The derivability problem for Lambek calculus with one division

Yury Savateev*
Department of Mathematical Logic and Theory of Algorithms
Moscow State University

25th November 2006

Abstract

In this paper we prove that the derivability problem for Lambek calculus with one division is decidable in polynomial time and present an algorithm for it.

Introduction

In [5] NP-completeness was proved for the derivability problem for Lambek calculus L. For the non-associative variant of Lambek calculus the derivability can be checked in polynomial time as shown in [3] (for the product-free fragment of the non-associative Lambek calculus this was proved already in [1]). The same problem for $\mathrm{L}^{\backslash}$, a part of associative Lambek calculus called Lambek calculus with one division, was an open problem.

Lambek calculus L

Lambek calculus was first introduced in [4]. One of its associative variants can be constructed as follows. Let $\mathbf{P} = \{p_0, p_1, \ldots\}$ be a countable set of what we call *primitive types*. Let Tp be the set of *types* constructed from primitive types with three binary connectives $/$, $\backslash$, and $\cdot$. We will denote primitive types by small letters (p, q, $r, \ldots$) and types by capital letters

*This research was supported in part by the Russian Foundation for Basic Research and by the Dutch-Russian cooperation program NWO/RFBR, project No. 047.017.014.

(A, B, $C, \ldots$). By capital greek letters (Π, Γ, $\Delta, \ldots$) we will denote finite (probably empty) sequences of types. Expressions like $\Pi \to A$ where Π is not empty are called L-formulas or *sequents*.

Axioms and rules of L:

$$A \to A$$

$$(\to /)\frac{\Pi A \to B}{\Pi \to B/A} \qquad (\to \backslash)\frac{A\Pi \to B}{\Pi \to A\backslash B}$$

$$(/ \to)\frac{\Pi \to A \quad \Gamma B\Delta \to C}{\Gamma(B/A)\Pi\Delta \to C} \qquad (\backslash \to)\frac{\Pi \to A \quad \Gamma B\Delta \to C}{\Gamma\Pi(A\backslash B)\Delta \to C}$$

$$(\cdot \to)\frac{\Gamma AB\Delta \to C}{\Gamma(A \cdot B)\Delta \to C} \qquad (\to \cdot)\frac{\Gamma \to A \quad \Delta \to B}{\Gamma\Delta \to A \cdot B}.$$

(Here Γ and Δ can be empty.)

In this paper we will only consider $\mathrm{L}^{\backslash}$, a similar calculus but with only one connective $\backslash$ (so it has a smaller set of types, $\mathrm{Tp}(\backslash)$). Hence it only has the axiom and two rules, $(\to \backslash)$ and $(\backslash \to)$. The fragment with one division has been considered, e.g., in [2].

Notational note: we will write $A\backslash B\backslash C$ instead of $(A\backslash(B\backslash C))$.

Representation of formulas as strings of atoms

Let Atn be the set of *atoms* or *primitive types with degrees*, $\{p_i^j | i, j \in \mathbb{N}\}$. For convenience we shall continue to use different letters to denote different primitive types, e.g., p^i, q^j, and so on. We will consider the set of *strings of atoms* Atn^*. We will denote strings by $\mathbb{A}$, $\mathbb{B}$, and so on; ε will denote the empty string. Let $(\cdot)^{+2}$ be the operation on Atn^* satisfying the following conditions:

$$(p^i)^{+2} = p^{(i+2)}$$

$$(\mathbb{A}\mathbb{B})^{+2} = (\mathbb{A})^{+2}(\mathbb{B})^{+2}.$$

We shall say that $\mathbb{A} \in \mathrm{Atn}^*$ is *higher than* n for $n \in \mathbb{N}$ if the degrees of all atoms in $\mathbb{A}$ are greater than n. We will denote this by $\mathbb{A} \succ n$. (For example, $\varepsilon \succ n$ for all $n \in \mathbb{N}$.)

Consider $\gamma, \overline{\gamma} : \mathrm{Tp}(\backslash) \to \mathrm{Atn}^*$, two mappings from types to strings of atoms defined by

$$\gamma(p) = p^1 \qquad \overline{\gamma}(p) = p^2$$
$$\gamma(A \backslash B) = \overline{\gamma}(A)\gamma(B) \qquad \overline{\gamma}(A \backslash B) = \overline{\gamma}(B)(\gamma(A))^{+2}.$$

For example $\gamma((s\backslash q)\backslash r\backslash p) = q^2 s^3 r^2 p^1$ and $\overline{\gamma}((s\backslash q)\backslash r\backslash p) = p^2 r^3 s^4 q^3$. Also define $\gamma(\Pi)$ for $\Pi \in \mathrm{Tp}^*(\backslash)$ as follows: if $\Pi = A_1 A_2 \dots A_n$, then $\gamma(\Pi) = \gamma(A_1)\gamma(A_2)\dots\gamma(A_n)$. It is readily seen that for any $n > 0$ the string $\gamma(A_1 \dots A_n)$ ends in p^1 and the total number of atoms of degree 1 in it equals n.

Now we define two subsets of Atn^* — *plus-strings* and *minus-strings*. We will write $\mathbb{A}^+$ and $\mathbb{B}^-$ to denote that $\mathbb{A}$ is a plus-string and $\mathbb{B}$ is a minus-string. These two sets satisfy the following recursive conditions:

$$(p^1)^+ \quad (p^2)^-$$
$$\mathbb{A}^+, \mathbb{B}^+ \Rightarrow (\mathbb{A}\mathbb{B})^+$$
$$\mathbb{A}^-, \mathbb{B}^+ \Rightarrow (\mathbb{A}\mathbb{B})^+$$
$$\mathbb{A}^-, \mathbb{B}^+ \Rightarrow (\mathbb{A}(\mathbb{B})^{+2})^-.$$

Lemma 1. *The mapping γ is a bijection between non-empty sequences from $\mathrm{Tp}^*(\backslash)$ and plus-strings. And $\overline{\gamma}$ is a bijection between $\mathrm{Tp}(\backslash)$ and minus-strings.*

Construction of subsidiary calculus S

In the calculus S the formulas are of the type $\to \mathbb{A}$, where $\mathbb{A}$ is from Atn^*. Axioms and rules of S are the following:

$$\to p^1 p^2$$
$$(\mathrm{S1})\frac{\to \mathbb{A}^+\mathbb{B}^+p^2}{\to \mathbb{B}p^2(\mathbb{A})^{+2}}$$
$$(\mathrm{S2})\frac{\to \mathbb{A}^+\mathbb{B}^- \qquad \to \mathbb{C}^+\mathbb{D}^+p^2}{\to \mathbb{C}\mathbb{A}\mathbb{B}\mathbb{D}p^2}.$$

Here $\mathbb{A}, \mathbb{B}$, and $\mathbb{D}$ must be non-empty.

By induction on the rules of S we can prove that if $\mathrm{S} \vdash\rightarrow \mathbb{A}$, then $\mathbb{A} = \mathbb{B}^+\mathbb{C}^-$ and such a fragmentation is unique.

Since γ and $\overline{\gamma}$ are bijections, to each $\mathrm{L}^{\backslash}$-formula $\Pi \rightarrow A$ we can assign the string $\gamma(\Pi)\overline{\gamma}(A)$ and vice versa to each string of the type $\mathbb{B}^+\mathbb{C}^-$ we can assign the $\mathrm{L}^{\backslash}$-formula $\gamma^{-1}(\mathbb{B}) \rightarrow \overline{\gamma}^{-1}(\mathbb{C})$.

Equivalence of $\mathrm{L}^{\backslash}$ and S

Lemma 2. *For every* $A \in \mathrm{Tp}(\backslash)$, $\Pi \in \mathrm{Tp}^*(\backslash)$ *we have*

$$\mathrm{L}^{\backslash} \vdash \Pi \rightarrow A \Longleftrightarrow \mathrm{S} \vdash\rightarrow \gamma(\Pi)\overline{\gamma}(A).$$

And conversely for $\mathbb{B}, \mathbb{C} \in \mathrm{Atn}^*$ *such that* $\mathbb{B}^+$ *and* $\mathbb{C}^-$ *we have*

$$\mathrm{S} \vdash\rightarrow \mathbb{B}\mathbb{C} \Longleftrightarrow \mathrm{L}^{\backslash} \vdash \gamma^{-1}(\mathbb{B}) \rightarrow \overline{\gamma}^{-1}(\mathbb{C}).$$

To prove this and some of the further statements we need the following lemma.

Lemma 3. $\mathrm{L}^{\backslash}$ *is equivalent to the calculus with the following axiom and rules:*

$$p \rightarrow p$$

$$\frac{A\Pi \rightarrow B}{\Pi \rightarrow A\backslash B}$$

$$\frac{\Pi \rightarrow A \quad \Gamma B \Delta \rightarrow p}{\Gamma\Pi(A\backslash B)\Delta \rightarrow p}$$

Proof. **Step 1**. Get rid of instances of the axiom of the form $A \rightarrow A$ where $A \notin \mathbf{P}$. We can obtain this by changing $C\backslash D \rightarrow C\backslash D$ to

$$(\rightarrow \backslash)\ \frac{(\backslash \rightarrow)\ \dfrac{C \rightarrow C \qquad D \rightarrow D}{C(C\backslash D) \rightarrow D}}{C\backslash D \rightarrow C\backslash D.}$$

If all instances of the axiom are of the form $p \rightarrow p$, then the length of the derivation of a formula (the number of applications of the rules $(\rightarrow \backslash)$ and $(\backslash \rightarrow)$) exactly equals the number of occurrences of $\backslash$ in it.

Step 2. All applications of the rule $(\backslash \rightarrow)$ must be of the form

$$\frac{\Pi \rightarrow A \quad \Gamma B \Delta \rightarrow p}{\Gamma\Pi(A\backslash B)\Delta \rightarrow p}.$$

The rule $(\to \backslash)$ is reversible in $L^{\backslash}$. Thus, if $L^{\backslash} \vdash \Gamma B \Delta \to C_n \backslash \ldots \backslash C_1 \backslash p$, then $L^{\backslash} \vdash C_1 \ldots C_n \Gamma B \Delta \to p$. Hence we can change

$$\frac{\Pi \to A \quad \Gamma B \Delta \to C_n \backslash \ldots \backslash C_1 \backslash p}{\Gamma \Pi (A \backslash B) \Delta \to C_n \backslash \ldots \backslash C_1 \backslash p}$$

to

$$\begin{array}{ll} & \Pi \to A \qquad C_1 \ldots C_n \Gamma B \Delta \to p \\ \cline{2-2} (\backslash \to) & C_1 \ldots C_n \Gamma \Pi (A \backslash B) \Delta \to p \\ \cline{2-2} (\to \backslash) & C_2 \ldots C_n \Gamma \Pi (A \backslash B) \Delta \to C_1 \backslash p \\ & \vdots \\ (\to \backslash) & \Gamma \Pi (A \backslash B) \Delta \to C_n \backslash \ldots \backslash C_1 \backslash p \end{array}$$

without making the derivation longer (the number of occurrences of \ remains the same).

Thus, the formula is derivable in the new calculus. □

Let us now prove Lemma 2. Suppose we have a derivation of the formula $\Gamma \to A$ in $L^{\backslash}$ with the new rules. If we change throughout the derivation

$$p \to p \qquad \text{to} \qquad \to p^1 p^2,$$

$$\begin{array}{ll} & C_1 \ldots C_n \Delta \to p \\ \cline{2-2} (\to \backslash) & C_2 \ldots C_n \Delta \to C_1 \backslash p \\ & \vdots \\ (\to \backslash) & \Delta \to C_n \backslash \ldots \backslash C_1 \backslash p \end{array} \qquad \text{to} \qquad (\mathrm{S1}) \ \frac{\to \gamma(C_1 \ldots C_n) \gamma(\Delta) p^2}{\to \gamma(\Delta) p^2 (\gamma(C_1 \ldots C_n))^{+2}},$$

$$(\backslash \to) \ \frac{\Pi \to B \quad \Phi C \Delta \to p}{\Phi \Pi (B \backslash C) \Delta \to p} \qquad \text{to} \qquad (\mathrm{S2}) \ \frac{\to \gamma(\Pi) \overline{\gamma}(B) \quad \to \gamma(\Phi) \gamma(C \Delta) p^2}{\to \gamma(\Phi) \gamma(\Pi) \overline{\gamma}(B) \gamma(C \Delta) p^2},$$

then we will get a derivation for $\to \gamma(\Gamma) \overline{\gamma}(A)$ in S.

If we have a derivation of $\to \gamma(\Gamma) \overline{\gamma}(A)$ in S, then we can obtain a derivation of $\Gamma \to A$ in $L^{\backslash}$ by making the reverse substitutions. Since γ and $\overline{\gamma}$ are bijections, the second part of Lemma 2 directly follows from the first part. Thus, Lemma 2 is proved completely.

Derivability criterion

Lemma 4. *Suppose* $\mathbb{A}, \mathbb{B} \in \mathrm{Atn}^*, \mathbb{A}^+$, *and* $\mathbb{B}^-$. *Then* $\mathrm{S} \vdash \to \mathbb{A}\mathbb{B}$ *iff all atoms in* $\mathbb{A}\mathbb{B}$ *can be divided into pairs satisfying the following conditions:*

1. *A pair consists of p^i and p^{i+1}. In other words, atoms in a pair correspond to the same primitive type, their degrees differ by 1, and the atom with lesser degree stays to the left (the string is thought to be written from left to right).*

2. *All lines connecting atoms in pairs can be drawn in one semiplane without intersections. In other words, atoms in any two pairs can only follow each other in one of the following orders:*

$$\dots p^i \dots p^{i+1} \dots q^j \dots q^{j+1} \dots$$
$$\dots p^i \dots q^j \dots q^{j+1} \dots p^{i+1} \dots$$

3. *If the degree of the leftmost atom in a pair equals $2l$ (is even), then there is an atom of degree $2l-1$ between the elements of the pair.*

Proof. ($\Rightarrow$). Consider a derivation of $\rightarrow \mathbb{A}\mathbb{B}$ in S. Divide the atoms into pairs throughout all the strings used in the derivation as follows: the atoms in one axiom form a pair, and the string resulting from an application of a rule has the same pairing as the premises (all atoms from the premises go over to the result). By induction on the rules of S one can easily prove that such a pairing satisfies all necessary conditions.

($\Leftarrow$) We will use induction on the length of the string $\mathbb{A}\mathbb{B}$. For strings of length 2 the proof is trivial. Suppose that $\mathbb{A}^+$, $\mathbb{B}^-$, and $\mathbb{A}\mathbb{B}$ has a pairing satisfying all necessary conditions. If $\mathbb{B}$ is of the form $p^2(\mathbb{C}^+)^{+2}$, then we consider $\mathbb{C}\mathbb{A}p^2$ instead of $\mathbb{A}\mathbb{B}$ ($\mathrm{S} \vdash\rightarrow \mathbb{C}\mathbb{A}p^2 \Leftrightarrow \mathrm{S} \vdash\rightarrow \mathbb{A}\mathbb{B}$). The string $\mathbb{C}\mathbb{A}p^2$ is of the form $\mathbb{D}q^1r^2\mathbb{E}s^1p^2$, where $\mathbb{E} \succ 1$. It is readily seen that if a pairing exists, then $s = p$, $r = q$, and these atoms form two pairs from the proper pairing. Hence, if $\mathbb{E} \succ 2$, then $\mathbb{D}q^1q^2\mathbb{E}$ is a shorter string having a proper pairing, $(\mathbb{D}q^1)^+$, and $(q^2\mathbb{E})^-$. Then $\mathrm{S} \vdash\rightarrow \mathbb{D}q^1q^2\mathbb{E}$. Then $\mathrm{S} \vdash\rightarrow \mathbb{D}q^1q^2\mathbb{E}p^1p^2$ (by rule S2) and finally $\mathrm{S} \vdash\rightarrow \mathbb{A}\mathbb{B}$ (by rule S1 if needed). The only case left is when $\mathbb{D}q^1q^2\mathbb{E}p^1p^2$ has an atom of degree 2 in $\mathbb{E}$. Let $\mathbb{E}$ be of the form $\mathbb{E}''t^2\mathbb{E}'$, where $\mathbb{E}'' \succ 2$. The paired atom for this t^2 is some t^1 standing to the left of q^1. Then the string is of the form $\mathbb{D}t^1\mathbb{D}''q^1q^2\mathbb{E}''t^2\mathbb{E}'p^1p^2$. Note that $(\mathbb{D}'t^1t^2\mathbb{E}'p^1)^+$, $(p^2)^-$, $(\mathbb{D}''q^1)^+$, $(q^2\mathbb{E}'')^-$, and the pairings for $\mathbb{D}t^1t^2\mathbb{E}'p^1p^2$ and $\mathbb{D}''q^1q^2\mathbb{E}''$ taken from the pairing for $\mathbb{D}t^1\mathbb{D}''q^1q^2\mathbb{E}''t^2\mathbb{E}'p^1p^2$ satisfy all necessary conditions. Then by the induction assumption we have $\mathrm{S} \vdash\rightarrow \mathbb{D}t^1t^2\mathbb{E}'p^1p^2$ and $\mathrm{S} \vdash\rightarrow \mathbb{D}''q^1q^2\mathbb{E}''$ and thus $\mathrm{S} \vdash\rightarrow \mathbb{A}\mathbb{B}$ (by rule S2 and then rule S1 if needed). $\square$

The algorithm for derivability checking for $\mathrm{L}^{\backslash}$

Consider an $\mathrm{L}^{\backslash}$-formula $\Gamma \to A$. Let the total number of occurrences of all primitive types be n. It takes $O(n^2)$ simple operations to construct the string $\mathbb{A} = \gamma(\Gamma)\overline{\gamma}(A)$. This string will consist of n atoms. Now we are to check whether it has a pairing satisfying the conditions of Lemma 4. By $\mathbb{A}(i)$ we will denote the i-th atom in the string. For $i, j \in \mathrm{N}$, where $i \leqslant n$ and $i{+}2j{-}1 \leqslant n$, we will compute a value $M(i;j) \in \{0;1\}$. If for the substring of $\mathbb{A}$ starting at $\mathbb{A}(i)$ and of length $2j$ there exists a pairing satisfying the conditions from Lemma 4, then $M(i;j) = 1$; else $M(i;j) = 0$. Then $\mathrm{L}^{\backslash} \vdash \Gamma \to A$ if and only if $M(1;n/2) = 1$.

We will compute $M(i;j)$ successively, starting from $M(i;0)$ for $0 < i \leqslant n$. Put $M(i;0) = 1$ for all i. How to compute $M(i;j)$ if for all pairs $(i;j\,')$ where $j\,' < j$ it is already computed? Let $\mathbb{A}(i)$ be some p^m. If m is odd, then there must exist $l = 2k \leqslant 2j$ such that $\mathbb{A}(i + l - 1) = p^{m+1}$, $M(i + 1; k - 2) = 1$, and $M(i + l; j - k) = 1$. If m is even, then there is an additional requirement that $\mathbb{A}(i + l') = p^{m-1}$ for some $l' < l$. This takes $O(j)$ operations. We must compute $M(i;j)$ for $O(n^2)$ pairs. Thus we need $O(n^3)$ operations to check derivability for the formula $\Gamma \to A$ in $\mathrm{L}^{\backslash}$.

Acknowledgements

I am most grateful to Prof. M. Pentus for his constant attention to my work.

References

[1] E. Aarts and K. Trautwein, Non-associative Lambek categorial grammar in polynomial time, *Mathematical logic Quarterly* **41** (1995) pp. 476–484.

[2] W. Buszkowski, The equivalence of unidirectional Lambek categorial grammars and context-free grammars, *Zeitschrift für mathematische Logik und Grundlagen der Mathematik* **31**, no. 4 (1985) pp. 369–384.

[3] Ph. de Groote, The non-associative Lambek calculus with product in polynomial time, in: *Automated Reasoning with Analytic Tableaux and Related Methods*, (N. V. Murray, ed.), Springer (1999) pp. 128–139.

[4] J. Lambek, The mathematics of sentence structure, *American Mathematical Monthly* **65** (3) (1958) pp. 154–170.

[5] M. Pentus, Lambek calculus is NP-complete, *Theoretical Computer Science* **357**, no. 1–3 (2006) pp. 186–201.