Contexts in Dynamic Predicate Logic

Albert Visser

Department of Philosophy, Utrecht University

Heidelberglaan 8, 3584 CS Utrecht, The Netherlands

email: `Albert.Visser@phil.ruu.nl`

Abstract

In this paper we introduce a notion of context for Groenendijk & Stokhof's Dynamic Predicate Logic *DPL*. We use these contexts to give a characterization of the relations on assignments that can be generated by composition from tests/conditions and random resettings in the case that we are working over an infinite domain. These relations are precisely the ones definable in *DPL* if we allow ourselves arbitrary tests as a starting point. We discuss some possible extensions of *DPL* and the way these extensions interact with our notion of context.

1 Introduction

Dynamic Predicate Logic (*DPL*) was invented by Jeroen Groenendijk and Martin Stokhof (see [1], see also our section 2) as a specification language (or better: as a module for a specification language) of meanings for fragments of natural language. Most of the research concerning *DPL* has gone into integrating it with versions of Montague Grammar (see [5]) and into integrating it with Frank Veltmans Update Semantics (see [6]).

DPL is a theory of testing and resetting of variables/registers. These are fundamental operations in computer science. Thus, apart from its use in Logical Semantics, *DPL* is a simple theory of these basic operations.[1]

DPL is a natural variant of Predicate Logic. It mainly differs in the treatment of the scope of the existential quantifier. Certain basic truths about variables in Predicate Logic, however, fail in *DPL* (see section 2, see also [2] for similar observations on *DPLE*). The study of *DPL* and its kin makes the dependence of these truths on the specific choice of scoping mechanisms in standard Predicate Logic visible.

In the light of the varied interest of *DPL*, it seems a good idea to make a closer study of its metamathematical properties. We focus on the closely related questions:

- Which relations between assignments are definable in *DPL* (in a sense we will specify later)?
- How does *DPL* treat its variables?

To throw some light on both questions a good notion of context in *DPL* is indispensable. When studying classical *DPL*, which is based on total assignments, contexts appear as *objets trouvés*. They are not part of the design of the language, but –as will become clear in the paper– can be viewed as the result of 'abstracting away' or 'erasing' certain properties of the predicate logical language (both vocabulary and structure), thus yielding an underspecified language. Underspecification simply means here that the denotations or meanings of the contexts are properties of relations, rather than relations.

[1] *DPL* is just one theory in a family of alternatives to predicate logic. In these alternatives 'resetting a register' is replaced by other related actions, like 'create a new register'. See section 7 for references.

We can view what happens in the paper in a different light. Our deeper interest is in such grand questions as:

- What should a general theory of information processing look like?
- What is the nature of the variable?

It seems to me that these questions are closely related and that ideas involving both Dynamics and Contexts should play a role in the answer. The study of *DPL*, here, is analogous to the study of a fruitfly in a laboratory situation. It allows us to focus on problems involving dynamics, variables, information processing. Still, these problems remain feasible.

We end this introduction with a brief sketch of the paper. Section 2 is a straightforward introduction to predicate logic. It contains all the *technical* material the reader needs to know. For a discussion of the applications to discourse phenomena, however, the reader should consult [1]. Section 3 presents our theory of contexts in *DPL*. We study contexts as mathematical objects in their own right and establish their connections to language and semantics. Some materials concerning the information ordering on contexts are placed in an appendix. In section 4, we treat the Switching Propery. This property is characteristic for the *DPL*-definable relations. Section 5 contains the main result of the paper: a relation over an infinite domain is *DPL*-definable iff it 'has' a context and satisfies the Switching Property. In the next section, we touch on the subject of extending the *DPL*-language with new operations such as conjunction and disjunction. We consider the question whether such extensions support a good theory of contexts. We will produce two extensions that are *complete* for all relations that have a context. In other words: all such relations are definable in those extensions. Our last section 7 is devoted to the idea of making the context *part of* the semantics.

2 What is Dynamic Predicate Logic?

We provide the basic definitions of *DPL*. Nothing in this section pretends to be original. We start by introducing some basic relational notions.

Definition 2.1 Let X be any non-empty set. *Rel(X)* is the set of binary relations on X, i.e., $Rel(X) := \wp(X \times X)$. Let $R, S \in Rel(X)$. We define:

1. The composition $R \circ S$ of R and S is defined by: $x(R \circ S)y :\Leftrightarrow \exists z\ xRzSy$. Note that composition is in the order of application.
2. The dynamic implication $(R \rightarrow S)$ between R and S is defined by:
$$x(R \rightarrow S)y :\Leftrightarrow x = y \textit{ and } \forall z(xRz \Rightarrow \exists u\ zSu).$$
Our use of $\rightarrow$ here overloads the symbol, since we also use it for implication in the objectlanguage. We write $\neg(R)$ for $(R \rightarrow \emptyset)$
3. id_X is the identity relation. R is a *condition* or *test* if $R \subseteq id_X$.
4. Consider $Y \subseteq X$. We define $diag(Y) := \{\langle y, y\rangle \in X \times X \mid y \in Y\}$.
5. $dom(R) := \{x \in X \mid \exists y\ xRy\}$ and $cod(R) := \{y \in X \mid \exists x\ xRy\}$.

□

The notion of dynamic implication was first introduced by Hans Kamp in his pioneering paper [4]. Note that: $\neg\neg(R) = (id_X \rightarrow R) = dom(R)$. The relations in the range of *diag* are precisely the conditions. Writing $(Y \rightarrow Z) := (X \setminus Y) \cup Z$, we have:

- $diag(X) = id_X$
- $diag(\emptyset) = \emptyset$
- $diag(Y \cap Z) = diag(Y) \circ diag(Z) = diag(Y) \cap diag(Z)$
- $diag(Y \cup Z) = diag(Y) \cup diag(Z)$
- $diag(Y \to Z) = diag(Y) \to diag(Z)$

Thus, *diag* is a homomorphic embedding of the structure $\langle \wp X; X, \emptyset, \cap, \cap, \cup, \to \rangle$ in the structure $\langle Rel(X); id_X, \emptyset, \circ, \cap, \cup, \to \rangle$. We will sometimes confuse, in the relational context, the set X with the relation $diag(X)$. We need some further relational notions specifically concerned with relations between assignments.

Definition 2.2 Let D be a non-empty domain and let *Var* be a set of variables. Let $R \in Rel(D^{Var})$, $Y \subseteq D^{Var}$, $f \in D^{Var}$ and $V \subseteq Var$. We define:

- $f^{d_1,\ldots,d_n}_{v_1,\ldots,v_n}$ is the result of changing the values of f on the v_i to d_i.
- $f \mathcal{I}_V g :\Leftrightarrow$ for all $v \in V$ $f(v) = g(v)$.
- $[V] := \mathcal{I}_{Var \setminus V}$. We write $[v]$ for: $[\{v\}]$.
- Y is an $\langle V \rangle$*-set* if $f \in Y$ and $f \mathcal{I}_V g \Rightarrow g \in Y$. Y is *finitely restricted* if Y is a $\langle I \rangle$-set for some *finite* I. A condition is finitely restricted iff it is the image of a finitely restricted set.

If we want to make the dependence of $\mathcal{I}$ or $[.]$ on *Var* or D visible, we add them as subscripts. ▯

We collect some simple facts concerning these notions. We have:

- $\mathcal{I}_\emptyset = [Var] = D^{Var} \times D^{Var}$, $\mathcal{I}_{Var} = [\emptyset] = id_{D^{Var}}$
- $\mathcal{I}_V \circ \mathcal{I}_W = \mathcal{I}_{V \cap W}$, $[V] \circ [W] = [V \cup W]$
- $\mathcal{I}_V \cap \mathcal{I}_W = \mathcal{I}_{V \cup W}$, $[V] \cap [W] = [V \cap W]$
- $\mathcal{I}_V \cup \mathcal{I}_W \subseteq \mathcal{I}_{V \cap W}$, $[V] \cup [W] \subseteq [V \cup W]$.

Note that the classical meaning of the existential quantifier as a 'cylindrification' can be given as: $\exists x(Y) := dom([x] \circ diag(Y))$. We turn to the definition of *DPL*.

Definition 2.3 A *DPL*-language $\mathcal{L}$ is a structure $\langle Pred, Ar, Var, Con \rangle$, where *Pred* is a set of predicate symbols; *Ar* is a function from *Pred* to the natural numbers (including 0); *Var* is a, possibly empty, set of variables, *Con* is a, possibly empty, set of constants. Let $Ref := Var \cup Con$ be the set of *referents*. We will use $v, w, \ldots$ for variables, $c, c', \ldots$ for constants and $r, s, \ldots$ for referents. The set of $\mathcal{L}$-*formulas*, $For_\mathcal{L}$, is the smallest set such that:

- $P(r_1, \cdots, r_n) \in For_\mathcal{L}$, for $P \in Pred$ with $Ar(P) = n$ and $r_1, \ldots, r_n \in Ref$
- $\top$, $\bot$, $r = s$, $\exists v$ are in $For_\mathcal{L}$ for $r, s \in Ref$ and $v \in Var$
- If $\phi, \psi \in For_\mathcal{L}$, then so are $\phi.\psi$ and $(\phi \to \psi)$

I feel that it is more faithful to the semantics to leave out the brackets in the formation rule for the dot *officially*, but nothing important hangs on this choice in this paper. We get an ambiguous syntax, but still unique meanings, since the operation of composition –the semantic counterpart of "."– is associative. An alternative notation for $\exists v$, is $[v :=?]$ (random reset). We use $\neg(\phi)$ and $\forall v(\phi)$ as abbreviations of, respectively, $(\phi \to \bot)$ and $(\exists v \to \phi)$. If $x \in Var$ and $r \in Ref$ and x and r are distinct, we write $[x := r]$ for: $\exists x.x = r$. ▯

Definition 2.4 A *DPL*-model $\mathfrak{M}$ for a *DPL*-language $\mathcal{L}$ is a structure $\langle D, I\rangle$, where D is a non-empty set, the *domain* of $\mathfrak{M}$; I is a function which assigns to each predicate symbol P of $Pred_{\mathcal{L}}$ an $Ar(P)$-ary relation on D and to each constant c an element of D. $Ass_{\mathfrak{M}}$, the set of *assignments* for $\mathfrak{M}$, is D^{Var}. Consider $r \in Ref$. We define:

$$|r|_{\mathfrak{M},f} := \begin{cases} f(r) & \text{if } r \in Var \\ I(r) & \text{if } r \in Con \end{cases}$$

The interpretation function $[.]_{\mathfrak{M}} : For_{\mathcal{L}} \to Rel(Ass_{\mathfrak{M}})$ is given as follows.

- $[P(r_1, \cdots, r_n)]_{\mathfrak{M}} := diag(\{f \in D^{Var} \mid \langle |r_1|_{\mathfrak{M},f}, \cdots, |r_n|_{\mathfrak{M},f}\rangle \in I(P)\})$
- $[\top]_{\mathfrak{M}} := id_{D^{Var}}$, $[\bot]_{\mathfrak{M}} := \emptyset$
- $[r = s]_{\mathfrak{M}} := diag(\{f \in D^{Var} \mid |r|_{\mathfrak{M},f} = |s|_{\mathfrak{M},f}\})$
- $[\exists v]_{\mathfrak{M}} := [v]_D$
- $[\phi.\psi]_{\mathfrak{M}} := [\phi]_{\mathfrak{M}} \circ [\psi]_{\mathfrak{M}}$
- $[(\phi \to \psi)]_{\mathfrak{M}} := ([\phi]_{\mathfrak{M}} \to [\psi]_{\mathfrak{M}})$

We write $\phi \equiv_{\mathfrak{M}} \psi$ for $[\phi]_{\mathfrak{M}} = [\psi]_{\mathfrak{M}}$. We define validity in *DPL* by:

$$\phi \models_{\mathfrak{M}} \psi \;:\Leftrightarrow\; \forall f, g(f[\phi]_{\mathfrak{M}} g \Rightarrow \exists h\, g[\psi]_{\mathfrak{M}} h).$$

As usual, $\phi \models \psi$ iff $\phi \models_{\mathfrak{M}} \psi$ for all models $\mathfrak{M}$ appropriate for the given language.

A binary relation R is *definable* in a *DPL*-model $\mathfrak{M}$ for a language $\mathcal{L}$ if there is an $\mathcal{L}$-formula ϕ, which defines R, i.e., $R = [\phi]_{\mathfrak{M}}$. ❑

We will often suppress the subscript $\mathfrak{M}$, when the model is clear from the context. We could extend the *DPL*-language with function symbols by copying the way this is done in ordinary predicate logic. However, for the kind of result we are after such an extension is immaterial, since the usual trick to eliminate function symbols works also in *DPL* —with a small twist. E.g., $P(f(g(x)))$ will be translated to: $\neg\neg(\exists u.G(x, u).\exists v.F(u, v).P(v))$.

We remind the reader of Geach's Donkey Sentence: *If a farmer owns a donkey, he beats it.* This sentence can be translated into *DPL* in a compositional way as:

$$(\exists x.farmer(x).\exists y.donkey(y).owns(x, y) \to beats(x, y)).$$

One striking feature of *DPL* is that it is not 'structural': the values the predicate symbols may assume are not all the possible meaning objects provided by the semantics; we only allow tests. A second striking feature is the *time symmetry* of resetting and composition, which constrast strongly with our *time asymetric intuition* about, say, the meaning of $P(x).\exists x.Q(x)$. The asymmetry of our intuition may be explained by the fact that we tend to think more in terms of *successful* resetting, i.e., $\mathfrak{M} \models P(x).\exists x.Q(x)$, than just in terms of what the resetting relation is.

Ordinary predicate logic can be interpreted in *DPL* as follows. We suppose that the predicate logical language has as connectives and quantifiers: $\top$, $\bot$, $\wedge$, $\to$, $\exists x$. We translate as follows:

- $(.)^*$ commutes with atomic formulas and with $\to$
- $(\phi \wedge \psi)^* = \phi^*.\psi^*$
- $(\exists x(\phi))^* = \neg\neg(\exists x.\phi^*)$

We find: $[\phi^*]_{\mathfrak{M}} = diag(\llbracket\phi\rrbracket_{\mathfrak{M}})$, where $\llbracket.\rrbracket$ is the usual valuation function of Predicate Logic. Our translation is compositional. It shows that we may consider Predicate Logic as a subsystem of *DPL*. There is also a kind of inverse translation $(.)^\circ$, which satisfies: $\llbracket\phi^\circ\rrbracket_{\mathfrak{M}} = dom([\phi^\circ])$. This translation involves renaming of variables and cannot be taken to show that *DPL* is a subsystem of Predicate Logic.

3 Contexts for Dynamic Predicate Logic

In this section, we study the notion of *context* and its connections with relations and language. We placed some materials concerning the information ordering on contexts in an appendix, since, on the one hand, they are conceptually relevant and have a clear place in the total picture, but, on the other hand, they have no direct bearing upon the main results of the paper.

3.1 Introductory remarks

To motivate our notion of contexts, we first give an intuitive discussion about substitution and kinds of variable occurrences in *DPL*.[2] In Predicate Logic variables may occur in a formula in two ways: freely and bound. The free variables admit (under certain conditions) substitution. The bound variables may be renamed salva significatione (α-conversion). Let's write $\sigma_x^t(\phi)$ for: the result of substituting t for x in ϕ. In Predicate Logic we have, e.g.,

$$f_x^{I(c)} \in [\![\phi]\!]_{\mathfrak{M}} \Leftrightarrow f \in [\![\sigma_x^c(\phi)]\!]_{\mathfrak{M}}.$$

What is the proper analogue of this fact for *DPL*? To simplify the discussion we will only treat a special case and refrain from giving official definitions. Consider the *DPL*-formula $P(x).\exists x.Q(x).\exists x.R(x)$. We have:

1. $f_x^{I(c)}[P(x).\exists x.Q(x).\exists x.R(x)]_{\mathfrak{M}}\, g \Leftrightarrow f[P(c).\exists x.Q(x).\exists x.R(x)]_{\mathfrak{M}}\, g$
2. $f[P(x).\exists x.Q(x).\exists x.R(x)]_{\mathfrak{M}}\, g_x^{I(c)} \Leftrightarrow f[P(x).\exists x.Q(x).\exists x.R(c)]_{\mathfrak{M}}\, g$

Meditation upon (1) and (2) suggests, that, in *DPL*, we have to distinguish two kinds of substitution *left* substitution and *right* substitution and corresponding to these kinds two kinds of 'free occurrence': *left* free and *right* free. We also speak of *input* occurrences and *output* occurrences. Following temporal intuitions —ignoring the essentially time-symmetric character of resetting and composition— we may also call the left free occurrences simply *free* and the right free occurrences *actively bound*. Now consider the following formula, say ϕ_0, in which we have tagged occurrences of x with superscript numerals.

$$P(x^1).\exists x^2.Q(x^3).\exists x^4.\neg\neg(\exists x^5.R(x^6)).S(x^7)$$

We see that x^1 is a (left) free or input occurrence. Left substitution for x will cause it to be replaced. If we form $T(x^0).\phi_0$, in the semantics the values assigned to x^0 and x^1 will be unified. If we form $\exists x.\phi_0$, x^1 will be 'bound' or 'initialized' by the new 'quantifier'. Symmetrically, x^7 is right free or actively bound. It will be in the scope of right substitution. If we form $\phi_0.T(x^0)$, the values of x^7 and x^0 will be unified, If we form $\phi_0.\exists x$, x^7 will be 'aborted'. Neither x^1 nor x^7 are open to α-conversion *salva significatione*. x^3 is not accessible for substitution, nor is it α-convertible: replacing x^2, x^3 and x^4 by, say, y, will result in a formula that resets y, which ϕ_0 doesn't do. We call x^3 a *garbage* occurrence: it is something that 'exists', but is no longer 'used'.[3] x^6 is also inaccessible for substitution, but in addition it can be α-converted: replacing x^5 and x^6 by y does not change the meaning of ϕ_0. We say that x^6 is *clasically* bound. Finally, we consider x^2, x^4 and x^5. These are 'occurrences' in a purely syntactical sense only: they do not represent 'files' carrying information, but just signal that incoming files labeled x should not be 'unified' with outgoing files labeled x. We say that these 'occurrences' are *blockers*. x^5 is not a blocker in ϕ_0 as a whole.[4]

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[2] The subject of kinds of variables and substitution would merit a far more extensive discussion. Regrettably, such a discussion is beyond the scope of the present paper.

[3] The notion of *garbage* is studied in [8] and in [9]

[4] Even if Vermeulens *DPLE* (see [7]) is quite close to *DPL*, the discussion of kinds of occurrences would be very different.

Contexts, in our present set-up, signal the presence of input occurrences, of blockers and of output occurrences.[5] They are abstract —in comparison with formulas— in the sense that they contain no information about the number or the place of these occurrences. Contexts can be studied independently from their connection with the logical language.

Contexts are familiar from Predicate Logic. There the context associated to a formula ϕ is simply the set F of free variables of ϕ.[6] A salient property of contexts in Predicate Logic is as follows. Suppose F is a context for ϕ, then $[\![\phi]\!]_{\mathfrak{M}}$ is F-restricted, i.e. $f \in [\![\phi]\!]_{\mathfrak{M}}$ and $f\mathcal{I}_F g$, implies $g \in [\![\phi]\!]_{\mathfrak{M}}$. Most of the work in this section will be devoted to proving the appropriate *DPL*-analogue of this property of Predicate Logic.

3.2 Contexts, considered by themselves

In this subsection, we treat contexts as mathematical objects in their own right. The natural connection with *DPL* will surface in the subsequent subsections.

Definition 3.1 A *DPL*-context is a triple $\langle I, B, O\rangle$, where I, B and O are finite sets of variables and where $I \backslash B = O \backslash B$, or, equivalently, $I \cup B = O \cup B$. The set I is the *input set*, i.e., the set on which the incoming assignments are constrained. The set O is the *output set*, i.e. the set on which the outgoing assignments are constrained. Finally, the set B is the set of blocks. This is the set of variables for which the identity between input and output value is cut through. The 'block' is a barrier between past and future, breaking the link between input- and output-value. We write $\downarrow$ for *is defined* or *converges*, and $\uparrow$ for *is undefined* or *diverges*. Define:

- $\mathfrak{id} := \langle \emptyset, \emptyset, \emptyset\rangle$
- $\langle I, B, O\rangle \bullet \langle I', B', O'\rangle := \langle I \cup (I'\backslash B), B \cup B', (O\backslash B') \cup O'\rangle$
- $\langle I, B, O\rangle \rightarrow \langle I', B', O'\rangle := \langle I \cup (I'\backslash B), \emptyset, I \cup (I'\backslash B)\rangle$
- $\langle I, B, O\rangle \leq \langle I', B', O'\rangle \Leftrightarrow I \subseteq I',\ O \subseteq O',\ B \subseteq B' \subseteq B \cup (I' \cap O')$
- $\cap$ is a partial operation on contexts, defined by:

$$\langle I, B, O\rangle \cap \langle I', B', O'\rangle := \begin{cases} \langle I \cap I', B \cap B', O \cap O'\rangle & \text{if } B \subseteq B' \cup (I \cap O) \\ & \text{and } B' \subseteq B \cup (I' \cap O') \\ \uparrow & \text{otherwise} \end{cases}$$

We will use $\mathfrak{c}$, $\mathfrak{d}$, ... as variables over contexts. We write $I_{\mathfrak{c}}$ for the first component of $\mathfrak{c}$, etcetera. ❑

The meanings of these objects, relations and operations will become apparent in subsection 3.3.

Lemma 3.2 The operations $\bullet$, $\rightarrow$ and $\cap$ are well defined. ❑

Proof

To see that $\bullet$ is well defined, note that:

$$\begin{aligned} (I \cup (I'\backslash B)) \cup (B \cup B') &= (I \cup B) \cup (I' \cup B') \\ &= (O \cup B) \cup (O' \cup B') \\ &= (O\backslash B') \cup O' \cup (B \cup B'). \end{aligned}$$

[5]In fact, there are good reasons also to put witnesses of *garbage* into the contexts. We will do not this in the present paper, since it is not necessery for our results here. Moreover, adding garbage leads to considerable complication of the framework and it necessitates bringing in Category Theory. We refer the reader further to [9].

[6]The reader is referred to [3] for a category-theoretical framework appropriate for the study of contexts in Predicate Logic.

It is trivial that $\rightarrow$ is well defined. For the proof that $\cap$ is well defined we refer the reader to the appendix. ❑

Theorem 3.3 *The contexts with* $\mathfrak{id}$ *and* $\bullet$ *form a* monoid. *Moreover,* $\leq$ *is a partial ordering.*

The proof of the theorem is easy. In the appendix we will show that $\cap$, if defined, is the infimum, w.r.t. $\leq$.

Consider the monoid of contexts. It can be represented in an alternative way, as follows. The monoid $\mathfrak{A}$ is the monoid on two generators $\mathfrak{a}$ and $\mathfrak{b}$, given by the equations: $\mathfrak{a} \bullet \mathfrak{a} = \mathfrak{a}$, $\mathfrak{b} \bullet \mathfrak{b} = \mathfrak{b}$ and $\mathfrak{b} \bullet \mathfrak{a} \bullet \mathfrak{b} = \mathfrak{b}$. The tabel of the monoidal operation $\bullet$ is a follows.

$\bullet$	$\mathfrak{e}$	$\mathfrak{a}$	$\mathfrak{b}$	$\mathfrak{ab}$	$\mathfrak{ba}$	$\mathfrak{aba}$
$\mathfrak{e}$	$\mathfrak{e}$	$\mathfrak{a}$	$\mathfrak{b}$	$\mathfrak{ab}$	$\mathfrak{ba}$	$\mathfrak{aba}$
$\mathfrak{a}$	$\mathfrak{a}$	$\mathfrak{a}$	$\mathfrak{ab}$	$\mathfrak{ab}$	$\mathfrak{aba}$	$\mathfrak{aba}$
$\mathfrak{b}$	$\mathfrak{b}$	$\mathfrak{ba}$	$\mathfrak{b}$	$\mathfrak{b}$	$\mathfrak{ba}$	$\mathfrak{ba}$
$\mathfrak{ab}$	$\mathfrak{ab}$	$\mathfrak{aba}$	$\mathfrak{ab}$	$\mathfrak{ab}$	$\mathfrak{aba}$	$\mathfrak{aba}$
$\mathfrak{ba}$	$\mathfrak{ba}$	$\mathfrak{ba}$	$\mathfrak{b}$	$\mathfrak{b}$	$\mathfrak{ba}$	$\mathfrak{ba}$
$\mathfrak{aba}$	$\mathfrak{aba}$	$\mathfrak{aba}$	$\mathfrak{ab}$	$\mathfrak{ab}$	$\mathfrak{aba}$	$\mathfrak{aba}$

The monoid of contexts is now given as the set of functions from *Var* to $\mathfrak{A}$ that are on all, but finitely many arguments equal to $\mathfrak{e}$. We put: $(f \bullet g)(v) := f(v) \bullet g(v)$. A triple $\langle I, B, O \rangle$ 'translates' to a function f with, e.g., $f(x) = \mathfrak{ab}$ iff $x \in I$, $x \in B$ and $x \notin O$, etcetera. A function f translates to a triple $\langle I, B, O \rangle$ with, e.g., $x \in I$ iff $f(x) \in \{\mathfrak{a}, \mathfrak{ab}, \mathfrak{aba}\}$, etcetera. It is easily seen that these 'translations' give us an isomorphism of monoids between the representations. $\mathfrak{A}$ is in fact isomorphic to the monoid of contexts in the case that $Var = \{x\}$. The alternative representation is possible by the fact that in our monoidal operation treats all variables 'independently'. It is not difficult to extend the structure on $\mathfrak{a}, \mathfrak{b}, \ldots$, to get a function representation also for $\rightarrow$, $\leq$ and $\cap$.

In this paper we will stick to the set representation, since this representation is closest to the relational notions we will need to formulate our theorem on contexts and relations —the theorem that tells us what the contexts *do*. The function representation, however, has two advantages. First, it is easier to use for doing computations 'in the head'. Secondly, its connection with the framework developed in [9] to study contexts, is more perspicuous.

3.3 Contexts and relations

We turn to the connection between contexts and relations.[7] We show that this connection 'commutes' w.r.t. $\bullet/\circ$, $\rightarrow/\rightarrow$ and $\leq/\subseteq$. We fix a non-empty domain D.

Definition 3.4 Consider a relation R on D^{Var}. R is an $\langle I, B, O \rangle$*-relation* if $R = (\mathcal{I}_I \circ R \circ \mathcal{I}_O) \cap [B]$. We say that $\mathfrak{c}$ is a context *for* R, if R is an $\mathfrak{c}$-relation. R is an *IBO-relation* if R is an $\langle I, B, O \rangle$-relation for some $\langle I, B, O \rangle$. We assign to each context $\mathfrak{c}$ the property or 'meaning' $[\![\mathfrak{c}]\!]_D$, the set of all $\mathfrak{c}$-relations on D^{Var}. (We will often suppress the subscript D.) ❑

The heuristic for this property is as follows. R is $\langle I, B, O \rangle$ means that R is only concerned with the values of the incoming assignment on I; R only cares about the values of the outgoing assignment on O; all this under the constraint that in going from input to output only values of variables in B are changed. Before proving some facts about the notion introduced above, we sample some immediate insights.

[7]The semantics for contexts given here certainly does not exhaust all possible uses of contexts. E.g., the problem of explaining what it is to be a variable occurrence of a certain kind is left untouched. Undoubtedly, contexts will play a role in solving this problem.

- $\{\emptyset, [B_{\mathfrak{c}}]\} \subseteq [\![\mathfrak{c}]\!]$
- $[\![\langle \emptyset, B, \emptyset \rangle]\!] = \{\emptyset, [B]\}$
- R is a $\langle I, \emptyset, I \rangle$-relation precisely if R is an $\langle I \rangle$-condition.

We show that $\leq$ on contexts describes the 'information ordering' on contexts. The idea is simply that $\mathfrak{c}$ is *more* informative than $\mathfrak{d}$ if $[\![\mathfrak{c}]\!] \subseteq [\![\mathfrak{d}]\!]$.

Theorem 3.5 *1.* $\mathfrak{c} \leq \mathfrak{d} \Rightarrow [\![\mathfrak{c}]\!] \subseteq [\![\mathfrak{d}]\!]$.

2. Suppose $|D| \geq 2$. *Then:* $[\![\mathfrak{c}]\!] \subseteq [\![\mathfrak{d}]\!] \Rightarrow \mathfrak{c} \leq \mathfrak{d}$.

Proof

Let $\mathfrak{c} = \langle I, B, O \rangle$ and $\mathfrak{d} = \langle I', B', O' \rangle$. We prove (1). Let $\mathfrak{c} \leq \mathfrak{d}$ and $R \in [\![\mathfrak{c}]\!]$. We want to show: $R = (\mathcal{I}_{I'} \circ R \circ \mathcal{I}_{O'}) \cap [B']$. Trivially, R is contained in $(\mathcal{I}_{I'} \circ R \circ \mathcal{I}_{O'}) \cap [B']$. We show $\mathcal{I}_{I'} \circ R \circ \mathcal{I}_{O'} \subseteq R$. Suppose $f' \mathcal{I}_{I'} f R g \mathcal{I}_{O'} g'$ and $f'[B']g'$. It is immediate, that $f' \mathcal{I}_I f R g \mathcal{I}_O g'$. We show that $f[B]g$. Suppose $v \in B' \backslash B$. Then, $v \in I' \cap O'$. We find: $f'(v) = f(v)$, since $v \in I'$. Moreover, $f(v) = g(v)$, since fRg, $R \subseteq [B]$ and $v \notin B$. Finally, $g(v) = g'(v)$, since $v \in O'$. Since R is a $\mathfrak{c}$-relation, we may conclude that $f'Rg'$.

We prove (2). We write X^c for $Var \backslash X$. Let $|D| \geq 2$ and $[\![\mathfrak{c}]\!] \subseteq [\![\mathfrak{d}]\!]$. Since, $[B] \in [\![\mathfrak{c}]\!]$, we have $[B] \in [\![\mathfrak{d}]\!]$. Hence, using the facts of page 3, we find:

$$\begin{aligned} [B] &= (\mathcal{I}_{I'} \circ [B] \circ \mathcal{I}_{O'}) \cap [B'] \\ &= [(I'^c \cup B \cup O'^c) \cap B']. \end{aligned}$$

Since, $|D| \geq 2$ it follows that: $B = (I'^c \cup B \cup O'^c) \cap B'$. Now it is immediate that: $B \subseteq B' \subseteq B \cup (I' \cap O')$.

The arguments that $I \subseteq I'$ and that $O \subseteq O'$ are analogous to one another. We give the argument for the I-case. Suppose, for a reductio, that $v \in I \backslash I'$. Let d and e be distinct elements of D. We write $[v = d]$ for the test: *is f(v)=d?*. Consider the relation $R := [v = d] \circ [B]$. Clearly, $R \in [\![\mathfrak{c}]\!]$ and, so, $R \in [\![\mathfrak{d}]\!]$. Consider f with $f(v) = d$. In case $v \in B'$, we have: $f_v^e \mathcal{I}_{I'} f R g \mathcal{I}_{O'} g$. Since, as we have already shown, $B \subseteq B'$, we find: $f_v^e [B'] g$. Hence, since $R \in [\![\mathfrak{d}]\!]$, we have $f_v^d R g$. Quod non. We turn to the case that $v \notin B'$. Since $I' \backslash B' = O' \backslash B'$, we find: $v \notin O'$. So: $f_v^e \mathcal{I}_{I'} f R g \mathcal{I}_{O'} g_v^e$. Since $B \subseteq B'$, we find $f_v^e [B'] g_v^e$. Thus, since $R \in [\![\mathfrak{d}]\!]$, we have $f_v^d R g_v^d$. Quod non. ❑

Theorem 3.5 shows clearly that contexts stand in a many-many relation to relations. Thus, the question *What is the context of R?* has no definite answer. In the appendix we show that, but for one notable exception, every relation has a most informative context. The next lemma may be used in some cases to simplify the verification that a relation is $\langle I, B, O \rangle$.

Lemma 3.6 R is a $\langle I, B, O \rangle$-relation iff $R = (\mathcal{I}_I \circ R \circ \mathcal{I}_{O \cap B}) \cap [B]$. ❑

Proof

It is clearly sufficient to show that for *any* $R \subseteq [B]$:

$$(\mathcal{I}_I \circ R \circ \mathcal{I}_O) \cap [B] = (\mathcal{I}_I \circ R \circ \mathcal{I}_{O \cap B}) \cap [B].$$

¿From left to right is immediate, since $\mathcal{I}_O \subseteq \mathcal{I}_{O \cap B}$. For the converse, suppose $f' \mathcal{I}_I f R g \mathcal{I}_{O \cap B} g'$ and $f'[B]g'$. We have to show: $g \mathcal{I}_O g'$. Consider $v \in O$. In case $v \in B$, we have $g(v) = g'(v)$. Suppose $v \in O \backslash B$. Then, also $v \in I \backslash B$. We have: $g'(v) = f'(v)$, since $v \notin B$. Moreover, $f'(v) = f(v)$, since $v \in I$. Also $f(v) = g(v)$, since fRg, $R \subseteq [B]$ and $v \notin B$. Putting the identities together, we find $g'(v) = g(v)$. ❑

The following lemma is quite useful in applications (see example 3.14). We write $A := I \cup B$.

Lemma 3.7 Suppose R is an $\langle I, B, O\rangle$-relation and $f'\mathcal{I}_I fRg$. Then there is a unique g', such that $f'Rg'\mathcal{I}_B g$. This g' has the following property: for any set of variables J, if $f'\mathcal{I}_J f$, then $g'\mathcal{I}_J g$. As a consequence, we find that $g'\mathcal{I}_I g$ and, hence, $g'\mathcal{I}_A g$. So, also, $g'\mathcal{I}_O g$. ❑

Proof

Let R be an $\langle I, B, O\rangle$-relation with $f'\mathcal{I}_I fRg$. Any g' with $f'Rg'\mathcal{I}_B g$, must satisfy: $f'[B]g'\mathcal{I}_B g$. So the only possible choice of such a g' is: $f' \restriction (Var\backslash B) \cup g \restriction B$. We verify that g', thus defined, satisfies $f'Rg'$. It is sufficient to show that $g\mathcal{I}_O g'$. This, in its turn, follows immediately from the property in the last part of the theorem.

Consider any set J such that $f'\mathcal{I}_J f$. Suppose $v \in J$. In case $v \in B$, we have $g(v) = g'(v)$. In case $v \notin B$, we have, $g(v) = f(v)$, since $v \notin B$, Moreover, $f(v) = f'(v)$, since $v \in J$. Finally $f'(v) = g'(v)$, since $v \notin B$. Putting things together, we find $g(v) = g(v')$, as desired. ❑

Note that in the lemma O plays no significant role. Due to the 'forward looking' and time asymmetric nature, however, of the definitions of implication and validity in *DPL*, it is sufficient for most applications. An immediate consequence of the lemma is that if R is an $\langle I, B, O\rangle$-relation, then *dom(R)* is an $\langle I\rangle$-condition and (by symmetry) *cod(R)* is an $\langle O\rangle$-condition.

Theorem 3.8 *Suppose R is a $\mathfrak{c}$-relation and S is a $\mathfrak{d}$-relation. Then $R \circ S$ is a $\mathfrak{c} \bullet \mathfrak{d}$-relation.*

Proof

It is easy to see that: $R \circ S \subseteq (\mathcal{I}_{I''} \circ (R \circ S) \circ \mathcal{I}_{O''}) \cap [B'']$ For the converse, suppose that $f'[B'']g'$, $f'\mathcal{I}_{I''} f$, $f(R \circ S)g$ and $g\mathcal{I}_{O''} g'$. We have to show: $f'(R \circ S)g'$. For some h, we have $fRhSg$. We partition Var into three sets $X_1 := O \cup I'$, $X_2 := B\backslash(O \cup I' \cup B')$ and $X_3 := Var\backslash((B\backslash B') \cup O \cup I')$. Define: $h' := h\restriction X_1 \cup g'\restriction X_2 \cup f'\restriction X_3$. We show that $f'Rh'Sg'$. We first prove that $f'Rh'$. We check the conditions for applying the fact that R is an $\langle I, B, O\rangle$-relation.

1. $f'\mathcal{I}_{I''} f$ and, hence, since $I \subseteq I''$, $f'\mathcal{I}_I f$.

2. fRh.

3. $h\mathcal{I}_{O \cup I'} h'$ and, hence, $h\mathcal{I}_O h'$.

4. We show that $f'[B]h'$. Consider a variable v not in B. We have to show $f'(v) = h'(v)$. We can only run into trouble in case v is not in X_3, i.e., if v is in $B\backslash B'$ or in $O \cup I'$. The first possibility is excluded, by the fact that v is not in B. Suppose v is in $O \cup I'$. Then: $h'(v) = h(v)$, by definition. R is an $\langle I, B, O\rangle$-relation, so $R \subseteq [B]$. We may conclude that $h(v) = f(v)$, since $v \notin B$. In case $v \in O$, we find: $v \in O\backslash B = I\backslash B \subseteq I \subseteq I''$. In case $v \in I'$, we find: $v \in I'\backslash B \subseteq I''$. So in both cases: $v \in I''$. Since $f'\mathcal{I}_{I''} f$, it follows that $f(v) = f'(v)$. Composing the identities, we find $h'(v) = f'(v)$, as desired.

By (1)-(4), we may conclude that $f'Rh'$. Next, we check the conditions for applying the fact that S is an $\langle I', B', O'\rangle$-relation.

1. We have $h\mathcal{I}_{O \cup I'} h'$ and, hence, $h'\mathcal{I}_{I'} h$.

2. hSg.

3. We have $g\mathcal{I}_{O''} g'$ and, hence, since $O' \subseteq O''$, $g\mathcal{I}_{O'} g'$.

4. We show that $h'[B']g'$. Consider a variable v not in B'. We have to show $h'(v) = g'(v)$. Inspecting the definition of h', we see that our desired identity can only fail if either $v \notin B$ or $v \in B$ and $v \in O \cup I' \cup B'$. We consider the case that $v \notin B$. We already showed that $f'[B]h'$. Moreover, we assumed $f'[B'']g'$. Hence: $h'[B]f'[B \cup B']g'$. So, $h'[B \cup B']g'$. Since $v \notin B \cup B'$, we find: $h'(v) = g'(v)$. Next, we consider the case that $v \in B$ and $v \in O \cup I' \cup B'$. Since, we choose v outside of B', we need only consider the possibility that $v \in O \cup I'$. We have, by definition, $h'(v) = h(v)$. Since $v \notin B'$, we find $h(v) = g(v)$. By our early assumption, $g\mathcal{I}_{O''}g'$, where $O'' = (O \backslash B') \cup O'$. If $v \in O$, then clearly $v \in O \backslash B'$, and we find $g(v) = g'(v)$. If $v \in I'$, we have $v \in I' \backslash B' = O' \backslash B' \subseteq O'$, hence, again, $g(v) = g'(v)$. Putting the identities together, we find $h'(v) = g'(v)$, as desired.

By (1)-(4), we have: $h'Sg'$. ❑

Theorem 3.9 *Suppose R is a $\mathfrak{c}$-relation and S is a $\mathfrak{d}$-relation. Then $R \rightarrow S$ is a $\mathfrak{c}\rightarrow\mathfrak{d}$-relation.*

Proof

Suppose R is a $\mathfrak{c}$-relation and S is a $\mathfrak{d}$-relation. Let $\mathfrak{c} = \langle I, B, O\rangle$, $\mathfrak{d} = \langle I', B', O'\rangle$ and $(\mathfrak{c} \rightarrow \mathfrak{d}) = \langle I'', \emptyset, I''\rangle$. Trivially, $(R \rightarrow S) \subseteq [\emptyset]$. Moreover, since $id \subseteq \mathcal{I}_{I''}$, $(R \rightarrow S) \subseteq (\mathcal{I}_{I''} \circ (R \rightarrow S) \circ \mathcal{I}_{I''})$

To prove the converse, suppose $f'\mathcal{I}_{I''}f$ and $f(R \rightarrow S)f$. We have to show that $f'(R \rightarrow S)f'$. Suppose $f'Rg'$. By lemma 3.7, there is a g such that $fRg\mathcal{I}_{I''\cup B}g'$. It follows that $g'\mathcal{I}_{I'}g$. Since $f(R \rightarrow S)f$, we can find an h, such that gSh. Again applying lemma 3.7, we find an h' with $g'Sh'$. ❑

We close this subsection with a language-free soundness result.

Definition 3.10 A relation on D^{Var} is *DPL*-definable over D iff it can be generated by composition from resettings $[v]$ and finitely restricted conditions over D ❑

Theorem 3.11 *Every DPL-definable relation over D is an IBO-relation.*

The proof is an obvious induction on the way the relation is generated.

3.4 Contexts and language

We turn to our discussion of how contexts are connected to formulas.

Definition 3.12 We assign to every *DPL*-formula ϕ a context $\mathfrak{c}_\phi$. Define:

- $\mathfrak{c}_{P(r_1,\cdots,r_n)} = \langle V, \emptyset, V\rangle$, where $V = \{r_1, \cdots, r_n\} \cap Var$
- $\mathfrak{c}_\top = \mathfrak{c}_\bot = \langle\emptyset, \emptyset, \emptyset\rangle$, $\mathfrak{c}_{r=s} = \langle\{r, s\} \cap Var, \emptyset, \{r, s\} \cap Var\rangle$, $\mathfrak{c}_{\exists v} = \langle\emptyset, \{v\}, \emptyset\rangle$
- $\mathfrak{c}_{\phi.\psi} = \mathfrak{c}_\phi \bullet \mathfrak{c}_\psi$ and $\mathfrak{c}_{(\phi\rightarrow\psi)} = (\mathfrak{c}_\phi \rightarrow \mathfrak{c}_\psi)$

We write I_ϕ for $I_{\mathfrak{c}_\phi}$, etcetera. ❑

Note that the definition correctly defines a function, by the associativity of $\bullet$. We now prove the main theorem of this section.

Theorem 3.13 *For every formula ϕ, $[\phi]$ is a $\mathfrak{c}_\phi$-relation on D^{Var}.*

Proof

The proof is by induction on ϕ using theorems 3.8 and 3.3. The atomic cases are easy. ❑

We obtain the following picture of the way contexts work: ϕ is mapped to $\mathfrak{c}_\phi$ by abstracting both from part of the vocabulary and part of the structure. $\mathfrak{c}_\phi$ is mapped to $[\![\mathfrak{c}_\phi]\!]_D$, a property of relations. Via a different route ϕ is mapped to the relation $[\phi]_\mathfrak{M}$. The two routes are connected by the theorem that $[\phi]_\mathfrak{M} \in [\![\mathfrak{c}_\phi]\!]_D$.

Note that $\mathfrak{c}_\phi$ is not always the $\leq$-minimal context of $[\phi]_\mathfrak{M}$, as is witnessed by the fact that $\mathfrak{c}_{x=x} = \langle\{x\}, \emptyset, \{x\}\rangle$ and that $[x = x]_\mathfrak{M}$ is the identity on D^{Var}, thus admitting the context $\langle\emptyset, \emptyset, \emptyset\rangle$

Example 3.14 We provide two examples of how theorem 3.13 in combination with lemma 3.7 can be used to verify a valid principle for *DPL*. We first prove:

$$\chi.\phi \models \phi, \text{ if } B_\phi \cap I_\phi = \emptyset.$$

Suppose $B_\phi \cap I_\phi = \emptyset$ and $f[\chi.\phi]g$. We have to produce an h with $g[\phi]h$. Since $f[\chi.\phi]g$, we can find a j with $j[\phi]g$. By theorem 3.13: $j[B_\phi]g$. Since, $B_\phi \cap I_\phi = \emptyset$, we find: $g\mathcal{I}_{I_\phi}j$. So, $g\mathcal{I}_{I_\phi}j[\phi]g$. By theorem 3.13 and lemma 3.7 we can find an h with $g[\phi]h$.

As a second example we prove:

$$\chi \models \phi.\psi \Rightarrow \chi \models \psi, \text{ if } B_\phi \cap I_\psi = \emptyset$$

Suppose $B_\phi \cap I_\psi = \emptyset$, $\chi \models \phi.\psi$ and $f[\chi]g$. We have to produce an h with $f[\psi]h$. By our assumptions, there are i and j such that $f[\phi]i[\psi]j$. Hence, $f[B_\phi]i$ and so $f\mathcal{I}_{I_\psi}i$. Thus, $f\mathcal{I}_{I_\psi}i[\psi]j$. We may conclude that there is an h with $f[\psi]h$.

The examples demonstrate the role contexts must play in the formulation of schematic principles for *DPL*. ❑

4 The Switching Property

In section 3 we introduced contexts or *IBO*'s as properties of relations and showed that every $[\phi]$ is an *IBO*-relations. A first conjecture for characterizing the *DPL*-definable relations would be that these are precisely the *IBO*-relations. We will see, however, that this conjecture is false. To characterize the *DPL*-definable relations we need one extra property: the Switching Property. In the present section we will prove that the *DPL*-definable relations *do* have the Switching Property (*soundness*). In section 5 will will show that every *IBO*-relation on an infinite domain that has the Switching Property is *DPL*-definable (*completeness*).

Definition 4.1 A relation R on D^{Var} has the *switching property* if it is either a condition or there are variables x and y (not necessarily distinct), such that $R = dom(R) \circ [x] \circ R \circ [y] \circ cod(R)$. If the second case obtains, we call the variables x and y involved a pair of *switching variables*. There might be more than one pair of switching variables. ❑

There are various other ways to define the Switching Property, but, I submit, the one presented here is the most natural one. In the lemma below, we collect some helpful insights.

Lemma 4.2 Suppose R is a relation on D^{Var}

1. $R = dom(R) \circ R \circ cod(R)$.
2. Suppose R is $C \circ T \circ C'$, where C and C' are conditions and where T is a relation. Then $dom(R) \circ C = dom(R)$ and $C' \circ cod(R) = cod(R)$.

3. Suppose C is a condition. Then, $C \circ [x] \circ C \circ [x] = C \circ [x]$ and $[x] \circ C \circ [x] \circ C = [x] \circ C$.

□

The easy proof is left to the reader.

Theorem 4.3 *Every DPL-definable relation over D has the switching property.*

Proof

In case R is a condition, we are done. Suppose R is not a condition. As is easily seen, R must be of the form $C \circ [x] \circ S \circ [y] \circ C'$, for some variables x, y, some conditions C, C', and some relation S. (In a formula ϕ defining R, x would correspond to the first existential quantifier occurring in ϕ, y to the last. Note that we allow x and y to be the same variable and even the first and last existential quantifier occurrence to be the same occurrence.) We have, using lemma 4.2:

$$\begin{aligned} \mathit{dom}(R) \circ [x] \circ C \circ [x] \circ S \circ [y] \circ C' \circ [y] \circ \mathit{cod}(R) &= \\ \mathit{dom}(R) \circ C \circ [x] \circ C \circ [x] \circ S \circ [y] \circ C' \circ [y] \circ C' \circ \mathit{cod}(R) &= \\ \mathit{dom}(R) \circ C \circ [x] \circ S \circ [y] \circ C' \circ \mathit{cod}(R) &= \\ \mathit{dom}(R) \circ R \circ \mathit{cod}(R) &= R \end{aligned}$$

□

The results of section 3 and of this section combine to the obvious 'soundness'-result:

Theorem 4.4 *Every DPL-definable relation over D is an IBO-relation with the Switching Property.*

Example 4.5 We show how to use the Switching Property to prove that certain relations are not *DPL*-definable. Suppose $|D| \geq 2$.

- Let $R := [x := y, y := x]$, where $f[x := y, y := x]g :\Leftrightarrow g = f_{x,y}^{f(y),f(x)}$. R is an *IBO*-relation, with context $\langle\{x,y\},\{x,y\},\{x,y\}\rangle$. Suppose R has the Switching Property. R is evidently not a condition. Let v, w be a pair of switching variables. Let fRg, $f(v) = d$, and $d \neq e$. Using the fact that the domain of R is the set of all assignments, we find:

  $$f_v^e(\mathit{dom}(R))f_v^e[v]fRg[w]g(\mathit{cod}(R))g.$$

  Hence, by the switching property: $f_v^e Rg$. But R is obviously injective. So we have a contradiction.

- Suppose our model $\mathfrak{M}$ is the usual structure of the natural numbers. Let $S := [x := x + 1]$, where $f[x := x + 1]g :\Leftrightarrow g = f_x^{f(x)+1}$. S is an *IBO*-relation with context $\langle\{x\},\{x\},\{x\}\rangle$. S does not have the Switching Property since: S is not a condition; S has as domain the set of all assignments; S is injective.

- Let $T := [(\exists x \vee \exists y)]$, where $[(\exists x \vee \exists y)] := [\exists x] \cup [\exists y]$. T is an *IBO*-relation. Surprisingly, the best context we can find for it is $\langle\{x,y\},\{x,y\},\{x,y\}\rangle$.[8] Suppose that T has the Switching Property. T is not a condition, so we can find switching variables v and w. By symmetry we may assume that $v \neq x$. We can find f and g with fSg and $f(x) \neq g(x)$. Choose d with $d \neq g(v)$. Using the fact that the domain of T is the set of all assignments, we find:

  $$f_v^d(\mathit{dom}(S))f_v^d[v]fTg[w]g(\mathit{cod}(T))g.$$

  By the Switching Property: $f_v^d Tg$. But we have two distinct variables x and v such that $f_v^d(x) = f(x) \neq g(x)$ and $f_v^d(v) = d \neq g(v)$. This is clearly impossible.

□

[8] We will discuss this fenomenon in more detail in subsection 6.3.

5 The DPL-definable relations on an infinite domain

In sections 3 and 4 we have seen that the *DPL*-definable relations are *IBO*-relations with the Switching Property. Here we show the converse —for the case that the domain, D, is infinite.

Theorem 5.1 *Let D be an* infinite *set. Then the DPL-definable relations over D are precisely the IBO-relations with the switching property on D^{Var}.*

Proof

One direction is by our previous results. Let R be an $\langle I,B,O\rangle$-relation with the switching property. In case R is a condition we are done. Suppose R is not a condition and let x, y be a pair of switching variables. By the switching property $R = dom(R) \circ [x] \circ R \circ [y] \circ cod(y)$. Note that *dom(R)* is an $\langle I\rangle$-condition and that *cod(R)* is an $\langle O\rangle$-condition. Thus, it is sufficient to show that $[x] \circ R \circ [y]$ is *DPL*-definable. $[x] \circ R \circ [y]$ is an $\langle I\backslash\{x\}, B, O\backslash\{y\}\rangle$-relation, where $x, y \in B$. After renaming, we see that it is sufficient to prove that any $\langle I,B,O\rangle$-relation R, with $x, y \in B$ and $x \notin I$ and $y \notin O$ is *DPL*-definable.

We will assume $x \neq y$. In case $x = y$, the proof is simpler. To increase readability, we will specify R in a *DPL*-language that we introduce along the way. Suppose $I = \{i_1, \cdots, i_m\}$, $B\backslash O = \{b_1, \cdots, b_n\}$ and $O \cap B = \{o_1, \cdots, o_p\}$. Here the i_k are supposed to be mutually distinct and similarly for the other sets. Since D is infinite there is a coding of finite sequences of elements of D in D. Par abus de langage, we will confuse this coding with our ordinary sequences of elements of D. Our language has an $(m+1)$-ary predicate symbol P, where:

$$\begin{aligned}\langle d_1, \cdots, d_m, e\rangle \in I(P) \Leftrightarrow \quad & \exists f, g \;\; fRg \text{ and } f(i_1) = d_1, \ldots, f(i_m) = d_m \text{ and} \\ & e = \langle g(o_1), \cdots, g(o_p)\rangle\end{aligned}$$

Remember that $[y := x]$ is short for $\exists y.y = x$. The formula ϕ is given by:

$$\exists x.P(i_1, \cdots, i_m, x).[y := x].\exists o_1. \cdots .\exists o_p.(y = \langle o_1, \cdots, o_p\rangle).\exists b_1. \cdots .\exists b_n$$

Here "$y = \langle o_1, \cdots, o_p\rangle$" stands for the obvious condition. Note that $[\phi]$ is an $\langle I,B,O\rangle$-relation. We claim that $R = [\phi]$. Suppose first that fRg. Take:

- $h_1 := f_x^{\langle g(o_1), \cdots, g(o_p)\rangle}$.

Remember that $x \notin \{i_1, \cdots, i_m\}$. We have:

$$\begin{aligned} h_1[P(i_1, \cdots, i_m, x)]h_1 \quad & \Leftrightarrow \quad \langle h_1(i_1), \cdots, h_1(i_m), h_1(x)\rangle \in I(P) \\ & \Leftrightarrow \quad \langle f(i_1), \cdots, f(i_m), \langle g(o_1), \cdots, g(o_p)\rangle\rangle \in I(P)\end{aligned}$$

Clearly, f and g witness that $\langle f(i_1), \cdots, f(i_m), \langle g(o_1), \cdots, g(o_p)\rangle\rangle$ is in $I(P)$. Next we set:

- $h_2 := (h_1)_y^{\langle g(o_1), \cdots, g(o_p)\rangle}$
- $h_3 := (h_2)_{o_1, \ldots, o_n}^{g(o_1), \ldots, g(o_n)}$
- $h_4 := (h_3)_{b_1, \ldots, b_n}^{g(b_1), \ldots, g(b_n)}$

We find (using $y \notin \{o_1, \cdots, o_p\}$):

$$\begin{array}{rll} f[\exists x]h_1 & h_1[P(i_1, \cdots, i_m, x)]h_1 & h_1[[y := x]]h_2 \\ h_2[\exists o_1. \cdots .\exists o_p]h_3 & h_3[y = \langle o_1, \cdots, o_p\rangle]h_3 & h_3[\exists b_1. \cdots .\exists b_n]h_4 \end{array}$$

Since $f[B]g$, it is easy to see that $g = h_4$.

For the converse, suppose $f'[\phi]g'$. Let h_1, h_2, h_3, be such that:

$$\begin{array}{rcc} f'[\exists x]h_1 & h_1[P(i_1,\cdots,i_m,x)]h_1 & h_1[[y := x]]h_2 \\ h_2[\exists o_1.\cdots.\exists o_p]h_3 & h_3[y = \langle o_1,\cdots,o_p\rangle]h_3 & h_3[\exists b_1.\cdots.\exists b_n]g' \end{array}$$

We are going to apply the fact that R is an $\langle I,B,O\rangle$-relation. By the fact that $h_1[P(i_1,\cdots,i_m,x)]h_1$ and by the definition of P, we can find f and g such that fRg, $f(i_1) = h_1(i_1)$,... $f(i_m) = h_1(i_m)$, and $\langle g(o_1),\cdots,g(o_p)\rangle = h_1(x)$. Since $x \notin \{i_1,\cdots,i_m\}$ and $f'[x]h_1$, it follows that $f'\mathcal{I}_I f$. Collecting what we have, we see:

- fRg.
- $f'[B]g'$
- $f'\mathcal{I}_I f$.

By lemma 3.6 we need to check:

- $g\mathcal{I}_{O\cap B}g'$.

Consider $v \in O \cap B$. We have: $h_1(x) = \langle g(o_1),\cdots g(o_p)\rangle$ and, hence, $h_2(y) = \langle g(o_1),\cdots g(o_p)\rangle$. Since $y \notin O$, we find: $h_3(y) = \langle g(o_1),\cdots g(o_p)\rangle$. Since $v \in \{o_1,\cdots,o_p\}$ and $h_3[y = \langle o_1,\cdots,o_p\rangle]h_3$, we find: $h_3(v) = g(v)$. Finally, v is not among the $b_1,\cdots,b_n$, and thus: $g'(v) = g(v)$.

Putting the itemized insights together, we may conclude: $f'Rg'$. ❑

6 Extensions of the DPL-language

We consider three extensions of the *DPL*-language. One with conjunction interpreted as intersection of relations, one with a new quantifier ∃ and one with disjunction interpreted as union of relations. We will show that our contexts work for each of these extensions. The contexts provided for disjunction are not optimally informative and intuitively queer, however. We will give some hints on how we think this apparent defect should be repaired. We show that ∃ is definable using ∧ and that in the system with ∃ all *IBO*-relations are definable.

6.1 Conjunction

We study the effect of adding intersection of relations to the *DPL*-repertoire. One way of of thinking about $R \cap S$ is as: *reset simultaneously via R and via S, and compare the results. If they are equal, make the output of our new relation the shared output, otherwise abort.* At the syntactical level, we reflect the new operation by extending the language of *DPL* by adding the clause:

- If $\phi, \psi \in \mathcal{L}$, then $(\phi \wedge \psi) \in \mathcal{L}$.

We will call the new language: $\mathcal{L}(\wedge)$. The semantic clause is: $[\phi\wedge\psi] := [\phi] \cap [\psi]$. We define intersection of contexts as follows.

- $\langle I,B,O\rangle \wedge \langle I',B',O'\rangle :=$
  $\langle I \cup I' \cup (O \cup O')\backslash(B \cap B'), B \cap B', O \cup O' \cup (I \cup I')\backslash(B \cap B')\rangle$

Note that:

$$\begin{aligned} I \cup I' \cup (O \cup O')\backslash(B \cap B') \cup (B \cap B') &= I \cup I' \cup O \cup O' \cup (B \cap B') \\ &= O \cup O' \cup (I \cup I')\backslash(B \cap B') \cup (B \cap B') \end{aligned}$$

So ∧ is a well-defined operation on contexts. We define:

- $\langle I,B,O\rangle \preceq \langle I',B',O'\rangle :\Leftrightarrow I' \subseteq I$ and $B \subseteq B'$ and $O' \subseteq O$

Note the difference between $\preceq$ and $\leq$. It is easy to see that $\wedge$ is precisely the *infimum* with respect to $\preceq$.

Theorem 6.1 *Let R be a $\mathfrak{c}$-relation and let S be a $\mathfrak{d}$-relation. Then, $R \cap S$ is a $(\mathfrak{c}\wedge\mathfrak{d})$-relation.*

Proof

Suppose that the conditions of the theorem are fulfilled. Let $\mathfrak{c} = \langle I,B,O\rangle$, $\mathfrak{d} = \langle I',B',O'\rangle$ and $(\mathfrak{c}\wedge\mathfrak{d}) = \langle I'',B'',O''\rangle$. Clearly, $f\mathcal{I}_{I''}f(R\cap S)g\mathcal{I}_{O''}$. Suppose $f(R\cap S)g$. It follows that $f([B]\cap[B'])g$, i.e., $f[B\cap B']g$. So:

$$(R\cap S) \subseteq (\mathcal{I}_{I''} \circ (R\cap S) \circ \mathcal{I}_{O''}) \cap [B''].$$

For the converse, suppose $f'\mathcal{I}_{I''}f$, $f(R\cap S)g$, $g\mathcal{I}_{O''}g'$, and $f'[B'']g'$. Evidently, $f'\mathcal{I}_I f$, fRg, $g\mathcal{I}_O g'$, and $f'[B]g'$. Ergo $f'Rg'$. Similarly $f'Sg'$. Hence $f'(R\cap S)g'$. ❑

We extend the definition of $\mathfrak{c}_\phi$ to the new language by adding the clause $\mathfrak{c}_{(\phi\wedge\psi)} := \mathfrak{c}_\phi\wedge\mathfrak{c}_\psi$. Theorem 6.1 immediately yields the next theorem.

Theorem 6.2 *$[\phi]$ is a $\mathfrak{c}_\phi$-relation for every $\phi\in\mathcal{L}(\wedge)$.*

We consider an example. Let $\mathfrak{c}_\phi = \langle I,B,O\rangle$. Suppose $B = \{b_1,\cdots,b_n\}$. Let $\psi := (\phi.\exists b_1.\cdots.\exists b_n\wedge\top)$. We have: $[\psi] = [\neg\neg(\phi)] = [(\top\to\phi)]$. We compute $\mathfrak{c}_\psi$.

$$\begin{aligned}\mathfrak{c}_\psi &= (\langle I,B,O\rangle \bullet \langle\emptyset,B,\emptyset\rangle) \cap \langle\emptyset,\emptyset,\emptyset\rangle \\ &= \langle I,B,O\backslash B\rangle \cap \langle\emptyset,\emptyset,\emptyset\rangle \\ &= \langle I\cup\emptyset\cup(O\backslash B\backslash\emptyset),\emptyset,O\backslash B\cup(I\backslash\emptyset)\rangle \\ &= \langle I,\emptyset,I\rangle\end{aligned}$$

Thus, in this example our conjunction on contexts, gives us the 'intuitive result', i.e., $[\psi]$ *is an I-condition.*

Let's say that the *DPL*($\wedge$)-relations over a given domain D are the relations on this domain generated by finitely restricted conditions and resettings using composition and intersection. The results of the present section show that the *DPL*($\wedge$)-relations are all *IBO*-relations. The results of the next section, will imply that, conversely, every *IBO*-relation is *DPL*($\wedge$).

6.2 A new existential quantifier

We define: $\exists x(R) := ([x]\circ R\circ[x])\cap\mathcal{I}_{\{x\}}$. In case R is an $\langle I,B,O\rangle$-relation, we see that $\exists x(R) := ([x]\circ R\circ[x])\cap[B\backslash\{x\}]$. It follows —by the result of subsection 6.1— that $\exists(R)$ is an $\langle I\backslash\{x\},B\backslash\{x\},O\backslash\{x\}\rangle$-relation.

We extend the language of *DPL* by adding the clause:

- If $\phi\in\mathcal{L}$ and $v\in$ *Var* then $\exists v(\phi)\in\mathcal{L}$

Note the overloading of notations. The new language will be $\mathcal{L}(\exists)$. The new semantical clause is the obvious: $[\exists v\phi]_{\mathfrak{M}} = \exists v[\phi]_{\mathfrak{M}}$. $\exists v$ is definable in $\mathcal{L}(\wedge)$ as follows. Suppose $\mathfrak{c}_\phi = \langle I,B,O\rangle$. Let $B\backslash\{x\} = \{b_1,\cdots,b_n\}$, then we can put $(\exists x.\phi.\exists x\wedge\exists b_1.\cdots\exists b_n)$ for $\exists(\phi)$.

Theorem 6.3 *For* any *non-empty domain D, the DPL($\exists$)-definable relations are precisely the IBO-relations.*

Proof

[Sketch] We have already seen that every *DPL(∃)*-definable relation is *IBO*. For the converse, suppose that R is an $\langle I, B, O\rangle$-relation. Let $I = \{i_1, \cdots, i_m\}$, $B\backslash O = \{b_1, \cdots, b_n\}$ and $O \cap B = \{o_1, \cdots, o_p\}$. Here the i_k are supposed to be mutually distinct and similarly for the other sets. Take a *DPL(∃)*-language with an $(m+p)$-ary predicate symbol P, where:

$$\langle d_1, \cdots, d_m, e_1, \cdots, e_p\rangle \in I(P) \Leftrightarrow \quad \exists f, g \;\; fRg \text{ and } f(i_1) = d_1, \ldots, f(i_m) = d_m \\ \text{and } f(o_1) = e_1, \ldots, f(o_p) = e_p$$

Let $u_1 \ldots u_p$ be variables disjoint from $I \cup B \cup O$. Let ϕ be given by:

$$\exists u_1(\cdots(\exists u_p(P(i_1, \cdots, i_m, u_1, \cdots, u_p).\exists o_1.\cdots.\exists o_p. \\ o_1 = u_1.\cdots.o_p = u_p.\exists b_1.\cdots.\exists b_n)\cdots)$$

Clearly, that $[\phi]$ is an $\langle I, B, O\rangle$-relation. The verification that $R = [\phi]$ is along the lines of the proof of theorem 5.1. ❑

Example 6.4 We show how to define the three relations of example 4.5. We do a bit more than the theorem promises, because we give explicit descriptions of the 'P'.

- $[x := y, y := x]$ can be defined by:

  $$\exists u(\exists v(x = u.y = v.\exists x.\exists y.y = u.x = v)).$$

  Note that this gives us the expected context: $\langle\{x, y\}, \{x, y\}, \{x, y\}\rangle$.

- $[x := x + 1]$ can be defined by: $\exists v(x = v.\exists x.x = v + 1)$. (Strictly speaking we are working in a relational language, so $x = v + 1$ is a suggestive notation for, say, $S(v, x)$.) The context produced by the formula is as expected.

- We can define $[(\exists x \vee \exists y)]$ by:

  $$\exists u(\exists v(\neg(\neg(x = u).\neg(y = v)).\exists x.\exists y.x = u.y = v)).$$

  This gives us the context $\langle\{x, y\}, \{x, y\}, \{x, y\}\rangle$. We will discuss this context in the next subsection.

❑

Since ∃ is definable using ∧, the 'expressive completeness' of ∃ w.r.t. the *IBO*-relations implies the 'expressive completeness' of ∧. Finally, we can translate Predicate Logic into *DPL(∧)*, by changing the ∃-clause of our earlier translation to: $(\exists(\phi))^* := \exists(\phi^*)$. Remarkably, the old and the new translation produce precisely the same relations at the semantical level.

Operations like →, ∧, ∨ and *thereis* are not themselves *actions* in the sense of our semantics. They are transformers of actions. Yet there is a tendency to understand $\exists x(\phi)$ dynamically as a sequence of actions: *reset x; execute φ; set x back to its original value.* The problem with this way of viewing things is summarized with the question: where do we store the original value of x, so that we can restore it at the end? *DPL*-semantics does not supply the right kind of 'memory' to realize $\exists(\phi)$ as a sequence of actions. We can do that (or, rather, something very much like it) in the richer semantics of Kees Vermeulen's *DPLE* (see [7]), where under a *variablename* we do not store just one value, but a stack of values. Here the original value of x is simply stored 'under' the new one.

6.3 Disjunction

In this section we have a brief look at the problem of adding disjunction/union to *DPL*. Adding disjunction/union evokes problems that are definitely beyond the scope of the present paper. So we can only offer some tantalizing remarks.

One way of of thinking about $R \cup S$ is as: *Choose between R and S, and reset via the relation chosen.* At the syntactical level, we reflect the new operation by extending the language of *DPL* by adding the clause:

- If $\phi, \psi \in \mathcal{L}$, then $(\phi \vee \psi) \in \mathcal{L}$.

We will call the new language: $\mathcal{L}(\vee)$. The semantic clause is: $[\phi \vee \psi] := [\phi] \cup [\psi]$.

What could be a context for $[x] \cup [y]$? Some experimentation shows that the best we can do is: $\langle \{x, y\}, \{x, y\}, \{x, y\} \rangle$. This seems a wasteful way to represent the variable handling of this relation. Our intuition tells us that $[x] \cup [y]$ is a pure resetter and not something that 'constrains' w.r.t. x and y. The resetting part of our contexts is somehow too crude to represent 'choice' well. The example does not tell us that in any strict sense our present framework is wrong. It just suggests that, possibly, we could do better. We might try out richer notions of context. The most obvious proposal is to take as a context in the new sense a set of contexts in the old sense, where the set is given 'disjunctive reading'.[9] So, e.g., we would have:

$$\mathfrak{c}_{(\exists x \vee \exists y).P(x,y)} = \{\langle \{y\}, \{x\}, \{x, y\} \rangle, \langle \{x\}, \{y\}, \{x, y\} \rangle\}.$$

Note that e.g. the second occurrence of x in $(\exists x \vee \exists y).P(x, y)$ seems to be ambiguous between free and actively bound. So what is an ambiguous occurrence and how do we handle it theoretically? We propose to addres this question elsewhere.

7 Relations in context

In *DPL* meanings are relations. The contexts we studied appear as properties of these relations. We could give an alternative semantics for *DPL* by building the context into the meaning. Thus we take as meanings pairs $\langle \mathfrak{c}, R \rangle$, where $R \in [\![\mathfrak{c}]\!]$. Let's call such a pair a $\mathfrak{c}$-*relation*. We define: $[\![\phi]\!]_{\mathfrak{M}} := \langle \mathfrak{c}_\phi, [\phi]_{\mathfrak{M}} \rangle$.[10] The new domain of meanings is, on the one hand, essentially richer than the old one, since the same relation falls under several contexts. On the other hand, we threw all non-*IBO*-relations away. We can 'lift' the notions intoduced in this paper to $\mathfrak{c}$-relations:

- $\langle \mathfrak{c}, R \rangle \bullet \langle \mathfrak{d}, S \rangle = \langle \mathfrak{c} \bullet \mathfrak{d}, R \circ S \rangle$.
- A $\mathfrak{c}$-condition is a $\mathfrak{c}$-relations of the form $\langle \langle I, \emptyset, I \rangle, R \rangle$.
- If $\mathcal{R} = \langle \langle I, B, O \rangle, R \rangle$, then $dom(\mathcal{R}) := \langle \langle I, \emptyset, I \rangle, diag(dom(R)) \rangle$. Similarly for *cod*.
- $[\![B]\!] := \langle \langle \emptyset, B, \emptyset \rangle, [B] \rangle$. We write $[\![v]\!]$ for $[\![\{v\}]\!]$.
- A $\mathfrak{c}$-relation $\mathcal{R}$ has the Switching Property if it is either a $\mathfrak{c}$-condition or there are variables v and w such that:
$$\mathcal{R} = dom(\mathcal{R}) \bullet [\![v]\!] \bullet \mathcal{R} \bullet [\![w]\!] \bullet cod(\mathcal{R})$$
- A $\mathfrak{c}$-relation is *DPL*-definable (over a given domain D) if it can be generated using $\bullet$ from $\mathfrak{c}$-conditions and resettings $[\![v]\!]$.

[9] In fact, I think, this proposal is in the right direction, but still not quite right.
[10] I am convinced that the enriched semantics is better than the usual one, but I will not argue the case here.

In a similar way we can upgrade $\wedge$ and $\exists$. Inspection of the proofs in this paper shows that, in case D is infinite, the *DPL*-definable $\mathfrak{c}$-relations are precisely the ones with the Switching Property. Moreover, all $\mathfrak{c}$-relations over the given domain —infinite or not— are *DPL(∃)*-definable. We consider an example. Remember that:

$$\langle\emptyset,\emptyset,\emptyset\rangle \leq \langle\{x\},\emptyset,\{x\}\rangle \leq \langle\{x\},\{x\},\{x\}\rangle.$$

Let a model with domain D be given. We assume that D has at least two elements. Let $id := id_{D^{Var}}$. We consider three $\mathfrak{c}$-relations with associate relation id.

1. $[\![\top]\!] = \langle\langle\emptyset,\emptyset,\emptyset\rangle, id\rangle$,
2. $[\![x = x]\!] = \langle\langle\{x\},\emptyset,\{x\}\rangle, id\rangle$,
3. $[\![\exists u(u = x.\exists x.u = x)]\!] = \langle\langle\{x\},\{x\},\{x\}\rangle, id\rangle$.

(1) and (2) are $\mathfrak{c}$-conditions and, hence *DPL*-definable. In fact, they can be defined in the language by $\top$, respectively $x = x$. In contrast, (3) is not a condition. It is easy to see that (3) does not have the Switching Property, since the domain of its internal relation id consists of all assignments and id is injective. Hence (3) is not *DPL*-definable. Note that $\langle\langle\emptyset,\{x\},\emptyset\rangle, id\rangle$ is not a $\mathfrak{c}$-relation at all.

A further step in modifying our semantics is to make the assignments 'local'. The idea is that the context 'provides' the files/discourse referents/variables on which the variables are defined. Thus, our meaning objects would be of the form $\langle\langle I, B, O\rangle, R\rangle$, where R would be a relation taking input assignments defined on I and yielding output assignments defined on O. This approach leads to a semantics very much like Vermeulen's Referent Systems (see [8]). One effect of this further modification is that it leads to a somewhat different view of contexts. In the local approach, contexts are the central 'engines' that manage the flow of the files in the interactions of meanings. This more dramatic view of contexts is elaborated in [9].

Concluding remarks

In this paper we introduced a notion of context and specified its connections with relational semantics and language. We used these contexts to prove a characterization of the *DPL*-definable relations. Moreover, we illustrated the usefulness of contexts both in formulating and in verifying valid sequents of *DPL*. We illustrated the fact that 'understanding of what is going on' is not automatically preserved if we extend the *DPL*-language. E.g., adding disjunction leads to ambiguous occurrences of variables. This observation tells us that the study of extensions will provide us clues regarding the question: *what is it to be a variable occurrence of a certain kind?*

So —apart from the concrete results— what general conclusions may we draw from the paper? A first one is, surely, that a study of the elementaria of *DPL* is both necessary and rewarding. Questions on the nature of variable occurrences, the proper notion of syntactic substitution, etcetera, appear in a new light. The fruitfulness of the study of *DPL* is independent of the question whether *DPL* is really the best choice as a medium for representing dynamic phenomena. One reason is that, in a sense, the relational semantics of *DPL* is very simple and that it is, therefore, easier to make progress. A second conclusion is that it is rewarding to engage in a study that stresses the *differences* between *DPL* and Predicate Logic. Much effort has gone into integrating *DPL* into the classical Montague framework. This project has unavoidably a conservative flavour. The result has been that the unfamiliarity, the strangeness of *DPL* has been underadvertised. Precisely mastering the strangeness provides us with better insight into the formerly familiar notions. My third conclusion is simply: contexts are essential in the study of *DPL* and its kin. We may want to vary the contexts, e.g., we may want to add 'garbage elements' or to ignore the O-component, but contexts per se are there to stay. Our third conclusion points to a larger programmatic point: the study of contexts and the way they are contexts of their contents should be one of the central endeavors of the study of Information Processing and Dynamics.

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A Appendix: Intersection of contexts

In this appendix we treat the properties of intersection of contexts.

Lemma A.1 $\cap$ is well defined. ❏

Proof

We have to prove: $(I \cap I')\backslash(B \cap B') = (O \cap O')\backslash(B \cap B')$ on the assumption that $B \subseteq B' \cup (I \cap O)$ and $B' \subseteq B \cup (I' \cap O')$. We prove

$$(I \cap I')\backslash(B \cap B') \subseteq (O \cap O')\backslash(B \cap B').$$

The converse direction is dual. Suppose $v \in I \cap I'$ and $v \notin B \cap B'$. We want to prove: $v \in O \cap O'$. We have: $v \notin B$ or $v \notin B'$. Suppose $v \notin B$. It follows that $v \in I\backslash B$ and, hence, $v \in O\backslash B$, so $v \in O$. If $v \notin B'$, we find $v \in O'$, and we are done. Suppose $v \in B'$. We assumed that $B' \subseteq B \cup (I' \cap O')$. Since $v \notin B$, we get: $v \in I' \cap O'$, and, hence, $v \in O'$. The case that $v \notin B'$ is similar. ❏

Next we prove that $\cap$ produces the $\leq$ minimum, whenever there is one.

Theorem A.2 *1.* $\mathfrak{z} \leq \mathfrak{c}$ *and* $\mathfrak{z} \leq \mathfrak{d} \Leftrightarrow \mathfrak{c} \cap \mathfrak{d} \downarrow,\ \mathfrak{z} \leq \mathfrak{c} \cap \mathfrak{d}$.

2. $\mathfrak{c} \cap \mathfrak{d}$*, whenever it exists, is the infinimum of* $\mathfrak{c}$ *and* $\mathfrak{d}$*.*

Proof

Let $\mathfrak{c} = \langle I, B, O \rangle$, $\mathfrak{d} = \langle I', B'O' \rangle$ and $\mathfrak{z} = \langle J, C, P \rangle$. Suppose $\mathfrak{z} \leq \mathfrak{c}$ and $\mathfrak{z} \leq \mathfrak{d}$. We have: $J \subseteq I \cap I'$, $P \subseteq O \cap O'$, $C \subseteq B \cap B'$, $B \subseteq C \cup (I \cap O)$ and $B' \subseteq C \cup (I' \cap O')$. It follows that $B \subseteq (B \cap B') \cup (I \cap O)$, and, thus, $B \subseteq B' \cup (I \cap O)$. Moreover, $B' \subseteq (B \cap B') \cup (I' \cap O')$, and so $B' \subseteq B \cup (I' \cap O')$. We may conclude that $\mathfrak{c} \cap \mathfrak{d} \downarrow$, $\mathfrak{c} \cap \mathfrak{d} \leq \mathfrak{c}$ and $\mathfrak{c} \cap \mathfrak{d} \leq \mathfrak{d}$.

For the converse, it is sufficient to prove that if $\mathfrak{c} \cap \mathfrak{d} \downarrow$, then $\mathfrak{c} \cap \mathfrak{d} \leq \mathfrak{c}$ and $\mathfrak{c} \cap \mathfrak{d} \leq \mathfrak{d}$. So, suppose $\mathfrak{c} \cap \mathfrak{d} \downarrow$. We verify $\mathfrak{c} \cap \mathfrak{d} \leq \mathfrak{c}$, the case of $\mathfrak{d}$ being similar. The only problematic case to check is: $B \subseteq (B \cap B') \cup (I \cap O)$. But this immediate from: $B \subseteq B' \cup (I \cap O)$. ❑

In the following theorem we show that one can always find a 'best' context for a given non-empty *IBO*-relation.

Theorem A.3 *Suppose* $|D| \geq 2$*. We have:*

1. *Suppose* $\emptyset \neq R \in [\![\mathfrak{c}]\!] \cap [\![\mathfrak{d}]\!]$*. Then* $\mathfrak{c} \cap \mathfrak{d} \downarrow$ *and* $R \in [\![\mathfrak{c} \cap \mathfrak{d}]\!]$*.*
2. $\mathfrak{c} \cap \mathfrak{d} \downarrow \Rightarrow [\![\mathfrak{c} \cap \mathfrak{d}]\!] = [\![\mathfrak{c}]\!] \cap [\![\mathfrak{d}]\!]$.
3. *Suppose* $\emptyset \neq R$*. Let* $\mathfrak{X} := \{\mathfrak{x} \mid R \in [\![\mathfrak{x}]\!]\}$*. Suppose* $\mathfrak{X} \neq \emptyset$*. Then* $\mathfrak{X}$ *has a minimum.*

Proof

Let $\mathfrak{c} = \langle I, B, O \rangle$ and $\mathfrak{d} = \langle I', B', O' \rangle$. We prove (1). Suppose $\emptyset \neq R \in [\![\mathfrak{c}]\!] \cap [\![\mathfrak{d}]\!]$. Suppose fRg. We first prove that $\mathfrak{c} \cap \mathfrak{d} \downarrow$, i.e. $B \subseteq B' \cup (I \cap O)$ and $B' \subseteq B \cup (I' \cap O')$. By symmetry, we only need to treat the first desideratum. Suppose, to obtain a contradiction, that for some v: $v \in B$, $v \notin I \cap O$ and $v \notin B'$. By the duality between I and O, we can restrict ourselves to the case that $v \notin I$. Pick d with $d \neq g(v)$. We have: $f_v^d \mathcal{I}_I f R g \mathcal{I}_O g$ and, since $R \subseteq [B]$ and $v \in B$, $f_v^d [B] g$. Hence $f_v^d R g$. It follows that $R \not\subseteq [B']$, a contradiction.

We show that $R \in [\![\mathfrak{c} \cap \mathfrak{d}]\!]$. Suppose $f' \mathcal{I}_{I \cap I'} f R g \mathcal{I}_{O \cap O'} g'$ and $f'[B \cap B']g'$. We have to show: $f' R g'$. Define:

- $f^* := f \restriction I \,\cup\, f' \restriction (Var \backslash I)$.
- $g^* := g \restriction O \,\cup\, g' \restriction (Var \backslash O)$.

Clearly, $f \mathcal{I}_I f^*$, and, since $f' \mathcal{I}_{I \cap I'} f$, $f^* \mathcal{I}_{I'} f'$. Similarly, $g \mathcal{I}_O g^*$ and $g^* \mathcal{I}_{O'} g'$.

We show $f^*[B]g^*$. Consider $v \notin B$. In case $v \in I$, we have $v \in I \backslash B$, and, hence, $v \in O \backslash B$ and, thus, $v \in O$. We find: $f^*(v) = f(v)$, since $v \in I$. $f(v) = g(v)$, since fRg and $R \subseteq [B]$. $g(v) = g^*(v)$, since $v \in O$. So $f^*(v) = g^*(v)$, as desired. In case $v \notin I$, we also have $v \notin O$, since, otherwise $v \in O \backslash B = I \backslash B$. By the definitions of f^* and g^* we find: $f^*(v) = f'(v)$ and $g'(v) = g^*(v)$. Moreover, by the fact that $f'[B \cap B']g'$, we get $f'(v) = g'(v)$. Hence, $f^*(v) = g^*(v)$.

Since $f'[B \cap B']g'$, we have, a fortiori, $f' B' g'$. Collecting all previous insights, we may conclude: $f^* \mathcal{I}_I f R g \mathcal{I}_O g^*$ and $f^*[B]g^*$. Hence $f^* R g^*$. It follows that $f' \mathcal{I}_{I'} f^* R g^* \mathcal{I}_{O'} g'$ and $f'[B']g'$. Hence $f' R g'$.

We turn to (2). Suppose $\mathfrak{c} \cap \mathfrak{d} \downarrow$. By theorem 3.5 and the fact that $\mathfrak{c} \cap \mathfrak{d} \leq \mathfrak{c}$ and $\mathfrak{c} \cap \mathfrak{d} \leq \mathfrak{d}$, we have: $[\![\mathfrak{c} \cap \mathfrak{d}]\!] \subseteq [\![\mathfrak{c}]\!] \cap [\![\mathfrak{d}]\!]$. For the converse, apply (1). Finally to prove (3), note that, since contexts are finite, $\leq$ is well-founded. Hence, $\mathfrak{X}$ has a minimal element. Moreover, (1) implies that $\mathfrak{X}$ is closed under $\cap$. So the minimal element must be the minimum. ❑

Our last theorem corresponds to the familiar fact of ordinary Predicate Logic that if a set of assignments F is finitely restricted, then one can find a $\subseteq$-minimal I, such that F is $\langle I \rangle$-restricted. Note that $\emptyset$ in the *DPL* case corresponds to many $\leq$-incomparable contexts. Thus, it is hopelessly ambiguous, in contrast to the predicate logical case.