Riemann's zetafunction and its relatives

Before the break:

Riemann's zetafunction

After the break:

The rest

Counting prime numbers

Define

$$\pi(x) := |\{p \text{ prime}, \ p \leq x\}|$$

and

$$\text{Li}(x) = \int_2^x \frac{dt}{\log t}$$

Prime Number Theorem

$$\pi(x) \sim \int_2^x \frac{dt}{\log t} \left(\sim \frac{x}{\log x}\right)$$

Proved in 1899 by Hadamard and
De la Valleé Poussin (independently).

Define

$$\psi(x) := \log(\text{lcm}(1, 2, 3, \ldots, [x])).$$

Following statement is equivalent to PNT:

$$\psi(x) \sim x$$

x	$\pi(x)$	$\frac{\text{Li}(x)-\pi(x)}{\sqrt{x}}$
10^2	25	0.500
10^3	168	0.316
10^4	1229	0.170
10^5	9592	0.120
10^6	78498	0.130
10^7	664579	0.107
10^8	5761455	0.075
10^9	50847534	0.053
10^{10}	455052511	0.031
10^{11}	4118054813	0.036
10^{12}	37607912018	0.038
10^{13}	346065536839	0.034
10^{14}	3204941750802	0.031
10^{15}	29844570422669	0.033
10^{16}	279238341033925	0.032

Theorem (Littlewood) $\text{Li}(x) - \pi(x)$ changes sign infinitely often.

Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

Absolute convergence for every $s: \ \Re(s) > 1$.

Euler product

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}.$$

Proof:

$$\prod_p \frac{1}{1 - p^{-s}} = \prod_p \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots\right)$$
$$= \left(1 + \frac{1}{2^s} + \frac{1}{2^{2s}} + \cdots\right)\left(1 + \frac{1}{3^s} + \frac{1}{3^{2s}} + \cdots\right)\cdots$$
$$= 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \cdots$$

Results from Riemann's 1859 memoir

(1) The function $\zeta(s)$ can be continued analytically to all of $\mathbb{C}$ except for a first order pole at $s = 1$ with residue 1.

(2) The function

$$Z(s) := \pi^{-s/2}\Gamma(s/2)\zeta(s)$$

satisfies the functional equation

$$Z(1-s) = Z(s).$$

(3) $\zeta(s)$ has a simple (trivial) zero at
$s = -2, -4, -6, \ldots$.

The functional equation

Start with

$$\Gamma\left(\frac{s}{2}\right) = \int_0^\infty e^{-x} x^{\frac{s}{2}-1}\ dx.$$

Replace x by $\pi n^2 x$,

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\frac{1}{n^s} = \int_0^\infty e^{-\pi n^2 x} x^{\frac{s}{2}-1}\ dx.$$

Sum over $n = 1, 2, 3, \ldots$

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \int_0^\infty x^{\frac{s}{2}-1}\Theta(x)\ dx$$

where

$$\Theta(x) = \sum_{n=1}^\infty e^{-\pi n^2 x}.$$

We have

$$1 + 2\Theta(x) = \frac{1}{\sqrt{x}}\left(1 + 2\Theta\left(\frac{1}{x}\right)\right)$$

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \int_1^\infty x^{\frac{s}{2}-1}\Theta(x)\ dx + \int_0^1 x^{\frac{s}{2}-1}\Theta(x)\ dx$$

Use functional equation of $\Theta(x)$ to obtain

$$Z(s) = \frac{1}{s(1-s)} + \int_1^\infty \left(x^{\frac{s}{2}-1} + x^{\frac{1-s}{2}-1}\right)\Theta(x)\ dx$$

where

$$Z(s) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

A modular form

The function

$$\begin{aligned}\theta(\tau) &:= 1 + 2\Theta(-i\tau) \\ &= \sum_{n\in\mathbb{Z}} e^{\pi i n^2 \tau}\end{aligned}$$

is a function on the complex upper half plane $\tau \in \mathbb{C} : \ \Im(\tau) > 0$ and satisfies the functional equations

$$\theta\left(-\frac{1}{\tau}\right) = \sqrt{\frac{\tau}{i}}\ \theta(\tau)$$

and

$$\theta(\tau + 2) = \theta(\tau).$$

The tranformations $\tau \to -\frac{1}{\tau}$ and $\tau \to \tau + 2$ generate the group Γ consisting of all $\tau \to \frac{a\tau+b}{c\tau+d}$ where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ modulo 2.}$$

For every such transformation we have

$$\theta\left(\frac{a\tau+b}{c\tau+d}\right) = \epsilon(c\tau+d)^{1/2}\theta(\tau)$$

where $\epsilon^8 = 1$. We see: $\theta(\tau)$ is a modular form of weight 1/2 with respect to the group Γ.

Riemann's questions

(1) The number of zeros in the box $0 < \Re(s) < 1, 0 < \Im(s) < T$ equals

$$\frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T)$$

(2) **For every non-trivial zero ρ we have**

$$\Re(\rho) = 1/2$$

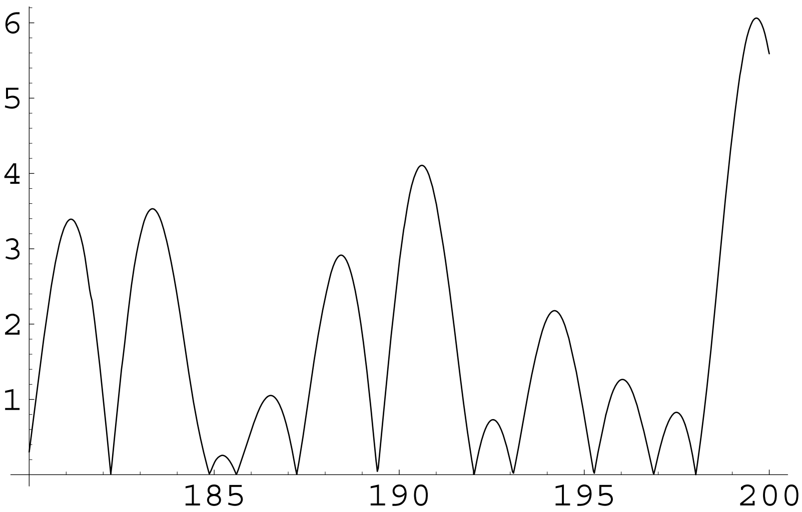
Statement (1) was proved by Von Mangoldt in 1895

Statement (2) (Riemann's hypothesis) is still open. Known results:

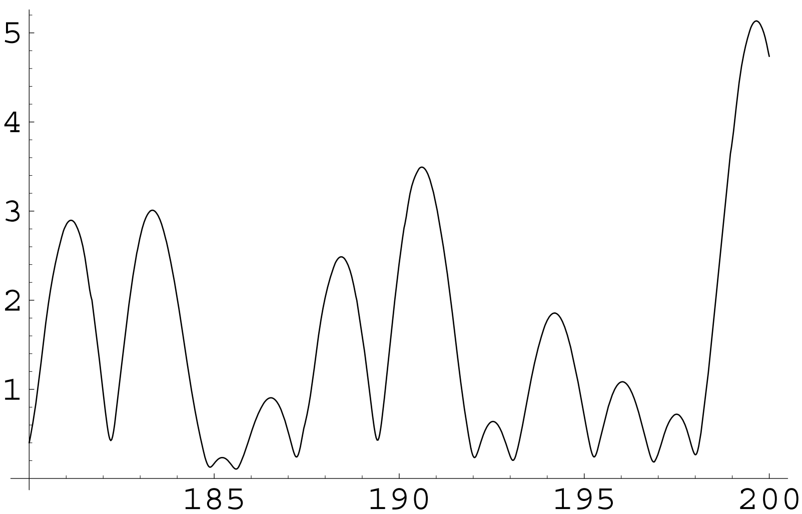
At least one third of all zeros lies on $\Re(s) = 1/2$ (N.Levinson, 1954)

The first 1.5 billion zeros lie on $\Re(s) = 1/2$ (Te Riele, Lioen, Van der Lune, 1984)

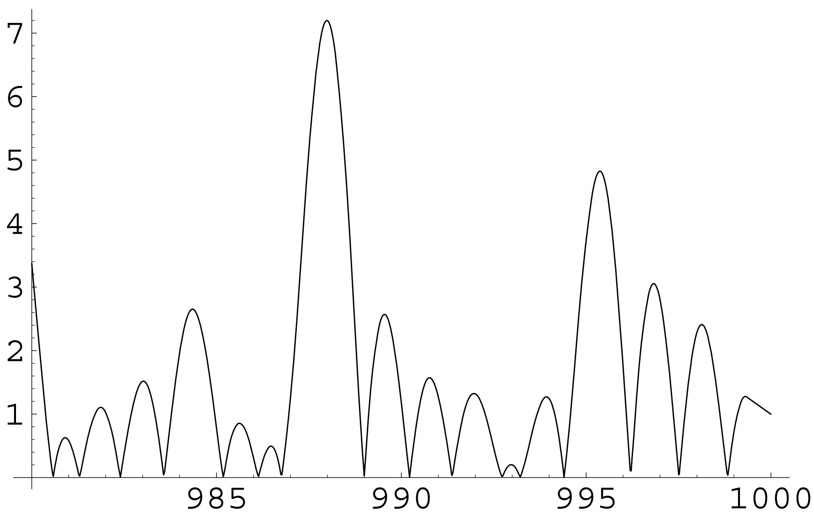
| ζ(0.5+it) |, 180<t<200



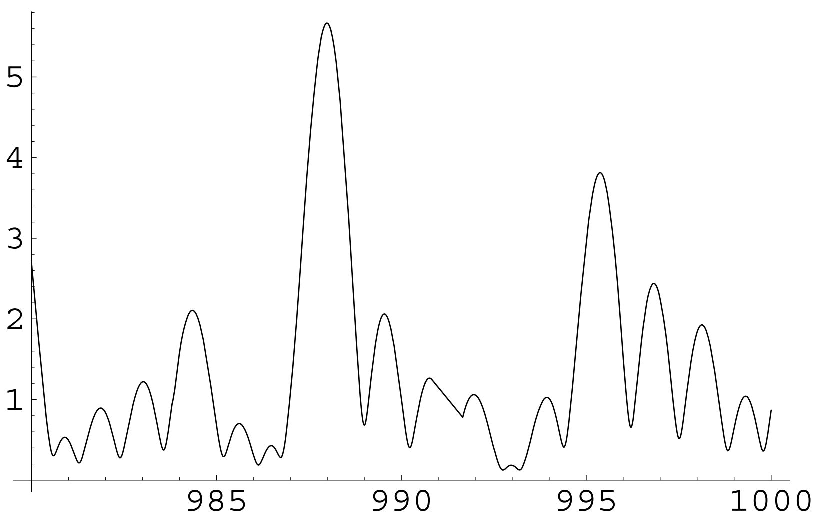
| ζ(0.6+it) |, 180<t<200



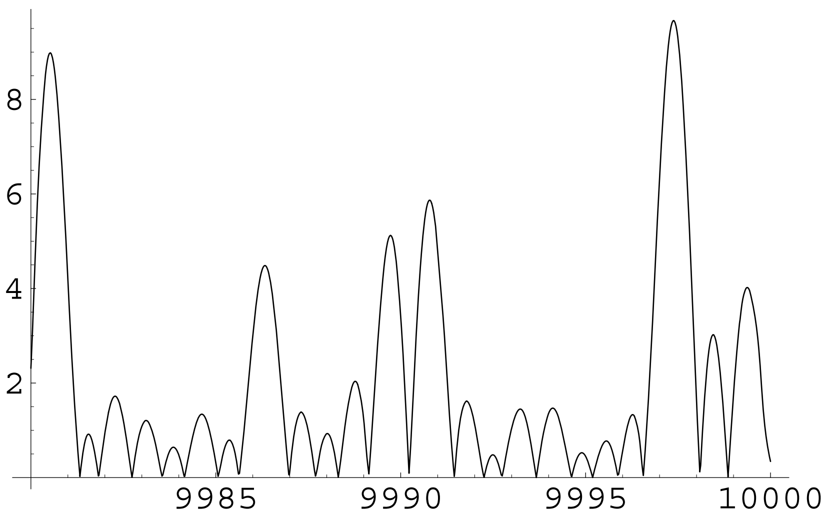
| ζ(0.5+it) | , 980<t<1000



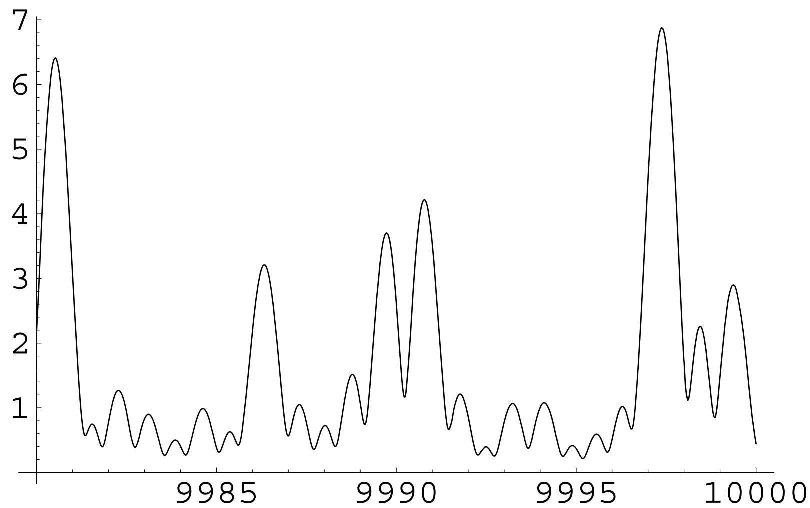
| ζ(0.6+it) | , 980<t1000



$|\zeta(0.5+it)|$, 9980<t<10000



$|\zeta(0.6+it)|$, 9980<t<10000



Von Mangoldt's formula

$$\psi(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} - C(x)$$

where

$$C(x) = -\frac{1}{2}\log(1 - 1/x^2) - \frac{\zeta'(0)}{\zeta(0)}$$

Corollary:
Suppose $\Re(\rho) \leq a$ for every zero. Then

$$\psi(x) = x + O(x^a(\log x)^2)$$

and

$$\pi(x) = \mathrm{Li}(x) + O(x^a \log x)$$

Remark: Converse also holds.

Remark: For the proof of PNT it suffices to know that $\zeta(1 + it)$ for all real t.

Montgomery's pair correlation

Assume that the Riemann hypothesis is true. Denote the zeros by $\frac{1}{2}+i\gamma$. Then

Conjecture (H.Montgomery, 1973):
Let $0<\alpha<\beta$. Then the number of pairs γ,γ' such that

$$\frac{2\pi\alpha}{\log T}\leq \gamma-\gamma'\leq \frac{2\pi\beta}{\log T}$$

is asymptotically equal to

$$\delta(\alpha,\beta)\frac{T}{2\pi}\log T$$

where

$$\delta(\alpha,\beta)=\int_{\alpha}^{\beta}\left(1-\left(\frac{\sin\pi u}{\pi u}\right)^2\right)\,du.$$

The GUE distribution (Gaussian unitary ensemble)

Let $A \in U(N)$. Normalise the distances between consecutive eigenvalues so that their average is one. Out of these spacings form probability measure $\mu(A, U(N))$ on $\mathbb{R}_{\geq 0}$ giving mass $1/N$ to each spacing. Define

$$\mu(U(N)) := \int_{U(N)} \mu(A, U(N)) \, dA$$

The measures $\mu(U(N))$ have a limiting measure $\mu(U)$ as $N \to \infty$ which is called the GUE measure (Wigner measure in physics).

Suspicion(A.Odlyzko). Let $0 \leq \alpha < \beta$. Denote the imaginary parts of zeta-zeros in increasing order by γ_n. Then the number of $n: \ N+1 \leq n \leq N+M$ such that

$$\frac{2\pi\alpha}{\log T} \leq \gamma_n - \gamma_{n+1} \leq \frac{2\pi\beta}{\log T}$$

asymptotically (as $N, M \to \infty$) equals

$$\int_{\alpha}^{\beta} \mu(U) \ dx$$

Tested in the range $n \in [10^{12}, 10^{12} + 10^5]$.

An extremely simple example

Fact: Let p be a prime $\neq 23$. The equation $x^2 \equiv -23 \pmod{p}$ is solvable if and only if p can be written in the form

$$p = x^2 + xy + 6y^2$$

or

$$p = 2x^2 + xy + 3y^2$$

Define for every prime p,

$$\chi(p) = \begin{cases} 1 & \text{if solvable} \\ \text{-1} & \text{if unsolvable} \\ 0 & \text{if } p = 23 \end{cases}$$

and

$$a(p) = \begin{cases} \text{-1} & \text{if } p = 2x^2 + xy + 3y^2 \\ 2 & \text{if } p = x^2 + xy + 6y^2 \\ 1 & \text{if } p = 23 \\ 0 & \text{else} \end{cases}$$

Consider the product

$$L(s) = \prod_{p \text{ prime}} \frac{1}{1 - a(p)p^{-s} + \chi(-p)p^{-2s}}$$

Expand

$$L(s) = \sum_{n=1}^{\infty} a(n)n^{-s}$$

Both product and series converge for all $\Re(s) > 1$.

Fact: $L(s)$ can be continued analytically to $\mathbb{C}$ and

$$\Lambda(1-s) = \Lambda(s)$$

where

$$\Lambda(s) = \left(\frac{2\pi}{\sqrt{23}}\right)^{-s} \Gamma(s)L(s).$$

Proof: write

$$(2\pi)^{-s}\Gamma(s)L(s) = \int_0^\infty x^{s-1} f(x)\ dx$$

where

$$f(x) = \sum_{n=1}^{\infty} a(n) e^{-2\pi n x}$$

Surprise: $f(-i\tau)$ is a modular form of weight one with respect to $\Gamma_0(23)$ (Langlands).

Also,

$$f\left(\frac{1}{23x}\right) = \sqrt{23}x f(x)$$

Moreover,

$$f(x) = q \prod_{n=1}^{\infty} (1 - q^n)(1 - q^{23n}), \qquad q = e^{-2\pi x}$$

The latter product equals

$$\begin{aligned}
& q - q^2 - q^3 + q^6 + q^8 - q^{13} - q^{16} + q^{23} \\
& -q^{24} + q^{25} + q^{26} + q^{27} - q^{29} - q^{31} \\
& +q^{39} - q^{41} - q^{46} - q^{47} + q^{48} + q^{49} \\
& -q^{50} - q^{54} + q^{58} + 2q^{59} + q^{62} + q^{64} \\
& -q^{69} - q^{71} - q^{73} - q^{75} - q^{78} - q^{81} \\
& +q^{82} + q^{87} + q^{93} + q^{94} - q^{98} + 2q^{101} \\
& -q^{104} - 2q^{118} + q^{121} + q^{123} - q^{127} - q^{128} \\
& -q^{131} + q^{138} - q^{139} + q^{141} + q^{142} + q^{146} \\
& -q^{147} + q^{150} - q^{151} + q^{162} - q^{163} + 2q^{167} \\
& +2q^{173} - q^{174} - 2q^{177} - q^{179} + q^{184} - q^{186} \\
& -q^{192} - q^{193} - q^{197} + q^{200} + \cdots
\end{aligned}$$

A second simple example

Consider the algebraic curve

$$y^2 - xy = x^3 + x^2$$

For every prime p let N_p be the number of solutions to

$$y^2 - xy \equiv x^3 + x^2 \pmod{p}$$

p	N_p	$a_p := p - N_p$
2	4	-2
3	4	-1
5	4	1
7	9	-2
11	10	1
13	9	4
17	15	-2
19	19	0
⋮	⋮	⋮

Theorem (Hasse) $|a_p| < 2\sqrt{p}$ for all primes p

Define

$$L(E,s) = \frac{1}{1-11^{-s}} \prod_{p\neq 11} \frac{1}{1-a_p p^{-s} + p^{1-2s}}$$

In expanded form:

$$L(E,s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

The function $L(E,s)$ can be continued analytically to $\mathbb{C}$ and we have

$$\Lambda(2-s) = \Lambda(s)$$

where

$$\Lambda(s) = \left(\frac{2\pi}{\sqrt{11}}\right)^{-s} \Gamma(s) L(s)$$

Proof: write

$$(2\pi)^{-s}\Gamma(s)L(s) = \int_0^\infty x^{s-1} f(x)\ dx$$

where

$$f(x) = \sum_{n=1}^{\infty} a(n) e^{-2\pi n x}$$

Surprise: $f(-i\tau)$ is a modular form of weight two with respect to $\Gamma_0(11)$ (Wiles).

Also,

$$f\left(\frac{1}{11x}\right) = 11x f(x)$$

Moreover,

$$f(x) = x \prod_{n=1}^{\infty} (1-q^n)^2(1-q^{11n})^2, \qquad q = e^{-2\pi x}$$

The latter product equals

$$\begin{aligned}
&q - 2q^2 - q^3 + 2q^4 + q^5 + 2q^6 - 2q^7 - 2q^9 \\
&-2q^{10} + q^{11} - 2q^{12} + 4q^{13} + 4q^{14} \\
&-q^{15} - 4q^{16} - 2q^{17} + 4q^{18} + 2q^{20} \\
&+2q^{21} - 2q^{22} - q^{23} - 4q^{25} - 8q^{26} \\
&+5q^{27} - 4q^{28} + 2q^{30} + 7q^{31} + 8q^{32} \\
&-q^{33} + 4q^{34} - 2q^{35} - 4q^{36} + 3q^{37} \\
&-4q^{39} - 8q^{41} - 4q^{42} - 6q^{43} + 2q^{44} \\
&-2q^{45} + 2q^{46} + 8q^{47} + 4q^{48} - 3q^{49} + O(q^{50})
\end{aligned}$$

General principle:

K finite Galois extension of $\mathbb{Q}$. Let

$$\rho : \mathrm{Gal}(K/\mathbb{Q}) \to GL(n, \mathbb{C})$$

be an irreducible representation of the Galois group of K. For every prime p there is a special element of Gal($K/\mathbb{Q}$), the *Frobenius element* F_p. Define

$$L(\rho, s) = \prod_p \frac{1}{\det(1 - F_p p^{-s})}$$

In our example we took K to be splitting field $x^3 - x - 1$ and the standard two dimensional representation of S_3 (isometries of triangle).

Conjecture (E.Artin). When ρ is non-trivial, $L(\rho, s)$ can be continued analytically to $\mathbb{C}$ and satisfies a functional equation for $s \leftrightarrow 1 - s$.

Proved in the case $n = 1$ (Dirichlet L-series) and partly for $n = 2$ (Langlands, Tunnell, Buzzard et al)

When $n = 2$ the Mellin transform is conjectured to be a modular form of weight one.

When $n > 2$, the L-series is conjecturally associated to representation theory of $GL(n)$.

More precisely/vaguely (Langlands correspondence):

Continuous representations of Gal($\overline{\mathbb{Q}}/\mathbb{Q}$)

$$\updownarrow$$

Cuspidal automorphic representations of classical reductive Lie-groups like $GL(n)$.