An Axiomatisation of Strong Negation and Relational Composition

Marco Hollenberg

Utrecht University, Department of Philosophy

Heidelberglaan 8, 3584 CS Utrecht, the Netherlands

hollenb@phil.ruu.nl

http://www.phil.ruu.nl/home/marco/

1 Introduction

In the tradition of dynamic semantics in the style of Dynamic Predicate Logic (DPL, [4]), formulas of standard logics are interpreted in a non-standard way, not on assignments, but on *pairs* of assignments. Formulas are thus no longer viewed as sets of assignments, but as relations on assignments. The four main ingredients of this approach are the following. First there are the constants $P(x_1, \ldots, x_n)$ and $\exists x$. The first is a *test*: it defines a subset of the diagonal identity relation, containing all pairs of assignments (f, f) for which $\langle f(x_1), \ldots, f(x_n)\rangle$ is in the interpretation of P. $\exists x$ is not a test: it holds between assignments f and g if f and g disagree on at most x. Next, we have two operators on relations, $\sim$ (strong negation) and ; (composition). The unary $\sim$ yields again a test. Given a relation R, $\sim R$ holds between two assignments f and g iff $f = g$ and f is not in the domain of R. Negation in DPL is interpreted by $\sim$. The binary operator ; is simply relational composition. Given two relations R and R', $R;R'$ holds between two assignments f and g iff there is a third, h, such that fRh and $hR'g$. It is used as the interpretation of conjunction in DPL.

In this paper, we study the operations $\sim$ and ; on their own, as operations on arbitrary relations, not just on relations between assignments. The *empty* relation, denoted by $\bot$, is definable in this fragment as $(\sim x); x$. Nevertheless, for expository's sake, we include $\bot$ in our signature.

Besides the fact that this is a natural fragment of DPL, much in the same way as propositional logic is a natural fragment of predicate logic, there is an additional motivation for studying it. Consider algebraic terms t_1 and t_2, constructed using $\bot$, $\sim$ and ;. Then t_1 and t_2 are equal for any possible uniform choice of relations assigned to the variables in these terms iff they are always equal whenever we substitute the variables in t_1 and t_2 uniformly to interpretations of DPL-formulas. In other words, equality in the general case corresponds exactly to *schematic equality* in DPL. This interesting result can be derived from the results in [10].

This paper will axiomatise these equalities, in an algebraic way, i.e. by giving a finite set of axioms, from which all other valid equalities can be derived simply by means of equational logic. Its main tools are from polymodal logic, namely the completeness theorem for minimal modal logic and an unravelling technique.

An important preliminary for the result in the present paper can be found in [2]. The main objective of the latter paper is to study the logic of *dynamic implication* $\Rightarrow$, which in our system can be defined by means of the equation $t \Rightarrow t' := \sim(t; \sim t')$. The first part of [2] studies the logic of $\Rightarrow$ in combination with the booleans, here interpreted as the usual set-theoretic operations on relations. An axiomatisation is given of what in the present setting corresponds to $\models t = \top$ (i.e. the interpretation of t is always the full square $S \times S$, when we are considering relations on S). This is an interesting and beautiful result, which establishes a connection between the static tradition (the booleans) and the dynamic one ($\Rightarrow$). This very strength can also be viewed as a weakness, as it does perhaps not do much justice to DPL, where $\sim$ and ; were presented as *alternatives* to the booleans. The second part of [2] addresses this issue and studies a proper fragment of our system, namely the $\{\Rightarrow, \bot\}$-fragment. $\sim t$ can be defined in this setting as $t \Rightarrow \bot$, but composition cannot be defined, hence this is a proper fragment of $\{\bot, \sim, ;\}$. A tableau system is presented, axiomatising for which t (in this restricted language) $\models t = \text{ID}$ holds, where ID is interpreted as the identity relation and defined as $\sim \bot$. The present paper may be viewed as an attempt to extend the results of [2] and to axiomatise the fragment where the important ; is also present.

2 Preliminary definitions

Let $\mathcal{T}$ be the set of *terms* constructed from an infinite set $\mathcal{V}$ of variables, a constant $\bot$, the unary operation $\sim$ and the binary ;. The binding strength of $\sim$ is assumed greater than ;. We have already seen some abbreviations of terms. We repeat these here and add an extra one:

$$\begin{array}{lcl} \text{ID} & := & \sim \bot \\ t \Rightarrow t' & := & \sim (t; \sim t') \\ t \vee t' & := & \sim (\sim t; \sim t') \end{array}$$

Both $\Rightarrow$ and $\vee$ are discussed in [4] as natural interpretations of implication and disjunction in natural language, the main motivation for dynamic semantics. Note that $t \vee t' = (\sim t) \Rightarrow t'$.

A **dynamic relation algebra** is an algebra for the signature $\{\bot, \sim, ;\}$, of the following form. Its domain is the powerset of a *square* (see [9] for this terminology). That is, it is of the form $\wp(S \times S)$ for some set S. The elements are thus binary relations on S. $\bot$ is interpreted as the empty relation $\emptyset$, $\sim$ as:

$$\sim R := \{(s, s) \mid s \in S \text{ and } \neg\exists t.(sRt)\}$$

and ; simply as relational composition. So a dynamic relation algebra is determined completely by a set S. The class of all dynamic relation algebras is denoted by DRA. We write $\mathsf{DRA} \models t_1 = t_2$, or just $\models t_1 = t_2$, when the equation is valid (under all assignments to variables) in all dynamic relation algebras.

The reader is warned that our definition of a dynamic relation algebra is not what is commonly referred to as a 'relation algebra' in the literature (originating in [8]): these more standard definitions also include the booleans (relation algebras are thus boolean algebras) and a *converse* operator. Strong negation can be defined in ordinary relation algebra by means of the equation:[1]

$$\sim R := \text{ID} \wedge \neg(R; \top)$$

(The identity relation *is* present in ordinary relation algebra.) Our quest can thus also be situated within the field of relation algebra: we study a small non-boolean fragment.

Let us consider what relations are denoted by our abbreviations. We have already remarked that ID is interpreted as $\{(s, s), \mid s \in S\}$, the identity-relation on S. $\Rightarrow$ and $\vee$ are more interesting:

- $R_1 \Rightarrow R_2 := \{(s, s) \mid s \in S \text{ and } \forall u(sR_1u \rightarrow \exists v.uR_2v)\}$. In words, $R_1 \Rightarrow R_2$ is a test, that succeeds only at states such that after every execution of R_1 from this state, we are able to continue with an R_2-execution.
- $R_1 \vee R_2 := \{(s, s) \mid s \in S \text{ and } \exists u.(sR_1u) \vee \exists v.(sR_2v)\}$. Thus $R_1 \vee R_2$ is again a test, that succeeds when we can proceed with either an R_1- or an R_2-step.

Table 1 contains a finite set of axioms, **AX**, that is intended to completely axiomatise equational validity in dynamic relation algebras. We write $\vdash t_1 = t_2$ if this equation is derivable from the equations in **AX** and the rules of equational logic. The main theorem of this paper will state that $\models t_1 = t_2$ iff $\vdash t_1 = t_2$.

[1] This also follows from a folklore result (attributed to Tarski), which states that any first order definable operation on binary relations that only uses at most three variables (free or bound) can already be expressed in relation algebra.

A0:	$\sim x; x = \bot$	(zero definition)
A1:	$x; \bot = \bot$	(zero right)
A2:	$\bot; x = \bot$	(zero left)
A3:	$\text{ID}; x = x$	(identity left)
A4:	$x; (y; z) = (x; y); z$	(associativity)
A5:	$\sim x; \sim y = \sim y; \sim x$	(test permutation)
A6:	$x = (\sim\sim x); x$	(domain test)
A7:	$\sim\sim (\sim x; \sim y) = \sim x; \sim y$	(test composition)
A8:	$\sim (x; y); x = (\sim (x; y); x); \sim y$	(modus ponens)
A9:	$\sim (x; (y \vee z)) = \sim ((x; y) \vee (x; z))$	(distribution)

Table 1: **AX.**

The result suggests that the dynamic way of looking at the step from propositional logic to predicate logic behaves better than in the static view. In the static view, first order logic is arrived at, algebraically, by moving from boolean algebras, via relation algebras, to cylindric algebras. In the dynamic camp this same sequence of steps would be as follows: we start again from boolean algebras, as an intermediate step we get dynamic relation algebras (that this is an important intermediate step, comparable to the role of relation algebras in the static view, follows from the results in [10]), and finally standard algebras for DPL (that is, algebras over relations between assignments, with standard interpretations for the DPL-constants). The advantage of the dynamic view here is that the intermediate step is a class of algebras that is finitely axiomatisable (the contribution of this paper), while that of the static view is not (cf. [7]).

Remark.
Some more needs to be said about the first step above, from boolean algebras to the intermediate setting. In the static view, this step is clear: relation algebras *are* boolean algebras, in an expanded signature. The step from boolean algebras to dynamic relation algebras is of a different nature. Roughly, boolean algebras live inside relation algebras, by means of an embedding. To be precise, any powerset boolean algebra $(\wp(S), -, \cap, \emptyset)$ can be embedded into the dynamic relation algebra $(\wp(S \times S), \sim, ;, \emptyset)$ via the function $\iota : A \mapsto \{(s, s) \mid s \in A\}$. Applying strong negation to any $R \subseteq S \times S$ gives you $\sim R = \iota(S - dom(R))$, i.e. an element in the range of ι. This implies that when $t_1 = t_2$ is a valid equation in the class of boolean algebras (in the signature given above), then $t_1^\circ = t_2^\circ$ must be a valid equation in DRA, where the translation $\circ$ is given by:

$$\begin{array}{lcl} x^\circ & := & \sim x \\ 0^\circ & := & \bot \\ (-t)^\circ & := & \sim t^\circ \\ (t_1 \cap t_2)^\circ & := & t_1^\circ; t_2^\circ \end{array}$$

This immediately gives us the axioms **A5** (via the valid boolean law $x \cap y = y \cap x$) and **A7** (via $- - (x \cap y) = x \cap y$). □

It is trivial to verify that **AX** is sound for DRA. Let us dwell on the axioms a while longer.

- **A0** is obviously valid. Order is important here, as $R; \sim R$ need *not* always be the empty relation.
- A *test* is any subset of the identity relation. Thus for any R, $\sim R$ is a test. Tests can also be characterised as those relations R for which $R = \sim\sim R$. $\sim\sim R$ tests whether an element is in the domain of R: it contains exactly those pairs (s, s) for which $s \in dom(R)$.

  Quite a few of our axioms are about the behaviour of tests:

  - **A5** says that tests permute.

- **A6** says that to do an x-step from s to t is the same as first *checking* (or testing) that s is in the domain of x and *then* taking the x-step to t.
- **A7** states that composition of two tests again yields a test. This axiom can also be viewed as the familiar De Morgan law: when we apply our definition for dynamic disjunction, it states that $\sim (x \vee y) = \sim x; \sim y$.
- Note that $dom(R_1 \vee R_2)$ is simply $dom(R_1 \cup R_2)$. So **A9** is actually an embodiment of the fact that $R_0; (R_1 \cup R_2) = (R_0; R_1) \cup (R_0; R_2)$ is always the case.

- **A8** is called 'modus ponens' because of the following instance of it, which will be its main use: $(x \Rightarrow y); x = (x \Rightarrow y); x; \sim\sim y$.

We list a few useful consequences of **AX**. In the proofs we treat ; as an associative operation, which is justified by **A4**. These derivable laws are given names, so that we may refer to them later.

- Identity right: $x; \textsc{id} = x$.

$$\begin{array}{rcll} x; \textsc{id} & = & \textsc{id}; x; \textsc{id} & (\textbf{A3}) \\ & = & \sim (x; \bot); x; \sim \bot & (\textbf{A1}) \\ & = & \sim (x; \bot); x & (\textbf{A8}) \\ & = & \textsc{id}; x & (\textbf{A1}) \\ & = & x & (\textbf{A3}) \end{array}$$

- Triple negation law: $\sim\sim\sim x = \sim x$.

$$\begin{array}{rcll} \sim\sim\sim x & = & \sim\sim (\sim \bot; \sim x) & (\textbf{A3}) \\ & = & \sim \bot; \sim x & (\textbf{A7}) \\ & = & \sim x & (\textbf{A3}) \end{array}$$

- Test idempotency: $\sim x; \sim x = \sim x$.

$$\begin{array}{rcll} \sim x; \sim x & = & \sim\sim\sim x; \sim x & (\text{triple negation}) \\ & = & \sim x & (\textbf{A6}) \end{array}$$

  So $x \vee x = \sim\sim x$ is provable, as $x \vee x$ is defined as $\sim (\sim x; \sim x)$.

- Range test: $\sim (x; y) = \sim (x; \sim\sim y)$.

$$\begin{array}{rcll} \sim (x; y) & = & \sim\sim\sim (x; y) & (\text{triple negation}) \\ & = & \sim ((x; y) \vee (x; y)) & (\text{idempotency}) \\ & = & \sim (x; (y \vee y)) & (\textbf{A9}) \\ & = & \sim (x; \sim\sim y) & (\text{idempotency}) \end{array}$$

Finally, we mention a few surprising (from the static logician's point of view) invalid equations: $x; x = x$ and $x \Rightarrow x = \textsc{id}$.

Remark. Throughout this paper, the terms 'equal', 'equivalent' and the like will be used frequently, although sometimes in different senses. Either they will be used in a *semantic* way, where 'equal' means 'equal under all assignments in all dynamic relation algebras', or in a *syntactic* way, where 'equal' means 'provably equal in **AX**'. We hope the sense intended will always be clear from the context. □

3 Modal techniques

Before we prove the completeness theorem, let us focus on our main techniques, which originate in modal logic.

Define MOD to be the set of all modal formulas constructed from $\bot$ (as a modal formula, not as a $\mathcal{T}$-term), unary modalities $\langle x \rangle$ (for $x \in \mathcal{V}$) and the booleans $\wedge$ and $\neg$. Standard abbreviations are used for other connectives: $\top := \neg\bot$, $\phi \to \psi := \neg(\phi \wedge \neg\psi)$, $\phi \leftrightarrow \psi := (\phi \to \psi) \wedge (\psi \to \phi)$, $\phi \vee \psi := \neg(\neg\phi \wedge \neg\psi)$ and $[x]\phi := \neg\langle x \rangle \neg\phi$.

A dynamic relation algebra determined by a set S together with a valuation to variables $\sigma : \mathcal{V} \to \wp(S \times S)$ can easily be seen as a Kripke model[2] $\mathcal{M} = (S, \stackrel{x}{\to})_{x \in \mathcal{V}}$, with $\stackrel{x}{\to} := \sigma(x)$ the accessibility relation for a modal diamond $\langle x \rangle$. We will write (S, σ) when we have this model in mind. The modal satisfaction relation $(S, \sigma), s \Vdash \phi$, for $s \in S$ and ϕ a modal formula is defined as usual. We omit (S, σ) when it is clear from the context.

Besides diamonds $\langle x \rangle$ for each variable, we could introduce diamonds $\langle t \rangle$ for all the other $\mathcal{T}$-terms as well. In a model (S, σ) such a diamond would have $\sigma(t)$ as its accessibility relation, where σ is here viewed as the obvious extension to $\mathcal{T}$-terms. This would not really give us any new modal formulas though, as $\langle \bot \rangle \phi$ is equivalent to $\bot$, $\langle \sim t \rangle \phi$ to $\phi \wedge [t]\bot$ and $\langle t; t' \rangle \phi$ to $\langle t \rangle \langle t' \rangle \phi$. So formulas that contain these new diamonds can be viewed as abbreviations for modal formulas only containing diamonds $\langle x \rangle$.

This immediately gives us decidability of the problem $\models t_1 = t_2$, because this is true iff for some proposition letter p, $\langle t_1 \rangle p$ is equivalent (in minimal polymodal logic) to $\langle t_2 \rangle p$, which is decidable. By the same reasoning we get the finite model property for nonvalid equations: any nonvalid equation $t_1 = t_2$ can be falsified in a finite dynamic relation algebra.

A **bisimulation** between two Kripke models (S, σ) and (T, τ) is a relation $Z \subseteq S \times T$ such that the following two *zigzag-conditions* hold:

Zig: aZb and $(a, a') \in \sigma(x)$ imply the existence of a $b' \in T$ with $a'Zb'$ and $(b, b') \in \tau(x)$.

Zag: Vice versa: aZb and $(b, b') \in \tau(x)$ imply the existence of an $a' \in S$ such that $a'Zb'$ and $(a, a') \in \sigma(x)$.

We write $Z : (S, \sigma) \underline{\leftrightarrow} (T, \tau)$ when Z satisfies these constraints. A **full** bisimulation is a bisimulation $Z : (S, \sigma) \underline{\leftrightarrow} (T, \tau)$ such that the domain of Z is S and its range is T. We say that two models are **bisimilar** when there is a full bisimulation between them.

The operations $\bot$, $\sim$ and ; are **safe for bisimulation** ([1]): if $Z : (S, \sigma) \underline{\leftrightarrow} (T, \tau)$ then the bisimulation clauses extend to the interpretation of any $\mathcal{T}$-term t:

Zig: aZb and $(a, a') \in \sigma(t)$ imply the existence of a $b' \in T$ with $a'Zb'$ and $(b, b') \in \tau(t)$.

Zag: Vice versa: aZb and $(b, b') \in \tau(t)$ imply that there is an $a' \in S$ such that $a'Zb'$ and $(a, a') \in \sigma(t)$.

In particular, when (S, σ) and (T, σ) are bisimilar then $\sigma(t)$ is nonempty iff $\tau(t)$ is also nonempty.

Examples of operations that are not safe are boolean intersection, complement and $\top$ (interpreted as $S \times S$).

A model (S, σ) is called **unravelled** when it satisfies the following:

1. The relation $\bigcup_{x \in \mathcal{V}} \sigma(x)$ is well-founded: there is no infinite decreasing sequence $s_0 \stackrel{x_0}{\leftarrow} s_1 \stackrel{x_1}{\leftarrow} \ldots$.

2. For any $s \in S$ there is at most one pair (s', x) such that $(s', s) \in \sigma(x)$. So if $s_1 \stackrel{x}{\to} s$ and $s_2 \stackrel{y}{\to} s$ then $s_1 = s_2$ and $x = y$.

These conditions ensure that (S, σ) behaves like a tree, locally.

Any model is bisimilar to an unravelled one. This is achieved as follows. Let (S, σ) be any model. Let $S^\diamond$ consist of all pairs $(s_0 \ldots s_n, x_1 \ldots x_n)$ with $s_0, \ldots, s_n \in S$ and $x_1, \ldots, x_n \in \mathcal{V}$ such that $s_0 \stackrel{x_1}{\to} s_1 \stackrel{x_2}{\to} \ldots \stackrel{x_n}{\to} s_n$ in (S, σ). We also define an assignment $\tau : \mathcal{V} \to \wp(S^\diamond \times S^\diamond)$. $\tau(x)$ contains precisely all pairs

$$((s_0 \ldots s_n, x_1 \ldots x_n), (s_0 \ldots s_{n+1}, x_1 \ldots x_n x))$$

[2] We will not be concerned with truth of modal formulas under all valuations, so we refrain from calling this a *frame*.

in $S^\diamond \times S^\diamond$ such that $s_n \stackrel{x}{\to} s_{n+1}$ (which is actually a superfluous remark, because otherwise the second pair would not be in $S^\diamond$ in the first place). Now $(S^\diamond, \tau)$ is unravelled and the function $f : S^\diamond \to S$ sending $(s_0 \ldots s_n, x_1 \ldots x_n)$ to s_n is a full bisimulation. (It is a functional bisimulation, or a **zigzagmorphism**, also known in the modal literature as a **p-morphism**). What is of most importance to this paper is that if $\sigma(t)$ is nonempty in some model (S, σ), we now know that there must exist an unravelled model (T, τ) where $\tau(t)$ is also nonempty. It is in fact the only use we make of the unravelling technique, and of the notion of bisimulation.

4 The completeness theorem

We will prove that the axiom-system **AX** displayed in table 1 is sound and complete with respect to equational validity in dynamic relation algebras.

Modal tests $\phi?$, where ϕ is a modal formula (as in the previous section), are defined as follows:

$$\bot? := \bot \qquad (\langle x\rangle\phi)? := \sim\sim(x;\phi?)$$
$$(\phi \wedge \psi)? := \phi?;\psi? \qquad (\neg\phi)? := \sim(\phi?)$$

Note that the following are now trivially derivable in **AX** (mostly just by definition, but in the last $[x]$-case by using the triple negation law):

$$\top? = \text{ID} \qquad (\phi \to \psi)? = \phi? \Rightarrow \psi?$$

$$(\phi \leftrightarrow \psi)? = (\phi? \Rightarrow \psi?);(\psi? \Rightarrow \phi?)$$

$$(\phi \vee \psi)? = \phi? \vee \psi? \qquad ([x]\phi)? = x \Rightarrow \phi?$$

Lemma 4.1 *A modal test $\phi?$ is provably equal to $\sim\sim\phi?$ (i.e. $\vdash \phi? = \sim\sim\phi?$).*

Proof.
This is proved by induction on ϕ. For the base case, we need to show that $\vdash \bot = \sim\sim\bot$. This is proved by the following reasoning:

$$\begin{array}{rcll} \sim\sim\bot & = & \sim\sim(\sim x;x) & (\mathbf{A0}) \\ & = & \sim\sim(\sim x;\sim\sim x) & (\text{range test}) \\ & = & \sim x;\sim\sim x & (\mathbf{A7}) \\ & = & \sim\sim x;\sim x & (\mathbf{A5}) \\ & = & \bot & (\mathbf{A0}) \end{array}$$

The induction step for $\wedge$ uses **A5**, while those for $\neg$ and $\langle x\rangle$ use the triple negation law. □

Lemma 4.2 *Any term $\sim\sim t$ of $\mathcal{T}$ is provably equivalent (in* **AX***) to $\phi?$ for some modal formula ϕ.*

Proof.
We prove this by induction on t.

- $\vdash \sim\sim x = \sim\sim(x;\text{ID})$ by the right identity law (see page 4). This latter term is the definition of $(\langle x\rangle\top)?$.
- By lemma 4.1 $\sim\sim\bot$ is provably equal to $\bot$, which is by definition the modal test $\bot?$.
- Suppose $\vdash \sim\sim t = \phi?$ (i.e. suppose the induction hypothesis holds for t) Then $\vdash \sim\sim\sim t = \sim(\phi?) = (\neg\phi)?$.
- Suppose the statement holds for t and t'. By range test, $\vdash \sim\sim(t;t') = \sim\sim(t;\sim\sim t')$. By the hypothesis, $\sim\sim t'$ must be provably equal to a modal test $\psi?$. We now prove by induction on t that $\sim\sim(t;\psi?)$ is also a modal test.

- $\vdash \sim\sim (x;\psi?) = (\langle x\rangle\psi)?$, simply by definition.
- We may use **A2** to show that $\sim\sim (\bot;\psi?)$ is equal to $\sim\sim \bot$, which has already been shown to be provably equal to the modal test $\bot?$.
- $\vdash \sim\sim (\sim t;\psi?) = \sim\sim (\sim\sim\sim t;\psi?)$, by the triple negation law. By the first induction hypothesis, $\vdash \sim\sim t = \phi?$, for some modal ϕ, so $\vdash \sim\sim (\sim t;\psi?) = \sim\sim (\sim \phi?;\psi?) = (\neg\neg(\neg\phi \wedge \psi))?$.
- $\sim\sim ((t_1;t_2);\psi?) = \sim\sim (t_1;(t_2;\psi?))$, by **A4**. Using range test, we get $\sim\sim (t_1;(t_2;\psi?)) = \sim\sim (t_1;\sim\sim (t_2;\psi?))$. By the second induction hypothesis $\sim\sim (t_2;\psi?)$ is provably equal to a modal test, say $\phi?$. Using the second induction hypothesis again, we get that $\sim\sim (t_1;\phi?)$ is a modal test and we are done. □

As a corollary of this lemma we note that, as any term $\sim t$ is provably equal to $\sim\sim\sim t$, any strongly negated term is in fact provably equal to a modal test.

It can easily be seen that for all modal formulas ϕ:

$$\models \phi \qquad \text{iff} \qquad \models \phi? = \mathrm{ID}$$

where $\models$ on the left is the usual modal validity in the class of all polymodal Kripke models, while the one on the right is validity in dynamic relation algebras. We assume that the reader will not confuse these two different validity relations (even though they are denoted by the same symbol $\models$) as modal validity only applies to modal formulas, while relational validity only to equations between $\mathcal{T}$-terms.

We already have an axiomatisation for validity of modal formulas:

Minimal modal logic axioms:

1.	$\phi \to (\psi \to \phi)$	(K)
2.	$(\phi \to (\psi \to \chi)) \to ((\phi \to \psi) \to (\phi \to \chi))$	(S)
3.	$(\phi_1 \wedge \phi_2) \to \phi_i$	(i-th projection, for $i \in \{1,2\}$)
4.	$\phi \to (\psi \to (\phi \wedge \psi))$	($\wedge$-introduction)
5.	$\neg\neg\phi \to \phi$	(double negation)
6.	$\bot \to \phi$	(falsum law)
7.	$\neg\phi \leftrightarrow (\phi \to \bot)$	(negation definition)
8.	$\langle x\rangle(\phi \vee \psi) \leftrightarrow (\langle x\rangle\phi \vee \langle x\rangle\psi)$	(distribution)

Rules:

1.	$\phi, \phi \to \psi / \psi$	(modus ponens)
2.	$\phi / [x]\phi$	(necessitation)
3.	$\phi \leftrightarrow \psi / \chi[\phi] \leftrightarrow \chi[\psi]$	(substitution of equivalents)

Lemma 4.3 *If a modal formula ϕ is derivable in the above system then $\vdash \phi? = \mathrm{ID}$. (So if $\models \phi$ then $\vdash \phi? = \mathrm{ID}$, by the completeness theorem for minimal modal logic.)*

First we prove another useful lemma:

Lemma 4.4 $\vdash (\phi \leftrightarrow \psi)? = \mathrm{ID}$ *iff* $\vdash \phi? = \psi?$.

Proof.
From left to right, assume $\vdash (\phi \leftrightarrow \psi)? = \mathrm{ID}$. Then:

$$\begin{array}{rcll}
\phi? & = & \mathrm{ID};\phi? & (\mathbf{A3}) \\
& = & (\phi? \Rightarrow \psi?);(\psi? \Rightarrow \phi?);\phi? & (\text{assumption}) \\
& = & (\psi? \Rightarrow \phi?);(\phi? \Rightarrow \psi?);\phi? & (\mathbf{A5}) \\
& = & (\psi? \Rightarrow \phi?);(\phi? \Rightarrow \psi?);\phi?;\psi? & (\mathbf{A8}) \\
& = & (\phi? \Rightarrow \psi?);(\psi? \Rightarrow \phi?);\psi?;\phi? & (\mathbf{A5}) \\
& = & (\phi? \Rightarrow \psi?);(\psi? \Rightarrow \phi?);\psi? & (\mathbf{A8}) \\
& = & \psi? & (\text{assumption}, \mathbf{A3})
\end{array}$$

From right to left, we need to show that $\phi? \Rightarrow \phi? = \text{ID}$, which we leave to the reader (use lemma 4.1). □

Proof of lemma 4.3

We first show that for all modal axioms ϕ, $\phi? = \text{ID}$ is provable.

1. K-axiom:
$$\begin{array}{rcll} \phi? \Rightarrow (\psi? \Rightarrow \phi?) & = & \sim(\phi?;\sim\sim(\psi?;\sim\phi?)) & \text{(definition of } \Rightarrow) \\ & = & \sim(\phi?;\psi?;\sim\phi?) & \text{(range test)} \\ & = & \sim(\sim\phi?;\phi?;\psi?) & (\mathbf{A5}) \\ & = & \sim(\bot;\psi?) & (\mathbf{A0}) \\ & = & \sim\bot & (\mathbf{A2}) \\ & = & \text{ID} & \end{array}$$
Note that the use in this proof of **A5** is justified, as all modal tests are equivalent to strongly negated terms, by lemma 4.1.

2. S-axiom:
$$\begin{array}{cll} & (\phi? \Rightarrow (\psi? \Rightarrow \chi?)) \Rightarrow ((\phi? \Rightarrow \psi?) \Rightarrow (\phi? \Rightarrow \psi?)) & \\ = & \sim((\phi? \Rightarrow (\psi? \Rightarrow \chi?));(\phi? \Rightarrow \psi?);\phi?;\sim\chi?) & \text{(range test)} \\ = & \sim((\phi? \Rightarrow (\psi? \Rightarrow \chi?));(\phi? \Rightarrow \psi?);\phi?;\psi?;\sim\chi?) & (\mathbf{A8}) \\ = & \sim((\phi? \Rightarrow (\psi? \Rightarrow \chi?));\phi?;\psi?;\sim\chi?;(\phi? \Rightarrow \psi?)) & (\mathbf{A5}) \\ = & \sim((\phi? \Rightarrow (\psi? \Rightarrow \chi?));\phi?;(\psi? \Rightarrow \chi?);\psi?;\chi?;\sim\chi?;(\phi? \Rightarrow \psi?)) & (\mathbf{A8}) \\ = & \sim((\phi? \Rightarrow (\psi? \Rightarrow \chi?));\phi?;(\psi? \Rightarrow \chi?);\psi?;\underline{\sim\chi?;\chi?};(\phi? \Rightarrow \psi?)) & (\mathbf{A5}) \\ = & \text{ID} & (\mathbf{A0\text{-}2}) \end{array}$$

3. Projection axioms. As $(\phi_1?;\phi_2?) \Rightarrow \phi_i?$ is the same as $\sim(\phi_1?;\phi_2?;\sim\phi_i?)$, simply apply **A5** and the $\bot$-axioms to arrive at ID.

4. Using range test, $\phi? \Rightarrow (\psi? \Rightarrow (\phi?;\psi?))$ is equivalent to $\sim(\phi?;\psi?;\sim(\phi?;\psi?))$. With **A0** and **A5** this is equal to ID.

5. Double negation axiom: by lemma 4.4, it suffices to show $\vdash \sim\sim\phi? = \phi?$. But this was already shown in lemma 4.1.

6. Falsum law: by **A2**, $\vdash \bot \Rightarrow \phi? = \sim(\bot;\sim\phi?) = \text{ID}$.

7. Negation definition: use the right identity law to show $\vdash \phi? \Rightarrow \bot = \sim(\phi?;\sim\bot) = \sim\phi?$.

8. Distribution. First observe that double strong negation distributes over disjunction:
$$\begin{array}{rcll} \sim\sim(t \vee t') & = & \sim\sim\sim(\sim t;\sim t') & \text{(definition of } \vee) \\ & = & \sim(\sim\sim\sim t;\sim\sim\sim t') & \text{(triple negation)} \\ & = & \sim\sim t \vee \sim\sim t' & \text{(definition of } \vee) \end{array}$$
Now:
$$\begin{array}{rcll} (\langle x\rangle(\phi \vee \psi))? & = & \sim\sim(x;(\phi? \vee \psi?)) & \text{(definition of ?)} \\ & = & \sim\sim((x;\phi?) \vee (x;\psi?)) & (\mathbf{A9}) \\ & = & \sim\sim(x;\phi?) \vee \sim\sim(x;\psi?) & (\sim\sim \text{ distributes over } \vee) \\ & = & (\langle x\rangle\phi \vee \langle x\rangle\psi)? & \text{(definition of ?)} \end{array}$$

What remains is to show closure under the rules.

1. Modus ponens. If $\vdash \phi? = \text{ID}$ and $\vdash \phi? \Rightarrow \psi? = \text{ID}$, substitution gives us $\vdash \text{ID} \Rightarrow \psi? = \text{ID}$. But $\vdash \text{ID} \Rightarrow \psi? = \sim(\text{ID};\sim\psi?) = \sim\sim\psi?$. The latter term is provably equal, by lemma 4.1, to $\psi?$, so we are done.

2. Necessitation. Suppose $\phi? = \text{ID}$ has been proved. Then:

$$\begin{array}{rcll} \sim(x;\sim\phi?) & = & \sim(x;\sim\text{ID}) & \text{(assumption)} \\ & = & \sim(x;\sim\sim\bot) & (\text{ID-definition}) \\ & = & \sim(x;\bot) & \text{(range test)} \\ & = & \sim\bot & (\mathbf{A1}) \\ & = & \text{ID} & (\text{ID-definition}) \end{array}$$

3. Substitution of equivalents: by lemma 4.4 this rule may be dealt with by means of the usual equational substitution axioms. □

Let us call a term t **empty** if $\models t = \bot$, i.e. if its interpretation in any dynamic relation algebra is always empty. We call a term t **nonempty** if in *some* dynamic relation algebra and for *some* assignment to variables t is interpreted as a nonempty relation. In other words, a term is nonempty if it is not empty.

Lemma 4.5 *If t is empty then $\vdash t = \bot$.*

Proof.
If $\models t = \bot$ then also $\models \sim\sim t = \bot$. $\sim\sim t$ is equivalent in **AX** to a modal test $\phi?$. Thus $\models \phi? = \bot$, and hence $\models \phi \leftrightarrow \bot$. By lemmas 4.4 and 4.3 this gives us $\vdash \phi? = \bot$ and thus $\vdash \sim\sim t = \bot$. By **A6**, $\vdash t = \sim\sim t; t$, so $\vdash t = \bot; t = \bot$. □

Lemma 4.6 *Any $\mathcal{T}$-term t is provably equivalent to a* **path-term**, *i.e. a formula of the form*

$$\phi_0?; x_1; \phi_1?; \ldots; x_n; \phi_n?$$

where n is any natural number (possibly 0, in which case a path-term is just a modal test).

Proof.
Any variable x is equal to $\text{ID}; x; \text{ID}$, which is the same as $\top?; x; \top?$ and thus a path-term. If $\phi_0?; x_1; \phi_1?; \ldots; x_n; \phi_n?$ and $\psi_0?; y_1; \psi_1?; \ldots; y_m; \psi_m?$ are two path-terms then

$$\phi_0?; x_1; \phi_1?; \ldots; x_n; (\phi_n \wedge \psi_0)?; y_1; \psi_1?; \ldots; y_m; \psi_m?$$

is equal to their composition. We have already seen that any strongly negated term is equivalent to a modal test, and thus to a path-term. The same remark applies to $\bot$. □

Lemma 4.7 *If $t_1 := \phi_0?; x_1; \phi_1?; \ldots; x_n; \phi_n?$ and $t_2 := \psi_0?; y_1; \psi_1?; \ldots; y_m; \psi_m?$ are two semantically equal, nonempty path-terms, then $n = m$, $x_1 = y_1, \ldots, x_n = y_n$.*

Proof.
t_1 is nonempty, so there must be a dynamic relation algebra $(\wp(S \times S), \sim, ;, \emptyset)$ and an assignment to variables σ such that $\sigma(t_1)$ is nonempty. Then, in the associated polymodal Kripke frame $\mathcal{M} = (S, \xrightarrow{x})_{x \in \mathcal{V}}$ (with $\xrightarrow{x} = \sigma(x)$) there are points $s_0, \ldots, s_n$ with $s_0 \xrightarrow{x_1} s_1 \ldots s_{n-1} \xrightarrow{x_n} s_n$ and $s_i \Vdash \phi_i$, for all $i \leq n$ (i.e. $(s_0, s_n) \in \sigma(t_1)$). By the remarks in section 3 we may in fact assume that $\mathcal{M}$ is unravelled. Then $s_0 \xrightarrow{x_1} s_1 \xrightarrow{x_2} \ldots \xrightarrow{x_n} s_n$ is the *unique* path from s_0 to s_n. By assumption $\sigma(t_1) = \sigma(t_2)$, so there will also be a t_2-path from s_0 to s_n. Together these imply the desirables. □

Lemma 4.5 and lemma 4.7 together *almost* give us the completeness theorem. For suppose $\models t_1 = t_2$. We may assume that they are path-terms. If these path-terms are empty, by lemma 4.5 they will both be provably equivalent to $\bot$, hence $\vdash t_1 = t_2$. And if t_1 and t_2 are nonempty then by lemma 4.7 they will be of the same shape. If we could now show that $\vdash \phi_i? = \psi_i?$ for each $i \leq n$ (where we assume that t_1 and t_2 are as in lemma 4.7), $\vdash t_1 = t_2$ would follow. This may not be the case though. Two prime examples are the following:

Up: Clearly $\models ([x]\phi)?; x; \psi? = ([x]\phi)?; x; (\phi \wedge \psi)?$, while ψ need not imply ϕ. We need to be able to move ϕ up the path. This is precisely the use of our 'modus ponens' axiom **A8**, which equates $(x \Rightarrow \phi?); x; \psi?$ to $(x \Rightarrow \phi?); x; \phi?; \psi?$ (which is provably equal to $([x]\phi)?; x; (\phi \wedge \psi)?$).

Down: Sometimes information may be required to move *down* a path as well. An example is in the semantic equality $\phi?; x; \psi? = (\phi \wedge \langle x \rangle \psi)?; x; \psi?$. Now we can use the 'domain test' axiom **A6**, which gives us that $\vdash \phi?; x; \phi? = \phi?; \sim\sim (x; \phi?); x; \phi?$, which is provably equal to $(\phi \wedge \langle x \rangle \psi)?; x; \psi?$.

These two tricks (the use of **A6** and **A8**) turn out to suffice, as we will see.

The **degree** $d(\phi)$ of a modal formula ϕ is defined as usual:

$$\begin{array}{lcl} d(\bot) & = & 0 \\ d(\langle x \rangle \phi) & = & d(\phi) + 1 \\ d(\phi \wedge \psi) & = & max(d(\phi), d(\psi)) \\ d(\neg \phi) & = & d(\phi) \end{array}$$

Intuitively, the degree of a formula tells you how deep down the model the formula can *see*, how much of the model is relevant to it. The degree of a path-term $\phi_0?; x_1; \phi_1?; \ldots; x_n; \phi_n?$ is defined as $d(\phi_0 \wedge \langle x_1 \rangle (\phi_1 \wedge \ldots \langle x_n \rangle (\phi_n) \ldots))$.

Given a finite set of variables $X \subset \mathcal{V}$ and a certain $n \in \mathbb{N}$, we may define $\text{MOD}_n(X)$, the set of all modal formulas ϕ of degree at most n and that only contain modalities $\langle x \rangle$ with $x \in X$. There are, up to equivalence, only finitely many formulas in $\text{MOD}_n(X)$. Thus there exists a finite set $\text{FIN}_n(X) \subset \text{MOD}_n(X)$ such that if $\phi \in \text{MOD}_n(X)$ then there is a formula $\psi \in \text{FIN}_n(X)$ equivalent to ϕ.

Whenever s is a point in a Kripke model (having relations $\xrightarrow{x}$ for each $x \in X$) the **total n-description** of s is defined as:

$$\bigwedge \{\phi \in \text{FIN}_n(X) \mid s \Vdash \phi\}$$

As there are only finitely many subsets of $\text{FIN}_n(X)$, there are only finitely many total descriptions.

Lemma 4.8 (Normal Form Lemma) *Let t be a path-term $\phi_0?; x_1; \phi_1?; \ldots; x_n; \phi_n?$, whose degree is at most m and whose variables occur among the finite set X. Then t is provably equivalent to a path-term of the form $\chi_0?; x_1; \chi_1?; \ldots; x_n; \chi_n?$, where*

1. *Each χ_i is a disjunction of total $(m - i)$-descriptions.*
2. *For each $i < n$: $\chi_i \models \langle x_{i+1} \rangle \chi_{i+1}$.*
3. *For each $i < n$, whenever $\psi \in \text{MOD}_{(m-i)-1}(X)$ and $\chi_i \models [x_{i+1}]\psi$, then $\chi_{i+1} \models \psi$.*

Proof.
Define:

$$\begin{array}{lll} \mu_n := \phi_n & \pi_n := \phi_n? & \\ \mu_i := \phi_i \wedge \langle x_{i+1} \rangle \mu_{i+1} & \pi_i := \phi_i?; x_{i+1}; \pi_{i+1} & (\text{if } i < n) \end{array}$$

Note that $\pi_0 = t$. Clearly $\models \sim\sim \pi_i = \mu_i?$. Furthermore, as $\mu_i \models \phi_i$, $\models \sim\sim \pi_i; \phi_i? = \mu_i?$. So $\vdash \sim\sim \pi_i; \phi_i? = \mu_i?$, by lemma 4.3. Thus for any $i < n$: $\vdash \pi_i = \sim\sim \pi_i; \pi_i = \sim\sim \pi_i; \phi_i?; x_{i+1}; \pi_{i+1} = \mu_i?; x_{i+1}; \pi_{i+1}$, by **A6**. Using this fact we get $\vdash t = \mu_0?; x_1; \mu_1?; \ldots; x_n; \mu_n?$. So we have already fulfilled the second of our desirables: for $i < n$, $\mu_i \models \langle x_{i+1} \rangle \mu_{i+1}$, by definition. The others need not yet be satisfied, so we cannot stop here.

We prove, by induction on $i \leq n$, the following: there are sets D_j of total $(m - j)$-descriptions (with $j \leq i$) such that

$$\vdash t = (\bigvee D_0)?; x_1; (\bigvee D_1)?; \ldots; x_i; (\bigvee D_i)?; x_{i+1}; \mu_{i+1}?; \ldots; x_n; \mu_n?$$

such that:

1. $\bigvee D_j \models \langle x_{j+1}\rangle \bigvee D_{j+1}$ for $j < i$.

2. $\bigvee D_i \models \langle x_{i+1}\rangle \mu_{i+1}$ if $i < n$.

3. If $\phi \in \mathrm{MOD}_{(m-j)-1}(X)$ (for $j < i$) and $\bigvee D_j \models [x_{j+1}]\phi$ then $\bigvee D_{j+1} \models \phi$.

First of all note that μ_0 has a degree of at most m, otherwise t would have a higher degree than m. So D_0 is readily picked, since any modal formula of depth at most m is equivalent to a disjunction of total m-descriptions. Thus $\vdash \mu_0? = (\bigvee D_0)?$, hence $\vdash t = (\bigvee D_0)?; x_1; \mu_1?; \ldots; x_n; \mu_n?$ The desirables obviously hold: we only need to check the second item and as $\mu_0 \models \langle x_1\rangle \mu_1$, this does not change when we change to an equivalent formula.

So suppose we have found an appropriate term for i and we are looking for one for $i+1$. Let N (for '*n*ecessary') be the unique minimal set of total $((m-i)-1)$-descriptions such that $\bigvee D_i \models [x_{i+1}] \bigvee N$. This set must exist. For the disjunction of the set A of *all* total $((m-i)-1)$-descriptions is a tautology, hence $\bigvee D_i \models [x_{i+1}] \bigvee A$ holds. Furthermore, if N_1 and N_2 are two minimal sets satisfying the requirement then $\bigvee D_i \models [x_{i+1}]((\bigvee N_1) \wedge (\bigvee N_2))$. Because N_1 and N_2 are sets of total descriptions (and hence *different* elements of N_1 and N_2 are incompatible), $(\bigvee N_1) \wedge (\bigvee N_2)$ is equivalent to $\bigvee (N_1 \cap N_2)$. Using the assumption that N_1 and N_2 are minimal, we must conclude that $N_1 = N_1 \cap N_2 = N_2$. Thus uniqueness is guaranteed.

Choose $D_{i+1} := \{\delta \in N \mid \delta \models \mu_{i+1}\}$. By lemma 4.3 and the fact that $\bigvee D_i$ implies $[x_{i+1}] \bigvee N$:

$$\vdash (\bigvee D_i)?; ([x_{i+1}] \bigvee N)? = (\bigvee D_i)?$$

Thus $(\bigvee D_i)?; x_{i+1}; \mu_{i+1}?$ is equal in **AX** to:

$$(\bigvee D_i)?;\ (x_{i+1} \Rightarrow (\bigvee N)?)\ ; x_{i+1}; \mu_{i+1}?$$

By **A8**, this equals:

$$(\bigvee D_i)?;\ (x_{i+1} \Rightarrow (\bigvee N)?)\ ; x_{i+1};\ (\bigvee N)?\ ; \mu_{i+1}?$$

But $(\bigvee N) \wedge \mu_{i+1}$ is modally equivalent to $\bigvee D_{i+1}$, hence the above term is provably equal to

$$(\bigvee D_i)?; x_{i+1}; (\bigvee D_{i+1})?$$

So:

$$\vdash t = (\bigvee D_0)?; x_1; (\bigvee D_1)?; \ldots; x_i; (\bigvee D_{i+1})?; x_{i+2}; \mu_{i+2}?; \ldots; x_n; \mu_n?$$

as desired.

What remains is to verify the desirables:

1. We have to show $\bigvee D_i \models \langle x_{i+1}\rangle \bigvee D_{i+1}$. By the induction hypothesis we have $\bigvee D_i \models \langle x_{i+1}\rangle \mu_{i+1}$. By definition of N, we have $\bigvee D_i \models [x_{i+1}] \bigvee N$. Combining these two statements we get that $\bigvee D_i \models \langle x_{i+1}\rangle((\bigvee N) \wedge \mu_{i+1})$. As $\bigvee D_{i+1}$ is equivalent to $(\bigvee N) \wedge \mu_{i+1}$, we are done.

2. Suppose that $i+1 < n$. Then we need to show that $\bigvee D_{i+1} \models \langle x_{i+2}\rangle \mu_{i+2}$. Each $\delta \in D_{i+1}$ implies μ_{i+1}, hence $\bigvee D_{i+1} \models \mu_{i+1}$, which by definition implies $\langle x_{i+2}\rangle \mu_{i+2}$.

3. Let ψ be a modal formula of degree at most $(m-i)-1$ and suppose $\bigvee D_i \models [x_{i+1}]\psi$. If $\bigvee N \not\models \psi$ then for some $\delta \in N$, $\delta \not\models \psi$ (and as δ is a total description, $\delta \models \neg\psi$). But then $\bigvee D_i \models [x_{i+1}] \bigvee (N - \{\delta\})$, which contradicts the minimality of N. So $\bigvee N$ *does* imply ψ. As $D_{i+1} \subseteq N$, $\bigvee D_{i+1}$ also implies ψ, which is what we had to prove.

The desired formula for the proof is thus $(\bigvee D_0)?; x_1; (\bigvee D_1)?; \ldots; x_n; (\bigvee D_n)?$. □

Let us call path-terms that satisfy the constraints of the above lemma in **normal form**. We have shown that any term can be proved equal to a normal form.

Theorem 4.9 (Completeness theorem) $\models t_1 = t_2$ *iff* $\vdash t_1 = t_2$.

Proof.
Soundness is trivial to verify. For completeness, suppose $\models t_1 = t_2$. By lemmas 4.6 and 4.8, we may assume that these terms are in normal form (with respect to $m = max(d(t_1), d(t_2))$ and an X containing the variables of t_1 and t_2). For reference, suppose:

$$\begin{aligned} t_1 &:= \phi_0?; x_1; \phi_1?; \ldots; x_n; \phi_n? \\ t_2 &:= \psi_0?; y_1; \psi_1?; \ldots; y_k; \psi_k? \end{aligned}$$

We may furthermore assume that t_1 and t_2 are nonempty, otherwise we have already seen that $\vdash t_1 = t_2$ (lemma 4.5). But if they are nonempty, by lemma 4.7 they must have the same shape: $n = k$, $x_1 = y_1, \ldots, x_n = y_n$. We will show that ϕ_i is semantically equivalent to ψ_i for each $i \leq n$, hence $\vdash \phi_i? = \psi_i?$ for such i (by lemmas 4.3 and 4.4). This is of course sufficient to show that $\vdash t_1 = t_2$.

Suppose $\phi_i \not\models \psi_i$. ϕ_i is the disjunction of a set D_i of total $(m-i)$-descriptions, so for some $\delta \in D_i$: $\delta \not\models \psi_i$. As ψ_i has degree $m-i$, δ in fact implies $\neg\psi_i$. We will show that this δ must occur at the i-th step of a t_1-path in some Kripke model.

Suppose that there exists no path (in any model) of the form $s_0 \xrightarrow{x_1} s_1 \xrightarrow{x_2} \ldots s_{i-1} \xrightarrow{x_i} s_i$ with $s_j \Vdash \phi_j$ for each $j < i$ and $s_i \Vdash \delta$. Then:

$$\phi_0 \wedge \langle x_1\rangle(\phi_1 \wedge \ldots \langle x_{i-1}\rangle(\phi_{i-1} \wedge \langle x_i\rangle\delta)\ldots)$$

must be a contradiction, and its negation:

$$\phi_0 \rightarrow [x_1](\phi_1 \rightarrow \ldots [x_{i-1}](\phi_{i-1} \rightarrow [x_i]\neg\delta)\ldots)$$

a tautology. Now $\phi := \phi_1 \rightarrow [x_2](\phi_2 \rightarrow \ldots [x_{i-1}](\phi_{i-1} \rightarrow [x_i]\neg\delta)\ldots)$ is of depth $m-1$ and $\phi_0 \models [x_1]\phi$. So, by the fact that t_1 is in normal form, $\phi_1 \models \phi$ and thus $\phi_1 \models [x_2](\phi_2 \rightarrow \ldots [x_{i-1}](\phi_{i-1} \rightarrow [x_i]\neg\delta)\ldots)$. Repeating this argument, we arrive at the conclusion that $\phi_i \models \neg\delta$. But then $\delta \models \neg\delta$, which is contradictory with the fact that δ is a total description and thus consistent.

So we have a path of the required form. It may still not be of the desired length (if $i \neq n$), but as ϕ_i implies $\langle x_{i+1}\rangle(\phi_{i+1} \wedge \ldots \langle x_n\rangle\phi_n \ldots)$ (due to the normal form of t_1), we can extend the path to a longer path $s_0 \xrightarrow{x_1} s_1 \xrightarrow{x_2} \ldots\ldots \xrightarrow{x_{n-1}} s_n$ with $s_j \models \phi_j$ for each $j \leq n$ and in particular $s_i \models \delta$. So this is a t_1-path. We may assume that it occurs in an unravelled model (see section 3). By the assumption that $\models t_1 = t_2$, this same path must also be a t_2-path. But then $s_i \Vdash \psi_i$, which is in contradiction with the assumption that $\delta \models \neg\psi_i$.

We have thus proved that $\phi_i \models \psi_i$. The proof that $\psi_i \models \phi_i$ is entirely analogous. □

5 Atomic Tests and Union

The operations $\sim$ and ; are not just of interest to dynamic semantics, but also to computer science, as far as the study of Propositional Dynamic Logic (PDL, [6]) is concerned. Van Benthem, in his [1], shows that the first order definable operations that are safe for bisimulation are precisely the ones that can be constructed from the following repertoire: atomic tests $p?$, the basic binary relations, strong negation, relational composition and union $\cup$. These are precisely the first order definable PDL-programs.[3]

Our technique naturally generalises to this broader language. Dynamic relation algebras are the same as before, except for the extra binary operation $\cup$ (simply interpreted as union of relations) and possibly infinitely many constants $p?$ which have to be interpreted as some subset of the diagonal.

The valid equations in this setting are axiomatised by **AX** in addition to the axioms displayed in table 2. **A5**, **A7** and **A9** may now be omitted, as they become derivable in the rest of the system.

[3] $\sim$ is usually not present in presentations of PDL, but it is implicitly present, as modal tests $\phi?$ for any modal ϕ are considered PDL-programs and $\sim t$ is equivalent to $([t]\bot)?$.

B1:	$p? = \sim\sim p?$	(test)
B2:	$x \cup (y \cup z) = (x \cup y) \cup z$	(associativity)
B3:	$x \cup y = y \cup x$	(commutativity)
B4:	$x \cup x = x$	(idempotency)
B5:	$(x \cup y); z = (x; z) \cup (y; z)$	(left-distribution)
B6:	$x; (y \cup z) = (x; y) \cup (x; z)$	(right-distribution)
B7:	$x \cup \bot = x$	($\cup$-unit)
B8:	$\sim (x \cup y) = \sim x; \sim y$	(De Morgan)
B9:	$\sim x \cup \sim y = \sim\sim (\sim x \cup \sim y)$	(test union)

Table 2: Extra axioms for $p?$ and $\cup$.

We sketch a proof of the completeness theorem. First of all, any term t can be proved equal to a union of path-terms. As an example, consider four path-terms $t_1, \ldots, t_4$. Then $(t_1 \cup t_2); (t_3 \cup t_4)$ is equivalent to $(t_1; t_3) \cup (t_1; t_4) \cup (t_1; t_3) \cup (t_1; t_4)$, using only **B5** and **B6**. The latter term is a union of path-terms, as desired.

Furthermore, we can make sure that the path-terms occurring in these unions only contain modal tests $\phi?$ where ϕ is a total description. For instance, suppose we have a path-term $t := \phi_1?; x; \phi_2?$ and an $m \geq d(t)$. Then ϕ_1 is equivalent to some disjunction $\bigvee D_1$ of total m-descriptions and ϕ_2 to a disjunction $\bigvee D_2$ of total $(m-1)$-descriptions. By **B9**, $(\bigvee D_i)? = \bigcup\{\delta? \mid \delta \in D_i\}$, so using **B5** and **B6** again we may derive that:

$$t = \bigcup\{\delta_1?; x; \delta_2? \mid \delta_1 \in D_1, \delta_2 \in D_2\}$$

Now if $\models t_1 = t_2$, $\vdash t_1 = \bigcup_{i \in I} \pi_i$ and $\vdash t_2 = \bigcup_{j \in J} \rho_j$, where the π_i and ρ_j are path-terms in the restricted format described above, then if π_i is a nonempty path-term, it must be actually equal to some ρ_j and vice versa (we again use unravelling here). So using **B7** to remove empty paths, and **B2-4** to order the unions as we please, $\vdash t_1 = t_2$. We see that the completeness proof is even simpler when we extend the language with union: the subtlety of the Normal Form Lemma 4.8 proves unnecessary.

6 The variety generated by DRA

We have proved that the variety generated by DRA ($\mathbf{V}(\mathsf{DRA})$) is the same as the set of algebras that satisfy the equations of **AX**. An interesting question is what $\mathbf{V}(\mathsf{DRA})$ actually looks like.

By Birkhoff's theorem (see, for instance [3]), $\mathbf{V}(\mathsf{DRA}) = \mathbf{HSP}(\mathsf{DRA})$. It is well-known that for standard relation algebra, this general statement can be sharpened somewhat: $\mathbf{V}(\mathsf{RA}) = \mathbf{SP}(\mathsf{RA})$, i.e. closure under homomorphic images may be left out. Can something similar be said about DRA?

It is immediate that $\mathbf{V}(\mathsf{DRA})$ is not DRA itself, as DRA is not closed under subalgebras. This is proved by a simple cardinality argument. If $\mathsf{DRA}(S)$ denotes the dynamic relation algebra given by the set S (i.e. its domain consists of all relations over S) then $\mathsf{DRA}(\{1, \ldots, n\})$ has exactly $2^{n \cdot n}$ elements. But the subalgebra of $\mathsf{DRA}(\{0, 1\})$ generated by the relation $\{(0, 1)\}$ has 5 elements, which is clearly not of the form $2^{n \cdot n}$ for any natural number n.

A more promising fact is that any product of dynamic relation algebras is a subalgebra of a dynamic relation algebra: $\mathbf{SP}(\mathsf{DRA}) = \mathbf{S}(\mathsf{DRA})$. For $\prod_{i \in I} \mathsf{DRA}(S_i)$ can be embedded into $\mathsf{DRA}(\biguplus_{i \in I} S_i)$ (where $\biguplus$ denotes disjoint union), via the embedding:

$$\iota : (R_i)_{i \in I} \mapsto \biguplus_{i \in I} R_i.$$

So we get that $\mathbf{V}(\mathsf{DRA}) = \mathbf{HS}(\mathsf{DRA})$. As of yet it is unclear whether $\mathbf{H}$ may be dropped from this equation. Should this be the case, it would give us a nice representation theorem: a theorem stating that all algebras satisfying the axioms in **AX** really *are* (up to isomorphism of course) spaces of relations.

7 Further research

In this paper we have shown that the $\{\bot, \sim, ;\}$-fragment may be axiomatised by essentially coding modal logic in the calculus, plus a few extra tricks to deal with proper paths. The notion of unravelling was especially useful in proving the completeness theorem. These same methods were used to prove a completeness theorem for an extended fragment that includes union and atomic tests. We list a few possible extensions that the results in this paper may lead to.

Extending our axiom systems to larger fragments including intersection $t \cap t'$ and iteration t^* call for different methods, intersection because it is not safe for bisimulation and our unravelling arguments will fail, and iteration because of its infinitary nature.

An alternative road to axiomatise validity in our fragment is to design a sequent calculus for these systems, where a sequent $\phi_1, \ldots, \phi_n \vdash \psi_1, \ldots, \psi_m$ corresponds to $\models ((\phi_1; \ldots; \phi_n) \Rightarrow (\psi_1; \ldots; \psi_m)) = \textsc{id}$, i.e. to the usual notion of validity in DPL (discussed in [4]). The work of [5] may be relevant to such a calculus.

Of most interest however is to extend the result to real DPL-equality, and axiomatise which DPL-formulas are equal as relations. Such an axiomatisation would have to have at least the following as extra axioms:

$$\begin{array}{lcl} P(x_1, \ldots, x_n) & = & \sim\sim P(x_1, \ldots, x_n) \\ \exists x; \exists y & = & \exists y; \exists x \\ \exists x; \exists x & = & \exists x \end{array}$$

where the variables in these equations are not algebraic variables: $P(x_1, \ldots, x_n)$ and $\exists x$ are viewed as constants. By the result in [10], only axioms that contain these constants need to be added.

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