}_{2.d(k)}"] & m'=m \\
 & k \arrow[u, "="] \arrow[r, dash, "\mathcal{Z}_{2.d(k)}"] & m \arrow[u, dashed, "="'] \\
k \arrow[ur, "="] \arrow[rr, dash, "\mathcal{Z}_{2.d(k)+1}"] & & p'=p \arrow[u, "\preceq_{\mathbb{M}}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
 & k' \arrow[r, dash, dashed, "\mathcal{Z}_{2.d(k')+1}"] & p'=m' \\
 & k \arrow[u, "\prec_{\mathbb{K}}"] \arrow[r, dash, "\mathcal{Z}_{2.d(k)}"] & m \arrow[u, dashed, "\preceq_{\mathbb{M}}"'] \\
k \arrow[ur, "="] \arrow[rr, dash, "\mathcal{Z}_{2.d(k)+1}"] & & p \arrow[u, "\preceq_{\mathbb{M}}"']
\end{tikzcd}$$

We check the zag-property. Suppose $k\mathcal{U}m \preceq m'$. We want a $k' \geq k$ and a p', where p' witnesses $k'\mathcal{U}m'$. We have $p \leq m \leq m'$ and hence $p \leq m'$. Since $k\mathcal{Z}_{2.d(k)+1}p$, it follows that for some $k' \geq k$: $k'\mathcal{Z}_{2.d(k)}m'$. There are two possibilities: $k' = k$ or $k' \succ k$. In case $k' = k$ take $p' := p$. In case $k' \succ k$, take $p' := m'$.

$$\begin{tikzcd}
k'=k \arrow[rr, dash, dashed, "\mathcal{Z}_{2.d(k)}"] & & m' \\
 & k \arrow[r, dash, "\mathcal{Z}_{2.d(k)}"] & m \arrow[u, "\preceq_{\mathbb{M}}"'] \\
k \arrow[uu, dashed, "="] \arrow[ur, "="] \arrow[rr, dash, "\mathcal{Z}_{2.d(k)+1}"] & & p'=p \arrow[u, "\preceq_{\mathbb{M}}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
k' \arrow[rr, dash, dashed, "\mathcal{Z}_{2.d(k')+1}"] & & p'=m' \\
 & k \arrow[r, dash, "\mathcal{Z}_{2.d(k)}"] & m \arrow[u, "\preceq_{\mathbb{M}}"'] \\
k \arrow[uu, dashed, "\prec_{\mathbb{K}}"] \arrow[ur, "="] \arrow[rr, dash, "\mathcal{Z}_{2.d(k)+1}"] & & p \arrow[u, "\preceq_{\mathbb{M}}"']
\end{tikzcd}$$

❑

Example 6.7 1. Consider the model $\mathbb{I}$ of example 5.1. Let $i \in \omega$. We have $2.d(2i+1) = 2.i$. Let $\mathcal{Z}$ be any ℓ-simulation between $\mathbb{I}$ and $\mathbb{I}$. Since $2i+1$ and $2i+2$ do not bisimulate, we cannot have: $(2i+1)\mathcal{Z}_{2i+1}(2i+2)$. On the other hand we do have: $(2i+1)E_{2i}(2i+2)$. So the index in theorem 6.6 cannot be lowered.

2. We illustrate that $\mathcal{U}$ with $k\mathcal{U}m :\Leftrightarrow k\mathcal{Z}_{2.d(k)+1}m$, is not generally a bisimulation and thus that the proof of theorem 6.6. cannot be simplified in this direction. Consider the weakly partially ordered $\emptyset$-models $\mathbb{K}$ and $\mathbb{M}$ with: $K := \{a\}$, $M := \{0, 1\}$, where the wpo on M is.generated by: $0 \prec 1$. Take $\mathcal{Z} := \{\langle a, 1, 0\rangle, \langle a, 0, 0\rangle, \langle a, 0, 1\rangle\}$. Clearly $\mathcal{Z}$ is a downward closed ℓ-simulation between $\mathbb{K}$ and $\mathbb{M}$. Evidently $\mathcal{Z}_{2.d(2)+1}$ is not a bisimulation.

❑

For the case of persistent models there is a variant of theorem 6.6, that is easier to apply.

Theorem 6.8 *We consider persistent and partially ordered models and nodes (pointed models) of the same kind. The relation $\mathcal{Z}$ between $\mathbb{K}$ and $\mathbb{M}$ defined by:*

$$k\mathcal{Z}m :\Leftrightarrow k \simeq_{2.d(k)} m \text{ and } k \preceq_{2.d(k)+1} m$$

is a bisimulation. It follows that: $k \preceq_{2.d(k)+1} m \Rightarrow k \preceq_{\infty} m$.

Proof

Consider k, m with $k\mathcal{Z}m$. It is clear that k and m force the same atoms. We check the zig-property. Suppose $k \preceq k'$. In case $k = k'$, we choose $m' := m$. In case $k \neq k'$, we apply $k \simeq_{2.d(k)} m$, to find an $m' \succeq m$ with $k' \simeq_{2.d(k)-1} m'$. Since, $2.d(k') + 1 \leq 2.(d(k) - 1) + 1 \leq 2.d(k) - 1$, we find, $k' \simeq_{2.d(k')+1} m'$. Thus, a fortiori, $k'\mathcal{Z}m'$. We check the zig-property. Suppose $m \preceq m'$. Since, $k \preceq_{2.d(k)+1} m$, there is an $k' \succeq k$ with $k' \simeq_{2.d(k)} m'$. In case $k = k'$, we have $k' = k \preceq_{2.d(k)+1} m'$. Thus, $k'\mathcal{Z}m'$. In case $k \neq k'$, we find $k' \simeq_{2.d(k')+1} m'$. Hence, $k'\mathcal{Z}m'$.

To prove the consequence, suppose $k \preceq_{2.d(k)+1} m$. Since $m \preceq m$, there is a $k' \succeq k$ with $k' \simeq_{2.d(k)} m$. In case $k = k'$, we find $k'\mathcal{Z}m$. In case $k \neq k'$, we find $k' \simeq_{2.d(k')+1} m$, and hence $k'\mathcal{Z}m$. In both cases, we have, $k' \simeq m$. ❑

7 Some basic facts about IPC

In this section we present some basic facts about and constructions in IPC. The present section is not a self contained introduction. It is intended to fix some notations and to establish soem convenient lemmas. $\vdash$ will stand for derivability in IPC in this section. Consider any $\mathcal{P}$. We define $\mathcal{L}^i(\mathcal{P})$ as the smallest set such that:

- $\mathcal{P} \subseteq \mathcal{L}^i(\mathcal{P})$, $\bot, \top \in \mathcal{L}^i(\mathcal{P})$
- if $A, B \in \mathcal{L}^i(\mathcal{P})$, then $(A \wedge B), (A \vee B), (A \to B) \in \mathcal{L}^i(\mathcal{P})$.

$\mathsf{PV}(A)$ is the set of propositional variables occurring in A. $\mathsf{Sub}(A)$ is the set of subformulas of A. A model is an IPC-*model* if it is transitive, reflexive, antisymmetric and persistent. Consider an IPC $\mathcal{P}$-model $\mathbb{K}$ we take $\models_i$ to be the smallest relation between K and $\mathcal{L}^i(\mathcal{P})$ such that:

- $k \models_i p :\Leftrightarrow k \models p$, $k \models_i \top$
- $k \models_i A \wedge B :\Leftrightarrow k \models_i A$ and $k \models_i B$
- $k \models_i A \vee B :\Leftrightarrow k \models_i A$ or $k \models_i B$
- $k \models_i A \to B :\Leftrightarrow \forall k' \succ k\ (k' \models_i A \Rightarrow k' \models_i B)$

We will omitt the subscript i, as long as it is sufficiently clear from the context that the persistent case is intended. Note that, by transitivity, the persistence for $\mathcal{P}$ extends to the persistence for $\mathcal{L}^i(\mathcal{P})$. Define further:

- $k \models \Gamma :\Leftrightarrow$ for all $A \in \Gamma : k \models A$
- $\mathbb{K} \models A :\Leftrightarrow$ for all $k \in K$ $k \models A$

A set X is $\mathcal{P}$-*adequate* if $X \subseteq \mathcal{L}^i(\mathcal{P})$ and X is closed under subformulas. A set Γ is X-saturated (for IPC) if:

1. $\Gamma \subseteq X$
2. $\Gamma \nvdash \bot$
3. $(\Gamma \vdash A$ and $A \in X) \Rightarrow A \in \Gamma$
4. $(\Gamma \vdash (B \vee C)$ and $(B \vee C) \in X) \Rightarrow (B \in \Gamma$ or $C \in \Gamma)$.

We describe the Henkin construction for IPC. To lighten our notational burdens we will assume in this section that we work with some fixed $\mathcal{P}$. Consider a $\mathcal{P}$-adequate set X. The Henkin model for X is the model $\mathbb{H} := \mathbb{H}_X$, where:

- $K_{\mathbb{H}} := \{\Delta \mid \Delta \text{ is } X\text{-saturated}\}$
- $\Gamma \prec \Delta :\Leftrightarrow \Gamma \subseteq \Delta$
- $\mathcal{P}_{\mathbb{H}} := \mathcal{P} \cap X$
- $\Gamma \models p :\Leftrightarrow p \in \Gamma$

It is easily verified that $\mathbb{H}$ is an IPC-model.

Theorem 7.1 *for all* $A \in X$: $\Gamma \models_{\mathbb{H}} A \Leftrightarrow A \in \Gamma$.

If X is finite, then $\mathbb{H}_X$ is finite. We say that $\mathbb{M}$ is a *rooted Henkin model* if it is of the form $\mathbb{H}_X[\Delta]$ for some X-saturated Δ. We have:

Theorem 7.2 (Kripke Completeness for IPC) *For* $\Gamma \subseteq \mathcal{L}^i(\mathcal{P})$ *and* $A \in \mathcal{L}^i(\mathcal{P})$*:*

$$\Gamma \vdash_{\mathcal{P}} A \Leftrightarrow \text{ for all } \mathcal{P}\text{-models } \mathbb{K} : \Gamma \models_{\mathbb{K}} A.$$

In case Γ *is finite, we can improve this to:*

$$\Gamma \vdash_{\mathcal{P}} A \Leftrightarrow \text{ for all finite } \mathcal{P}\text{-models } \mathbb{K} : \Gamma \models_{\mathbb{K}} A.$$

Since depth of (nodes in) models will be important in this paper, we remark that a slightly different notion of Henkin model reduces depth in the IPC-case. Define $\mathbb{G}_X$ like $\mathbb{H}_X$, except:

- $\Gamma \prec \Delta :\Leftrightarrow \Gamma \subseteq \Delta$ and for some $(C \to D) \in X : C \notin \Gamma,\ C \in \Delta$ and $D \notin \Delta$

The standard argument that for all $A \in X$: $\Gamma \models A \Leftrightarrow A \in \Gamma$, works without change for our alternative model. Note that e.g. for $X = \{p\}$ the depth of $\mathbb{H}_X$ is 1 and the depth of $\mathbb{G}_X$ is 0.

For IPC we have a distinctive result involving downward extensions of models. We first introduce the necessary machinery. Let K be a set of IPC-models. $\mathbb{M} := \mathsf{M}(\mathsf{K})$ is the IPC-model with :

- $M := \{\langle k, \mathbb{K}\rangle \mid k \in K_{\mathbb{K}} \text{ and } \mathbb{K} \in \mathsf{K}\}$
- $\langle k, \mathbb{K}\rangle \prec \langle m, \mathbb{M}\rangle :\Leftrightarrow \mathbb{K} = \mathbb{M}$ and $k \prec_{\mathbb{K}} m$
- $\mathcal{P}_{\mathbb{M}} := \bigvee\{\mathcal{P}_{\mathbb{K}} \mid \mathbb{K} \in \mathsf{K}\}$.
- $\langle k, \mathbb{K}\rangle \models p :\Leftrightarrow k \models_{\mathbb{K}} p$

In practice we will forget the second components of the new nodes, pretending the domains to be disjoint already. Let $\mathbb{K}$ be a IPC $\mathcal{P}$-model. $\mathsf{B}(\mathbb{K})$ is the (rooted) IPC $\mathcal{P}$-model obtained by adding a new bottom $\mathfrak{b}$ to $\mathbb{K}$ and by taking: $\mathfrak{b} \models p :\Leftrightarrow \mathbb{K} \models p$. Finally we define $\mathsf{Glue}(\mathsf{K}) := \mathsf{B}(\mathsf{M}(\mathsf{K}))$.

Theorem 7.3 (Push Down Lemma) *Let* X *be adequate. Suppose* Δ *is* X*-saturated and* $\mathbb{K}$ *is an* IPC*-model with* $\mathbb{K} \models \Delta$*. Then* $\mathsf{Glue}(\mathbb{H}_X[\Delta], \mathbb{K}) \models \Delta$.

Proof

We show by induction on $A \in X$ that $\mathfrak{b} \models A \Leftrightarrow A \in \Delta$. The cases of atoms, conjunction and disjunction are trivial. If $(B \to C) \in X$ and $\mathfrak{b} \models (B \to C)$, then $\Delta \models (B \to C)$ and, hence, $(B \to C) \in \Delta$. Conversely suppose $(B \to C) \in \Delta$. If $\mathfrak{b} \not\models B$, we are easily done. If $\mathfrak{b} \models B$, then, by the Induction Hypothesis: $B \in \Delta$, hence $C \in \Delta$ and, by the induction hypothesis: $\mathfrak{b} \models C$. ❑

Instead of using the Push Down Lemma we could have employed the Kleene slash. We say that Δ is *$\mathcal{P}$-prime* if it is consistent and for every $(C \vee D) \in \mathcal{L}^i(\mathcal{P})$: $\Delta \vdash (C \vee D) \Rightarrow \Delta \vdash C$ or $\Delta \vdash D$. A formula A is *$\mathcal{P}$-prime* if $\{A\}$ is $\mathcal{P}$-prime. As usual, we will suppress the $\mathcal{P}$.

Theorem 7.4 *Suppose X is adequate and Δ is X-saturated. then Δ is prime.*

Proof

Δ is consistent by definition. Suppose $\Delta \vdash C \vee D$ and $\Delta \nvdash C$ and $\Delta \nvdash D$. Suppose $\mathbb{K} \models \Delta$, $\mathbb{K} \not\models C$, $\mathbb{M} \models \Delta$ and $\mathbb{M} \not\models D$. Consider $\mathsf{Glue}(\mathbb{H}_X(\Delta), \mathbb{K}, \mathbb{M})$. By the Push Down Lemma (theorem 7.3) we have: $\mathfrak{b} \models \Delta$. On the other hand by persistence: $\mathfrak{b} \not\models C$ and $\mathfrak{b} \not\models D$. Contradiction. ❑

Theorem 7.5 *Consider any formula A. The formula A can be written (modulo* IPC*-provable equivalence) as a disjunction of prime formulas C. Moreover these C are conjunctions of implications and propositional variables in* $\mathsf{Sub}(A)$.

Proof

Consider a $\mathsf{Sub}(A)$-saturated Δ. Let $\mathsf{IP}(\Delta)$ be the set of implications and atoms of Δ. It is easily seen that $\mathsf{IPC} \vdash \bigwedge \mathsf{IP}(\Delta) \leftrightarrow \bigwedge \Delta$. Take:

$$D := \bigvee \{ \bigwedge \mathsf{IP}(\Delta) \mid \Delta \text{ is } \mathsf{Sub}(A)\text{-saturated and } A \in \Delta \}.$$

Trivially: $\mathsf{IPC} \vdash D \to A$. On the other hand if $\mathsf{IPC} \nvdash A \to D$, then by a standard construction there is a $\mathsf{Sub}(A)$-saturated set Γ such that $A \in \Gamma$ and $\Gamma \nvdash D$. Quod non. ❑

8 Formula Classes and Model Descriptions for IPC

In this section all models will be IPC models. Define $\mathfrak{i} : \mathcal{L}^i(\mathcal{P}) \to \omega$, by:

- $\mathfrak{i}(p) := \mathfrak{i}(\bot) := \mathfrak{i}(\top) := 0$
- $\mathfrak{i}(A \wedge B) := \mathfrak{i}(A \vee B) := max(\mathfrak{i}(A), \mathfrak{i}(B))$
- $\mathfrak{i}(A \to B) := max(\mathfrak{i}(A), \mathfrak{i}(B)) + 1$
- $I_n(\mathcal{P}) := \{A \in \mathcal{L}^i(\mathcal{P}) \mid \mathfrak{i}(A) \leq n\}$
- $I_\infty(\mathcal{P}) := \mathcal{L}^i(\mathcal{P})$

By an easy induction on n we may prove the following theorem.

Theorem 8.1 *$I_n(\vec{p})$ is finite modulo* IPC*-provable equivalence.*

Define for $X \subseteq \mathcal{L}^i(\mathcal{P})$:

- $\mathsf{Th}_X(k) := \{A \in X \mid k \models A\}$
- For $\mathbb{K}$ pointed with point k: $\mathsf{Th}_X(\mathbb{K}) := \mathsf{Th}_X(k)$
- $\mathsf{Th}(k) := \mathsf{Th}_{\mathcal{L}^i(\mathcal{P})}(k)$

Theorem 8.2 *Suppose that $\mathcal{Z}$ is an ℓ-simulation between the $\mathcal{P}$-models $\mathbb{K}$ and $\mathbb{M}$. Then:*

$$k\mathcal{Z}_\alpha m \Rightarrow \mathsf{Th}_{I_\alpha(\mathcal{P})}(k) = \mathsf{Th}_{I_\alpha(\mathcal{P})}(m).$$

Proof

By induction on A in I_α. Suppose $k\mathcal{Z}_\alpha m$. The cases of atoms, conjunction and disjunction are trivial. Suppose, e.g., $k \not\models (B \to C)$. Then, for some $k' \succ k$, $k' \models B$ and $k' \not\models C$. There is an $m' \succ m$, such that $k'\mathcal{Z}_{\alpha-1}m'$ and hence by the induction hypothesis (applied for $\alpha - 1$, noting that if $A \in I_\alpha(\mathcal{P})$, then $B, C \in I_{\alpha-1}(\mathcal{P})$): $m' \models B$ and $m' \not\models C$. Ergo $m \not\models (B \to C)$. ❑

Theorem 8.3 *Suppose k and m are $\mathcal{P}$-nodes. Then:*

$$k \preceq_\alpha m \Rightarrow \mathsf{Th}_{I_\alpha(\mathcal{P})}(k) \subseteq \mathsf{Th}_{I_\alpha(\mathcal{P})}(m).$$

Proof

In case $\alpha = 0$, this is trivial. Suppose $\alpha > 0$ and $k \preceq_\alpha m$. The proof is a simple induction on $A \in I_\alpha(\mathcal{P})$. The cases of atoms, $\wedge$, $\vee$ are trivial. Suppose $A = (B \to C)$ and $m \not\models (B \to C)$. Then for some $m' \succeq m$: $m' \models B$ and $m' \not\models C$. There is a $k' \succeq k$, such that $k' \simeq_{\alpha-1} m'$ and, hence, by theorem 8.2: $k' \models B$ and $k' \not\models C$. Ergo $k \not\models (B \to C)$. ❑

We formulate a partial converse for theorem 8.3. It is well known that the converse for the case of ∞, i.e. for the case where one would like to infer bisimulation from the relation of forcing the same formulas of the full language, does not go through. There is a lot of work (for the analogous case of modal logic) on better converses than the one given here. We refer the reader to [7] and [9].

Theorem 8.4 *Suppose k and m are $\vec{p}$-nodes. Then:*

$$\mathsf{Th}_{I_n(\vec{p})}(k) \subseteq \mathsf{Th}_{I_n(\vec{p})}(m) \Rightarrow k \preceq_n m.$$

Proof

Suppose k and m are $\vec{p}$-nodes, and $\mathsf{Th}_{I_n(\vec{p})}(k) \subseteq \mathsf{Th}_{I_n(\vec{p})}(m)$. We want to prove: $k \preceq_n m$. In case $n = 0$ this is trivial. Suppose $n > 0$. Define, for k' in the model corresponding to k and m' in the model corresponding to m:

$$k'\mathcal{Z}_i m' :\Leftrightarrow \mathsf{Th}_{I_i(\vec{p})}(k') = \mathsf{Th}_{I_i(\vec{p})}(m').$$

We check that $\mathcal{Z}$ is an ℓ-simulation and that for every $k' \succeq k$ there is an $m' \succeq m$ with $k'\mathcal{Z}_n m'$.

Suppose $i > 0$ and $k'\mathcal{Z}_i m'$. Clearly k' and m' force the same atoms. We verify e.g. the zig-property. Suppose $k' \preceq k''$. Let:

$$\eta_i(k'') := (\bigwedge\{B \in I_{i-1}(\vec{p}) \mid k'' \models B\} \to \bigvee\{C \in I_{i-1}(\vec{p}) \mid k'' \not\models C\}).$$

Clearly $k' \not\models \eta_i(k'')$ and $\eta_i(k'') \in I_i(\vec{p})$. Ergo $m' \not\models \eta_i(k'')$. But then for some $m'' \geq m'$:

$$m'' \models \bigwedge\{B \in I_{i-1}(\vec{p}) \mid k'' \models B\} \text{ and } m'' \not\models \bigvee\{C \in I_{i-1}(\vec{p}) \mid k' \not\models C\}.$$

It follows that $k''\mathcal{Z}_{i-1}m''$.

To show that for any $m' \succeq m$ there is a $k' \succeq k$ with $k'\mathcal{Z}_n m'$. Note that $m \not\models \eta_n(m')$, ergo $k \not\models \eta_n(m')$, and, thus, for some k':

$$k' \models \bigwedge\{B{\in}I_{n-1}(\vec{p}) \mid m' \models B\} \text{ and } k' \not\models \bigvee\{C{\in}I_{n-1}(\vec{p}) \mid m' \not\models C\}.$$

Hence: $k\mathcal{Z}_{n-1}m$. ❑

Let k be a $\vec{p}$-node. Define:

- $\mathsf{Y}_{n,k} := \mathsf{Y}_{n,k}(\vec{p}) := \bigwedge\{C{\in}I_n(\vec{p}) \mid k \models C\}$
- $\mathsf{N}_{n,k} := \mathsf{N}_{n,k}(\vec{p}) := \bigvee\{D{\in}I_n(\vec{p}) \mid k \not\models D\}$

Theorem 8.5 $k \models \mathsf{Y}_{n,k}$ *and* $k \not\models \mathsf{N}_{n,k}$.

Let m be a $\vec{p}$-node. We have:

Theorem 8.6 $k \preceq_n m \Leftrightarrow m \models \mathsf{Y}_{n,k} \Leftrightarrow k \not\models \mathsf{N}_{n,m}$.

Theorem 8.7 *For* $n \leq n'$*:*

1. $\mathsf{IPC} \vdash \mathsf{Y}_{n',k} \to \mathsf{Y}_{n,k}$.
2. $\mathsf{IPC} \vdash \mathsf{N}_{n,k} \to \mathsf{N}_{n',k}$.

Theorem 8.8 *We have:*

1. $k \preceq_n m \Leftrightarrow \mathsf{IPC} \vdash \mathsf{Y}_{n,m} \to \mathsf{Y}_{n,k}$
2. $k \preceq_n m \Leftrightarrow \mathsf{IPC} \vdash \mathsf{N}_{n,m} \to \mathsf{N}_{n,k}$

Proof

(1) "$\Rightarrow$" Suppose $k \preceq_n m$. Let r be any $\vec{p}$-node with $r \models \mathsf{Y}_{n,m}$. It follows that $m \preceq_n r$ and, hence, $k \preceq_n r$. Ergo, $r \models \mathsf{Y}_{n,k}$. "$\Leftarrow$" Suppose $\mathsf{IPC} \vdash \mathsf{Y}_{n,m} \to \mathsf{Y}_{n,k}$. Since $m \models \mathsf{Y}_{n,m}$, it follows that $m \models \mathsf{Y}_{n,k}$, and, hence, $k \preceq_n m$.

(2) "$\Rightarrow$" Suppose $k \preceq_n m$. Let r be any $\vec{p}$-node with $r \not\models \mathsf{N}_{n,k}$. It follows that $r \preceq_n k$ and, hence, $r \preceq_n m$. Ergo: $r \not\models \mathsf{N}_{n,m}$. "$\Leftarrow$" Suppose $\mathsf{IPC} \vdash \mathsf{N}_{n,m} \to \mathsf{N}_{n,k}$. Since $k \not\models \mathsf{N}_{n,k}$, it follows that $k \not\models \mathsf{N}_{n,m}$ and hence: $k \preceq_n m$. ❑

Suppose k is a $\vec{p}$-node of finite depth. Define:

- $\mathsf{Y}_k := \mathsf{Y}_{2.d(k)+1,k}$
- $\mathsf{N}_k := \mathsf{N}_{2.d(k)+2,k}$

Let m also be a $\vec{p}$-node.

Theorem 8.9 1. *Suppose k is a $\vec{p}$-node of finite depth.* $k \preceq_\infty m \Leftrightarrow m \models \mathsf{Y}_k$.

2. $k \preceq_\infty m \Leftrightarrow k \not\models \mathsf{N}_m$.

Proof

(1) "$\Rightarrow$" Suppose $k \preceq_\infty m$. It follows that $k \preceq_{2.d(k)+1} m$. Hence, $m \models \mathsf{Y}_k$. "$\Leftarrow$" Suppose $m \models \mathsf{Y}_k$. It follows that $k \preceq_{2.d(k)+1} m$. Ergo by theorem 6.8, $k \preceq_\infty m$.

(2) "$\Rightarrow$" Easy. "$\Leftarrow$" Suppose $k \not\models \mathsf{N}_m$. It follows that $k \preceq_{2.d(m)+2} m$. Hence, since $m \preceq m$, for some $k' \succeq k$, $k' \simeq_{2.d(m)+1} m$. It follows that $k' \simeq m$. ❑

By inspecting the model of example 5.1 one can show that the esimate of theorem 8.9(1) is optimal. Is it possible to improve upon (2)?

Theorem 8.10 *1.* $k \preceq_\infty m \Leftrightarrow \mathsf{IPC} \vdash \mathsf{Y}_m \to \mathsf{Y}_k$.

2. $k \preceq_\infty m \Leftrightarrow \mathsf{IPC} \vdash \mathsf{N}_m \to \mathsf{N}_k$.

Theorem 8.11 *For all n:*

1. $\mathsf{IPC} \vdash \mathsf{Y}_k \to \mathsf{Y}_{n,k}$.

2. $\mathsf{IPC} \vdash \mathsf{N}_{n,k} \to \mathsf{N}_k$.

Proof

(1) $m \models \mathsf{Y}_k \Rightarrow k \preceq_\infty m \Rightarrow k \preceq_n m \Rightarrow m \models \mathsf{Y}_{n,k}$. (2) Similar. ❑

Theorem 8.12 $\mathsf{Y}_{n,k}$ *is a prime formula.*

Proof

It is easily seen that $\mathsf{Y}_{n,k}$ is $I_n(\vec{p})$-saturated. Apply theorem 7.4. ❑

9 Ordinary interpolation for IPC

We show ordinary interpolation by a quick and simple proof. The present proof has two sources. In 1980 I learned from Wim Ruitenburg a Henkin style proof of the ordinary interpolation theorem. Much later in 1993, when I started thinking about interpolation, I saw how to refine the proof to get a bound on the complexity of the interpolant. In the meantime Gleit and Goldfarb in their [6] used characteristic formulas to prove interpolation for Löb's Logic. The present proof is close to the Henkin style proof, but even closer to Gleit and Golfarb's proof, the main difference being the use of simulations rather than characteristic formulas.

Lemma 9.1 Consider $\langle \mathbb{K}, k_0\rangle \in \mathsf{Pmod}(\mathcal{Q}, \vec{p})$ and $\langle \mathbb{M}, m_0\rangle \in \mathsf{Pmod}(\vec{p}, \mathcal{R})$, where $\mathcal{Q}$, $\vec{p}$, and $\mathcal{R}$ are pairwise disjoint. Let $X \subseteq \mathcal{L}(\mathcal{Q}, \vec{p})$ and $Y \subseteq \mathcal{L}(\vec{p}, \mathcal{R})$ be finite adequate sets. Take $\vec{q} := \mathcal{Q} \cap X$ and $\vec{r} := \mathcal{R} \cap Y$. Let:

$$\mu := |\{C \in X \mid C \text{ is an implication}\}| + |\{C \in Y \mid C \text{ is an implication}\}|$$

Suppose that $k_0 \simeq_{\mu,\vec{p}} m_0$. Then there is a finite, pointed $\vec{q}, \vec{p}, \vec{r}$-model $\langle \mathbb{N}, n_0\rangle$ such that: $\mathsf{Th}_X(n_0) = \mathsf{Th}_X(k_0)$ and $\mathsf{Th}_Y(n_0) = \mathsf{Th}_Y(m_0)$. ❑

Proof

Let $\mathcal{Z}$ be a downwards closed witness of: $k_0 \simeq_{\mu,\vec{p}} m_0$. Define Φ_X from $\mathbb{K}$ to the Henkin model $\mathbb{G}_X$ and Φ_Y from $\mathbb{M}$ to the Henkin model $\mathbb{G}_Y$ as follows:

- $\Phi_X(k) := \Gamma(k) := \{B \in X \mid k \models B\}$
- $\Phi_Y(m) := \Delta(m) := \{B \in Y \mid m \models B\}$

Warning: Φ_X need not be order-preserving! Define further for k in $\mathbb{K}$ and m in $\mathbb{M}$:

- $d_X(k) := d_{\mathbb{G}_X}(\Gamma(k))$
- $d_Y(m) := d_{\mathbb{G}_Y}(\Delta(m))$
- $\zeta(k,m) := d_X(k) + d_Y(m)$

Note that for all k in K and m in M: $\zeta(k,m) \leq \mu$. Consider a pair $\langle\Gamma,\Delta\rangle$, where Γ is in $\mathbb{G}_X$ and Δ is in $\mathbb{G}_Y$. Define: k, m is a *witnessing pair* for $\langle\Gamma,\Delta\rangle$ if $k \in K$, $m \in M$, $\Gamma = \Gamma(k)$, $\Delta = \Delta(m)$ and $k\mathcal{Z}_{\zeta(k,m)}m$. Define $\mathbb{N}$ and n_0 as follows:

- $N := \{\langle\Gamma,\Delta\rangle \mid \text{there is a witnessing pair for } \langle\Gamma,\Delta\rangle\}$
- $n_0 := \langle\Gamma(k_0),\Delta(m_0)\rangle$
- $\langle\Gamma,\Delta\rangle \prec \langle\Gamma',\Delta'\rangle :\Leftrightarrow \Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$
- $\langle\Gamma,\Delta\rangle \models_{\mathbb{N}} s :\Leftrightarrow s \in \Gamma \cup \Delta$

By assumption, $k_0\mathcal{Z}_\mu m_0$. Moreover: $\zeta(k_0,m_0) = d_X(k_0) + d_Y(m_0) \leq \mu$. Hence: $k_0\mathcal{Z}_{\zeta(k_0,m_0)}m_0$. So we can take k_0, m_0 as witnessing pair for n_0.

We show that for $A \in X$: $\langle\Gamma,\Delta\rangle \models_{\mathbb{N}} A \Leftrightarrow A \in \Gamma$ by induction on A. The case of Y and Δ is similar. The atomic case is easy, noting that the existence of a witnessing pair guarantees that for $p \in \vec{p}$: $p \in \Gamma \Leftrightarrow p \in \Delta$. The cases of $\wedge$ and $\vee$ are trivial. Suppose $A = (B \to C)$. First suppose $(B \to C) \in \Gamma$. Consider $\langle\Gamma',\Delta'\rangle \succeq \langle\Gamma,\Delta\rangle$ and suppose $\langle\Gamma',\Delta'\rangle \models B$. By the Induction Hypothesis we have: $B \in \Gamma'$ and hence, since $(B \to C) \in \Gamma'$: $C \in \Gamma'$. Again by the Induction Hypothesis we find: $\langle\Gamma',\Delta'\rangle \models C$. So we may conclude: $\langle\Gamma,\Delta\rangle \models B \to C$. Conversely, suppose that $(B \to C) \notin \Gamma$. In case $B \in \Gamma$, we are easily done. Suppose not. Let k, m be a witnessing pair for $\langle\Gamma,\Delta\rangle$. We find that $k \not\models B \to C$ and hence there is a k' such that $k' \models B$ and $k' \not\models C$. Since $k\mathcal{Z}_{\zeta(k,m)}m$, there is an $m' \geq m$ such that $k'\mathcal{Z}_{\zeta(k,m)-1}m'$. Consider $\langle\Gamma',\Delta'\rangle := \langle\Gamma(k'),\Delta(m')\rangle$. Since $B' \in \Gamma'$ and $C \notin \Gamma'$, evidently in $\mathbb{G}_X$: $\Gamma \prec \Gamma'$ and hence $\zeta(k',m') \leq \zeta(k,m) - 1$. Hence by downward closure k', m' is a witnessing pair for $\langle\Gamma',\Delta'\rangle$. So $\langle\Gamma',\Delta'\rangle$ is in $\mathbb{N}$ and by the Induction Hypothesis: $\langle\Gamma',\Delta'\rangle \models B$ and $\langle\Gamma',\Delta'\rangle \not\models C$. It follows that $\langle\Gamma,\Delta\rangle \not\models B \to C$. ❑

Theorem 9.2 (Ordinary interpolation) *Suppose $A \in \mathcal{L}^i(\vec{q},\vec{p})$ and $B \in \mathcal{L}^i(\vec{p},\vec{r})$ and* $\mathsf{IPC} \vdash A \to B$. *Let:*

$$\mu := |\{C \in \mathsf{Sub}(A) \mid C \textit{ is an implication}\}| + |\{C \in \mathsf{Sub}(B) \mid C \textit{ is an implication}\}|.$$

Then there is an $I \in I_\mu(\vec{p})$ with: $\mathsf{IPC} \vdash A \to I$ *and* $\mathsf{IPC} \vdash I \to B$.

Proof

Take:

$$I := \bigvee\{C\in I_\mu(\vec{p}) \mid \mathsf{IPC}\vdash D\to B\}.$$

Clearly $\mathsf{IPC}\vdash I\to B$. Suppose to get a contradiction that $\mathsf{IPC}\nvdash A\to I$. Let m be any $\vec{p},\vec{r}$-node with $m\models A$ and $m\not\models I$. Let $\mathsf{Y} := \mathsf{Y}_{\mu,m[\vec{p}]}$ and $\mathsf{N} := \mathsf{N}_{\mu,m[\vec{p}]}$. We claim that: $\mathsf{Y}\nvdash \mathsf{N}\vee B$. Note that, by theorem 8.12, Y is prime. So if $\mathsf{Y}\vdash\mathsf{N}\vee B$, then $\mathsf{Y}\vdash\mathsf{N}$ or $\mathsf{Y}\vdash B$. Since, $\mathsf{Y}\nvdash\mathsf{N}$, it follows that $\mathsf{Y}\vdash B$. But then, by definition, $\mathsf{Y}\vdash I$. Quod non, since $m\models\mathsf{Y}$ and $m\not\models I$. Let k be any $\vec{q},\vec{p}$-node such that: $k\models\mathsf{Y}$ and $k\not\models\mathsf{N}\vee B$. We find that $k\simeq_{\mu,\vec{p}} m$. Apply lemma 9.1 with $\mathsf{Sub}(A)$ in the role of Y and $\mathsf{Sub}(B)$ in the role of X to find a $\vec{q},\vec{p},\vec{r}$-node n with: $\mathsf{Th}_{\mathsf{Sub}(A)}(m)=\mathsf{Th}_{\mathsf{Sub}(A)}(n)$ and $\mathsf{Th}_{\mathsf{Sub}(B)}(k)=\mathsf{Th}_{\mathsf{Sub}(B)}(n)$. It follows that $n\models A$, but $n\not\models B$. A contradiction. ❑

Lemma 9.1 lacks, in a sense, purity since it combines the method of simulations with a Henkin style argument. The impurity has the advantage of efficiency. Of course we can at some cost find a pure formulation of the lemma. Here we formulate a purified weak version that follows immediately from lemma 9.1 itself.

Corollary 9.3 Consider two models $\mathbb{K}\in\mathsf{Rmod}(\mathcal{Q},\vec{p})$ and $\mathbb{M}\in\mathsf{Rmod}(\vec{p},\mathcal{R})$, where $\mathcal{R}$, $\vec{p}$ and $\mathcal{R}$ are pairwise disjoint. Take $\vec{q}:=\mathcal{Q}\cap X$ and $\vec{r}:=\mathcal{R}\cap Y$. Then:

$$\forall k\,\exists\mu(\mathbb{K}(\vec{p})\simeq_\mu\mathbb{M}(\vec{p})\Rightarrow\exists\mathbb{N}\in\mathsf{Rmod}(\vec{q},\vec{p},\vec{r})(\mathbb{K}\simeq_k\mathbb{N}(\vec{q},\vec{p})\text{ and }\mathbb{M}\simeq_k\mathbb{N}(\vec{p},\vec{r}))).$$

❑

10 Uniform Interpolation for IPC

Uniform Interpolation was proved for GL by V. Shavrukov (see: [20]). Shavrukov used the method of characters as developed by Z. Gleit and W. Goldfarb, who proved the Fixed Point Theorem of Provability Logic and the ordinary Interpolation Theorem employing characters (see: [6]). The methods of Gleit & Goldfarb and later of Shavrukov can be viewed as model theoretical. For IPC, A. Pitts proved Uniform Interpolation by proof theoretical methods, using proof systems allowing efficient cut-elimination (see: [14]), developed, independently, by J. Hudelmaier (see: [10]) and R. Dyckhoff (see: [2]). Later S. Ghilardi and M. Zawadowski (see: [4]), and, independently but later, A. Visser, found a model theoretical proof for Pitt's result using bounded bisimulations.

We prove an amalgamation lemma. Note that the proof of lemma 9.1 follows the pattern of theorem 6.2. The proof of the present lemma is like the the proof of lemma 9.1, replacing the the argument in the style of theorem 6.2 by an argument in the style of theorem 6.6.

In this section, we will use $\preceq$ for the weak partial orderings and $\prec$ for the associated strict orderings.

Lemma 10.1 Consider disjoint sets of propositional variables $\mathcal{Q}$, $\vec{p}$ and $\mathcal{R}$. Let $X\subseteq\mathcal{L}^i(\mathcal{Q},\vec{p})$ be a finite adequate set. Let $\langle\mathbb{K},k_0\rangle\in\mathsf{Pmod}(\mathcal{Q},\vec{p})$, $\langle\mathbb{M},m_0\rangle\in\mathsf{Pmod}(\vec{p},\mathcal{R})$. Let:

$$\nu := |\{C\in X \mid C\text{ is a propositional variable or an implication}\}|.$$

Suppose that $k_0\simeq_{2.\nu+1,\vec{p}} m_0$. Then there is a $\mathcal{Q}$-extension $\langle\mathbb{N},n_0\rangle$ of $\langle\mathbb{M},m_0\rangle$ such that $\mathsf{Th}_X(n_0)=\mathsf{Th}_X(k_0)$. ❑

Proof

Let $\mathcal{Z}$ be a downwards closed witness of $k_0\simeq_{2.\nu+1,\vec{p}} m_0$. Define Φ_X from $\mathbb{K}$ to the Henkin model $\mathbb{H}:=\mathbb{H}_X$ as follows: $\Phi_X(k):=\Delta(k):=\{B\in X\mid k\models B\}$. Define further for k in $\mathbb{K}$: $d_X(k):=d_{\mathbb{H}}(\Delta(k))$. Note that: $d_X(k)\leq\nu$.

Consider a pair $\langle \Delta, m\rangle$ for Δ in H and m in $\mathbb{M}$. We say that k', k, m' is a *witnessing triple* for $\langle \Delta, m\rangle$ if:

$$\Delta = \Delta(k) = \Delta(k'),\ k' \preceq k,\ m' \preceq m,\ k'\mathcal{Z}_{2.d_X(k')+1}m',\ k\mathcal{Z}_{2.d_X(k')}m.$$

$$\begin{tikzcd}
& \Delta \arrow[rr, dash, "\Phi_X"] & & k \arrow[r, dash, "\mathcal{Z}_{2d_X(k')}"] & m \\
\Delta \arrow[ur, "="] \arrow[rr, dash, "\Phi_X"] & & k' \arrow[ur, "\preceq"] \arrow[rr, dash, "\mathcal{Z}_{2d_X(k')+1}"] & & m' \arrow[u, "\preceq"]
\end{tikzcd}$$

Define:

- $N := \{\langle \Delta, m\rangle \mid$ there is a witnessing triple for $\langle \Delta, m\rangle\}$
- $n_0 := \langle \Delta(k_0), m_0\rangle$
- $\langle \Delta, m\rangle \preceq_{\mathbb{N}} \langle \Gamma, n\rangle :\Leftrightarrow \Delta \preceq_{\mathbb{H}} \Gamma$ and $m \preceq_{\mathbb{M}} n$
- $\langle \Delta, m\rangle \models_{\mathbb{N}} s :\Leftrightarrow \Delta \models_{\mathbb{H}} s$ or $m \models_{\mathbb{M}} s$

Note that by assumption $k_0\mathcal{Z}_{2\nu+1}m_0$. Moreover: $2.d_X(k_0)+1 \leq 2.\nu+1$. Hence: $k_0\mathcal{Z}_{2d_X(k_0)+1}m_0$. So we can take k_0, k_0, m_0 as witnessing triple for n_0. Let k', k, m' be a witnessing triple for $\langle \Delta, m\rangle$. Note that for $p \in \vec{p} \cap X$: $\Delta \models p \Leftrightarrow k \models p \Leftrightarrow m \models p$, and hence: $\langle \Delta, m\rangle \models p \Leftrightarrow \Delta \models p \Leftrightarrow m \models p$. We claim:

Claim 1 $n_0 \simeq_{\vec{p},\mathcal{R}} m_0$.

Claim 2 For $B \in X : \langle \Delta, m\rangle \models B \Leftrightarrow B \in \Delta$.

Evidently the lemma is immediate from the claims.

We prove Claim 1. Take as bisimulation $\mathcal{B}$ with $\langle \Delta, m\rangle\mathcal{B}m$. Clearly, $\mathsf{Th}_{\vec{p},\mathcal{R}}(\langle \Delta, m\rangle) = \mathsf{Th}_{\vec{p},\mathcal{R}}(m)$. Moreover, $\mathcal{B}$ has the zig-property. We check that $\mathcal{B}$ has the zag-property. Suppose $\langle \Delta, m\rangle\mathcal{B}m \preceq n$. We are looking for a pair $\langle \Gamma, n\rangle$ in N such that $\Delta \preceq \Gamma$. Let k', k, m' be a witnessing triple for $\langle \Delta, m\rangle$. Since $k'\mathcal{Z}_{2.d_X(k')+1}m' \preceq n$, there is a h such that $k' \preceq h\mathcal{Z}_{2.d_X(k')}n$. We take $\Gamma := \Delta(h)$. We need a witnessing triple k'^*, k^*, m'^* for $\langle \Gamma, n\rangle$ We distinguish two possibilities. First, $\Delta = \Gamma$. In this case we can take: $k'^* := k'$, $k^* := h$, $m'^* := m'$.

$$\begin{tikzcd}
\Gamma = \Delta \arrow[rr, dashed, dash, "\Phi_X"] & & k^* = h \arrow[rr, dashed, dash, "\mathcal{Z}_{2d_X(k')}"] & & n \\
& \Delta \arrow[rr, dash, "\Phi_X"] & & k \arrow[r, dash, "\mathcal{Z}_{2d_X(k')}"] & m \arrow[u, "\preceq"] \\
\Delta \arrow[uu, dashed, "="] \arrow[ur, "="] \arrow[rr, dash, "\Phi_X"] & & k'^* = k' \arrow[uu, dashed, "\preceq"] \arrow[ur, "\preceq"] \arrow[rr, dash, "\mathcal{Z}_{2d_X(k')+1}"] & & m'^* = m' \arrow[u, "\preceq"]
\end{tikzcd}$$

Secondly, $\Delta \neq \Gamma$. In this case we can take: $k'^* := h$, $k^* := h$, $m'^* := n$. To see this, note that, since $k' \preceq h$, we have: $\Delta = \Delta(k') \prec \Gamma$. Ergo $d_X(h) < d_X(k')$. It follows that: $2.d_X(h) + 1 \leq 2.d_X(k')$. So,

$h\mathcal{Z}_{2.d_X(k')+1}n$ (and by downward closure also $h\mathcal{Z}_{2.d_X(k')}n$).

$$\begin{tikzcd}
\Gamma \arrow[rr, dashed, dash, "\Phi_X"] & & k'^* = k^* = h \arrow[rr, dashed, dash, "\mathcal{Z}_{2d_X(h)+1}"] & & m'^* = n \\
& \Delta \arrow[rrr, dash, "\Phi_X"] & & k \arrow[r, dash, "\mathcal{Z}_{2d_X(k')}"] & m \arrow[u, "\preceq"] \\
\Delta \arrow[uu, dashed, "\prec"] \arrow[ur, "="] \arrow[rr, dash, "\Phi_X"] & & k' \arrow[uu, dashed, "\preceq"] \arrow[ur, "\preceq"] \arrow[rr, dash, "\mathcal{Z}_{2d_X(k')+1}"] & & m' \arrow[u, "\preceq"]
\end{tikzcd}$$

Finally, clearly, $b_{\mathbb{N}}\mathcal{B}b_{\mathbb{M}}$.

We prove Claim 2. The proof is by induction on X. The cases of atoms, conjunction and disjunction are trivial. We treat the case of implication. Suppose $(C \to D) \in X$. Consider the node $\langle \Delta, m\rangle$ with witnessing triple k', k, m'.

Suppose $\Delta \not\models (C \to D)$. In case $\Delta \models C$ and $\Delta \not\models D$, by the Induction Hypothesis, $\langle \Delta, m\rangle \models C$ and $\langle \Delta, m\rangle \not\models D$. So, $\langle \Delta, m\rangle \not\models (C \to D)$. Suppose $\Delta \not\models C$. Clearly, $k \not\models (C \to D)$, so there is an $h \succeq k$ with $h \models C$ and $h \not\models D$. Let $\Gamma := \Delta(h)$. Since, $\Delta \not\models C$, we find: $\Delta \prec \Gamma$ and, thus, $k \prec h$. Note that it follows that $2.d_X(k') \geq 2$. Since $k\mathcal{Z}_{2.d_X(k')}m$ and $k \preceq h$, there is an $n \succeq m$ with $h\mathcal{Z}_{2.d_X(k')-1}n$. Moreover: $2.d_X(h) + 1 \leq 2.d_X(k') - 1$. Ergo: $h\mathcal{Z}_{2.d_X(h)+1}n$. So h, h, n is a witnessing triple for $\langle \Gamma, n\rangle$. Clearly, $\langle \Delta, m\rangle \preceq \langle \Gamma, n\rangle$. By the Induction Hypothesis: $\langle \Gamma, n\rangle \models C$ and $\langle \Gamma, n\rangle \not\models D$. Hence, $\langle \Delta, m\rangle \not\models (C \to D)$.

$$\begin{tikzcd}
& \Gamma \arrow[rrr, dash, "\Phi_X"] & & k'^* = k^* = h \arrow[r, dashed, dash, "\mathcal{Z}_{2d_X(h)+1}"] & m'^* = n \\
& \Delta \arrow[u, "\prec"] \arrow[rrr, dash, "\Phi_X"] & & k \arrow[u, "\preceq"] \arrow[r, dash, "\mathcal{Z}_{2d_X(k')}"] & m \arrow[u, dashed, "\preceq"] \\
\Delta \arrow[ur, "="] \arrow[rr, dash, "\Phi_X"] & & k' \arrow[ur, "\preceq"] \arrow[rr, dash, "\mathcal{Z}_{2d_X(k')+1}"] & & m' \arrow[u, "\preceq"]
\end{tikzcd}$$

Suppose $\langle \Delta, m\rangle \not\models (C \to D)$. There is a $\langle \Gamma, n\rangle$ in $\mathbb{N}$ with $\langle \Delta, m\rangle \preceq \langle \Gamma, n\rangle$ and $\langle \Gamma, n\rangle \models C$ and $\langle \Gamma, n\rangle \not\models D$. Clearly $\Delta \preceq \Gamma$. By the Induction Hypothesis $\Gamma \models C$ and $\Gamma \not\models D$. Ergo $\Delta \not\models (C \to D)$. Thus we have proved Claim 2. ❑

Theorem 10.2 (Pitts' Uniform Interpolation Theorem) *Here is our version of Pitts' Uniform Interpolation Theorem.*

1. *Consider any formula A and any finite set of variables $\vec{q}$. Let*

$$\nu := \nu_{\mathsf{Sub}(A)} := |\{C \in \mathsf{Sub}(A) \mid C \textit{ is a propositional variable or an implication}\}|$$

There is a formula $\exists\vec{q}.A$ such that:

(a) $\mathsf{PV}(\exists\vec{q}.A) \subseteq \mathsf{PV}(A) \setminus \vec{q}$

(b) $\mathsf{i}(\exists\vec{q}.A) \leq 2.\nu + 2$

(c) *For all* $B \in \mathcal{L}^i$ *with* $\mathsf{PV}(B) \cap \vec{q} = \emptyset$*, we have:*

$$\mathsf{IPC} \vdash A \to B \Leftrightarrow \mathsf{IPC} \vdash \exists\vec{q}.A \to B.$$

2. *Consider any formula* B *and any finite set of variables* $\vec{q}$*. Let* $\nu := \nu_{\mathsf{Sub}(B)}$*. There is a formula* $\forall\vec{q}.B$ *such that:*

(a) $\mathsf{PV}(\forall\vec{q}.B) \subseteq \mathsf{PV}(B) \setminus \vec{q}$

(b) $\mathsf{i}(\forall\vec{q}.B) \leq 2.\nu + 1$

(c) *For all* $A \in \mathcal{L}^i$ *with* $\mathsf{PV}(A) \cap \vec{q} = \emptyset$*, we have:*

$$\mathsf{IPC} \vdash A \to B \Leftrightarrow \mathsf{IPC} \vdash A \to \forall\vec{q}.B.$$

Note that in (1) we have estimate $2.\nu + 2$ and in (2) $2.\nu + 1$. Somehow I find this annoying. In theorem 10.3 we will show how to get the marginal improvement to $2.\nu + 1$ also for (1).

Proof

(1) Consider A and $\vec{q}$. Let $\vec{p} := \mathsf{PV}(A) \setminus \vec{q}$. Take:

$$\exists\vec{q}.A := \bigwedge\{C \in I_{2.\nu+2}(\vec{p}) \mid \mathsf{IPC} \vdash A \to C\}.$$

Clearly $\exists\vec{q}.A$ satisfies (a) and (b). Moreover, $\mathsf{IPC} \vdash A \to \exists\vec{q}.A$. Hence all we have to prove is that for all B with $\mathsf{PV}(B) \cap \vec{q} = \emptyset$:

$$\mathsf{IPC} \vdash A \to B \Rightarrow \mathsf{IPC} \vdash \exists\vec{q}.A \to B.$$

Suppose, to the contrary, that for some B: $\mathsf{PV}(B) \cap \vec{q} = \emptyset$ and $\mathsf{IPC} \vdash A \to B$ and $\mathsf{IPC} \nvdash \exists\vec{q}.A \to B$. Take $\vec{r} := \mathsf{PV}(B) \setminus \vec{p}$. Note that $\vec{p}, \vec{q}, \vec{r}$ are pairwise disjoint, $\mathsf{PV}(A) \subseteq \vec{q} \cup \vec{p}$ and $\mathsf{PV}(B) \subseteq \vec{p} \cup \vec{r}$.

Let m be any $\vec{p}, \vec{r}$-node with $m \models \exists\vec{q}.A$ and $m \not\models B$. Let $\mathsf{Y} := \mathsf{Y}_{2.\nu+1,m[\vec{p}]}$ and $\mathsf{N} := \mathsf{N}_{2.\nu+1,m[\vec{p}]}$ (see section 8). We claim that: $A, \mathsf{Y} \nvdash \mathsf{N}$. If it did, we would have: $A \vdash \mathsf{Y} \to \mathsf{N}$. And hence by definition: $\exists\vec{q}.A, \mathsf{Y} \vdash \mathsf{N}$. Quod non, since $m \models \exists\vec{q}.A, \mathsf{Y}$ and $m \not\models \mathsf{N}$. Let k be any $\vec{q}, \vec{p}$-node such that: $k \models A, \mathsf{Y}$ and $k \not\models \mathsf{N}$. We find that $k \simeq_{2.\nu+1,\vec{p}} m$. Apply lemma 10.1 with $\mathsf{Sub}(A)$ in the role of X to find a $\vec{q}, \vec{p}, \vec{r}$-node n with: $m \simeq_{\vec{p},\vec{r}} n$ and $\mathsf{Th}_{\mathsf{Sub}(A)}(k) = \mathsf{Th}_{\mathsf{Sub}(A)}(n)$. It follows that $n \not\models B$, but $n \models A$. A contradiction.

(2) Consider B and $\vec{q}$. Let $\vec{p} := \mathsf{PV}(B) \setminus \vec{q}$. Take: $\forall\vec{q}.B := \bigvee\{D \in I_{2.\nu+1}(\vec{p}) \mid \mathsf{IPC} \vdash D \to B\}$. Clearly $\forall\vec{q}.B$ satisfies (a) and (b). Moreover $\mathsf{IPC} \vdash \forall\vec{q}.B \to B$. Hence all we have to prove is that for all A with $\mathsf{PV}(A) \cap \vec{q} = \emptyset$:

$$\mathsf{IPC} \vdash A \to B \Rightarrow \mathsf{IPC} \vdash A \to \forall\vec{q}.B.$$

Suppose that, to the contrary, for some A: $\mathsf{PV}(A) \cap \vec{q} = \emptyset$ and $\mathsf{IPC} \vdash A \to B$ and $\mathsf{IPC} \nvdash A \to \forall\vec{q}.B$. Take $\vec{r} := \mathsf{PV}(A) \setminus \vec{p}$. Note that $\vec{p}, \vec{q}, \vec{r}$ are pairwise disjoint, $\mathsf{PV}(B) \subseteq \vec{q}, \vec{p}$ and $\mathsf{PV}(A) \subseteq \vec{p}, \vec{r}$.

Let m be any $\vec{p}, \vec{r}$-node with $m \models A$ and $m \not\models \forall\vec{q}.B$. Let $\mathsf{Y} := \mathsf{Y}_{2.\nu+1,m[\vec{p}]}$ and $\mathsf{N} := \mathsf{N}_{2.\nu+1,m[\vec{p}]}$. We claim that: $\mathsf{Y} \nvdash \mathsf{N} \vee B$. Note that, by theorem 8.12, Y is prime. So if $\mathsf{Y} \vdash \mathsf{N} \vee B$, then $\mathsf{Y} \vdash \mathsf{N}$ or $\mathsf{Y} \vdash B$. Since $\mathsf{Y} \nvdash \mathsf{N}$, it follows that $\mathsf{Y} \vdash B$. But then by definition: $\mathsf{Y} \vdash \forall\vec{q}.B$. Quod non, since $m \models \mathsf{Y}$ and $m \not\models \forall\vec{q}.B$. Let k be any $\vec{q}, \vec{p}$-node such that: $k \models \mathsf{Y}$ and $k \not\models \mathsf{N} \vee B$. We find that $k \simeq_{2.\nu+1,\vec{p}} m$. Apply lemma 10.1 with $\mathsf{Sub}(B)$ in the role of X to find a $\vec{q}, \vec{p}, \vec{r}$-node n with: $m \simeq_{\vec{p},\vec{r}} n$ and $\mathsf{Th}_{\mathsf{Sub}(B)}(k) = \mathsf{Th}_{\mathsf{Sub}(B)}(n)$. It follows that $n \models A$, but $n \not\models B$. A contradiction. ❑

Theorem 10.3 (An improvement of theorem 10.2) *We can replace the estimate* $2.\nu + 2$ *in theorem 10.2(1), by* $2.\nu + 1$.

Proof

By theorem 7.5 we can write A as a disjunction of prime formulas D with $\mathsf{PV}(D) \subseteq \mathsf{PV}(A)$ and $\nu_{\mathsf{Sub}(D)} \leq \nu_{\mathsf{Sub}(A)}$. Say the set of these disjuncts is $\mathcal{D}$. For B with $\mathsf{PV}(B) \cap \vec{q} = \emptyset$ we have:

$$\begin{aligned} \mathsf{IPC} \vdash A \to B \quad &\Leftrightarrow \quad \text{for all } D \in \mathcal{D} \;\; \mathsf{IPC} \vdash D \to B \\ &\Leftrightarrow \quad \text{for all } D \in \mathcal{D} \;\; \mathsf{IPC} \vdash \exists \vec{q}.D \to B \\ &\Leftrightarrow \quad \mathsf{IPC} \vdash \bigvee \{\exists \vec{q}.D \mid D \in \mathcal{D}\} \to B \end{aligned}$$

Ergo we may take $\exists \vec{q}.A := \bigvee \{\exists \vec{q}.D \mid D \in \mathcal{D}\}$. It follows that it is sufficient to prove theorem 10.2(1) with $2\nu + 1$ for prime A.

So suppose A is prime. Let $\vec{p} := \mathsf{Sub}(A) \setminus \vec{q}$. Define:

$$\exists \vec{q}.A := \bigwedge \{C \in I_{2.\nu+1}(\vec{p}) \mid \mathsf{IPC} \vdash A \to C\}.$$

Suppose that for some B: $\mathsf{PV}(B) \cap \vec{q} = \emptyset$ and $\mathsf{IPC} \vdash A \to B$ and $\mathsf{IPC} \nvdash \exists \vec{q}.A \to B$. Define further:

- $Y := \mathsf{Sub}(A) \cup I_{2\nu+1}(\vec{p})$, $\Delta := \{C \in Y \mid \mathsf{IPC} \vdash A \to C\}$
- $Z := \mathsf{Sub}(B) \cup I_{2.\nu+1}(\vec{p})$, $\Gamma := \{D \in Z \mid \mathsf{IPC} \vdash \exists \vec{q}.A \to D\}$.

Note that since $\{C \in I_{2.\nu+1}(\vec{p}) \mid \mathsf{IPC} \vdash A \to C\}$ is $I_{2.\nu+1}(\vec{p})$-saturated, $\exists \vec{q}.A$ is prime. We find that Δ is Y-saturated and Γ is Z-saturated. Take: $\langle \mathbb{K}, k \rangle := \mathbb{H}_Y[\Delta]$, $\langle \mathbb{M}, m \rangle := \mathbb{H}_Z[\Gamma]$. It follows that for $C \in I_{2.\nu+1}(\vec{p})$:

$$\Delta \models C \Leftrightarrow \Delta \vdash C \Leftrightarrow A \vdash C \Leftrightarrow \exists \vec{q}.A \vdash C \Leftrightarrow \Gamma \models C.$$

Ergo $k \simeq_{2.\nu+1} m$, $k \models A$, $m \models \exists \vec{q}.A$ and $m \not\models B$. From this point on the argument proceeds as in the proof of 10.2(1). ❑

Theorem 10.4 (Semantical Characterization of Pitts' Quantifiers) *Consider a node m. Suppose $A \in \mathcal{L}^i$. We have:*

1. $m \models \exists q.A \Leftrightarrow \exists n\; m \simeq_{[q]} n$ *and* $n \models A$.
2. $m \models \forall q.A \Leftrightarrow$ *for all n with* $m \simeq_{[q]} n$, $n \models A$.

Proof

(1) "$\Leftarrow$" Trivial. "$\Rightarrow$" Let $\vec{p} := \mathsf{PV}(A) \setminus \{q\}$ and $\nu := \nu_{\mathsf{Sub}(A)}$. Suppose $m \models \exists q.A$, where m is an $\mathcal{R}$-node with $\vec{p} \subseteq \mathcal{R}$. Let $\mathsf{Y} := \mathsf{Y}_{2.\nu+1,m[\vec{p}]}$ and $\mathsf{N} := \mathsf{N}_{2.\nu+1,m[\vec{p}]}$. As in theorem 10.2(2), $A, \mathsf{Y} \nvdash \mathsf{N}$. Let k be any $q, \vec{p}$-node such that: $k \models A, \mathsf{Y}$ and $k \not\models \mathsf{N}$. We find that $k \simeq_{2.\nu+1,\vec{p}} m$. Apply lemma 10.1 to k and $m[\mathcal{R} \setminus \{q\}]$ with $\mathsf{Sub}(A)$ in the role of X, $\{q\}$ in the role of $\mathcal{Q}$, $\vec{p}$ in the role of $\vec{p}$, $\mathcal{R} \setminus (\vec{p} \cup \{q\})$ in the role of $\mathcal{R}$, to find a $q, \vec{p}, \mathcal{R}$-node n with: $m \simeq_{[q]} n$ and $\mathsf{Th}_{\mathsf{Sub}(A)}(k) = \mathsf{Th}_{\mathsf{Sub(A)}}(n)$, and, thus, $n \models A$.

(2) The proof of (2) is similar. ❑

Theorem 10.2 is not formulated entirely in terms of ℓ-simulations. The reason is that such a form does not provide a very sharp estimate on uniform interpolants. But if we do not want to worry about precise complexities a watered down version can be pleasant to have. By applying theorem 10.2 to $X := I_n(\vec{p}, \vec{q})$ we find:

Corollary 10.5 For all disjoint $\vec{q}, \vec{p}$ and numbers s, there is an N (multi-exponential in $|\vec{q}, \vec{p}| + s$), such that: for all $k \in \mathsf{Pmod}(\vec{q}, \vec{p})$, and all $m \in \mathsf{Pmod}$ with $\vec{q} \cap \mathcal{P}_{\mathbb{M}} = \emptyset$ and $\vec{p} \subseteq \mathcal{P}_m$, we have:

$$k \simeq_{N,\vec{p}} m \Rightarrow \text{ there is an } n \in \mathsf{Pmod}(\vec{q}, \mathcal{P}_m) \text{ with } n \simeq_{s,\vec{q},\vec{p}} k \text{ and } n \simeq_{\mathcal{P}_m} m.$$

❑

We end this section with a result from [25]. We illustrate that the increase of implicational complexity in going to a uniform interpolant is unavoidable. It is an interesting problem to find both better upper and lower bounds.

Theorem 10.6 *Every formula of $\mathcal{L}^i$ is equivalent to an I_2-formula preceded by existential quantifiers and to an I_3-formula preceded by universal quantifiers.*

Proof

Suppose $A \in \mathcal{L}^i(\vec{p})$. Let $\vec{q}$ be a set of variables disjoint from $\vec{p}$ that is in 1-1 correspondence with the subformulas of the form $(B \to C)$ of A. Let the correspondence be $\mathfrak{q}$. We define a mapping $\mathcal{T}$ as follows:

- $\mathcal{T}$ commutes with atoms, conjunction and disjunction
- $\mathcal{T}(B \to C) := \mathfrak{q}(B \to C)$

Define:

- $\mathsf{EQ} := \bigwedge\{\mathfrak{q}(B \to C) \leftrightarrow (\mathcal{T}(B) \to \mathcal{T}(C)) \mid (B \to C) \in \mathsf{Sub}(A)\}$

Note that EQ is I_2. Finally we put:

- $A^{\#} := \exists\vec{q}(\mathsf{EQ} \wedge \mathcal{T}(A))$
- $A^{\$} := \forall\vec{q}(\mathsf{EQ} \to \mathcal{T}(A))$

By elementary reasoning in second order propositional logic we find: $\vdash A \leftrightarrow A^{\#}$ and $\vdash A \leftrightarrow A^{\$}$. ❑

11 Propositional Quantifiers for IPC

There is a variety of ways to introduce propositional quantifiers in IPC. A first idea is proof theoretical. We add the obvious analogues of the rules for the quantifiers for Predicate Logic to IPC. E.g.:

$\forall$**R** $\Gamma \vdash A \Rightarrow \Gamma \vdash \forall p.A$, provided that p does not occur in Γ

$\forall$**L** $\Gamma, A[p := B] \vdash C \Rightarrow \Gamma, \forall p.A \vdash C$

$\exists$**R** $\Gamma \vdash A[p := B] \Rightarrow \Gamma \vdash \exists p.A$

$\exists$**L** $\Gamma, A \vdash C \Rightarrow \Gamma, \exists p.A \vdash C$, provided that p does not occur in Γ, C

We call the theory thus obtained IPC^2. In a clear sense the quantifiers thus introduced are minimal. IPC^2 is undecidable, as is shown by Löb in [13]. See also the papers by Gabbay, [3] and by Smoryński, [21]. Gabbay gives a semantics of sorts for these quantifiers. Note that the undecidability of the minimal system is not preserved by extensions that are conservative over IPC: IPC with the Pitts quantifiers is a decidable conservative extension! A salient property of the minimal quantifiers, which is inherited by all extensions, is the definability of all connectives in terms of $\forall$ and $\to$. The definition is as follows:

- $\bot := \forall p.p$, $\top := \forall p(p \to p)$
- $A \wedge B := \forall p((A \to (B \to p)) \to p)$
- $A \vee B := \forall p((A \to p) \wedge (B \to p)) \to p)$
- $\exists q.A := \forall p(\forall q(A \to p) \to p)$

The topological interpretation for second order IPC is given as follows. Let $\mathcal{O}$ be a topological space. An assignment f sends the propositional variables to the open sets of $\mathcal{O}$. Define:

- $[\![p]\!]\, f = f(p)$, $[\![\top]\!]\, f = \mathcal{O}$, $[\![\bot]\!]\, f = \emptyset$
- $[\![A \wedge B]\!]\, f = [\![A]\!]\, f \cap [\![B]\!]\, f$
- $[\![A \vee B]\!]\, f = [\![A]\!]\, f \cup [\![B]\!]\, f$
- $[\![A \to B]\!]\, f = \mathsf{int}(([\![A]\!]\, f)^c \cup [\![B]\!]\, f)$
- $[\![\exists p.A]\!]\, f = \bigcup \{ [\![A]\!]\, f[p := P] \mid P \text{ is open}\}$
- $[\![\forall p.A]\!]\, f = \mathsf{int}(\bigcap \{ [\![A]\!]\, f[p := P] \mid P \text{ is open}\})$

For information about the topological interpretation, we refer the reader to the work by Tarski (see [23]), by Kreisel (see [11]), by Troelstra (see [24]), by Połacik (see [16], [17], [15]). Tomasz Połacik shows that the topological interpretation is not identical to the Pitts interpretation.

A 'subsemantics' of the topological interpretation is the semantics where one quantifies over upward persistent sets in Kripke models. This semantics is studied by Philip Kremer (see [12]). He shows that the valid principles of this interpretation are recursively isomorphic to full second order predicate logic.

We turn to the Pitts quantifiers. First let us note that, since Pitts quantifiers can be defined in the language of IPC the Pitts quantifiers can be compared with any other quantifiers. In fact, we have: $\mathsf{IPC}^2 \vdash \exists p.A \to \exists_{pitts} p.A$ and $\mathsf{IPC}^2 \vdash \forall_{pitts} p.A \to \forall p.A$.

Let UC_{pitts} be the universal closure of a formula with Pitts quantifiers. Clearly for $A \in \mathcal{L}^i$, $\mathsf{UC}_{pitts}(A)$ translates to a closed IPC-formula. Thus it can —modulo provable equivalence— be only $\top$ or $\bot$. In fact we have $\mathsf{IPC} \vdash A \Leftrightarrow \mathsf{UC}_{pitts}(A) \equiv \top$ and $\mathsf{IPC} \nvdash A \Leftrightarrow \mathsf{UC}_{pitts}(A) \equiv \bot$.

It is clear that the translations of the Pitts quantifiers are computable. It follows that the extension of IPC^2, valid under the Pitts interpretation is decidable.[7] Thus, Pitts quantification does not give the same valid principles as the upwards closed sets interpretation, by the result of Kremer. Similarly, it cannot be IPC^2 itself by the result of Löb. These non-identities can also be established directly. Consider $\forall p(p \vee \neg p)$. This formula is $\bot$ under the Pitts interpretation. In other words $\neg\forall p(p \vee \neg p)$ is valid under the Pitts interpretation. Under the interpretation studied by Kremer, however, $\forall p(p \vee \neg p)$ just defines the top nodes of Kripke models.[8] Thus, $\neg\forall p(p \vee \neg p)$ is not valid under the upwards closed sets interpretation and, a fortiori, not under the topological or the minimal interpretation. We end this section by verifying semantically a striking principle (present in Pitts' paper) valid for the Pitts interpretation.

Theorem 11.1 *Consider k. We have:*

$$k \models \forall p(B \vee C) \Rightarrow k \models \forall p.B \text{ or } k \models \forall p.C.$$

[7] IPC is PSPACE-complete. This is shown by Statman in [22]. I would guess that the principles valid under the Pitts interpretation are also PSPACE, but nobody, to my knowledge, took the trouble of checking the complexity of the algorithm given by Pitts.

[8] Note that being a top node is *not* bisimulation-invariant.

Proof

We reason by contraposition. Suppose $k \not\models \forall p.B$ and $k \not\models \forall p.C$. It follows that there are nodes m and n, such that $k \simeq_{[p]} m \not\models B$ and $k \simeq_{[p]} n \not\models C$. Let $\mathbb{M}$ and $\mathbb{N}$ be the models of, respectively, m and n. Consider $\mathbb{P} := \mathsf{Glue}(\mathbb{M}[m], \mathbb{N}[n])$. Let the new bottom be b. It is easily seen that $k \simeq_{[p]} b$ and $b \not\models (B \vee C)$ ❑

12 Formula Classes and Model Descriptions in Modal Logic

We briefly treat the connection between modal propositional formulas and bounded bisimulations. Since these facts are similar to, but simpler than the corresponding facts for IPC I just state the results without the proofs.

Let $\mathfrak{b}(A)$ be the box-depth of a formula. $B_k(\vec{p})$ is the set of formulas in the variables $\vec{p}$ with box-depth $\leq k$. $B_k(\vec{p})$ is finite *modulo provable equivalence.*

Consider $\vec{p}$-nodes k and m. Then:

$$k \simeq_n m \Leftrightarrow \mathsf{Th}_{B_n(\vec{p})}(k) = \mathsf{Th}_{B_n(\vec{p})}(m)$$

Let k be a $\vec{p}$-node. Define: $\mathsf{Y}_{n,k} := \bigwedge \mathsf{Th}_{B_n(\vec{p})}(k)$. Clearly, for any $\vec{p}$-node m: $k \simeq_n m \Leftrightarrow m \models \mathsf{Y}_{n,k}$. We have: $k \simeq_n m \Leftrightarrow \mathsf{K} \vdash \mathsf{Y}_{n,m} \leftrightarrow \mathsf{Y}_{n,k}$

Suppose k is a transitive $\vec{p}$-node of finite depth. Define: $\mathsf{Y}_k := \mathsf{Y}_{2.d(k)+1,k}$ Let m also be a transitive $\vec{p}$-node. We have: $k \simeq m \Leftrightarrow m \models \mathsf{Y}_k$. It follows that: $k \simeq m \Leftrightarrow \mathsf{K4} \vdash \mathsf{Y}_m \leftrightarrow \mathsf{Y}_k$.

13 Uniform Interpolation for K

Before considering uniform interpolation for more complicated modal systems like $\mathsf{S4Grz}$, we do the relatively easy proof for K. This theorem was first proved by Silvio Ghilardi. Uniform interpolation for K follows from the amalgamation lemma below.

Lemma 13.1 Consider pairwise disjoint sets of propositional variables $\mathcal{Q}$, $\vec{p}$ and $\mathcal{R}$. Let $\langle \mathbb{K}, k_0 \rangle \in \mathsf{Pmod}(\mathcal{Q}, \vec{p})$ and $\langle \mathbb{M}, m_0 \rangle \in \mathsf{Pmod}(\vec{p}, \mathcal{R})$. Suppose that $k_0 \simeq_{\alpha,\vec{p}} m_0$. Then there is a $\mathcal{Q}$-extension $\langle \mathbb{N}, n_0 \rangle$ of $\langle \mathbb{M}, m_0 \rangle$ such that $n_0 \simeq_\alpha k_0$. ❑

Proof

Let $\mathcal{Z}$ be a downwards closed witness of $k_0 \simeq_{\alpha,\vec{p}} m_0$. We add a 'virtual top' $\top$ to $\mathbb{K}$ and stipulate that $\top$ satisfies no atoms. Let's call the new model $\mathbb{K}^\top$. We extend ω^∞ with a new bottom $\perp$ to $\omega^{\infty,\perp}$. Define $\mathsf{Pd}(n+1) := n$, $\mathsf{Pd}(0) := \mathsf{Pd}(\perp) := \perp$, $\mathsf{Pd}(\infty) = \infty$. Now define the following model $\mathbb{N}$:

- $N := \mathcal{Z} \cup \{\langle \top, \perp, m \rangle \mid m \in M\}$
- $\langle k, \alpha, m \rangle \prec_{\mathbb{N}} \langle k', \alpha', m' \rangle :\Leftrightarrow k \prec_{\mathbb{K}^\top} k'$ and $\alpha' = \mathsf{Pd}(\alpha)$ and $m \prec_{\mathbb{M}} m'$
- $\langle k, \alpha, m \rangle \models s :\Leftrightarrow k \models_{\mathbb{K}} s$ or $m \models_{\mathbb{M}} s$

We claim:

Claim 1 $n_0 \simeq_{\vec{p},\mathcal{R}} m_0$.

Claim 2 $n_0 \simeq_{\alpha,(\mathcal{Q},\vec{p})} k_0$.

We prove Claim 1. Take as bisimulation $\mathcal{B}$, with $\langle k,\alpha,m\rangle\mathcal{B}m' :\Leftrightarrow m=m'$. Clearly, if $n\mathcal{B}m$ then $\mathsf{Th}_{\vec{p},\mathcal{R}}(n)=\mathsf{Th}_{\vec{p},\mathcal{R}}(m)$. Moreover, $\mathcal{B}$ trivially has the zig-property. We check that $\mathcal{B}$ has the zag-property. Suppose $\langle k,\alpha,m\rangle\mathcal{B}m\prec m'$. If $\alpha\in\{0,\bot\}$, we can finish the diagram with $\langle\top,\bot,m'\rangle$. If $\alpha=\alpha'+1$ for $\alpha'\in\omega^\infty$, we have $k\mathcal{Z}_\alpha m$ and, hence, there is a k' such that $k\prec_{\mathbb{K}}k'$ and $k'\mathcal{Z}_{\alpha'}m'$. So we can finish the diagram with $\langle k',\alpha',m'\rangle$.

We prove Claim 2. Take as layered bisimulation $\mathcal{S}$, with $\langle k,\alpha,m\rangle\mathcal{S}_\alpha k'\Leftrightarrow k=k'$ (for $\alpha\in\omega^\infty$). Clearly, if $n\mathcal{S}_\alpha k$ then $\mathsf{Th}_{\mathcal{Q},\vec{p}}(n)=\mathsf{Th}_{\mathcal{Q},\vec{p}}(k)$. We check that $\mathcal{S}$ has the zag-property. The zig-property is analogous. Suppose $\langle k,\alpha+1,m\rangle\mathcal{S}_{\alpha+1}k\prec k'$. Since $k\mathcal{Z}_{\alpha+1}m$, there exists $m'\succ m$ such that $k'\mathcal{Z}_\alpha m'$. Hence $\langle k',\alpha,m'\rangle\succ\langle k,\alpha+1,m\rangle$, and $\langle k',\alpha,m'\rangle\mathcal{S}_\alpha k'$. ❑

Theorem 13.2 (Uniform Interpolation) *We prove uniform interpolation for* K

1. *Consider any formula A and any finite set of variables $\vec{q}$. Let $\nu:=\mathfrak{b}(A)$. There is a formula $\exists\vec{q}.A$ such that:*
   (a) $\mathsf{PV}(\exists\vec{q}.A)\subseteq\mathsf{PV}(A)\setminus\vec{q}$
   (b) $\mathfrak{b}(\exists\vec{q}.A)\leq\nu$
   (c) *For all $B\in\mathcal{L}^m$ with $\mathsf{PV}(B)\cap\vec{q}=\emptyset$, we have:*
   $$\mathsf{K}\vdash A\to B\Leftrightarrow\mathsf{K}\vdash\exists\vec{q}.A\to B.$$
2. *Consider any formula B and any finite set of variables $\vec{q}$. Let $\nu:=\mathfrak{b}(B)$. There is a formula $\forall\vec{q}.B$ such that:*
   (a) $\mathsf{PV}(\forall\vec{q}.B)\subseteq\mathsf{PV}(B)\setminus\vec{q}$
   (b) $\mathfrak{b}(\forall\vec{q}.B)\leq\nu$
   (c) *For all $A\in\mathcal{L}^m$ with $\mathsf{PV}(A)\cap\vec{q}=\emptyset$, we have:*
   $$\mathsf{K}\vdash A\to B\Leftrightarrow\mathsf{K}\vdash A\to\forall\vec{q}.B.$$

Proof

We just prove (1). The proof of (2) is analogous. (Alternatively, we may take $(\forall\vec{q}.B):=(\neg\exists\vec{q}\neg B)$.) Consider A and $\vec{q}$. Let $\vec{p}:=\mathsf{PV}(A)\setminus\vec{q}$. Take:

$$\exists\vec{q}.A:=\bigwedge\{C\in I_\nu(\vec{p})\mid\mathsf{K}\vdash A\to C\}.$$

Clearly $\exists\vec{q}.A$ satisfies (a) and (b). Moreover, $\mathsf{K}\vdash A\to\exists\vec{q}.A$. Hence, all we have to prove is that for all B with $\mathsf{PV}(B)\cap\vec{q}=\emptyset$:

$$\mathsf{K}\vdash A\to B\Rightarrow\mathsf{K}\vdash\exists\vec{q}.A\to B.$$

Suppose, to the contrary, that for some B: $\mathsf{PV}(B)\cap\vec{q}=\emptyset$ and $\mathsf{K}\vdash A\to B$ and $\mathsf{K}\nvdash\exists\vec{q}.A\to B$. Take $\vec{r}:=\mathsf{PV}(B)\setminus\vec{p}$. Note that $\vec{p},\vec{q},\vec{r}$ are pairwise disjoint, $\mathsf{PV}(A)\subseteq\vec{q}\cup\vec{p}$ and $\mathsf{PV}(B)\subseteq\vec{p}\cup\vec{r}$.

Let m be any $\vec{p},\vec{r}$-node with $m\models\exists\vec{q}.A$ and $m\not\models B$. Let $\mathsf{Y}:=\mathsf{Y}_{\nu,m[\vec{p}]}$ and We claim that: A,Y is consistent. If it were not, we would have: $A\vdash\neg\mathsf{Y}$. And, hence, by definition: $\exists\vec{q}.A\vdash\neg\mathsf{Y}$. Quod non, since $m\models\exists\vec{q}.A,\mathsf{Y}$ and $\mathfrak{b}(\neg\mathsf{Y})=\nu$. Let k be any $\vec{q},\vec{p}$-node such that: $k\models A,\mathsf{Y}$. We find that $k\simeq_{\nu,\vec{p}}m$. Apply lemma 13.1 to find a $\vec{q},\vec{p},\vec{r}$-node n with: $m\simeq_{\vec{p},\vec{r}}n$ and $m\simeq_{\nu,(\vec{p},\vec{r})}n$. It follows that $n\not\models B$, but $n\models A$. A contradiction. ❑

The proof of the following theorem is fully analogous to the the proof of its twin for the case of IPC.

Theorem 13.3 *Consider a node m. Suppose $A\in\mathcal{L}^m$. We have:*

1. $m\models\exists q.A\Leftrightarrow\exists n\ m\simeq_{[q]}n$ *and* $n\models A$.
2. $m\models\forall q.A\Leftrightarrow$ *for all n with* $m\simeq_{[q]}n$, $n\models A$.

14 Uniform Interpolation for GL

In this section we prove Uniform Interpolation for GL. It is well known that GL is sound and complete for upward wellfounded Kripke models and that it has the finite model property. Since GL-models are irreflexive we use '$\prec$' for their accessibility relation and '$\preceq$' for the corresponding weak partial order. '$\vdash$' will stand for GL-derivability.

Let X be a finite, adequate set of formulas. *Adequate* means: closed under subformulas. The GL Henkin model $\mathbb{H}_X$ for X is constructed in the following way.

- The nodes are the subsets Δ of X that are *saturated*, i.e. if Δ proves some finite disjunction of elements of X then some disjunct is in Δ.[9]
- $\Delta \prec \Delta'$ iff $\Box A \in \Delta \Rightarrow A, \Box A \in \Delta'$

Note that this model may contain *non-trivial loops!* and, thus is not a GL-model. (It is easy to remove these loops, but for the present purposes, we need to keep them.) The height of a model is the maximal depth. The height of the Henkin model is $\leq 2.|\{C{\in}X \mid C \text{ is boxed}\}|$. To see this, consider $\Delta_0 \prec^+ \Delta_1 \prec^+ \Delta_2$. Clearly, going up the set of boxed formulas in the Δ_i increases. Suppose we had the same boxed formulas in Δ_0, Δ_1 and Δ_2. Suppose $\Box A \in \Delta_2$. Then, ex hypothesi, $\Box A \in \Delta_0$. Hence, $A, \Box A \in \Delta_1$. We may conclude that $\Delta_2 \prec \Delta_1$. Quod non. So, necessarily, the boxed formulas increase by at least one in going from Δ_0 to Δ_2. It follows that if we have a strictly ascending chain of length $2.n$, then there are at least n boxed subformulas.

As in for IPC and K we start with an amalgamation lemma. Consider disjoint sets of propositional variables $\mathcal{Q}$, $\vec{p}$ and $\mathcal{R}$. Let $\langle \mathbb{K}, k_0 \rangle \in \mathsf{Pmod}(\mathcal{Q}, \vec{p})$ and $\langle \mathbb{M}, m_0 \rangle \in \mathsf{Pmod}(\vec{p}, \mathcal{R})$ be pointed GL-models.

Lemma 14.1 Let $X \subseteq \mathcal{L}^m(\mathcal{Q}, \vec{p})$ be a finite adequate set. Let:

$$\nu := 2.|\{C{\in}X \mid C \text{ is boxed}\}|.$$

Suppose that $k_0 \simeq_{2.\nu+1,\vec{p}} m_0$. Then there is a $\mathcal{Q}$-extension $\langle \mathbb{N}, n_0 \rangle$ of $\langle \mathbb{M}, m_0 \rangle$ such that $\mathbb{N}$ is a GL-model and $\mathsf{Th}_X(n_0) = \mathsf{Th}_X(k_0)$. $\Box$

Proof

Let $\mathcal{Z}$ be a downwards closed witness of $k_0 \simeq_{2.\nu+1,\vec{p}} m_0$. Define Φ_X from $\mathbb{K}$ to the Henkin model $\mathbb{H} := \mathbb{H}_X$ as follows: $\Phi_X(k) := \Delta(k) := \{B{\in}X \mid k \models B\}$. Define further for k in $\mathbb{K}$: $d_X(k) := d_{\mathbb{H}}(\Delta(k))$. Note that: $d_X(k) \leq \nu$.

Consider a pair $\langle \Delta, m \rangle$ for Δ in H and m in $\mathbb{M}$. Consider k', k, m'. Let $\Delta' := \Phi_X(k')$. We say that k', k, m' is a *witnessing triple* for $\langle \Delta, m \rangle$ if:

$$\Delta' \approx \Delta,\ k' \preceq k,\ m' \preceq m,\ k' \mathcal{Z}_{2.d_X(k')+1} m',\ k \mathcal{Z}_{2.d_X(k')} m.$$

$$\begin{tikzcd}
& \Delta \arrow[rr, dash, "\Phi_X"] & & k \arrow[rr, dash, "\mathcal{Z}_{2d_X(k')}"] & & m \\
\Delta' \arrow[ur, "\approx"] \arrow[rr, dash, "\Phi_X"] & & k' \arrow[ur, "\preceq"] \arrow[rrr, dash, "\mathcal{Z}_{2d_X(k')+1}"] & & & m' \arrow[u, "\preceq"']
\end{tikzcd}$$

Define:

- $N := \{\langle \Delta, m \rangle \mid \text{there is a witnessing triple for } \langle \Delta, m \rangle\}$

[9]We consider $\bot$ as the empty disjunction. So saturation implies consistency.

- $n_0 := \langle \Delta(k_0), m_0 \rangle$
- $\langle \Delta, m \rangle \prec_{\mathbb{N}} \langle \Gamma, n \rangle :\Leftrightarrow \Delta \prec_{\mathbb{H}} \Gamma$ and $m \prec_{\mathbb{M}} n$
- $\langle \Delta, m \rangle \models_{\mathbb{N}} s :\Leftrightarrow \Delta \models_{\mathbb{H}} s$ or $m \models_{\mathbb{M}} s$

Note that by assumption $k_0 \mathcal{Z}_{2\nu+1} m_0$. Moreover: $2.d_X(k_0)+1 \leq 2.\nu+1$. Hence: $k_0 \mathcal{Z}_{2d_X(k_0)+1} m_0$. So we can take k_0, k_0, m_0 as witnessing triple for n_0. Let k', k, m' be a witnessing triple for $\langle \Delta, m \rangle$. Note that for $p \in \vec{p} \cap X$: $\Delta \models p \Leftrightarrow k \models p \Leftrightarrow m \models p$, and hence: $\langle \Delta, m \rangle \models p \Leftrightarrow \Delta \models p \Leftrightarrow m \models p$. It is easy to see that $\mathbb{N}$ is a GL-model (even if $\mathbb{H}_X$ need not be one). We claim:

Claim 1 $n_0 \simeq_{\vec{p},\mathcal{R}} m_0$.

Claim 2 For $B \in X : \langle \Delta, m \rangle \models B \Leftrightarrow B \in \Delta$.

Evidently the lemma is immediate from the claims.

We prove Claim 1. Take as bisimulation $\mathcal{B}$ with $\langle \Delta, m \rangle \mathcal{B} m$. Clearly, $\mathsf{Th}_{\vec{p},\mathcal{R}}(\langle \Delta, m \rangle) = \mathsf{Th}_{\vec{p},\mathcal{R}}(m)$. Moreover, $\mathcal{B}$ has the zig-property. We check that $\mathcal{B}$ has the zag-property. Suppose $\langle \Delta, m \rangle \mathcal{B} m \prec n$. We are looking for a pair $\langle \Gamma, n \rangle$ in N such that $\Delta \prec \Gamma$. Let k', k, m' be a witnessing triple for $\langle \Delta, m \rangle$. We write $\Delta' := \Delta(k')$. Since $k' \mathcal{Z}_{2.d_X(k')+1} m' \prec n$, there is a h such that $k' \prec h \mathcal{Z}_{2.d_X(k')} n$. We take $\Gamma := \Delta(h)$. Clearly $\Delta \prec \Gamma$. We need a witnessing triple k'^*, k^*, m'^* for $\langle \Gamma, n \rangle$ We distinguish two possibilities. First, $\Delta \approx \Gamma$. In this case we can take: $k'^* := k'$, $k^* := h$, $m'^* := m'$.

$$\begin{tikzcd}
\Gamma \arrow[rr, dash, dashed, "\Phi_X"] & & k^* = h \arrow[rr, dash, dashed, "\mathcal{Z}_{2d_X(k')}"] & & n \\
& \Delta \arrow[rr, dash, "\Phi_X"] & & k \arrow[r, dash, "\mathcal{Z}_{2d_X(k')}"] & m \arrow[u, "\prec"'] \\
\Delta' \arrow[uu, dashed, "\approx"] \arrow[ur, "\approx"] \arrow[rr, dash, "\Phi_X"] & & k'^* = k' \arrow[uu, dashed, "\prec"] \arrow[ur, "\preceq"] \arrow[rr, dash, "\mathcal{Z}_{2d_X(k')+1}"] & & m'^* = m' \arrow[u, "\preceq"']
\end{tikzcd}$$

Secondly, $\Delta \not\approx \Gamma$. In this case we can take: $k'^* := h$, $k^* := h$, $m'^* := n$. To see this, note that, since $k' \prec h$, we have: $\Delta \approx \Delta' \prec \Gamma$ and, hence, $\Delta \prec^+ \Gamma$. Ergo $d_X(h) < d_X(k')$. It follows that: $2.d_X(h) + 1 \leq 2.d_X(k')$. So, $h \mathcal{Z}_{2.d_X(k')+1} n$.

$$\begin{tikzcd}
\Gamma \arrow[rr, dash, dashed, "\Phi_X"] & & k'^* = k^* = h \arrow[rr, dash, dashed, "\mathcal{Z}_{2d_X(h)+1}"] & & m'^* = n \\
& \Delta \arrow[rr, dash, "\Phi_X"] & & k \arrow[r, dash, "\mathcal{Z}_{2d_X(k')}"] & m \arrow[u, "\prec"'] \\
\Delta' \arrow[uu, dashed, "\prec^+"] \arrow[ur, "\approx"] \arrow[rr, dash, "\Phi_X"] & & k' \arrow[uu, dashed, "\prec"] \arrow[ur, "\preceq"] \arrow[rr, dash, "\mathcal{Z}_{2d_X(k')+1}"] & & m' \arrow[u, "\preceq"']
\end{tikzcd}$$

Finally, clearly, $b_{\mathbb{N}} \mathcal{B} b_{\mathbb{M}}$.

We prove Claim 2. The proof is by induction on X. The cases of atoms, conjunction and disjunction are trivial. We treat the only non-trivial case: the left-to-right case of the box. Consider $\Box C \in X$ and consider the node $\langle \Delta, m\rangle$ with witnessing triple k', k, m'. Suppose $\Box C \notin \Delta$. Clearly, $k \not\models \Box C$, so there is an $h' \succ k$ with $h' \not\models C$. Let h be maximal in $\mathbb{K}$ with $h \succ k$ and $h \not\models C$. By maximality, we find: $h \models \Box C$. Let $\Gamma := \Delta(h)$. Since, $\Box C \notin \Delta$ and $\Box C \in \Gamma$, we find: $\Delta \prec^+ \Gamma$. Note that it follows that $d_X(k') \geq 1$. Since, $k\mathcal{Z}_{2.d_X(k')}m$ and $k \prec h$, there is an $n \succ m$ with $h\mathcal{Z}_{2.d_X(k')-1}n$. Moreover: $2.d_X(h)+1 \leq 2.d_X(k')-1$. Ergo: $h\mathcal{Z}_{2.d_X(h)+1}n$. So we can take $k'^* := h$, $k^* := h$, $m'^* := n$ to witness $\langle \Gamma, n\rangle$. Clearly, $\langle \Delta, m\rangle \prec \langle \Gamma, n\rangle$. By the Induction Hypothesis: $\langle \Gamma, n\rangle \not\models C$. Hence, $\langle \Delta, m\rangle \not\models \Box C$.

$$\begin{tikzcd}
 & \Gamma \arrow[rr, dash, "\Phi_X"] & & k'^* = k^* = h \arrow[r, dash, dashed, "\mathcal{Z}_{2d_X(h)+1}"] & m'^* = n \\
 & \Delta \arrow[u, "\prec^+"] \arrow[rr, dash, "\Phi_X"] & & k \arrow[u, "\prec"] \arrow[r, dash, "\mathcal{Z}_{2d_X(k')}"] & m \arrow[u, dashed, "\prec"'] \\
\Delta \arrow[ur, "\approx"] \arrow[rr, dash, "\Phi_X"] & & k' \arrow[ur, "\preceq"] \arrow[rr, dash, "\mathcal{Z}_{2d_X(k')+1}"] & & m' \arrow[u, "\preceq"']
\end{tikzcd}$$

Thus we have proved Claim 2. ❑

We formulate Uniform Interpolation for GL. Its proof is fully analogous to the one of Uniform Interpolation for K.

Theorem 14.2 (Uniform Interpolation) *We state uniform interpolation for* GL

1. *Consider any formula A and any finite set of variables $\vec{q}$. Let $\nu := 2.|\{C \in \mathsf{Sub}(A) \mid C \text{ is boxed}\}|$. There is a formula $\exists\vec{q}.A$ such that:*

   (a) $\mathsf{PV}(\exists\vec{q}.A) \subseteq \mathsf{PV}(A) \setminus \vec{q}$

   (b) $\mathfrak{b}(\exists\vec{q}.A) \leq 2.\nu + 1$

   (c) For all $B \in \mathcal{L}^m$ with $\mathsf{PV}(B) \cap \vec{q} = \emptyset$, we have:

   $$\mathsf{GL} \vdash A \to B \Leftrightarrow \mathsf{GL} \vdash \exists\vec{q}.A \to B.$$

2. *Consider any formula B and any finite set of variables $\vec{q}$. Let $\nu := 2.|\{C \in \mathsf{Sub}(B) \mid C \text{ is boxed}\}|$. There is a formula $\forall\vec{q}.B$ such that:*

   (a) $\mathsf{PV}(\forall\vec{q}.B) \subseteq \mathsf{PV}(B) \setminus \vec{q}$

   (b) $\mathfrak{b}(\forall\vec{q}.B) \leq 2.\nu + 1$

   (c) For all $A \in \mathcal{L}^m$ with $\mathsf{PV}(A) \cap \vec{q} = \emptyset$, we have:

   $$\mathsf{GL} \vdash A \to B \Leftrightarrow \mathsf{GL} \vdash A \to \forall\vec{q}.B.$$

The semantical interpretation of the propositional quantifiers is fully analogous to the case of K.

15 Uniform Interpolation for S4Grz

$\mathsf{S4Grz}$, a logic called after Andrzej Gregorczyk, is K extended with:

T $\vdash \Box A \to A$

4 $\vdash \Box A \to \Box\Box A$

Grz $\vdash \Box(\Box(A \to \Box A) \to A) \to A$

It is easy to see that T is superfluous. Note also that over $\mathsf{KT4}$ ($= \mathsf{S4}$), *Grz* is equivalent to:

Grz' $\Box(\Box(A \to \Box A) \to A) \to \Box A$

The logic is sound for weak partial orderings such that the associated strict ordering is upward wellfounded. We will show that the completeness of the logic in finite partial orderings. Since we deal with reflexive structures in this section, we will use '$\preceq$' for these relations. In case our relation is a weak partial ordering we write '$\prec$' for the associated strict ordering. For weak partial preorderings we will use $\prec^+$ for the associated strict version to stress the fact that also non-trivial loops are removed. '$\vdash$' will stand for $\mathsf{S4Grz}$-provability.

Let X be a finite adequate set. We construct a Henkin model $\mathbb{J}_X$ as follows. Let

$$X^+ := X \cup \{(B \to \Box B), \Box(B \to \Box B) \mid \Box B \in X\}.$$

Clearly, X is again adequate. Define:

- The domain J is the set of X^+-saturated sets Δ.
- $\Delta \preceq \Delta' :\Leftrightarrow \Delta = \Delta'$ or for all $\Box C \in \Delta$, $\Box C \in \Delta'$ and for some $\Box D \in \Delta'$, $\Box D \notin \Delta$)
- $\Delta \models p :\Leftrightarrow p \in \Delta$

It is easily seen that $\mathbb{J}_X$ is a finite partial order. We show that for all A in X, $\Delta \models A \Leftrightarrow A \in \Delta$. The crucial feature here is that we do *not* prove this fact for all A in X^+! The proof is by induction on A. We consider the only interesting case. Suppose that A is $\Box B$ and that $\Box B \notin X$. We show $\Delta \not\models \Box B$. We have to produce a Δ' with $\Delta' \succeq \Delta$ and $\Delta' \not\models B$. In case $B \notin \Delta$, and, hence, by the Induction Hypothesis, $\Delta \not\models B$, we are immediately done. So suppose $B \in \Delta$. Note that $\Box(B \to \Box B)$ cannot be in Δ, since, if it were, $\Box B$ would be in Δ. We claim:

$$\{\Box C \mid \Box C \in \Delta\} \cup \{\Box(B \to \Box B)\} \not\vdash B.$$

If it did, it would follow by $\mathsf{S4}$-reasoning that:

$$\{\Box C \mid \Box C \in \Delta\} \vdash \Box(\Box(B \to \Box B) \to B).$$

Hence by *Grz'*, $\{\Box C \mid \Box C \in \Delta\} \vdash \Box B$, and, thus $\Delta \vdash \Box B$. Quod non. By the usual methods we can construct an X^+-saturated set Δ' such that $\{\Box C \mid \Box C \in \Delta\} \cup \{\Box(B \to \Box B)\} \subseteq \Delta'$ and $B \notin \Delta'$. It follows that $\Delta \preceq \Delta'$ (with $\Box(B \to \Box B)$ in the role of the D of the definition). Since $B \notin \Delta'$, we have, by the Induction Hypothesis, $\Delta' \not\models B$.

For our proof of Uniform Interpolation we will use a different Henkin model $\mathbb{H}_X$, which is defined like $\mathbb{J}_X$, dropping the clause involving D, which excludes non-trivial loops. The height of $\mathbb{H}_X$ is estimated by the number of boxed formulas in X^+, which is two times the number of boxed formulas in X.

We start with a amalgamation lemma. Consider disjoint sets of propositional variables $\mathcal{Q}$, $\vec{p}$ and $\mathcal{R}$. Let $\langle \mathbb{K}, k_0 \rangle \in \mathsf{Pmod}(\mathcal{Q}, \vec{p})$ and $\langle \mathbb{M}, m_0 \rangle \in \mathsf{Pmod}(\vec{p}, \mathcal{R})$ be $\mathsf{S4Grz}$-models.

Lemma 15.1 Let $X \subseteq \mathcal{L}^m(\mathcal{Q}, \vec{p})$ be a finite adequate set. Let:

$$\nu := 2.|\{C \in X \mid C \text{ is boxed}\}|.$$

Suppose that $k_0 \simeq_{2.\nu+1,\vec{p}} m_0$. Then there is a $\mathcal{Q}$-extension $\langle \mathbb{N}, n_0 \rangle$ of $\langle \mathbb{M}, m_0 \rangle$ such that $\mathbb{N}$ is a $\mathsf{S4Grz}$-model and $\mathsf{Th}_X(n_0) = \mathsf{Th}_X(k_0)$. ❑

Proof

Let $\mathcal{Z}$ be a downwards closed witness of $k_0 \simeq_{2.\nu+1,\vec{p}} m_0$. Define Φ_X from $\mathbb{K}$ to the Henkin model $\mathbb{H} := \mathbb{H}_X$ as follows: $\Phi_X(k) := \Delta(k) := \{B \in X^+ \mid k \models B\}$. Define further for k in $\mathbb{K}$: $d_X(k) := d_{\mathbb{H}}(\Delta(k))$. Note that: $d_X(k) \leq \nu$.

Consider a pair $\langle \Delta, m\rangle$ for Δ in H and m in $\mathbb{M}$. Consider k', k, m'. Let $\Delta' := \Phi_X(k')$. We say that k', k, m' is a *witnessing triple* for $\langle \Delta, m\rangle$ if:

$$\Delta' \approx \Delta,\ k' \preceq k,\ m' \preceq m,\ k'\mathcal{Z}_{2.d_X(k')+1}m',\ k\mathcal{Z}_{2.d_X(k')}m.$$

$$\begin{tikzcd}
 & \Delta \arrow[r, dash, "\Phi_X"] & k \arrow[r, dash, "\mathcal{Z}_{2d_X(k')}"] & m \\
\Delta' \arrow[ur, "\approx"] \arrow[r, dash, "\Phi_X"] & k' \arrow[ur, "\preceq"] \arrow[rr, dash, "\mathcal{Z}_{2d_X(k')+1}"] & & m' \arrow[u, "\preceq"']
\end{tikzcd}$$

Define:

- $N := \{\langle \Delta, m\rangle \mid$ there is a witnessing triple for $\langle \Delta, m\rangle\}$
- $n_0 := \langle \Delta(k_0), m_0\rangle$
- $\langle \Delta, m\rangle \preceq_{\mathbb{N}} \langle \Gamma, n\rangle :\Leftrightarrow \langle \Delta, m\rangle = \langle \Gamma, n\rangle$ or $(\Delta \preceq_{\mathbb{H}} \Gamma$ and $m \prec_{\mathbb{M}} n)$ or $(\Delta \prec^+_{\mathbb{H}} \Gamma$ and $m \preceq_{\mathbb{M}} n)$
- $\langle \Delta, m\rangle \models_{\mathbb{N}} s :\Leftrightarrow \Delta \models_{\mathbb{H}} s$ or $m \models_{\mathbb{M}} s$

Note that by assumption $k_0\mathcal{Z}_{2\nu+1}m_0$. Moreover: $2.d_X(k_0)+1 \leq 2.\nu+1$. Hence: $k_0\mathcal{Z}_{2d_X(k_0)+1}m_0$. So we can take k_0, k_0, m_0 as witnessing triple for n_0. Let k', k, m' be a witnessing triple for $\langle \Delta, m\rangle$. Note that for $p \in \vec{p} \cap X$: $\Delta \models p \Leftrightarrow k \models p \Leftrightarrow m \models p$, and hence: $\langle \Delta, m\rangle \models p \Leftrightarrow \Delta \models p \Leftrightarrow m \models p$. It is easy to see that $\mathbb{N}$ is a $\mathsf{S4Grz}$-model (even if $\mathbb{H}_X$ need not be one). We claim:

Claim 1 $n_0 \simeq_{\vec{p},\mathcal{R}} m_0$.

Claim 2 For $B \in X : \langle \Delta, m\rangle \models B \Leftrightarrow B \in \Delta$.

Evidently the lemma is immediate from the claims.

We prove Claim 1. Take as bisimulation $\mathcal{B}$ with $\langle \Delta, m\rangle\mathcal{B}m$. Clearly, $\mathsf{Th}_{\vec{p},\mathcal{R}}(\langle \Delta, m\rangle) = \mathsf{Th}_{\vec{p},\mathcal{R}}(m)$. Moreover, $\mathcal{B}$ has the zig-property. We check that $\mathcal{B}$ has the zag-property. Suppose $\langle \Delta, m\rangle\mathcal{B}m \preceq n$. We are looking for a pair $\langle \Gamma, n\rangle$ in N such that $\Delta \preceq \Gamma$. In case $m = n$, we take $\langle \Gamma, n\rangle := \langle \Delta, m\rangle$. Suppose $m \neq n$ and, hence, $m \prec n$. Let k', k, m' be a witnessing triple for $\langle \Delta, m\rangle$. We write $\Delta' := \Delta(k')$. Since $k'\mathcal{Z}_{2.d_X(k')+1}m' \preceq n$, there is a h such that $k' \prec h\mathcal{Z}_{2.d_X(k')}n$. We take $\Gamma := \Delta(h)$. Clearly $\Delta \preceq \Gamma$. We need a witnessing triple k'^*, k^*, m'^* for $\langle \Gamma, n\rangle$ We distinguish two possibilities. First, $\Delta \approx \Gamma$. In this case we can take: $k'^* := k'$, $k^* := h$, $m'^* := m'$.

$$\begin{tikzcd}
\Gamma \arrow[rr, dash, dashed, "\Phi_X"] & & k^* = h \arrow[rr, dash, dashed, "\mathcal{Z}_{2d_X(k')}"] & & n \\
 & \Delta \arrow[rr, dash, "\Phi_X"] & & k \arrow[r, dash, "\mathcal{Z}_{2d_X(k')}"] & m \arrow[u, "\prec"'] \\
\Delta' \arrow[uu, dashed, "\approx"] \arrow[ur, "\approx"] \arrow[rr, dash, "\Phi_X"] & & k'^* = k' \arrow[uu, dashed, "\preceq"] \arrow[ur, "\preceq"] \arrow[rr, dash, "\mathcal{Z}_{2d_X(k')+1}"] & & m'^* = m' \arrow[u, "\preceq"']
\end{tikzcd}$$

Secondly, $\Delta \not\approx \Gamma$. In this case we can take: $k'^* := h$, $k^* := h$, $m'^* := n$. To see this, note that, since $k' \preceq h$, we have: $\Delta \approx \Delta' \preceq \Gamma$ and, hence, since $\Delta \not\approx \Gamma$, $\Delta \prec^+ \Gamma$. Ergo $d_X(h) < d_X(k')$. It follows that: $2.d_X(h) + 1 \leq 2.d_X(k')$. So, $h\mathcal{Z}_{2.d_X(k')+1}n$.

$$
\begin{tikzcd}
\Gamma \arrow[rr, dashed, dash, "\Phi_X"] & & k'^* = k^* = h \arrow[rr, dashed, dash, "\mathcal{Z}_{2d_X(h)+1}"] & & m'^* = n \\
& \Delta \arrow[rr, dash, "\Phi_X"] & & k \arrow[r, dash, "\mathcal{Z}_{2d_X(k')}"] & m \arrow[u, "\prec"'] \\
\Delta' \arrow[uu, dashed, "\prec^+"] \arrow[ur, "\approx"] \arrow[rr, dash, "\Phi_X"] & & k' \arrow[uu, dashed, "\preceq"] \arrow[ur, "\preceq"] \arrow[rr, dash, "\mathcal{Z}_{2d_X(k')+1}"] & & m' \arrow[u, "\preceq"']
\end{tikzcd}
$$

Finally, clearly, $n_0\mathcal{B}m_0$.

We prove Claim 2. The proof is by induction on X. The cases of atoms, conjunction and disjunction are trivial. We treat the only non-trivial case: the right-to-left case for the box. Consider $\Box C \in X$ and consider the node $\langle\Delta, m\rangle$ with witnessing triple k', k, m'. Suppose $\Box C \notin \Delta$.

In case $C \notin \Delta$, we have, by the Induction Hypothesis, $\langle\Delta, m\rangle \not\models C$ and, hence, $\langle\Delta, m\rangle \not\models \Box C$.

Suppose $C \in \Delta$. It follows that $\Box(C \to \Box C)$ is not in Δ, since, otherwise, $\Box C$ would be in Δ. Clearly, $k \not\models \Box C$, so there is an $h' \succeq k$ with $h' \not\models C$. Let h be maximal in $\mathbb{K}$ with $h \succeq k$ and $h \not\models C$. By maximality, we find: $h \models \Box(C \to \Box C)$. Let $\Gamma := \Delta(h)$. Since, $\Box(C \to \Box C) \notin \Delta$ and $\Box(C \to \Box C) \in \Gamma$, we find: $\Delta \prec^+ \Gamma$. Note that it follows that $d_X(k') \geq 1$. Since, $k\mathcal{Z}_{2.d_X(k')}m$ and $k \preceq h$, there is an $n \succeq m$ with $h\mathcal{Z}_{2.d_X(k')-1}n$. Moreover: $2.d_X(h) + 1 \leq 2.d_X(k') - 1$. Ergo: $h\mathcal{Z}_{2.d_X(h)+1}n$. So we can take $k'^* := h$, $k^* := h$, $m'^* := n$ to witness $\langle\Gamma, n\rangle$. Clearly, $\langle\Delta, m\rangle \preceq \langle\Gamma, n\rangle$. By the Induction Hypothesis: $\langle\Gamma, n\rangle \not\models C$. Hence, $\langle\Delta, m\rangle \not\models \Box C$.

$$
\begin{tikzcd}
& \Gamma \arrow[rr, dash, "\Phi_X"] & & k'^* = k^* = h \arrow[r, dashed, dash, "\mathcal{Z}_{2d_X(h)+1}"] & m'^* = n \\
& \Delta \arrow[u, "\prec^+"] \arrow[rr, dash, "\Phi_X"] & & k \arrow[u, "\preceq"] \arrow[r, dash, "\mathcal{Z}_{2d_X(k')}"] & m \arrow[u, dashed, "\preceq"] \\
\Delta \arrow[ur, "\approx"] \arrow[rr, dash, "\Phi_X"] & & k' \arrow[ur, "\preceq"] \arrow[rr, dash, "\mathcal{Z}_{2d_X(k')+1}"] & & m' \arrow[u, "\preceq"]
\end{tikzcd}
$$

Thus we have proved Claim 2. ❑

The statement of uniform interpolation and the semantical interpretation of the propositional quantifiers are fully analogous to the case of GL.

We show that Uniform Interpolation for $\mathsf{S4Grz}$ implies Uniform Interpolation for IPC. By itself this is not so important, since we proved Uniform Interpolation for IPC directly. I feel, however, that the methodology of such transfers is interesting by itself.

Define $\mathsf{Nec}(A) := \bigwedge\{\Box(p \to \Box p) \mid p \in \mathsf{PV}(A)\}$. The Gödel Translation $(.)^*$ from $\mathcal{L}^i$ to $\mathcal{L}^m$ is specified as follows.

- $(.)^*$ commutes with atoms, $\wedge$ and $\vee$
- $(A \to B)^* := \Box(A^* \to B^*)$

Lemma 15.2 1. $\mathsf{IPC} \vdash A \Leftrightarrow \mathsf{S4Grz} \vdash \mathsf{Nec}(A) \to A^*$.

2. $\mathsf{S4Grz} \vdash (\mathsf{Nec}(A) \wedge A) \to \Box A \Rightarrow$ for some $A^i \in \mathcal{L}^i$, $\mathsf{S4Grz} \vdash \mathsf{Nec}(A) \to (A \leftrightarrow A^{i*})$.

Proof

(1) and (2) are a well know facts. (1) is due to Gödel. (2) is probably first due to Rybakov. We prove (2). The proof is by induction on the length of A. Suppose $\mathsf{S4Grz} \vdash (\mathsf{Nec}(A) \wedge A) \to \Box A$. We rewrite A to conjunctive normal form treating the boxed formulas as atoms. Schematically, this form is: $\bigwedge\{\bigvee\{\Box B, \neg\Box C, p, \neg q\}\}$. We find, in $\mathsf{S4Grz} + \mathsf{Nec}(A)$:

$$\begin{aligned} A &\leftrightarrow \bigwedge\{\bigvee\{\Box B, \neg\Box C, p, \neg q\}\} \\ &\leftrightarrow \Box\bigwedge\{\bigvee\{\Box B, \neg\Box C, p, \neg q\}\} \\ &\leftrightarrow \bigwedge\{\Box\bigvee\{\Box B, \neg\Box C, p, \neg q\}\} \\ &\leftrightarrow \bigwedge\{\Box(\bigwedge\{\Box C, q\} \to \bigvee\{\Box B, p\})\} \\ &\leftrightarrow \bigwedge\{\Box(\bigwedge\{(\Box C)^{i*}, q\} \to \bigvee\{(\Box B)^{i*}, p\})\} \end{aligned}$$

So we can take $A^i := \bigwedge\{(\bigwedge\{(\Box C)^i, q\} \to \bigvee\{(\Box B)^i, p\})\}$. ❑

Theorem 15.3 *Uniform Interpolation for* $\mathsf{S4Grz}$ *implies Uniform Interpolation for* IPC

Proof

Consider A in $\mathcal{L}^i$. Let $\vec{q}$ be some subset of $\mathsf{PV}(A)$. Let $\tilde{A}$ be the post-interpolant w.r.t. $\vec{q}$ of $\mathsf{Nec}(A) \wedge A^*$ in $\mathsf{S4Grz}$. Note that: $\mathsf{S4Grz} \vdash (\mathsf{Nec}(A) \wedge A^*) \to \Box\tilde{A}$. Hence, by the properties of the post-interpolant: $\mathsf{S4Grz} \vdash \tilde{A} \to \Box\tilde{A}$. Thus, we can find an $\mathcal{L}^i$-formula $\tilde{A}^i$, such that $\mathsf{S4Grz} \vdash \mathsf{Nec}(\tilde{A}) \to (\tilde{A} \leftrightarrow \tilde{A}^{i*})$. We show that $\tilde{A}^i$ is the desired post-interpolant. Note that, $\mathsf{S4Grz} \vdash (\mathsf{Nec}(A) \wedge A^*) \to \tilde{A}^{i*}$. We may conclude: $\mathsf{IPC} \vdash A \to \tilde{A}^i$.

Suppose $\mathsf{IPC} \vdash A \to B$, where the shared variables of A and B are in $\vec{q}$. It follows that: $\mathsf{S4Grz} \vdash \mathsf{Nec}(A \to B) \to (A^* \to B^*)$. Hence, $\mathsf{S4Grz} \vdash (\mathsf{Nec}(A) \wedge A^*) \to (\mathsf{Nec}(B) \to B^*)$. Thus: $\mathsf{S4Grz} \vdash \tilde{A}^{i*} \to (\mathsf{Nec}(B) \to B^*)$. And so, $\mathsf{S4Grz} \vdash (\mathsf{Nec}(\tilde{A}^i \to B) \wedge \tilde{A}^{i*}) \to B^*$. Ergo, $\mathsf{IPC} \vdash \tilde{A}^i \to B$.

We turn to pre-interpolants. Consider B in $\mathcal{L}^i$. Let $\vec{q}$ be some subset of $\mathsf{PV}(B)$. Let B' be the pre-interpolant w.r.t. $\vec{q}$ of $\mathsf{Nec}(B) \to B^*$ in $\mathsf{S4Grz}$. Take $\breve{B} := \Box B'$. We can find an $\mathcal{L}^i$-formula $\breve{B}^i$, such that $\mathsf{S4Grz} \vdash \mathsf{Nec}(\breve{B}) \to (\breve{B} \leftrightarrow \breve{B}^{i*})$. We show that $\breve{B}^i$ is the desired pre-interpolant. Note that, $\mathsf{S4Grz} \vdash (\mathsf{Nec}(B) \wedge \breve{B}^{i*}) \to B^*$. We may conclude: $\mathsf{IPC} \vdash \breve{B}^i \to B$.

Suppose $\mathsf{IPC} \vdash A \to B$, where the shared variables of A and B are in $\vec{q}$. It follows that: $\mathsf{S4Grz} \vdash \mathsf{Nec}(A \to B) \to (A^* \to B^*)$. Hence, $\mathsf{S4Grz} \vdash (\mathsf{Nec}(A) \wedge A^*) \to (\mathsf{Nec}(B) \to B^*)$. Thus: $\mathsf{S4Grz} \vdash (\mathsf{Nec}(A) \wedge A^*) \to B'$. And so, $\mathsf{S4Grz} \vdash (\mathsf{Nec}(A) \wedge A^*) \to \breve{B}$ (since $(\mathsf{Nec}(A) \wedge A^*)$ is self-necessitating). So, finally, $\mathsf{S4Grz} \vdash (\mathsf{Nec}(A) \wedge A^*) \to \breve{B}^{i*}$. Ergo, $\mathsf{IPC} \vdash A \to \breve{B}^i$. ❑

It would be interesting to find a similar argument to prove Uniform Interpolation for $\mathsf{S4Grz}$ from Uniform Interpolation for GL.

16 S4 does not have Uniform Interpolation

In their paper [5] Ghilardi and Zawadowski show that S4 does not satisfy uniform interpolation. We provide a version of the proof. In this section we use $\vdash$ for S4-provability. The models we consider will be S4-models.

Theorem 16.1 *1. The uniform interpolants —if they exist— are semantically quantifiers w.r.t. bisimulation extension for* finite *models. Consider, e.g., a formula $A(\vec{p},\vec{q})$. Suppose A has a uniform post-interpolant $\tilde{A}(\vec{q})$ for $\vec{q}$. Then, for any* finite *$\vec{q}$-model $\mathbb{K}$ and $k \in \mathbb{K}$, $k \models \tilde{A}$ iff there is a $\vec{p},\vec{q}$-model $\mathbb{M}$ and an $m \in \mathbb{M}$ such that $k \simeq_{[\vec{p}]} m$ and $m \models A$. (We can, but need not, restrict the extending models to finite models).*

2. Suppose S4 *satisfies Uniform Interpolation. The uniform interpolants are semantically quantifiers w.r.t. bisimulation extension. Consider, e.g., a formula $A(\vec{p},\vec{q})$. Let the uniform post-interpolant of A for $\vec{q}$ be $\tilde{A}(\vec{q})$. Then, for any $\vec{q}$-model $\mathbb{K}$ and $k \in \mathbb{K}$, $k \models \tilde{A}$ iff there is a $\vec{p},\vec{q}$-model $\mathbb{M}$ and an $m \in \mathbb{M}$ such that $k \simeq_{[\vec{p}]} m$ and $m \models A$.*

Proof

We prove (1) and (2) simultaneously, plugging in the extra assumptions of finiteness and full Uniform Interpolation, where needed. We treat the case of the post-interpolant. Consider A, $\tilde{A}$, a uniform post-interpolant of A for $\vec{q}$, $\mathbb{K}$ and k. Suppose first that there is a $\vec{p},\vec{q}$-model $\mathbb{M}$ and an $m \in \mathbb{M}$ such that $k \simeq_{[\vec{p}]} m$ and $m \models A$. Clearly, since $A \vdash \tilde{A}$, we find: $m \models \tilde{A}$. Since $\tilde{A} \in \mathcal{L}^m(\vec{q})$ and since k and m bisimulate w.r.t. $\mathcal{L}^m(\vec{q})$, we have: $k \models \tilde{A}$.

For the converse, suppose $k \models \tilde{A}$. Let X be the set of subformulas of A. Let, for $k' \in \mathbb{K}$, $\Delta(k') := \{B{\in}\mathcal{L}^m(\vec{q}) \mid k' \models B\}$. We say that Γ' of X, k'*-saturated* if

1. $\Gamma' \subseteq X$
2. $\Delta(k'), \Gamma' \nvdash \bot$
3. $(\Delta(k'), \Gamma' \vdash B$ and $B \in X) \Rightarrow B \in \Gamma'$
4. $(\Delta(k'), \Gamma' \vdash (B{\vee}C)$ and $B, C \in X) \Rightarrow (B \in \Gamma'$ or $C \in \Gamma')$.

We specify $\mathbb{M}$.

- $M := \{\langle k', \Gamma'\rangle \mid \Gamma'$ is X, k'-saturated$\}$
- $\langle k', \Gamma'\rangle \preceq \langle k'', \Gamma''\rangle :\Leftrightarrow k' \preceq k''$ and for all $\Box B \in \Gamma'$, we have: $\Box B \in \Gamma''$
- $\langle k', \Gamma'\rangle \models r :\Leftrightarrow k \models r$ or $r \in \Gamma'$

Clearly $\mathbb{M}$ is an S4-model, assuming that M is non-empty. We show that there is a Γ, consistent with $\Delta(k)$, such that $A \in \Gamma$. Consider $\Delta(k), A$. If this set were inconsistent there would be a D in $\Delta(k)$ (and, hence, in $\mathcal{L}^m(\vec{q})$), such that $A \vdash \neg D$. It follows, by the properties of the post-interpolant, that $\tilde{A} \vdash \neg D$. But, this is impossible, since $k \models \tilde{A}$. By the usual methods we may extend $\{A\}$ to an X, k-saturated Γ. Thus we have a node $m := \langle k, \Gamma\rangle$ in M. Note that if $\mathbb{K}$ is finite, then so is $\mathbb{M}$.

Define $k'\mathcal{B}\langle k'', \Gamma''\rangle :\Leftrightarrow k' = k''$. We show that $\mathcal{B}$ is a bisimulation between $\mathbb{K}$ and $\mathbb{M}(\vec{q})$. It is easy to see that $\mathcal{B}$ preserves the forcing on $\vec{q}$ and satisfies the zag-property. We verify the zig-property. Suppose $k'\mathcal{B}\langle k', \Gamma'\rangle$ and $k' \preceq k''$. We claim that $\Delta(k'') \cup \{\Box B{\in}X \mid \Box B \in \Gamma'\}$ is consistent. If it were not, we could find a D in $\Delta(k'')$ such that $D, \{\Box B{\in}X \mid \Box B \in \Gamma'\}$ is inconsistent. Clearly, $\Diamond D$ must be in $\Delta(k')$, and, hence, it would follow that $\Delta(k'), \Gamma'$ is inconsistent. Quod non. By the usual methods we can extend $\{\Box B{\in}X \mid \Box B \in \Gamma'\}$ to an X, k''-saturated Γ''. Thus we find: $k''\mathcal{B}\langle k'', \Gamma''\rangle$ and $\langle k', \Gamma'\rangle \preceq \langle k'', \Gamma''\rangle$.

Finally we show that for all C in X: $\langle k', \Gamma' \rangle \models C \Leftrightarrow C \in \Gamma'$. The proof is by induction on C. We treat the only non-tivial case, viz., $C = \Box E$ from left to right. Suppose, for all $\langle k'', \Gamma'' \rangle \succeq \langle k', \Gamma' \rangle$, we have $\langle k'', \Gamma'' \rangle \models E$. By the usual arguments, it follows that:

$$\text{for all } k'' \succeq k' : \ \Delta(k'') \cup \{\Box B \in X \mid \Box B \in \Gamma'\} \vdash E.$$

Let $F := (\bigwedge\{\Box B \in X \mid \Box B \in \Gamma'\} \to E)$. We show that there is a $\vec{q}$-formula $\breve{F}$, such that for all $k'' \succeq k'$ we have $\Delta(k'') \vdash \breve{F}$ and $\breve{F} \vdash F$. Here we split cases between (1) and (2) of the theorem. Ad (1): suppose $\mathbb{K}$ is finite. By compactness, for each $k'' \succeq k'$ there is a $D(k'') \in \Delta(k'')$ such that $D(k'') \vdash F$. We can take $\breve{F} := \bigvee\{D(k'') \mid k'' \succeq k'\}$. Ad (2): suppose $\mathsf{S4}$ has full Uniform Interpolation. Then, F has a pre-interpolant (w.r.t. $\vec{q}$). We take this pre-interpolant as $\breve{F}$. At this point the proofs of (1) and (2) merge again. We have: $\breve{F}, \{\Box B \in X \mid \Box B \in \Gamma'\} \vdash E$. It follows that $k' \models \Box\breve{F}$, and, hence, that $\Delta(k'), \Gamma' \vdash \Box E$. Ergo, $\Box E \in \Gamma$.

Thus, we may conclude: $k \simeq_{\vec{p}} \langle k, \Gamma \rangle$ and $\langle k, \Gamma \rangle \models A$. ❑

Exercise 16.2 Prove the converse of part (2) of our theorem: if $\mathsf{S4}$ has quantifier elimination for the bisimulation extension quantifiers, then $\mathsf{S4}$ has uniform interpolation. ❑

Consider the following formula $A(p, q, r)$:

$$p \wedge \Box(p \to \Diamond q) \wedge \Box(q \to \Diamond p) \wedge \Box(p \to r) \wedge \Box(q \to \neg r)$$

Suppose $\tilde{A}(r) := \exists p \exists q. A(p, q, r)$ exists. We have:

Theorem 16.3 *Consider any finite r-model $\mathbb{K}$ and k in $\mathbb{K}$. Then: $k \models \tilde{A}$ iff there is infinite sequence $k = k_0 \preceq k_1 \preceq \ldots$, such that $k_{2i} \models r$ and $k_{2i+1} \models \neg r$. (Our sequence will contain loops, so the underlying* set *of the k_i need not be infinite.)*

Proof

Suppose $k \models \tilde{A}$. Then, by theorem 16.1, there is a p, q, r-model $\mathbb{M}$ and a node m in $\mathbb{M}$ such that: $k \simeq_{[p,q]} m \models A$. We construct simultaneously sequences $(k_i)_{i \in \omega}$ and $(m_i)_{i \in \omega}$ such that: $m = m_0$, $k_i \simeq_{[p,q]} m_i$, $m_{2i} \models p$, $m_{2i+1} \models q$, $k_i \preceq k_{i+1}$, $m_i \preceq m_{i+1}$. Note that we get, $m \preceq m_i$, and so: $m_i \models (p \to r)$ and $m_i \models (q \to \neg r)$. Hence $m_{2i} \models r$ and so $k_{2i} \models r$ (as promised). Similarly, $m_{2i+1} \models \neg r$ and so $k_{2i+1} \models \neg r$. First note that $k_0 = k \simeq_{[p,q]} m = m_0 \models p$. Suppose we have, e.g., constructed k_{2i}, m_{2i} satisfying the conditions. We have $m \preceq m_{2i}$ and, hence, $m_{2i} \models (p \to \Diamond q)$. Since, by assumption, $m_{2i} \models p$, there is an $m_{2i+1} \succeq m_{2i}$, such that $m_{2i+1} \models q$. Moreover, since $k_{2i} \simeq m_{2i}$, we can find a $k_{2i+1} \succeq k_{2i}$, such that $k_{2i+1} \simeq_{[p,q]} m_{2i+1}$. The $(2i+1)$-case is similar.

For the converse, suppose $\mathbb{K}$ contains an infinite sequence $k = k_0 \preceq k_1 \preceq \ldots$, such that $k_{2i} \models r$ and $k_{2i+1} \models \neg r$. We extend the forcing of $\mathbb{K}$ with p, q as follows:

- $k' \models p :\Leftrightarrow k' = k_{2i}$ for some i.
- $k' \models q :\Leftrightarrow k' = k_{2i+1}$ for some i.

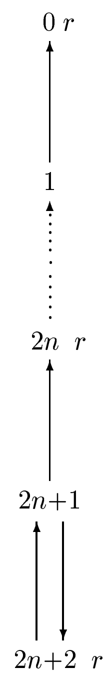
It is easy to see that the specified model is the desired extension. ❑

Theorem 16.4 *There is no formula $B(r) \in \mathcal{R}^m(r)$, such that for all finite models $\mathbb{K}$ and all k in $\mathbb{K}$: $k \models B$ iff there is infinite sequence $k = k_0 \preceq k_1 \preceq \ldots$, such that $k_{2i} \models r$ and $k_{2i+1} \models \neg r$.*

Proof

Suppose there is such a B. Suppose the box-depth of B is smaller or equal than $2n$. Consider the model following model $\mathbb{N}$:

- $N := \{0, 1, \ldots, 2i, 2n+1, 2n+2\}$
- $k \preceq k' :\Leftrightarrow k \geq k'$ or $k \in \{2i+1, 2i+2\}$
- $k \models r :\Leftrightarrow k$ is even



Define further: $k\,\mathcal{B}_i\,k' :\Leftrightarrow k = k'$ or $(k \equiv k' \pmod 2$ and $i \leq \min(k, k'))$. We check that $\mathcal{B}$ is a layered bisimulation. Suppose $k_1\,\mathcal{B}_i\,k_2$. In case $k_1 = k_2$ we are easily done, so suppose, e.g. $k_1 < k_2$. It follows that $i \leq k_1$. Clearly, k_1 cannot be in $\{2n+1, 2n+2\}$ and, so, $k_1 \succ k_2$. Preservation of atoms is immediate. Suppose $i > 0$. We check the zig-property. If $k_1' \succeq k_1$, then $k_1' \succeq k_2$. So we can finish the zig-diagram by taking $k_2' := k_1'$. We check the zag-property. Suppose $k_2' \succeq k_2$. In case $k_2' \leq k_1$, we can take $k_1' := k_2'$. Suppose $k_2' > k_1$. In this case we take $k_1' := k_1$ if $k_1 \equiv k_2' \pmod 2$ and $k_1' := k_1 - 1$ if $k_1 \not\equiv k_2' \pmod 2$. (Remember that $0 < i \leq k_1$). It is easy to see that $i - 1 \leq k_1 - 1 = k_1' = \min(k_1', k_2')$.

It follows that $2n\,\mathcal{B}_{2n}\,2n+2$. From our assumption we get that $2n+2 \models B$. It follows that $2n \models B$. A contradiction. ❑

We may conclude that S4 does not have uniform interpolation.

Open Question 16.5 In [1] it is shown that in theories having Uniform Interpolation every finite substitution yields an exact formula. Is there a finite substitution in S4, that does not yield an exact formula? ❑

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References

[1] D.H.J. de Jongh and A. Visser. *Embeddings of Heyting algebras*. Logic Group Preprint Series 97. Department of Philosophy, Utrecht University, Utrecht, 1993.

[2] R. Dyckhoff. Contraction-free sequent calculi for intuitionistic logic. *Journal of Symbolic Logic*, 57:795–807, 1992.

[3] D.M. Gabbay. *Semantical investigations in Heyting's intuitionistic logic*. Synthese Library, Studies in Epistemology, Logic, Methodology and the Philosophy of Science, vol. 148. Reidel, Dordrecht, 1981.

[4] S. Ghilardi and M. Zawadowski. A sheaf representation and duality for finitely presented Heyting algebras. *Journal of Symbolic Logic*, 60:911–939, 1995.

[5] S. Ghilardi and M. Zawadowski. Undefinability of Propositional Quantifiers in the Modal System S4. *Studia Logica*, 55:259–271, 1995.

[6] Z. Gleit and W. Goldfarb. Characters and Fixed Points in Provability Logic. *Notre Dame Journal of Formal Logic*, 31:26–36, 1990.

[7] R. Goldblatt. Saturation and the Hennessy-Milner Property. In A. Ponse, M. de Rijke, and Y. Venema, editors, *Modal Logic and Process Algebra, a Bisimulation Perspective*, CSLI Lecture Notes, no. 53, pages 107–129. Center for the Study of Language and Information, Stanford, 1995.

[8] W. Hodges. *Model theory*. Encyclopedia of Mathematics and its Applications, vol. 42. Cambridge University Press, Cambridge, 1993.

[9] M. Hollenberg. Hennessy-Milner Classes and Process Algebra. In A. Ponse, M. de Rijke, and Y. Venema, editors, *Modal Logic and Process Algebra, a Bisimulation Perspective*, CSLI Lecture Notes, no. 53, pages 187–216. Center for the Study of Language and Information, Stanford, 1995.

[10] J. Hudelmaier. *Bounds for cut elimination in intuitionistic propositional logic*. Ph.D. Thesis. University of Tübingen, Tübingen, 1989.

[11] G. Kreisel. Monadic operators defined by means of propositional quantification in intuitionistic logic. *Reports on Mathematical Logic*, 12:9–15, 1981.

[12] P. Kremer. On the complexity of propositional quantification in intuitionistic propositional logic, ??

[13] M.H. Löb. Embedding First Order Predicate Logic in Fragments of Intuitionistic Logic. *Journal of Symbolic Logic*, 41:705–718, 1976.

[14] A. Pitts. On an interpretation of second order quantification in first order intuitionistic propositional logic. *Journal of Symbolic Logic*, 57:33–52, 1992.

[15] T. Połacik. Pitts' quantifiers are not the topological interpretation, ??

[16] T. Połacik. Operators defined by propositional quantification and their interpretation over Cantor space. *Reports on Mathematical Logic*, 27:67–79, 1993.

[17] T. Połacik. Second order propositional operators over Cantor space. *Studia Logica*, 53:93–105, 1993.

[18] A. Ponse, M. de Rijke, and Y. Venema, editors. *Modal Logic and Process Algebra, a Bisimulation Perspective.* CSLI Lecture Notes, no. 53. Center for the Study of Language and Information, Stanford, 1995.

[19] V.V. Rybakov. Rules of inference with parameters for intuitionistic logic. *Journal of Symbolic Logic*, 57:912–923, 1992.

[20] V. Yu. Shavrukov. *Subalgebras of diagonalizable algebras of theories containing arithmetic.* Dissertationes Mathematicae CCCXXIII. Polska Akademia Nauk, Mathematical Institute, Warszawa, 1993.

[21] C. Smoryński. The undecidability of the 2nd Order Intuitionistic Propositional Calculus, ??

[22] R. Statman. Intuitionistic propositional logic is polynomial-space complete. *Theoretical Computer Science*, 9:67–72, 1979.

[23] A. Tarski. Der aussagenkalkül und die topologie. *Fundamenta Mathematicae*, 31:103–134, 1938.

[24] A. Troelstra. On a second order propositional operator in intuitionistic logic. *Studia Logica*, 40:113–139, 1981.

[25] A. Visser, J. van Benthem, D. de Jongh, and G. Renardel de Lavalette. NNIL, a Study in Intuitionistic Propositional Logic. In A. Ponse, M. de Rijke, and Y. Venema, editors, *Modal Logic and Process Algebra, a Bisimulation Perspective*, CSLI Lecture Notes, no. 53, pages 289–326. Center for the Study of Language and Information, Stanford, 1995.