On the Craig interpolation and the fixed point property for **GLP**

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Abstract

We prove the Craig interpolation and the fixed point property for **GLP** by finitary methods.

Konstantin Ignatiev [4], among other things, established the Craig interpolation and the fixed point property for Japaridze's polymodal provability logic **GLP**. However, it remained open if these results could be established by finitary methods formalizable in Peano arithmetic PA. (The question concerning the Craig interpolation was stated e.g. in [3].)

In this note we provide such proofs. These proofs are based on our previous paper [1] where a complete Kripke semantics for **GLP** is given. In that paper, using only finitary methods, the system **GLP** is reduced to a certain natural subsystem, denoted **J**.[1] **J** is sound and complete w.r.t. a natural class of finite Kripke frames [1]. (It is well-known that **GLP** is not complete w.r.t. *any* class of Kripke frames.) We establish the Craig interpolation and the fixed point properties for **J**, which also enables us to extend them to **GLP**. Apart from the reduction of **GLP** to **J** established in ref. [1] our methods are very standard.

1 Preliminaries

The system **J** is given by the following axiom schemata and inference rules.

Axioms: (i) Boolean tautologies;

(ii) $[n](\varphi \to \psi) \to ([n]\varphi \to [n]\psi)$;

[1]Similarly, Ignatiev [4] used a reduction of **GLP** to a weaker subsystem **I**, however the reducibility has not been established by finitary methods.

(iii) $[n]([n]\varphi \to \varphi) \to [n]\varphi$;

(iv) $[m]\varphi \to [n][m]\varphi$, for $m \leq n$.

(v) $\langle m\rangle\varphi \to [n]\langle m\rangle\varphi$, for $m < n$.

(vi) $[m]\varphi \to [m][n]\varphi$, for $m \leq n$.

Rules: modus ponens, $\varphi \vdash [n]\varphi$.

GLP is obtained from **J** by adding the *monotonicity schema*

$$[m]\varphi \to [n]\varphi, \quad \text{for } m \leq n.$$

Ignatiev's logic **I** is obtained from **J** by deleting axiom schema (vi).

The system **J** is sound and complete w.r.t. the class of Kripke frames satisfying the following conditions:

- R_k is a upwards well-founded, transitive ordering relation on $\mathcal{W}$, for each $k \geq 0$;
- $\forall x, y\, (xR_n y \Rightarrow \forall z\, (xR_m z \Leftrightarrow yR_m z))$ if $m < n$; (I)
- $\forall x, y\, (xR_m y \,\&\, yR_n z \Rightarrow xR_m z)$ if $m \leq n$. (J)

Such frames will be called **J**-frames.

One of the main results of ref. [1] states that **GLP** is reducible to **J** as follows. Let

$$M(\varphi) := \textstyle\bigwedge_{i<s}([m_i]\varphi_i \to [m_i + 1]\varphi_i),$$

where $[m_i]\varphi_i$ for $i < s$ are all subformulas of φ of the form $[k]\psi$. Let

$$\Box^+\varphi := \varphi \wedge \bigwedge_{i\leq n} [i]\varphi,$$

where $n := \max_{i<s} m_i$, and let $M^+(\varphi) := \Box^+ M(\varphi)$. Then,

$$\mathbf{GLP} \vdash \varphi \iff \mathbf{J} \vdash M^+(\varphi) \to \varphi. \qquad (Red)$$

This result is proved by finitary methods based on Kripke semantics.

2 Craig interpolation theorem for J

In the proof of the Craig interpolation theorem we shall use notation similar to Tait-style sequent calculus, that is:

- *Formulas* are built-up from constants $\bot$, $\top$, propositional variables p_i, $i \geq 0$, and their negations $\bar{p}_i$ using $\wedge$, $\vee$, and modalities $\langle n \rangle$, $[n]$, for each $n \geq 0$;

- *Sequents* are finite sets of formulas (denoted Γ, Δ, etc.) understood as disjunctions of their elements. We write $\vdash \Gamma$ if $\mathbf{J} \vdash \bigvee \Gamma$.

- *Negation* $\neg\varphi$ of a formula φ is defined by de Morgan's rules and the following identities: $\neg[n]\varphi := \langle n \rangle \neg\varphi$, $\neg\langle n \rangle\varphi := [n]\neg\varphi$. Implication $\varphi \to \psi$ is defined by $\neg\varphi \vee \psi$.

As usual we write Γ, Δ for $\Gamma \cup \Delta$ and Γ, φ for $\Gamma \cup \{\varphi\}$. We use the following abbreviations: $\langle n \rangle\Gamma := \{\langle n \rangle\varphi : \varphi \in \Gamma\}$, $[n]\Gamma := \{[n]\varphi : \varphi \in \Gamma\}$.

$\Diamond_{\geq n}\Gamma$ denotes the result of prefixing each formula from Γ by a modality of the form $\langle m \rangle$ for some $m \geq n$ (m can be different for each formula from Γ). $\Box_{\geq n}\Gamma$ is similarly defined.

Lemma 2.1 *Suppose Δ is a set of formulas of the form $\langle m \rangle\psi$ and $[m]\psi$ with $m < n$. If*

$$\vdash \Delta, \Diamond_{\geq n}\Gamma, \Gamma, \langle n \rangle\neg\varphi, \varphi$$

then $\vdash \Delta, \langle n \rangle\Gamma, [n]\varphi$.

Proof. Assume $\mathbf{J} \vdash \bigvee(\Delta, \Gamma, \Diamond_{\geq n}\Gamma, \langle n \rangle\neg\varphi, \varphi)$, then by propositional logic

$$\mathbf{J} \vdash (\bigwedge \neg\Delta \wedge \bigwedge \neg\Gamma \wedge \bigwedge \Box_{\geq n}\neg\Gamma) \to ([n]\varphi \to \varphi).$$

Denoting $\varphi_1 := \bigwedge \neg\Delta$, $\varphi_2 := \bigwedge \neg\Gamma$, and $\varphi_3 := \bigwedge \Box_{\geq n}\neg\Gamma$ we obtain:

$$\begin{aligned} \mathbf{J} \vdash [n](\varphi_1 \wedge \varphi_2 \wedge \varphi_3) &\to [n]([n]\varphi \to \varphi) && (1) \\ &\to [n]\varphi, \quad \text{by (iii)}. && (2) \end{aligned}$$

However, if $[m]\psi \in \neg\Delta$ then $\mathbf{J} \vdash [m]\psi \to [n][m]\psi$ by (iv), and if $\langle m \rangle\psi \in \neg\Delta$, then $\mathbf{J} \vdash \langle m \rangle\psi \to [n]\langle m \rangle\psi$ by (v). Hence,

$$\mathbf{J} \vdash \varphi_1 \to [n]\varphi_1.$$

Similarly, if $[k]\psi \in \Box_{\geq n}\neg\Gamma$ then $\mathbf{J} \vdash [n]\psi \to [n](\psi \wedge [k]\psi)$, by (vi). Hence,

$$\mathbf{J} \vdash [n]\varphi_2 \to [n](\varphi_2 \wedge \varphi_3).$$

We conclude

$$\begin{aligned} \mathbf{J} \vdash \varphi_1 \wedge [n]\varphi_2 &\to [n](\varphi_1 \wedge \varphi_2 \wedge \varphi_3) \\ &\to [n]\varphi, \quad \text{by (2)}. \end{aligned}$$

It follows that $\mathbf{J}$ derives $\bigvee(\Delta, \langle n\rangle\Gamma, [n]\varphi)$, as required. ⊠

Let $\mathrm{Var}(\varphi)$ denote the set of variables occurring in φ and $\mathrm{Var}(\Gamma) := \bigcup_{\varphi\in\Gamma}\mathrm{Var}(\varphi)$.

We say that θ *interpolates* a pair of sequents $(\Gamma;\Delta)$ if $\mathrm{Var}(\theta)\subseteq\mathrm{Var}(\Gamma)\cap\mathrm{Var}(\Delta)$ and

$$\vdash \Gamma, \theta \text{ and } \vdash \neg\theta, \Delta.$$

$(\Gamma;\Delta)$ is *inseparable* if it does not have an interpolant.

The following theorem subsumes both the completeness theorem for $\mathbf{J}$ and the Craig interpolation theorem.

Theorem 1 *The following statements are equivalent:*

(i) $(\Gamma;\Delta)$ *has an interpolant;*

(ii) $\vdash\Gamma,\Delta$*;*

(iii) For all (finite) $\mathbf{J}$*-models* $\mathcal{W}$*,* $\mathcal{W}\vDash\bigvee(\Gamma\cup\Delta)$*.*

Proof. The implications (i)⇒(ii) and (ii)⇒(iii) are easy. We prove (iii)⇒(i).

Call a finite set Φ of formulas *adequate* if it is closed under subformulas, negation, the following operation:

$$[n]\varphi,\ [m]\psi\in\Phi \Rightarrow [m]\varphi\in\Phi,$$

and for each variable $p\in\Phi$ contains $p\wedge\bar{p}$. Let $\mathrm{Op}(\Phi) = \{n\in\omega : [n]\varphi\in\Phi, \text{ for some } \varphi\}$. Clearly, every finite set of formulas Ψ can be extended to a finite adequate set $\Phi\supseteq\Psi$ such that $\mathrm{Op}(\Phi)=\mathrm{Op}(\Psi)$ and $\mathrm{Var}(\Phi)=\mathrm{Var}(\Psi)$.

Let us fix some finite adequate Φ. Below we shall only consider sequents Γ over Φ, that is, $\Gamma\subseteq\Phi$. An inseparable pair $(\Gamma_1;\Gamma_2)$ is *maximal* if for any other inseparable pair $(\Delta_1;\Delta_2)$ such that $\Gamma_1\subseteq\Delta_1$ and $\Gamma_2\subseteq\Delta_2$ one has $\Gamma_1=\Delta_1$ and $\Gamma_2=\Delta_2$.

Lemma 2.2 *Suppose* $(\Gamma_1;\Gamma_2)$ *is maximal inseparable. Then, for all* $\varphi,\psi\in\Phi$*, and* $i=1,2$*:*

(i) $(\varphi\wedge\psi)\in\Gamma_i\Rightarrow\varphi\in\Gamma_i$ or $\psi\in\Gamma_i$;

(ii) $(\varphi\vee\psi)\in\Gamma_i\Rightarrow\varphi\in\Gamma_i$ and $\psi\in\Gamma_i$;

(iii) If $\mathrm{Var}(\varphi)\subseteq\mathrm{Var}(\Gamma_i)$ then either $\varphi\in\Gamma_i$ or $\neg\varphi\in\Gamma_i$;

(iv) For no φ both φ, $\neg\varphi \in \Gamma_i$.

Proof. By the obvious symmetry it is sufficient to prove both claims for $i = 1$.

(i) Assume $\varphi, \psi \notin \Gamma_1$. We claim that at least one of the following two pairs is inseparable: $(\Gamma_1, \varphi; \Gamma_2)$ and $(\Gamma_1, \psi; \Gamma_2)$. Indeed, if θ_1 interpolates the first pair and θ_2 interpolates the second pair, then

$$\begin{array}{ll} \vdash \Gamma_1, \varphi, \theta_1 & \vdash \neg\theta_1, \Gamma_2 \\ \vdash \Gamma_1, \psi, \theta_2 & \vdash \neg\theta_2, \Gamma_2, \end{array}$$

whence

$$\vdash \Gamma_1, \varphi \wedge \psi, \theta_1 \vee \theta_2 \qquad \vdash \neg\theta_1 \wedge \neg\theta_2, \Gamma_2.$$

Hence, $\theta_1 \vee \theta_2$ interpolates $(\Gamma_1; \Gamma_2)$, a contradiction. It follows that $(\Gamma_1; \Gamma_2)$ is not maximal.

(ii) Assume $\varphi \notin \Gamma_1$ then $(\Gamma_1, \varphi; \Gamma_2)$ is inseparable. Otherwise, if θ interpolates this pair, then

$$\vdash \Gamma_1, \varphi, \theta \ \text{ and } \vdash \neg\theta, \Gamma_2$$

hence

$$\vdash \Gamma_1, \varphi \vee \psi, \theta \ \text{ and } \vdash \neg\theta, \Gamma_2,$$

that is, θ interpolates $(\Gamma_1; \Gamma_2)$, a contradiction. It follows that $(\Gamma_1; \Gamma_2)$ is not maximal. The case $\psi \notin \Gamma_1$ is similar.

(iii) Assume $\mathrm{Var}(\varphi) \subseteq \mathrm{Var}(\Gamma_1)$ and $\varphi, \neg\varphi \notin \Gamma_1$. Then one of the pairs $(\Gamma_1, \varphi; \Gamma_2)$ and $(\Gamma_1, \neg\varphi; \Gamma_2)$ is inseparable. Otherwise, if

$$\begin{array}{ll} \vdash \Gamma_1, \varphi, \theta_1 & \vdash \neg\theta_1, \Gamma_2, \\ \vdash \Gamma_1, \neg\varphi, \theta_2 & \vdash \neg\theta_2, \Gamma_2, \end{array}$$

then

$$\vdash \Gamma_1, \theta_1 \vee \theta_2 \qquad \vdash \neg\theta_1 \wedge \neg\theta_2, \Gamma_2.$$

Hence, $\theta_1 \vee \theta_2$ interpolates $(\Gamma_1; \Gamma_2)$, a contradiction. It follows that $(\Gamma_1; \Gamma_2)$ is not maximal.

(iv) If $\varphi, \neg\varphi \in \Gamma_1$ then $\vdash \Gamma_1, \bot$ and $\vdash \top, \Gamma_2$, which is impossible. ⊠

Consider the following Kripke frame. Let

$$\mathcal{W} := \{(\Gamma_1; \Gamma_2) : (\Gamma_1; \Gamma_2) \text{ is maximal inseparable over } \Phi\}.$$

For any $x = (\Gamma_1; \Gamma_2)$ and $y = (\Delta_1; \Delta_2)$ in $\mathcal{W}$ let xR_ny if the following conditions hold, for $i = 1, 2$:

1. $\mathrm{Var}(\Gamma_i) = \mathrm{Var}(\Delta_i)$;

2. For any $\langle n\rangle\varphi \in \Gamma_i$, $\varphi \in \Delta_i$ and
$$\forall k \geq n\, (k \in \mathrm{Op}(\Phi) \Rightarrow \langle k\rangle\varphi \in \Delta_i);$$

3. For any $m < n$,
$$\langle m\rangle\varphi \in \Gamma_i \iff \langle m\rangle\varphi \in \Delta_i;$$

4. For some $j \in \{1,2\}$, there is a $\langle n\rangle\varphi \in \Delta_j$ such that $\langle n\rangle\varphi \notin \Gamma_j$.

Lemma 2.3 *$\mathcal{W}$ is a **J**-frame.*

Proof. Condition 4 guarantees the irreflexivity of the relations R_n.

Assume $(\Gamma_1;\Gamma_2)R_n(\Delta_1;\Delta_2)$, $(\Gamma_1;\Gamma_2)R_m(\Sigma_1;\Sigma_2)$ and $m < n$; we prove $(\Delta_1;\Delta_2)R_m(\Sigma_1;\Sigma_2)$. Indeed, $\mathrm{Var}(\Delta_i) = \mathrm{Var}(\Gamma_i) = \mathrm{Var}(\Sigma_i)$, for $i = 1,2$. If $\langle m\rangle\varphi \in \Delta_i$ then $\langle m\rangle\varphi \in \Gamma_i$, since $m < n$. Hence $\varphi, \langle k\rangle\varphi \in \Sigma_i$, for $k \geq m$, $k \in \mathrm{Op}(\Phi)$. If $k < m$ then $\langle k\rangle\varphi \in \Delta_i \Leftrightarrow \langle k\rangle\varphi \in \Gamma_i \Leftrightarrow \langle k\rangle\varphi \in \Sigma_i$. Finally, we have $\langle m\rangle\psi \in \Sigma_j$, $\langle m\rangle\psi \notin \Gamma_j$, for some ψ, j. Hence, $\langle m\rangle\psi \notin \Delta_j$ because $m < n$.

Assume $(\Gamma_1;\Gamma_2)R_n(\Delta_1;\Delta_2)$, $(\Delta_1;\Delta_2)R_m(\Sigma_1;\Sigma_2)$ and $m \leq n$; we prove $(\Gamma_1;\Gamma_2)R_m(\Sigma_1;\Sigma_2)$. Indeed, if $\langle m\rangle\varphi \in \Gamma_i$ then $\langle m\rangle\varphi \in \Delta_i$, since $m \leq n$. Hence $\varphi, \langle k\rangle\varphi \in \Sigma_i$, for $k \geq m$, $k \in \mathrm{Op}(\Phi)$. If $k < m$ then $\langle k\rangle\varphi \in \Gamma_i \Leftrightarrow \langle k\rangle\varphi \in \Delta_i \Leftrightarrow \langle k\rangle\varphi \in \Sigma_i$. Finally, we have $\langle m\rangle\psi \in \Sigma_j$, $\langle m\rangle\psi \notin \Delta_j$, for some ψ, j. Hence, $\langle m\rangle\psi \notin \Gamma_j$ because $m \leq n$.

Assume $(\Gamma_1;\Gamma_2)R_m(\Delta_1;\Delta_2)$, $(\Delta_1;\Delta_2)R_n(\Sigma_1;\Sigma_2)$ and $m \leq n$; we prove $(\Gamma_1;\Gamma_2)R_m(\Sigma_1;\Sigma_2)$. Let $k \in \mathrm{Op}(\Phi)$. If $m \leq k \leq n$, then $\langle m\rangle\varphi \in \Gamma_i$ implies $\langle k\rangle\varphi \in \Delta_i$ and $\langle k\rangle\varphi \in \Sigma_i$. If $k \geq n$, then $\langle m\rangle\varphi \in \Gamma_i$ implies $\langle n\rangle\varphi \in \Delta_i$ and $\varphi, \langle k\rangle\varphi \in \Sigma_i$. Finally, there is a ψ such that $\langle m\rangle\psi \in \Delta_j$, $\langle m\rangle\psi \notin \Gamma_j$. Since $m \leq n$ we also have $\langle m\rangle\psi \in \Sigma_j$, and we are done. ⊠

We define the evaluation of propositional variables on $\mathcal{W}$ by letting
$$(\Gamma_1;\Gamma_2) \Vdash p \iff p \notin \Gamma_1 \cup \Gamma_2. \qquad (*)$$

Lemma 2.4 *For any $\varphi \in \Gamma_1 \cup \Gamma_2$ one has $(\Gamma_1;\Gamma_2) \nVdash \varphi$.*

Proof. Induction on the length of φ. We consider the following cases.

Case 1: $\varphi = \top$. If $\top \in \Gamma_1$ then $\vdash \Gamma_1, \bot$ and $\vdash \top, \Gamma_2$, hence $(\Gamma_1;\Gamma_2)$ is not inseparable. Thus, $\top \notin \Gamma_1$ and similarly $\top \notin \Gamma_2$.

Case 2: $\varphi = \bot$. We always have $(\Gamma_1;\Gamma_2) \nVdash \bot$.

CASE 3: $\varphi = p$. By $(*)$.

CASE 4: $\varphi = \bar{p}$. Suppose $\bar{p} \in \Gamma_1$. If $p \in \Gamma_1$, then $\vdash \Gamma_1, \bot$ and $\vdash \top, \Gamma_2$, a contradiction. If $p \in \Gamma_2$, then $\vdash \Gamma_1, p$ and $\vdash \bar{p}, \Gamma_2$, also contradicting the inseparability of $(\Gamma_1; \Gamma_2)$. Hence, $p \notin \Gamma_1 \cup \Gamma_2$ which entails $(\Gamma_1; \Gamma_2) \Vdash p$ and $(\Gamma_1; \Gamma_2) \nVdash \bar{p}$.

CASE 5: $\varphi = \varphi_1 \wedge \varphi_2$. If $\varphi \in \Gamma_i$ then by Lemma 2.2 either $\varphi_1 \in \Gamma_i$ or $\varphi_2 \in \Gamma_i$. Hence, $(\Gamma_1; \Gamma_2) \nVdash \varphi_1$ or $(\Gamma_1; \Gamma_2) \nVdash \varphi_2$. Therefore, $(\Gamma_1; \Gamma_2) \nVdash \varphi_1 \wedge \varphi_2$.

CASE 6: $\varphi = \varphi_1 \vee \varphi_2$. This is established dually by the same lemma.

CASE 7: $\varphi = \langle n \rangle \varphi_0$. Assume $\varphi \in \Gamma_1$. If $(\Gamma_1; \Gamma_2) R_n (\Delta_1; \Delta_2)$ then $\varphi_0 \in \Delta_1$ and by the induction hypothesis $(\Delta_1; \Delta_2) \nVdash \varphi_0$. Since this holds for all such $(\Delta_1; \Delta_2)$, we have $(\Gamma_1; \Gamma_2) \nVdash \langle n \rangle \varphi_0$.

CASE 8: $\varphi = [n]\varphi_0$. This is the central case.

Assume $[n]\varphi_0 \in \Gamma_1$. Let Δ_i for $i = 1, 2$ denote the union of the following sets of formulas:

1. $\Phi_1^i := \{\langle m \rangle \psi : \langle m \rangle \psi \in \Gamma_i,\ m < n\}$;
2. $\Phi_2^i := \{[m]\psi : [m]\psi \in \Gamma_i,\ m < n\}$;
3. $\Phi_3^i := \{\langle k \rangle \psi, \psi : \langle n \rangle \psi \in \Gamma_i,\ k \geq n, k \in \mathrm{Op}(\Phi)\}$;
4. $\Phi_4^i := \{p \wedge \bar{p} : p \in \mathrm{Var}(\Gamma_i)\}$.

We show that the pair

$$(\Delta_1, \langle n \rangle \neg \varphi_0, \varphi_0\,;\, \Delta_2)$$

is inseparable. Assume otherwise, then for some θ,

$$\vdash \Delta_1, \langle n \rangle \neg \varphi_0, \varphi_0, \theta \ \text{ and } \ \vdash \neg \theta, \Delta_2,$$

where

$$\mathrm{Var}(\theta) \subseteq \mathrm{Var}(\Delta_1, \langle n \rangle \neg \varphi_0, \varphi_0) = \mathrm{Var}(\Gamma_1)$$

and

$$\mathrm{Var}(\theta) \subseteq \mathrm{Var}(\Delta_2) = \mathrm{Var}(\Gamma_2).$$

The equalities hold because of the components Φ_4^i. Since $\bigvee \Phi_4^i$ is equivalent to $\bot$ and can be dropped from a disjunction, we obviously have

$$\vdash \Phi_1^1, \Phi_2^1, \Phi_3^1, \langle n \rangle \theta, \theta, \langle n \rangle \neg \varphi_0, \varphi_0$$

and hence

$$\vdash \Phi_1^1, \Phi_2^1, \{\langle n \rangle \psi : \psi \in \Gamma_1\}, \langle n \rangle \theta, [n]\varphi_0,$$

by Lemma 2.1. All the formulas in this sequent except for $\langle n\rangle\theta$ belong to Γ_1, hence $\vdash \Gamma_1, \langle n\rangle\theta$.

On the other hand, from $\vdash \neg\theta, \Delta_2$ we similarly obtain

$$\vdash \Phi_1^2, \Phi_2^2, \{\langle n\rangle\psi : \psi \in \Gamma_2\}, [n]\neg\theta$$

and hence $\vdash \Gamma_2, \neg\langle n\rangle\theta$. It follows that $\langle n\rangle\theta$ interpolates $(\Gamma_1; \Gamma_2)$, which is impossible.

Thus, $(\Delta_1, \langle n\rangle\neg\varphi_0, \varphi_0\,;\, \Delta_2)$ is inseparable and can be extended to a maximal inseparable pair $(\Delta_1'; \Delta_2')$ such that $\mathrm{Var}(\Delta_i') = \mathrm{Var}(\Delta_i) = \mathrm{Var}(\Gamma_i)$ for $i = 1, 2$. We observe that $(\Gamma_1; \Gamma_2)R_n(\Delta_1'; \Delta_2')$. Indeed, Conditions 1, 2 and 4 are obviously satisfied. Also, if $\langle m\rangle\psi \in \Gamma_i$ and $m < n$, then $\langle m\rangle\psi \in \Delta_i \subseteq \Delta_i'$. On the other hand, if $m < n$ and $\langle m\rangle\psi \in \Delta_i'$ then $\mathrm{Var}(\langle m\rangle\psi) \subseteq \mathrm{Var}(\Gamma_i)$ and hence either $\langle m\rangle\psi \in \Gamma_i$ or $\neg\langle m\rangle\psi \in \Gamma_i$, by Lemma 2.2 (iii). Yet, $[m]\neg\psi \in \Gamma_i$ implies $[m]\neg\psi \in \Delta_i$, whence Δ_i' contains both $\langle m\rangle\psi$ and its negation contradicting Lemma 2.2 (iv). Thus, we conclude $\langle m\rangle\psi \in \Gamma_i$, as required.

Since $\varphi_0 \in \Delta_1'$, by the induction hypothesis we obtain $(\Delta_1'; \Delta_2') \nVdash \varphi_0$. Hence, $(\Gamma_1; \Gamma_2) \nVdash [m]\varphi_0$. ⊠

From the previous lemma we obtain a proof of Theorem 1 in a standard way. Assume $(\Gamma; \Delta)$ is inseparable. Extend $\Gamma \cup \Delta$ to a finite adequate set Φ and build the corresponding model $\mathcal{W}$. Let x be any maximal inseparable pair of sequents over Φ containing $(\Gamma; \Delta)$. By Lemma 2.4, $\mathcal{W}, x \nVdash \bigvee(\Gamma \cup \Delta)$. ⊠

Corollary 2.5 (Craig interpolation for J) *If* $\mathbf{J} \vdash \varphi \to \psi$*, then there is a formula* θ *such that* $\mathrm{Var}(\theta) \subseteq \mathrm{Var}(\varphi) \cap \mathrm{Var}(\psi)$ *and*

$$\mathbf{J} \vdash \varphi \to \theta \quad \textit{and} \quad \mathbf{J} \vdash \theta \to \psi.$$

Corollary 2.6 *Craig interpolation property holds for* **GLP**.

Proof. If $\mathbf{GLP} \vdash \varphi \to \psi$ then $\mathbf{J} \vdash M^+(\varphi \to \psi) \to (\varphi \to \psi)$ by (*Red*). Since every subformula $[i]\xi$ of $\varphi \to \psi$ belongs either to φ or to ψ, we have

$$\mathbf{J} \vdash M^+(\varphi) \wedge \varphi \to (M^+(\psi) \to \psi).$$

Let θ be the corresponding interpolant. Then obviously $\mathbf{GLP} \vdash \varphi \to \theta$, $\mathbf{GLP} \vdash \theta \to \psi$, and $\mathrm{Var}(\theta) \subseteq \mathrm{Var}(\varphi) \cap \mathrm{Var}(\psi)$. ⊠

Open questions:

1. Can we also obtain an interpolant θ satisfying an additional condition $\mathrm{Op}(\theta) \subseteq \mathrm{Op}(\varphi) \cap \mathrm{Op}(\psi)$, that is, if modalities occurring in θ occur both in φ and in ψ? The given proof of Theorem 1 only implies that $\mathrm{Op}(\theta)$ is contained in $\mathrm{Op}(\varphi) \cup \mathrm{Op}(\psi)$. The stronger interpolation theorem obviously fails for the **GLP**, as the example $[0]p \to [1]p$ shows.

2. Does the sequential inference rule formulated in Lemma 2.1 provide a complete cut-free sequent calculus for **J**, taken together with a standard Tait-style axiomatization of propositional logic?

3. Does **J** satisfy uniform interpolation? Lindon interpolation?

3 Fixed points

As a standard corollary of interpolation we obtain Beth definability property for **J** and **GLP**.

Corollary 3.1 (Beth definability for J) *If q does not occur in $\varphi(p)$ and*

$$\mathbf{J} \vdash \varphi(p) \land \varphi(q) \to (p \leftrightarrow q),$$

then there is a ψ such that $\mathrm{Var}(\psi) = \mathrm{Var}(\varphi(p)) \setminus \{p\}$ and

$$\mathbf{J} \vdash \varphi(p) \to (p \leftrightarrow \psi).$$

Proof. Let ψ be the interpolant of the implication

$$\mathbf{J} \vdash \varphi(p) \land p \to (\varphi(q) \to q),$$

$\boxtimes$

A similar property obviously holds for **GLP**.

We obtain the fixed point property for **J** and **GLP** using the method of Smoryński and Bernardi (cf. [2]). First, we prove the so-called Bernardi lemma for **J**.

Lemma 3.2 *Suppose q does not occur in $\varphi(p)$ and p only occurs in $\varphi(p)$ within the scope of a modality. Then* **J** *proves the following formula B_φ:*

$$\Box^+(p \leftrightarrow \varphi(p)) \land \Box^+(q \leftrightarrow \varphi(q)) \to (p \leftrightarrow q).$$

Proof. We show that B_φ is valid in all finite **J**-models $\mathcal{W}$. With every $x \in \mathcal{W}$ we associate a sequence of numbers

$$D(x) := \langle d_0(x), d_1(x), \ldots, d_n(x) \rangle,$$

where $d_i(x)$ denotes the depth of x in $\mathcal{W}$ w.r.t. relation R_i inductively defined by

$$d_i(x) := sup\{d_i(y) + 1 : xR_i y\},$$

and n is the maximal number such that R_n is non-empty on $\mathcal{W}$. We consider a lexicographic ordering of such sequences.

Lemma 3.3 *For all $x, y \in \mathcal{W}$ and any k, if $xR_k y$ then $D(x) < D(y)$.*

Proof. Suppose $xR_k y$. For each $i < k$ we have $d_i(x) = d_i(y)$, since by (I) the same points z are R_i-accessible from x and from y. Also, obviously $d_k(x) > d_k(y)$, hence the result. ⊠

Suppose $\mathcal{W}$ is given and $\mathcal{W} \nvDash B_\varphi$. By considering a suitable generated submodel we may assume that

$$\mathcal{W} \vDash p \leftrightarrow \varphi(p), q \leftrightarrow \varphi(q) \qquad (**)$$

and $\mathcal{W} \nvDash p \leftrightarrow q$. Select $x \in \mathcal{W}$ such that p and q have have different evaluations at x and $D(x)$ is the minimal possible. By $(**)$ we have that $\varphi(p)$ and $\varphi(q)$ have different evaluations at x. Since p only occurs within the scope of modality in $\varphi(p)$, $\varphi(p)$ is a boolean combination of formulas of the form $[k]\psi(p)$ and variables different from p, q. Hence, there must exist a subformula $[k]\psi(p)$ of $\varphi(p)$ such that $[k]\psi(p)$ and $[k]\psi(q)$ have different evaluations at x. It follows that for some y such that $xR_k y$ the formulas $\psi(p)$ and $\psi(q)$ have different evaluations at y.

Let $\mathcal{W}_y$ denote the submodel of $\mathcal{W}$ generated by y. For each $z \in \mathcal{W}_y$ one has $xR_i z$, for some i. (If $yR_m z$ and $m < k$ then $xR_m z$ by (I), and if $m \geq k$ then $xR_k z$ by (J).) Hence, for all $z \in \mathcal{W}_y$, $D(z) < D(x)$. Therefore, by the choice of x, $\mathcal{W}_y \vDash p \leftrightarrow q$. It follows that for all subformulas $\theta(p)$ of $\varphi(p)$, $\mathcal{W}_y \vDash \theta(p) \leftrightarrow \theta(q)$. In particular,

$$\mathcal{W}, y \Vdash \psi(p) \leftrightarrow \psi(q),$$

a contradiction. ⊠

Corollary 3.4 (Fixed points in J) *Suppose q does not occur in $\varphi(p)$ and p only occurs in $\varphi(p)$ within the scope of a modality. Then there is a ψ (a fixed point of $\varphi(p)$) such that* $\mathrm{Var}(\psi) = \mathrm{Var}(\varphi(p)) \setminus \{p\}$ *and* $\mathbf{J} \vdash \psi \leftrightarrow \varphi(\psi)$. *Moreover, any two fixed points of $\varphi(p)$ are provably equivalent in* **J**.

Proof. Apply Beth definability property for the formula $\Box^+(p \leftrightarrow \varphi(p))$. Then we obtain a formula ψ such that

$$\mathbf{J} \vdash \Box^+(p \leftrightarrow \varphi(p)) \to (p \leftrightarrow \psi).$$

We show that ψ is the required fixed point.

Lemma 3.5 $\mathbf{J} \vdash \Box^+(p \leftrightarrow \psi) \to (p \leftrightarrow \varphi(p))$.

Proof. Consider a finite $\mathbf{J}$-model $\mathcal{W}$ and a node $x \in \mathcal{W}$ with the minimal $D(x)$ such that $\mathcal{W}, x \Vdash \Box^+(p \leftrightarrow \psi)$ and $\mathcal{W}, x \nVdash p \leftrightarrow \varphi(p)$. As before, we obviously have $\mathcal{W}_x \vDash p \leftrightarrow \psi$. Let p' be a fresh variable evaluated as follows: $\mathcal{W}, y \Vdash p'$ iff $\mathcal{W}, y \Vdash p$, for all $y \neq x$, and $\mathcal{W}, x \Vdash p'$ iff $\mathcal{W}, x \nVdash p$.

If $y \in \mathcal{W}_x$ and $y \neq x$ then $\mathcal{W}, y \Vdash p' \leftrightarrow \varphi(p')$, since p' and p have the same evaluation above x and $D(x)$ was chosen minimally. Since p occurs within the scope of a modality in $\varphi(p)$ we have $\mathcal{W}, x \Vdash \varphi(p)$ iff $\mathcal{W}, x \Vdash \varphi(p')$. Therefore, $\mathcal{W}, x \Vdash p' \leftrightarrow \varphi(p')$, since p' and p have opposite evaluations at x.

We conclude that $\mathcal{W}, x \Vdash \Box^+(p' \leftrightarrow \varphi(p'))$ and by the choice of ψ we must have $\mathcal{W}, x \Vdash p' \leftrightarrow \psi$. This implies $\mathcal{W}, x \Vdash p \leftrightarrow \psi \leftrightarrow p'$, quod non. ⊠

As an immediate corollary of this lemma (substituting ψ for p) we obtain

$$\mathbf{J} \vdash \psi \leftrightarrow \varphi(\psi).$$

If ψ_1 and ψ_2 are two fixed points of $\varphi(p)$, then obviously

$$\mathbf{J} \vdash \Box^+(\psi_i \leftrightarrow \varphi(\psi_i)), \text{ for } i = 1, 2.$$

Hence, by Bernardi's lemma $\mathbf{J} \vdash \psi_1 \leftrightarrow \psi_2$. ⊠

Corollary 3.6 *The fixed-point property holds for* **GLP**.

Proof. Given a formula $\varphi(p)$ in which p only occurs within the scope of a modality, we obtain a ψ such that $\mathbf{J} \vdash \psi \leftrightarrow \varphi(\psi)$. Obviously, the same equivalence also holds in a stronger system **GLP**.

To show the uniqueness, assume $\mathbf{GLP} \vdash \psi_1 \leftrightarrow \varphi(\psi_1)$, for another formula ψ_1. Denoting $\theta := \psi_1 \leftrightarrow \varphi(\psi_1)$ we obtain by (*Red*):

$$\mathbf{J} \vdash M^+(\theta) \to (\psi_1 \leftrightarrow \varphi(\psi_1)).$$

It follows that

$$\mathbf{J} \vdash \Box^+ M^+(\theta) \to \Box^+(\psi_1 \leftrightarrow \varphi(\psi_1)).$$

Since we also have $\mathbf{J} \vdash \Box^+(\psi \leftrightarrow \varphi(\psi))$, this implies

$$\mathbf{J} \vdash \Box^+ M^+(\theta) \to (\psi_1 \leftrightarrow \psi),$$

by Bernardi's lemma. Taking into account that $\mathbf{GLP} \vdash \Box^+ M^+(\theta)$, for any formula θ, this implies $\mathbf{GLP} \vdash \psi \leftrightarrow \psi_1$. ⊠

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