Delimiting diagrams

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Abstract

We introduce the unifying notion of delimiting diagram. Hitherto unrelated results such as: Minimality of the internal needed strategy for orthogonal first-order term rewriting systems, maximality of the limit strategy for orthogonal higher-order pattern rewrite systems (with maximality of the strategy F_∞ for the λ-calculus as a special case), and uniform normalisation of balanced weak Church–Rosser abstract rewriting systems, all are seen to follow from the property that any pair of diverging steps can be completed into a delimiting diagram. Apart from yielding simple uniform proofs of those results, the same methodology yields a proof of maximality of the strategy F_∞ for the $\lambda\mathtt{x}^-$-calculus. As far was we know, this is the first time that a strategy has been proven maximal for a λ-calculus with explicit substitutions.

Throughout notation is from and unqualified references are to [5]. Let $\to,\rightarrowtriangle$ be ARSs (Def. 8.2.2).

Definition. $\rightarrowtriangle$ *delimits* $\to$ if for all $b \leftarrow^n a \rightarrowtriangle^m c$, for some k,l,d, $b \rightarrowtriangle^k d \leftarrow^l c$ and $n+k \leq m+l$.

Local delimitation, obtained by restricting n,m to 1, does not imply delimitation as witnessed by $\circlearrowleft a \leftarrow b \rightleftarrows c \to d \circlearrowright$ with $\rightarrowtriangle = \to$, the problem being (Fig. 1.3) that tiling with locally delimiting diagrams may not terminate. If it *does*, the completed diagram is delimiting, cf. the Extension Lemma (EL) below. First we prove the Progress Lemma (PL). These will be our tools.

Lemma (Progress). If $\rightarrowtriangle$ locally delimits $\to$, and $b \leftarrow^n a \rightarrowtriangle^m c$, then $d \leftarrow^{n-m} c$ for some d.

Proof. By induction on n.[1] In case $n \leq m$, setting $d = c$ works, and in case $m = 0$, $d = b$ does. Supposing $n > m \geq 1$ (see the figure below), there exist b' and c' such that $b \leftarrow^{n-1} b' \leftarrow a \rightarrowtriangle c' \rightarrowtriangle^{m-1} c$. By local delimitation, there are $n' \leq m'$, d' such that $b' \rightarrowtriangle^{n'} d' \leftarrow^{m'} c'$. By the IH for

$$\begin{tikzcd}
a \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{LD}"] & c' \arrow[r, "m-1"] \arrow[d] \arrow[dd, bend left, "n-1"] \arrow[dr, phantom, "\mathrm{IH}"] & c \arrow[d, "n-m"] \\
b' \arrow[r, "n' \leq m'"] \arrow[d, "n-1"'] \arrow[dr, phantom, "\mathrm{IH}"] & d' \arrow[d, "(n-1)-n'"'] \arrow[d, phantom, shift left=4, "\mathrm{A}"] & d \\
b & d'' &
\end{tikzcd}$$

LD = local delimitation
IH = induction hypothesis
A = arithmetic for natural numbers

$b \leftarrow^{n-1} b' \rightarrowtriangle^{n'} d'$, there is a d'' such that $d'' \leftarrow^{(n-1)-n'} d'$, hence $d'' \leftarrow^{((n-1)-n')+m'} c'$. By calculating $((n-1)-n')+m' \geq ((n-1)-m')+m' = \max(n-1,m') \geq n-1$ we find a reduction of length $n-1$ from c'. By the IH for it and $c' \rightarrowtriangle^{m-1} c$, there is a d such that $d \leftarrow^{(n-1)-(m-1)} c$. $\square$

Lemma (Extension). If $\to,\rightarrowtriangle$-normal forms coincide and $\rightarrowtriangle$ locally delimits $\to$, then for c normal, $b \leftarrow^n a \rightarrowtriangle^m c$ is completable into a delimiting diagram, i.e. $b \rightarrowtriangle^k c$ with $n+k \leq m$.

Proof. By induction on m. In case $m = 0$, then $n = 0$ by the assumption that $\to,\rightarrowtriangle$-normal forms coincide. In case $n = 0$, setting $k = m$ works. Supposing $n,m \geq 1$, there exist b' and c' such that $b \leftarrow^{n-1} b' \leftarrow a \rightarrowtriangle c' \rightarrowtriangle^{m-1} c$. By local delimitation, there are $n' \leq m'$, d' such that $b' \rightarrowtriangle^{n'} d' \leftarrow^{m'} c'$. By the IH for $d' \leftarrow^{m'} c' \rightarrowtriangle^{m-1} c$, $d' \rightarrowtriangle^{k'} c$ with $m'+k' \leq m-1$, hence $b' \rightarrowtriangle^{n'+k'} c$. By $n'+k' \leq m'+k' \leq m-1$ the IH may be applied to $b \leftarrow^{n-1} b' \rightarrowtriangle^{n'+k'} c$, yielding $b \rightarrowtriangle^k c$ with $(n-1)+k \leq n'+k'$, which implies $n+k \leq m$ as desired. $\square$

[1] M. Bezem has shown that formalising the induction step in Geometric Logic [1] its proof is automatable.

Note that the lemmata still hold if the disjunction of the matrix of local delimitation with 'or $\leftarrow^{\infty} c$' is taken (in PL proceed to the second application of the IH; in EL this case can be excluded using PL). In the following, we first assume $\to$ to be a (Def. 9.1.1) strategy for $\rightarrowtriangle$, next, dually, $\rightarrowtriangle$ to be a strategy for $\to$, and finally both, i.e. that $\to$ and $\rightarrowtriangle$ coincide. By the definition of strategy (Def. 9.1.1), the normal forms of $\to$ and $\rightarrowtriangle$ coincide each time. So, first, let $\to$ be a strategy for $\rightarrowtriangle$. Then we leave $\rightarrowtriangle$ implicit, and say $\to$ is *locally minimal* instead of $\rightarrowtriangle$ locally delimits $\to$.

Lemma (Internal Needed). The internal needed strategy is locally minimal for OTRSs.

Proof. [2] A step is internal needed (Def. 9.4.4) if it is both *needed* (it cannot be eliminated by erasure alone; at least one of its residuals has to be contracted in any reduction to normal form) and *internal* (none of its residuals nests a needed redex). By Lem. 9.4.5, the internal needed redexes coincide with the innermost redexes among the needed ones. We distinguish cases on the relative positions of the redexes contracted at p, q in a peak $s \leftarrow_p t \rightarrowtriangle_q u$:
(=) Then $s = u$, so the $\to$-step is locally minimal.
($\|$) (Not. 2.1.8) Then $s \rightarrowtriangle_q v \leftarrowtriangle_p u$, for some term v. Since the unique residual of a needed redex is needed, in fact $v \leftarrow_p u$ and we are ok again.
($<$) Then q is non-needed. Hence if u has an $\to$-reduction to normal form, contracting the same redexes from s yields a $\rightarrowtriangle$-reduction to normal form again and of the same length, so we are done.
($>$) Then $s \rightarrowtriangle_q v \twoheadleftarrow u$ for some v (Lem. 4.7.8). We may partition $v \twoheadleftarrow u$ as $v \twoheadleftarrow v' \twoheadleftarrow u$, where the former is non-needed and the latter is internal needed and moreover non-empty as p was needed. Since taking the residual of any needed reduction from v' along $v \twoheadleftarrow v'$ yields a reduction from v of exactly the same length, the result follows from confluence (or non-termination). $\square$

Minimality and normalisation (Defs. 9.4.1, 9.1.12) of the internal needed strategy follow:

Theorem (Minimality). If $\to$ is locally minimal, then it is minimal and normalising.

Proof. Suppose $a \rightarrowtriangle^m b$ with b a normal form. Then, by the EL, m exceeds the length of any $\to$-reduction from a (normalisation), in particular of any $\to$-reduction from a to b (minimality). $\square$

Another application is the Gross-Knuth strategy $\mathrel{-\!\!\circ\!\!\!\to}_{GK}$, contracting all redexes in one go: $\mathrel{-\!\!\circ\!\!\!\to}_{GK}$ is locally minimal, since if $s \mathrel{\leftarrow\!\!\!\circ\!\!-}_{GK} t \mathrel{-\!\!\circ\!\!\!\to} u$, then either $s = u$ or $s \mathrel{\leftarrow\!\!\!\circ\!\!-}_{GK} u$ or $\exists v\ s \mathrel{-\!\!\circ\!\!\!\to} v \mathrel{\leftarrow\!\!\!\circ\!\!-}_{GK} u$. Next, let $\rightarrowtriangle$ be a strategy for $\to$. We leave $\to$ implicit and say $\rightarrowtriangle$ is *locally maximal* instead of $\rightarrowtriangle$ locally delimits $\to$. How to find such a $\rightarrowtriangle$? For fully-extended orthogonal higher-order pattern rewrite systems (PRSs, cf. p. 604 and [4, Def. 10]) external steps (Def. 9.2.31) would fit the bill, but for their failure to deal with erasure: e.g. for the λ-calculus leftmost-outermost (lmo) steps are external but local maximality fails for $(\lambda x.y)N' \leftarrow (\lambda x.y)N \rightarrowtriangle y$. The limit strategy[3] (Def. 9.5.5) solves this by calling itself on erased arguments, instead of taking the external step in such cases.

Lemma (Limit). The limit strategy is locally maximal for fully-extended orthogonal PRSs.[4]

Proof. Distinguish cases on the relative positions of the redexes contracted at p, q in $s \leftarrow_p t \rightarrowtriangle_q u$:
(=) Then $s = u$, so local maximality holds.
($\|$) Then $s \rightarrowtriangle_q v \leftarrow_p u$, for some term v, and we are ok again.
($<$) Then by definition of $\rightarrowtriangle$ and orthogonality, p must be a redex erasing q, so $s \leftarrow_p u$.
($>$) Then $s \to_q v \mathrel{\leftarrow\!\!\!\circ\!\!-} u$ for some v by Thm. 11.6.29. If $s \to_q v$ is non-erasing, then $s \rightarrowtriangle_q v$. Otherwise,[5] $\rightarrowtriangle$-reduce each erased argument in turn to its normal form, before contracting the (then limit) redex at q to v. By Thm. 11.6.29, u reduces to v by performing for each erased argument of s, the same steps on its (non-empty) set of residuals in u. To guarantee that this reduction has at least the same length as that from s, it suffices to perform the residual reduction according to the inside-out order of the residuals. The only exception to this is when the $\rightarrowtriangle$-reduction from s be infinite, but then u allows an infinite reduction as well. $\square$

[2] Apart from separating out local minimality, this proof also corrects the flawed proof of Thm. 9.4.7.
[3] The strategy F_∞ for the λ-calculus is a special case of a limit strategy (Rem. 9.5.4).
[4] Existence of external steps needs full-extendedness. The result extends Thm. 9.5.7 to higher-order (Rem. 9.5.4).
[5] In the λ-calculus this is witnessed by e.g. $(\lambda x.y)N \leftarrow (\lambda x.(\lambda z.y)x)N \rightarrowtriangle (\lambda z.y)N$. In that case, we should have first $\rightarrowtriangle$-reduced N to normal form, say N', before to proceed with $(\lambda x, y)N' \rightarrowtriangle y$

The λ-calculus with explicit substitutions $\lambda\mathtt{x}^-$ is a left-linear and left-normal (Def. 8.5.59) PRS [4, Def. 13]. So lmo-redexes are external, inducing a limit strategy known as F_∞.

Lemma ($\lambda\mathtt{x}^-$-limit). F_∞ is locally maximal for $\lambda\mathtt{x}^-$.

Proof. It suffices to extend the case analysis in the proof of the Limit Lemma with overlap:
(#) Then $C[M\langle x{:=}N\rangle\langle y{:=}P\rangle] \leftarrow_p C[(\lambda x.M)N\langle y{:=}P\rangle] \rightarrowtriangle_q C[(\lambda x.M)\langle y{:=}P\rangle N\langle y{:=}P\rangle]$. As in the ($>$)-case we simulate a $\rightarrowtriangle$-reduction from s, by a reduction from u which is at least as long, giving the desired result by confluence of $\lambda\mathtt{x}^-$. Start simulating with $u \to C[M\langle y{:=}P\rangle\langle x{:=}N\langle y{:=}P\rangle\rangle] = u'$. Then the idea is that the $\langle x{:=}N\rangle\langle y{:=}P\rangle$-closure in s is only ever replicated in its entirety, in two consecutive $\rightarrowtriangle$-steps, which are matched on u' each time by replicating the corresponding $\langle y{:=}P\rangle\langle x{:=}N\langle y{:=}P\rangle\rangle$-closure. Only if a variable is reached, a further case distinction is needed:
(x) Then we conclude from $x\langle x{:=}N\rangle\langle y{:=}P\rangle \rightarrowtriangle N\langle y{:=}P\rangle \leftarrow^2 x\langle y{:=}P\rangle\langle x{:=}N\langle y{:=}P\rangle\rangle$.
(y) Then $\rightarrowtriangle$ recurs on N and we go via $y\langle x{:=}N'\rangle\langle y{:=}P\rangle \rightarrowtriangle y\langle y{:=}P\rangle \rightarrowtriangle P$ to P', with N', P' the normal forms of N, P. This is matched by a reduction from $y\langle y{:=}P\rangle\langle x{:=}N\langle y{:=}P\rangle\rangle$ via $y\langle y{:=}P\rangle\langle x{:=}N'\langle y{:=}P\rangle\rangle \to P\langle x{:=}N'\langle y{:=}P\rangle\rangle$ to $P'\langle x{:=}N'\langle y{:=}P\rangle\rangle$ to P' (in at least one step), using that x occurs in P nor P', and that P' is in normal form so does not contain closures.
(z) As in the previous case but simpler as we end up just in z (if $\rightarrowtriangle$ terminates at all). $\square$

These strategies being locally maximal implies their maximality and perpetuality (p. 530), by substituting the latters for minimality and normalisation in (the proof of) the Minimality Theorem:

Theorem (Maximality). If $\rightarrowtriangle$ is locally maximal, then it is maximal and perpetual.

This answers the open question whether F_∞ is maximal [2, Rem. 3.18] for $\lambda\mathtt{x}^-$, in the affirmative.[6] Finally, we let $\to$ coincide with $\rightarrowtriangle$, in which case we speak of $\to$ being *locally self-delimiting*. Joining minimality and maximality into *equidistance*, i.e. all reductions from a given object a to a normal form b have the same length, and normalisation and perpetuality into uniform normalisation (*uniformity* for short [4, Def. 2]), i.e. for all objects a, WN(a) implies SN(a), we have:

Theorem (Self-delimitation). if $\to$ is *locally self-delimiting*, then it is equidistant and uniform.

This covers many notions and results in the literature: Balanced WCR [6] requires $k = l$ (in the definition of $\to$ local delimitation $\to$), and its generalisation balanced SCR [3] requires the natural numbers and object chosen for a given a peak and its symmetric version to be identical. Linear biclosed [4] requires $k = 0$ or $k = 1 = l$, and its generalisation SCR$^{\geq 1}$ [3] $k = 0 = l$ or $k \leq 1 \leq l$. Our generalisation is proper in all these cases as witnessed by $a \to b \to e \rightleftarrows f$ and $a \to c \to d \to e$.

Note that for any abstract rewriting system, if $b \leftarrow a \to c$, then also $c \leftarrow a \to b$. Hence local self-delimitation requires that in such a case we not only have $b \to^n d \leftarrow^m c$, for some d and some $n \leq m$, but also $b \to^{n'} d' \leftarrow^{m'} c$, for some d' and some $n' \geq m'$ (cf. this to Exc. 1.3.11).

Further research What about essentially non-confluent systems, say systems having choice?

Acknowledgments We would like to thank F.-J. de Vries for the initial question and him, M. Bezem, E. Bonelli, B. Gramlich, Z. Khasidashvili, A. Middeldorp, and Y. Toyama for comments.

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[6] As our proof is a more precise version of the proof [4, Lem. 6] of preservation of strong normalisation (PSN) for $\lambda\mathtt{x}^-$, which only used perpetuality (not maximality), we expect it works for PSN λ-calculi with explicit substitutions.