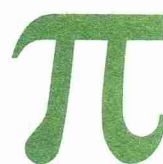


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# Lazy & Quarrelsome Brackets

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# *Lazy & Quarrelsome Brackets*

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**ABSTRACT:** In this paper we study two (kinds of) systems of brackets in an algebraic way. Lazy brackets have the same effect as introducing or eliminating ‘a sufficient amount’ of ordinary brackets at the same time. Quarrelsome brackets are brackets corresponding to different types of ‘levels’: think e.g. of term levels versus sentence levels. A modest framework is proposed to study these kinds of brackets simultaneously.

**1 Introduction:** In this paper I present a mildly abstract framework to introduce and study stacking cells of different kinds. The idea of a stacking cell is simply this: one way of viewing a bracket is as an instruction to introduce or throw away a *file* or *level* or *discourse object*. A stacking cell is just the combined action of a number of brackets, where we divide out subactions that cancel each other.

Stacking cells can be seen as closely related to the integers: an integer can be seen as the action resulting from a number of additions and subtractions. The difference is that in the case of stacking cells we do not abstract away from the order of the actions. In an earlier paper Visser[92a], I studied stacking cells algebraically in context where (analogues of) the integers are present. In this paper we look at cases where stacking cells are sensible objects, but no decent integer analogues exist.<sup>1</sup>

My two examples are lazy brackets and quarrelsome ones. A system of lazy brackets is simply a system where some brackets have more weight than others. Let e.g. the round brackets be the ones of least weight and let square brackets be heavier than round ones. For example consider the formula of propositional logic:  $((p \wedge q) \vee r) \rightarrow (p \vee s)$ . With the help of lazy brackets we could rewrite it to e.g.:  $[p \wedge q] \vee r \rightarrow p \vee s$ . The idea is that e.g.  $[$  can be viewed as ‘a sufficient amount of (‘s’.

I would conjecture that lazy brackets can be useful in theories of incremental generation

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<sup>1</sup> This is not a technical claim: I don’t pretend to have a good analysis of what an integer analogue is supposed to be. So a fortiori I have no *proof* that e.g. the ordinals with their usual addition and ordering do not admit a good integer analogue. I just don’t *see* how there could be one.

of grammar. For example consider the sentence:

*A child's mother takes good care of it.*

One would imagine that this sentence could be parsed as:

*((a child's) mother) takes (good care) (of it))*

Note that at the beginning we do not only have to realize that Speaker starts a new sentence, but we also have to conjecture precisely how many levels are required. Or: we have to backtrack later to add the appropriate number of ('s (i.e.: level introductions or pushes). Also at the end we have to carefully close all the brackets that are still left open (or: pop all the levels that are still left over). Clearly the obvious lazy representation doesn't ask for all this effort:

*[a child's) mother) takes (good care) (of it]*

The [ can be seen as *the meaning of the capital*: imagine a sufficient amount of (stacked) levels to be present to store the information that will be produced in the forthcoming sentence. The ] can be seen as *the meaning of the point*: throw away all the levels that have not been thrown away yet.<sup>2</sup>

Quarrelsome brackets can be used when we feel a distinction between kinds of levels is important. Consider e.g. the sentence:

*The quick brown fox that jumped over the lazy dog wanted the rabbit that ran.*

In round bracket notation we could parse this as e.g.:

*((the quick brown fox ((that) jumped (over the lazy dog))) wanted (the rabbit ((that) ran)))*

But suppose that we wanted to keep sentence levels and term levels strictly apart. In this case we could e.g. write:

*{(the quick brown fox {(that) jumped (over the lazy dog)}) wanted (the rabbit {(that) ran})}*

Here parentheses enclose sentence levels and round brackets term levels. If an opening ( is closed with a } we count this ungrammatical. Semantically we represent this by going into the error state 0.<sup>3</sup>

In this paper I study stacking cells as interesting beasts in their own right. How to apply them will be explained in my forthcoming paper *Meanings in Time*. The idea is roughly as follows. (i) The basic format of meanings is context+content. (ii) One function of contexts is to control the process of merging of contents. Thus 'dynamic

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<sup>2</sup> This example is inspired by some questions by Henk Zeevat. Kees Vermeulen suggested the bracketlike interaction of Capital and Point.

<sup>3</sup> A more surprising and subtle use of (an extension of) quarrelsome brackets can be found in Kees Vermeulen's forthcoming paper on incremental semantics for propositional texts.

aspects' are located in the context. (iii) Files or discourse objects play a central role in storing information. (iv) Both anaphoric machinery and syntactic structure can be viewed as providing instructions for handling files. Thus the contexts will contain certain traces of syntax or bits of syntactic memory. Stacking cells are such bits of syntactic memory or at least central ingredients of such bits of syntactic memory.

**2 The monoids of lazy brackets and of quarrelsome brackets:** We start with introducing a monoid for lazy brackets. Let  $\mathfrak{LB}$  be the free monoid on generators  $(,), [, ]$ , satisfying the following equations:  $[] = [ (= [ ( [] ) ] = ]$ ,  $[] = () = 1$ . The idea is that e.g.  $[$  stands for an infinite number of 's:  $\dots((($ . Note that we 'count' our brackets here from right to left. Similarly  $]$  stands for:  $)))\dots$ . Now we count from left to right.

Obviously a term rewriting system can be based on the monoidal rules and our equations. Just as clearly this system has the Church Rosser Property and is strongly normalizing. An example of a normal form is:  $]]]]]]]]))(((([[[[[[$ .

Using the normal forms as representations of the objects of our monoid, here is an example of an identity:  $]]]](([[ \bullet ]]])([ = ]]])([$ .

Our next example is a monoid for quarrelsome brackets. Let  $\mathfrak{QB}$  be the free monoid on generators  $(,), \{, \}$  with extra constant  $0$  satisfying the following equations

$$0 \bullet x = x \bullet 0 = 0, () = \{ \} = 1, \{ \} = () = 0.$$

The idea is that  $0$  is the error state.

A term rewriting system can be based on the monoidal rules and our equations. Just as clearly this system has Church Rosser and is strongly normalizing. An example of a normal form is:  $))))))\} \{ (((((($ .

Using the normal forms as representations of the objects of our monoid, here are two examples of identities:  $))\} \{ \{ (\bullet) \} \} ((\{ = \} \} \} ((\{ , \} \} \{ \{ (\bullet) \} \} ((\{ = 0$ .

**3 Constructing stacking cells from an L-monoid:** Consider a structure  $\mathfrak{M} = (M, 1, \bullet, \leftarrow)$ , where  $\mathfrak{M} = (M, 1, \bullet)$  is a monoid. Define:

$x \leq y : \Leftrightarrow$  for some  $u$ :  $u \bullet y = x$ . We demand the following properties:

L1  $x \bullet z = y \bullet z \Rightarrow x = y$ ,

$$L2 \quad x \bullet y = 1 \Rightarrow y = 1,$$

$$L3 \quad x \bullet y \leq z \Leftrightarrow x \leq z \leftarrow y.$$

We call an  $\mathcal{M}$  satisfying these properties an L-monoid.  $\leftarrow$  is the binary operation *post-implication*.

Sometimes it is pleasant to add an ‘error-state’ 0 to an L-monoid. We then obtain an  $L^+$ -monoid. An  $L^+$ -monoid is a structure  $\mathcal{M} = \langle M, 1, 0, \bullet, \leftarrow \rangle$ , satisfying:  $\mathcal{M} = \langle M, 1, \bullet \rangle$  is a monoid,  $0 \in M$ ,  $L^+2 := L2$ ,  $L^+3 := L3$  and:

$$L^+1 \quad z \neq 0 \text{ and } x \bullet z = y \bullet z \Rightarrow x = y,$$

$$L^+4 \quad x \bullet 0 = 0 \bullet x = 0 \quad (0 \text{ is the annihilator for } \bullet)$$

$$L^+5 \quad x \bullet y = 0 \Rightarrow x = 0 \text{ or } y = 0,$$

Since the results on L-monoids can be obtained from the ones on  $L^+$ -monoids by just omitting the things pertaining to 0, we now assume that we are working in an  $L^+$ -monoid.

We will construct our stacking cells from  $\mathcal{M}$ . But first we need some assorted facts on  $L^+$ -monoids.

We will sometimes write:  $u : x \leq y$ , for  $u \bullet y = x$ . Note that if  $y \neq 0$  and  $u : x \leq y$ , then  $u$  is the unique witness of  $x \leq y$ .

### 3.1 Fact

- i)  $\leq$  is a partial ordering with top 1 and bottom 0.
- ii)  $v \bullet y \leq y$ ,
- iii)  $u \leq v \Rightarrow u \bullet x \leq v \bullet x$ ; if  $x \neq 0$ :  $u \bullet x \leq v \bullet x \Rightarrow u \leq v$ .

**Proof:** (i) We have:

$$u : x \leq y, v : y \leq z \Rightarrow u \bullet v : x \leq y,$$

$$1 : x \leq x, x : x \leq 1, 0 : 0 \leq x.$$

Suppose  $u : x \leq y, v : y \leq x$ . Then  $u \bullet v : x \leq x$ . If  $u \bullet v = 0$ , then  $x = 0$  and hence, since  $v \bullet x = y$ ,  $y = 0$  and hence  $x = y$ . If  $u \bullet v \neq 0$ , then  $(u \bullet v) \bullet x = 1 \bullet x$  and hence  $u \bullet v = 1$ , so  $v = 1$  and  $x = y$ .

ii)  $v : v \bullet y \leq y$ . (iii) Suppose  $w : u \leq v$ , then  $w : u \bullet x \leq v \bullet x$ . Suppose  $x \neq 0$  and  $w' : u \bullet x \leq v \bullet x$ . Then  $w' \bullet v \bullet x = u \bullet x$  and hence  $w' \bullet v = u$ , so  $w' : u \leq v$ .  $\square$

### 3.2 Fact

- i)  $(x \leftarrow 1) = x, (1 \leftarrow x) = 1, (x \leftarrow 0) = 1, (x \leftarrow x) = 1$ , if  $x \neq 0$ :  $(0 \leftarrow x) = 0$ ,

- ii) if  $z \neq 0$ :  $(y \bullet z \leftarrow x \bullet z) = (y \leftarrow x)$ ,
- iii)  $x \leq y \Leftrightarrow (y \leftarrow x) = 1$ ,
- iv)  $(z \leftarrow x \bullet y) = ((z \leftarrow y) \leftarrow x)$ .

**Proof:** E.g (ii): Suppose  $z \neq 0$ , then:

$$\begin{aligned} u \leq (y \bullet z \leftarrow x \bullet z) &\Leftrightarrow u \bullet x \bullet z \leq y \bullet z \\ &\Leftrightarrow u \bullet x \leq y \\ &\Leftrightarrow u \leq (y \leftarrow x) \end{aligned}$$

□

**3.3 Fact:** We have (finite) infima:  $x \wedge y := (y \leftarrow x) \bullet x$ .

**Proof:**

$$\begin{aligned} z \leq (y \leftarrow x) \bullet x &\Leftrightarrow \text{for some } u: u \bullet (y \leftarrow x) \bullet x = z \\ &\Leftrightarrow \text{for some } v: v \bullet x = z \text{ and } v \leq (y \leftarrow x) \\ &\Leftrightarrow \text{for some } v: v \bullet x = z \text{ and } v \bullet x \leq y \\ &\Leftrightarrow z \leq x \text{ and } z \leq y. \end{aligned}$$

□

**3.4 Fact**

- i)  $((y \wedge z) \leftarrow x) = ((y \leftarrow x) \wedge (z \leftarrow x))$
- ii)  $(x \wedge y) \bullet z = x \bullet z \wedge y \bullet z$ .
- iii)  $x \leq y \Rightarrow (x \leftarrow y) \bullet x \leq y$ .

**Proof:** E.g. (ii): If  $z=0$  this is trivial. So suppose  $z \neq 0$ . We have:

$$(x \wedge y) \bullet z = (y \leftarrow x) \bullet x \bullet z = (y \bullet z \leftarrow x \bullet z) \bullet x \bullet z = x \bullet z \wedge y \bullet z.$$

$$\text{iii) } (x \leftarrow y) \bullet y = x \wedge y = x.$$

□

**3.5 Remark:** We can characterize  $\leftarrow$  in terms of  $\wedge$ , for note that:

$$(y \leftarrow x) = (y \leftarrow x) \wedge 1 = (y \leftarrow x) \wedge (x \leftarrow x) = ((x \wedge y) \leftarrow x),$$

So in case  $(x \wedge y) \neq 0$ :  $y \leftarrow x$  is the unique witness of  $(x \wedge y) \leq x$ . In case  $(x \wedge y) = 0$ :  $y \leftarrow x$  is 1 if  $x=0$  and 0 otherwise.

Now suppose that we were given a structure  $(M, 1, 0, \bullet, \wedge)$ , satisfying  $L^+1, L^+2, L^+4, L^+5$  and:

$$L'3 \quad x \leq y \wedge z \Leftrightarrow x \leq y \text{ and } x \leq z.$$

Define:

$$\begin{aligned}(y \leftarrow x) &:= \text{the unique } u: (x \wedge y) \leq x \text{ if } (x \wedge y) \neq 0, \\ &:= 0 \text{ if } (x \wedge y) = 0 \text{ and } x \neq 0 \\ &:= 1 \text{ if } x = 0.\end{aligned}$$

Note that in all cases:  $(x \wedge y) = (y \leftarrow x) \cdot x$ .

We verify L3: in case  $(x \wedge y) \neq 0$ :

$$\begin{aligned}v \cdot x \leq y &\Leftrightarrow v \cdot x \leq (x \wedge y) = (y \leftarrow x) \cdot x \\ &\Leftrightarrow \text{for some } w: v \cdot x = w \cdot (y \leftarrow x) \cdot x \\ &\Leftrightarrow \text{for some } w: v = w \cdot (y \leftarrow x) \\ &\Leftrightarrow v \leq (y \leftarrow x).\end{aligned}$$

In case  $(x \wedge y) = 0$  and  $x \neq 0$ :

$$\begin{aligned}v \cdot x \leq y &\Leftrightarrow v \cdot x \leq (x \wedge y) = 0 \\ &\Leftrightarrow v \leq 0 = (y \leftarrow x).\end{aligned}$$

In case  $x = 0$ :

$$v \cdot x = 0 \leq y \Leftrightarrow v \leq 1 = (y \leftarrow x). \quad \square$$

**3.6 Fact:** if  $x \neq 0$ , then:  $y \cdot z \leftarrow x = (y \leftarrow (x \leftarrow z)) \cdot (z \leftarrow x)$

**Proof:** Suppose  $x \neq 0$ , then:

$$\begin{aligned}(y \leftarrow (x \leftarrow z)) \cdot (z \leftarrow x) \cdot x &= (y \leftarrow (x \leftarrow z)) \cdot (x \wedge z) \\ &= (y \leftarrow (x \leftarrow z)) \cdot (x \leftarrow z) \cdot z \\ &= ((x \leftarrow z) \wedge y) \cdot z \\ &= (x \leftarrow z) \cdot z \wedge y \cdot z \\ &= z \wedge x \wedge y \cdot z \\ &= x \wedge y \cdot z \\ &= (y \cdot z \leftarrow x) \cdot x.\end{aligned}$$

So:  $y \cdot z \leftarrow x = (y \leftarrow (x \leftarrow z)) \cdot (z \leftarrow x). \quad \square$

Now we are ready and set to introduce stacking cells.

**3.7 Construction:** Let  $SC := ((M \setminus \{0\}) \times (M \setminus \{0\})) \cup \{(0, 0)\}$  and define  $\cdot$  on SC as follows:  $\langle x', x \rangle \cdot \langle y', y \rangle := \langle (y' \leftarrow x) \cdot x', (x \leftarrow y') \cdot y \rangle$ . Note that if  $(x \wedge y') = 0$ :  $\langle x', x \rangle \cdot \langle y', y \rangle = \langle 0, 0 \rangle$  and if  $(x \wedge y') \neq 0$ , then  $\langle x', x \rangle \cdot \langle y', y \rangle \in (M \setminus \{0\}) \times (M \setminus \{0\})$ . (If we would allow

arbitrary elements of  $M \times M$  we would get some undesired effects: e.g. for  $x' \neq 0$  we would have:  $\langle x', 0 \rangle \bullet \langle 0, 0 \rangle := \langle x', 0 \rangle$ , so  $\langle 0, 0 \rangle$  would not be an annihilator.)

Let  $1 := \langle 1, 1 \rangle$ ,  $0 := \langle 0, 0 \rangle$ ,  $\mathfrak{S}\mathfrak{C}_{\mathfrak{M}} := \langle \mathfrak{S}\mathfrak{C}, 1, 0, \bullet \rangle$ .

**3.8 Theorem:**  $\mathfrak{S}\mathfrak{C}_{\mathfrak{M}}$  is a monoid with annihilator 0.

**Proof:** the only non-trivial part is the associativity of  $\bullet$ . Consider:  $\langle x', x \rangle \bullet \langle y', y \rangle \bullet \langle z', z \rangle$ . If one of  $\langle x', x \rangle$ ,  $\langle y', y \rangle$ ,  $\langle z', z \rangle$  is  $\langle 0, 0 \rangle$ , associativity is trivial. Otherwise we get by 3.6:

$$\begin{aligned} (z' \leftarrow ((x \leftarrow y') \bullet y)) \bullet (y' \leftarrow x) \bullet x' &= ((z' \leftarrow y) \leftarrow (x \leftarrow y')) \bullet (y' \leftarrow x) \bullet x' \\ &= ((z' \leftarrow y) \bullet y' \leftarrow x) \bullet x'. \end{aligned}$$

The other case is dual.  $\square$

Note that associativity would be spoiled if we admitted elements of the form  $\langle 0, u \rangle$ ,  $\langle u, 0 \rangle$ , where  $u \neq 0$ . Suppose e.g. in the situation above  $x=y'=z'=0$  and  $x'=y=1$ . We get:  $(z' \leftarrow ((x \leftarrow y') \bullet y)) \bullet (y' \leftarrow x) \bullet x' = 0$  and  $((z' \leftarrow y) \bullet y' \leftarrow x) \bullet x' = 1$ .

Evidently if we start with an L-monoid we can obtain by our construction an ordinary monoid without annihilator.

Note that:

$$\begin{aligned} \langle 1, x \rangle \bullet \langle 1, y \rangle &= \langle (1 \leftarrow x) \bullet 1, (x \leftarrow 1) \bullet y \rangle = \langle 1, x \bullet y \rangle, \\ \langle x', 1 \rangle \bullet \langle y', 1 \rangle &= \langle (y' \leftarrow 1) \bullet x', (1 \leftarrow y') \bullet 1 \rangle = \langle y' \bullet x', 1 \rangle, \\ \langle x', 1 \rangle \bullet \langle 1, y \rangle &:= \langle (1 \leftarrow 1) \bullet x', (1 \leftarrow 1) \bullet y \rangle = \langle x', y \rangle, \\ \langle 1, x \rangle \bullet \langle y', 1 \rangle &:= \langle (y' \leftarrow x) \bullet 1, (x \leftarrow y') \bullet 1 \rangle = \langle y' \leftarrow x, x \leftarrow y' \rangle. \end{aligned}$$

In the next section we show how lazy brackets are an application of our construction.

**4 An ordinal representation of lazy brackets:** lazy brackets are in fact just stacking cells on  $\omega^2$ . Here we develop stacking cells (as monoid) on an arbitrary limit ordinal  $\lambda$ .

Let an arbitrary limit ordinal  $\lambda$  be given. Define  $\mathfrak{M}_\lambda = \langle M, 1, \bullet, \leftarrow \rangle$ , as follows:

$$M := \{ \alpha \mid \alpha <_{\text{ord}} \lambda \}, \quad 1 := 0_{\text{ord}}, \quad \alpha \bullet \beta := \beta + \alpha.$$

Note that we get  $\alpha \leq \beta \Leftrightarrow \beta \leq_{\text{ord}} \alpha$  and  $\alpha \wedge \beta := \max_{\text{ord}}(\alpha, \beta)$ . As is easily seen  $\langle M, 1, \bullet, \wedge \rangle$  satisfies  $L^+1, L^+2, L^+3, L^+4, L^+5$  and hence we can define  $\leftarrow$  as in remark 3.5, thus

obtaining an L-monoid.

If we want to think in ‘ordinal’ terms and not algebraically we write  $\alpha \dot{-} \beta$  (cut-off subtraction) for  $\alpha \leftarrow \beta$ . We have:  $\alpha \leq_{\text{ord}} \beta + \gamma \Leftrightarrow \alpha \dot{-} \beta \leq_{\text{ord}} \gamma$ . Note that 3.6 translates to:

$$(\alpha + \beta) \dot{-} \gamma = (\alpha \dot{-} \gamma) + (\beta \dot{-} (\gamma \dot{-} \alpha)) \quad (*)$$

Let

$$\begin{aligned} \chi_{\leq}(\gamma, \alpha) &:= 1_{\text{ord}} \text{ if } \gamma \leq_{\text{ord}} \alpha \\ &:= 0_{\text{ord}} \text{ if } \alpha <_{\text{ord}} \gamma. \end{aligned}$$

In other words: let  $\chi_{\leq}$  be the characteristic function of  $\leq_{\text{ord}}$ . Note that:

$$\chi_{\leq}(\gamma, \alpha) = (1_{\text{ord}} \dot{-} (\gamma \dot{-} \alpha)).$$

Comparing this last form of  $\chi_{\leq}$  with (\*) for  $\beta := 1_{\text{ord}}$ , suggests the following alternative definition for  $\dot{-}$  by recursion:

$$\begin{aligned} 0 \dot{-} \gamma &:= 0, \\ (\alpha + 1) \dot{-} \gamma &:= (\alpha \dot{-} \gamma) + \chi_{\leq}(\gamma, \alpha), \\ \mu \dot{-} \gamma &:= \sup_{\text{ord}} \{ \alpha \dot{-} \gamma \mid \alpha <_{\text{ord}} \mu \} \text{ for limit ordinals } \mu <_{\text{ord}} \lambda, \end{aligned}$$

(The sup-step is correct since *in the algebra* for any set of elements  $Y$ : if  $\inf(Y)$  exists, then  $(\inf(Y) \leftarrow x) = \inf(\{y \leftarrow x \mid y \in Y\})$ .)

We will simply identify  $\mathcal{M}_{\lambda}$  with  $\lambda$ . Consider  $\mathfrak{S}\mathfrak{C}_{\lambda}$ . We sometimes write element  $x$  of  $\mathfrak{S}\mathfrak{C}_{\lambda}$  as:  $\langle \text{pop}_x, \text{push}_x \rangle$ . Our monoidal operation on the stacking cells becomes:

$$\langle \text{pop}_x, \text{push}_x \rangle \bullet \langle \text{pop}_y, \text{push}_y \rangle = \langle \text{pop}_x + (\text{pop}_y \dot{-} \text{push}_x), \text{push}_y + (\text{push}_x \dot{-} \text{pop}_y) \rangle.$$

From this point on in this section we will work in ‘ordinal’ as opposed to ‘algebraic’ terms. Consequently we drop the subscript  $_{\text{ord}}$ .

#### 4.1 Examples

$$\begin{aligned} \langle 0, k \rangle \bullet \langle 0, \omega \cdot n \rangle &= \langle 0 + (0 \dot{-} k), \omega \cdot n + (k \dot{-} 0) \rangle = \langle 0, \omega \cdot n + k \rangle, \\ \langle \alpha, 0 \rangle \bullet \langle 0, \beta \rangle &= \langle \alpha + (0 \dot{-} \beta), \beta + (0 \dot{-} \alpha) \rangle = \langle \alpha, \beta \rangle. \end{aligned}$$

#### 4.2 Theorem: $\mathfrak{S}\mathfrak{C}_{\omega^2}$ is isomorphic to $\mathfrak{LB}$ .

**Proof:** Consider  $f: \mathfrak{LB} \rightarrow \mathfrak{S}\mathfrak{C}_{\omega^2}$  generated by:

$$(\mapsto \langle 0, 1 \rangle, ) \mapsto \langle 1, 0 \rangle, [ \mapsto \langle 0, \omega \rangle, ] \mapsto \langle \omega, 0 \rangle.$$

Note that:

$$\begin{aligned} f[(\langle 0, \omega \rangle \bullet \langle 0, 1 \rangle) &= \langle 0 + (0 \dot{-} \omega), 1 + (\omega \dot{-} 0) \rangle = \langle 0, \omega \rangle = f[, \\ f[ &= \langle 0, \omega \rangle \bullet \langle 1, 0 \rangle = \langle 0 + (1 \dot{-} \omega), 0 + (\omega \dot{-} 1) \rangle = \langle 0, \omega \rangle = f[, \end{aligned}$$

and dually for  $)]$ ,  $([$ . By easy inductions one shows that

$$f[.]..)]..)(..([ [ = \langle \omega.n'+k', \omega.n+k \rangle,$$

where  $n', k', n, k$  are respectively the numbers of  $]'$ 's,  $)'$ 's,  $[$ 's,  $('$ 's.

So  $f$  is a bijection and hence an isomorphism.  $\square$

**5 A string representation of quarrelsome brackets:** Let  $A$  be a finite alphabet and let  $A^*$  be the set of all strings of elements of  $A$ . Let  $\bullet$  be concatenation on  $A^*$  and let  $1 := \emptyset :=$  the empty string. Note that  $w \leq u$  means:  $u$  is an endstring of  $w$ . Add a new element  $0$  to  $A^*$ . We stipulate:  $w \bullet 0 = 0 \bullet w = 0$ . It is easily seen that the structure thus defined admits a residuation operation:

$$\begin{aligned} w \leftarrow v &:= u \text{ if } w = u \bullet v \\ &:= 1 \text{ if } v = u' \bullet w \text{ for some } u' \\ &:= 0 \text{ otherwise.} \end{aligned}$$

(Alternatively note that  $0 \neq w$ ,  $w \leq u$  and  $w \leq v$  implies that  $v \leq u$  or  $u \leq v$ . So if the only  $w$  below  $u$  and  $v$  is  $0$ , take  $u \wedge v := 0$ ; if there is some non-zero  $w$  below  $u$  and  $v$  take  $u \wedge v := u$  if  $u \leq v$  and  $u \wedge v := v$  if  $v \leq u$ .)

The stacking cells defined on our structure model precisely the quarrelsome brackets. For example take  $A := \{a, b\}$ . We can assign:

$$( \mapsto \langle \emptyset, a \rangle, ) \mapsto \langle a, \emptyset \rangle, \{ \mapsto \langle \emptyset, b \rangle, \} \mapsto \langle b, \emptyset \rangle.$$

Closely analogous (in our context) to strings are the morphisms of the simplicial category with the operation  $+$  as defined in Mac Lane[71], p171. We can take:  $1 := \emptyset: 0 \rightarrow 0$ ,  $f \bullet g = f + g$  or  $f \bullet g := g + f$  and add  $0$  in the obvious way. Both choices of  $\bullet$  yield an  $L^+$ -monoid (with the additional property:  $0 \neq w$ ,  $w \leq u$ ,  $w \leq v$  implies that  $u \leq v$  or  $v \leq u$ ). I admit that I have no idea how to apply the stacking cells obtained from these monoids.

**6 Stacking cells as an update algebra:** The notion of update algebra was introduced in Visser[92a]. Consider any monoid  $\mathfrak{M}$ . Let  $S$  be a subset of  $N$ , the domain of  $\mathfrak{M}$ , with  $1 \in S$ .  $S$  is considered as a set of ‘states’ in  $\mathfrak{M}$ . To each element  $n$  of  $\mathfrak{M}$  we assign an update function  $\Phi_a: S \rightarrow S$  as follows (in post-fix notation):

$$s\Phi_a := s \bullet a \text{ if } s \bullet a \in S, s\Phi_a \text{ is undefined otherwise.}$$

$(\mathfrak{M}, S)$  is called an *update algebra* if  $\Phi_{a \bullet b} = \Phi_a \circ \Phi_b$  ( $\circ$  is function composition read in the order of application:  $s\Phi_a \circ \Phi_b = (s\Phi_a)\Phi_b$ ). We have:

**6.1 Theorem:**  $\langle \mathcal{M}, S \rangle$  is an update algebra iff  $\langle \mathcal{M}, S \rangle$  satisfies the OTAT Principle:

for all  $x, y$ :  $x \bullet y \in S \Rightarrow x \in S$ .

(OTAT means: *once a thief, always a thief*. It tells us that if something is not a state, then whatever you add later, it never will become a state. The principle can be viewed as a principle of *error propagation*.)

**Proof:** See Visser[92a] or [92b].  $\square$

Now consider an  $L^+$ -monoid  $\mathcal{M}$ . We take as states the non-zero elements of  $M$ . Clearly the algebra thus obtained is an update algebra. Now consider  $\mathcal{S}\mathcal{C}_{\mathcal{M}}$ . As states we take  $S := \{ \langle 1, s \rangle \mid s \neq 0 \}$ . We have:

**6.2 Theorem:**  $\langle \mathcal{S}\mathcal{C}_{\mathcal{M}}, S \rangle$  is an update algebra.

**Proof:** Suppose  $\langle x', x \rangle \bullet \langle y', y \rangle$  is in  $S$ , then:  $(y' \leftarrow x) \bullet x' = 1$ . So  $1 \leq (y' \leftarrow x) \bullet x' \leq x'$ , so  $x' = 1$ . Ergo  $\langle x', x \rangle$  is in  $S$ .  $\square$

**7 Stacking cells as partial isomorphisms:** For purposes of application we need to describe what the ‘levels’ present in a stacking cell are and how these levels move when two stacking cell are merged. To do this we first associate certain partial bijections to stacking cells.

Let  $\mathcal{M}$  be an  $L^+$ -monoid. To a stacking cell  $\langle x, y \rangle$  on  $\mathcal{M}$  we associate a partial bijection

$\Theta_{\langle x, y \rangle} := f: M \setminus \{0\} \rightarrow M \setminus \{0\}$  by stipulating (in post-fix notation):

zf is defined if  $z \leq x$ ,  $(u \bullet x)f := u \bullet y$  (for  $u \neq 0$ ).

$\Theta$  is well defined because of  $L^+1$ . Clearly  $\Theta := \lambda a. \Theta_a$  is injective. If  $f = \Theta_{\langle x, y \rangle}$ , we will sometimes call this  $x \text{ pop}_f$  and  $y \text{ push}_f$ .

**7.1 Fact:**  $\Theta$  is an injective morphism from  $\mathcal{S}\mathcal{C}_{\mathcal{M}}$  to the monoid of partial bijections on  $M \setminus \{0\}$  with operation  $\circ$ , with as unit the identity function and with annihilator the nowhere defined function  $\emptyset$ .

**Proof:** We leave the verification that  $\langle 1, 1 \rangle$  is mapped to the identity function and that  $\langle 0, 0 \rangle$  is mapped to the nowhere defined function to the industrious reader. Suppose  $a, b$  are non-zero stacking cells. Let  $f := \Theta_a$  and  $g := \Theta_b$ . We find:

$$\begin{aligned}
x(\text{fog}) \downarrow &\Leftrightarrow \text{for some } u: x=u \bullet \text{pop}_f \text{ and } u \bullet \text{push}_f \leq \text{pop}_g \\
&\Leftrightarrow \text{for some } u: x=u \bullet \text{pop}_f \text{ and } u \leq (\text{pop}_g \leftarrow \text{push}_f) \\
&\Leftrightarrow x \leq (\text{pop}_g \leftarrow \text{push}_f) \bullet \text{pop}_f.
\end{aligned}$$

$$\begin{aligned}
(u \bullet (\text{pop}_g \leftarrow \text{push}_f) \bullet \text{pop}_f)(\text{fog}) &= (u \bullet (\text{pop}_g \leftarrow \text{push}_f) \bullet \text{push}_f)g \\
&= (u \bullet (\text{push}_f \leftarrow \text{pop}_g) \bullet \text{pop}_g)g \\
&= u \bullet (\text{push}_f \leftarrow \text{pop}_g) \bullet \text{push}_g. \quad \square
\end{aligned}$$

Define  $a \subseteq b : \Leftrightarrow$  for some  $u: u:\text{pop}_a \leq \text{pop}_b, u:\text{push}_a \leq \text{push}_b$ . Define also:  $\langle x, y \rangle^\wedge := \langle y, x \rangle$ .

## 7.2 Fact

- i)  $a \subseteq b \Leftrightarrow \Theta_a \subseteq \Theta_b$ ,
- ii)  $\Theta_{a^\wedge}$  is the inverse of  $\Theta_a$ .

**Proof:** (ii) is trivial. We prove (i):

“ $\Rightarrow$ ” Suppose:  $u:\text{pop}_a \leq \text{pop}_b$  and  $u:\text{push}_a \leq \text{push}_b$ . Consider  $x \leq \text{pop}_a$ , then certainly  $x \leq \text{pop}_b$ . Say  $x=v \bullet \text{pop}_a$ . We have:

$$x \Theta_b = (v \bullet \text{pop}_a) \Theta_b = (v \bullet u \bullet \text{pop}_b) \Theta_b = v \bullet u \bullet \text{push}_b = v \bullet \text{push}_a = (v \bullet \text{pop}_a) \Theta_a = x \Theta_a.$$

“ $\Leftarrow$ ” Suppose  $\Theta_a \subseteq \Theta_b$ , then clearly for some  $u: u:\text{pop}_a \leq \text{pop}_b$ . We find:

$$\text{push}_a = \text{pop}_a \Theta_a = (u \bullet \text{pop}_b) \Theta_b = u \bullet \text{push}_b. \quad \square$$

We will conveniently confuse stacking cells with the associated bijections. The ordering on stacking cells will be  $\subseteq$ . We state some simple facts about  $\bullet, (\cdot)^\wedge$  and  $\subseteq$ .

## 7.3 Fact

- a)  $(\cdot)^\wedge$  is monotonic and  $\bullet$  is monotonic in both arguments.
- b)  $(a \bullet b)^\wedge = b^\wedge \bullet a^\wedge$ .
- c)  $a^\wedge^\wedge = a$ .
- d)  $a \bullet a^\wedge \subseteq 1, a^\wedge \bullet a \subseteq 1$ .

**Proof:** Trivial.  $\square$

Note that (d), implies  $1^\wedge = 1 \bullet 1^\wedge \subseteq 1$  and hence by (c), (a)  $1 = 1^\wedge^\wedge \subseteq 1^\wedge$ . Ergo  $1 = 1^\wedge$ .

**8 Levels:** We want to assign levels/files/discourse objects to our stacking cells

and explain how these levels travel when cells are fused. This assignment will work via the association with partial bijections of section 7. Let an arbitrary non empty set  $X$  be given with  $0 \notin X$ . We assign sets of files to partial bijections on  $X$ . Define for any partial bijection  $R$  on  $X$ :

$$R\mathfrak{L} := R \cup \{(0,0)\} \cup \{(x,0) | x \notin \text{dom}(R)\} \cup \{(0,y) | y \notin \text{range}(R)\}.$$

The idea is that every element of  $X$  is a potential ‘file’ both on the input and on the output side.  $R$  links up or identifies certain input and output files. The identified input-output files are represented just by the pairs of  $R$ . Files not in the domain of  $R$  are input files that come in, but are not passed on: i.o.w. they are popped. These files are represented by  $\langle x,0 \rangle$ : the second component is ‘error’ to symbolize that  $x$  is not passed on. Similarly the  $y$  not in the range of  $R$  are pushed. They are represented by:  $\langle 0,y \rangle$ . Finally  $\langle 0,0 \rangle$  is an ideal element to smooth the presentation. Generally this construction produces far more levels than needed: these levels should be viewed as just virtually present or als sleeping.

We may call (as in Visser[92b]):

$$R =: \text{STEM}_R, \{(x,0) | x \notin \text{dom}(R)\} =: \text{POP}_R, \{(0,x) | x \notin \text{range}(R)\} =: \text{PUSH}_R, \\ \{(0,0)\} =: \text{GARB}_R.$$

Let  $\mathfrak{R} := \Theta \circ \mathfrak{L}$  (i.e. first  $\Theta$ , then  $\mathfrak{L}$ ). So  $\mathfrak{R}$  assigns levels or files to stacking cells.

A morphism  $\phi: a \rightarrow a'$  between stacking cells is a pair of stacking cells  $\langle b, b' \rangle$  that satisfies certain conditions. This pair represents a function  $\phi\mathfrak{R}$  between the corresponding sets of levels. The conditions on  $\langle b, b' \rangle$  can be best explained by reflecting on the specification of this function.

For any partial bijection  $G$  on  $M \setminus \{0\}$ , let  $G^+$  on  $M$  be given by:

$$xG^+ := xG \text{ if } x \neq 0 \text{ and } xG \text{ is defined} \\ := 0 \text{ otherwise.}$$

Note that  $G\mathfrak{L} = G^+ \cup G^{\wedge+}$ . An easy case-checking yields:  $(G \circ H)^+ = G^+ \circ H^+$ .

Let  $F := b\Theta$  and  $F' := b'\Theta$ ,  $R := a\Theta$ ,  $R' := a'\Theta$ . Define:  $\langle x, x' \rangle (\phi\mathfrak{R}) := \langle xF^+, x'F'^+ \rangle$ . For this definition to make sense we need:  $\phi\mathfrak{R}: a\mathfrak{R} \rightarrow a'\mathfrak{R}$ .

**8.1 Motivating Fact:**  $\phi\mathfrak{R}:a\mathfrak{R}\rightarrow a'\mathfrak{R}$  precisely if for all  $x, x'$  in  $R$ :

- a) If  $xF, x'F'$  are defined, then  $\langle xF, x'F' \rangle \in R'$ ,
- b) If  $xF$  is defined and  $(xF)R'y'$ , then  $x'F'=y'$ ,
- c) If  $x'F'$  is defined and  $yR'x'F'$ , then  $xF=y$ .

**Proof:** From left to right is trivial. Suppose for the converse that our conditions are fulfilled. Consider  $\langle x, x' \rangle$  in  $R$ . In case  $xF^+ \neq 0$  and  $x'F'^+ \neq 0$ , we find that  $xF$  and  $x'F'$  are defined, so  $\langle xF, x'F' \rangle \in R' \subseteq R'\mathcal{Q}$ . Suppose e.g.  $xF^+ = 0$ . In case  $x'F'^+$  is also 0, we are done since  $\langle 0, 0 \rangle \in R'\mathcal{Q}$ . So suppose  $x'F'^+ = y' \neq 0$ , and thus  $x'F'=y'$ . If  $y'$  were in the range of  $R'$ , say  $yR'y'$ , then by our third condition  $xF=y$ . Quod non. So  $y'$  is not in the range of  $R'$  and thus  $\langle xF^+, x'F'^+ \rangle = \langle 0, y' \rangle \in R'\mathcal{Q}$ .  $\square$

Note that 8.1(a) is equivalent with:  $F \wedge_o R_o F' \subseteq R'$ ; that 8.1(b) means that if  $x(F_o R')y'$ , then  $x(R_o F')y'$  and thus  $F_o R' \subseteq R_o F'$ ; that 8.1(c) means:  $F'_o R'^\wedge \subseteq R^\wedge_o F$ . Thus we are lead to the following definition. A morphism  $\phi:a \rightarrow a'$  is a pair of SSC's  $\langle b, b' \rangle$  such that: (a)  $b^\wedge \bullet a \bullet b' \subseteq a'$ , (b)  $b \bullet a' \subseteq a \bullet b'$ , (c)  $b' \bullet a'^\wedge \subseteq a^\wedge \bullet b$ . (N.B.: full specification of a morphism requires its dom and cod.)

Let  $\langle b, b' \rangle: a \rightarrow a'$  and  $\langle c, c' \rangle: a' \rightarrow a''$ . Define:  $\langle b, b' \rangle_o \langle c, c' \rangle := \langle b \bullet c, b' \bullet c' \rangle$ . Note that:

$$b \bullet c \bullet a'' \subseteq b \bullet a' \bullet c' \subseteq a \bullet b' \bullet c',$$

$$b' \bullet c' \bullet a''^\wedge \subseteq b' \bullet a'^\wedge \bullet c \subseteq a^\wedge \bullet b \bullet c,$$

$$(b \bullet c)^\wedge \bullet a \bullet (b' \bullet c') \subseteq c^\wedge \bullet (b^\wedge \bullet a \bullet b) \bullet c' \subseteq c^\wedge \bullet a' \bullet c' \subseteq a'',$$

So  $\langle b \bullet c, b' \bullet c' \rangle$  is indeed a morphism. Clearly morphisms are associative in the usual way.

Let  $\text{id}(a)$  be given by  $\langle 1, 1 \rangle$ . We leave the simple check that  $\text{id}(a)$  functions as an identity morphism to the reader.

Clearly we have defined a category  $\mathfrak{D} := \mathfrak{C}(\mathfrak{S}\mathfrak{C}_{\mathfrak{M}})$ .

**8.2 Fact:**  $\mathfrak{R}$  is a functor from  $\mathfrak{D}$  to the category of Sets with functions as morphisms.

**Proof:**  $\mathfrak{R}$  commutes with  $_o$ , since:  $((b \bullet c)\Theta)^+ = (b\Theta)^+_o(c\Theta)^+ \quad \square$

**8.3 Fact:**  $\phi:a \rightarrow b$  is an isomorphism iff  $a=b$  and  $\phi=id(a)$ .

**Proof:** trivial. □

Define  $in_1(a,b) := \langle 1,b \rangle$ ,  $in_2(a,b) := \langle a^{\wedge}, 1 \rangle$ .  $in_1(a,b):a \rightarrow a \bullet b$ , since:

$$1^{\wedge} \bullet a \bullet b \subseteq a \bullet b,$$

$$1 \bullet (a \bullet b) \subseteq a \bullet b \text{ and } b \bullet (a \bullet b)^{\wedge} \subseteq a^{\wedge} \bullet 1.$$

Moreover:  $in_2(a,b):b \rightarrow a \bullet b$ , since:

$$a^{\wedge} \bullet b \bullet 1 \subseteq a \bullet b.$$

$$a^{\wedge} \bullet (a \bullet b) \subseteq b \bullet 1 \text{ and } 1 \bullet (a \bullet b)^{\wedge} \subseteq b^{\wedge} \bullet a^{\wedge}.$$

**8.4 Fact:**  $(a \mathfrak{f}) \circ (in_1(a,b) \mathfrak{f}) \cup (b \mathfrak{f}) \circ (in_2(a,b) \mathfrak{f}) = (a \bullet b) \mathfrak{f}$

**Proof:** Let  $F:=a \Theta$ ,  $G:=b \Theta$ . Note that:

$$\begin{aligned} \langle x,y \rangle \in (a \mathfrak{f}) \circ (\langle 1,b \rangle \mathfrak{f}) &\Leftrightarrow \text{for some } z,u: x(ID^{+ \wedge})z(F \mathfrak{L})u(G^{+})y \\ &\Leftrightarrow x(F^{+} \circ G^{+})y \text{ or } x(F^{\wedge+ \wedge} \circ G^{+})y. \end{aligned}$$

(Since  $F \mathfrak{L} = F^{+} \cup F^{\wedge+ \wedge}$ .) Similarly:

$$\begin{aligned} \langle x,y \rangle \in (b \mathfrak{f}) \circ (\langle a^{\wedge}, 1 \rangle \mathfrak{f}) &\Leftrightarrow \text{for some } z,u: x(F^{\wedge+ \wedge})z(G \mathfrak{L})u(ID^{+})y \\ &\Leftrightarrow x(F^{\wedge+ \wedge} \circ G^{+})y \text{ or } x(F^{\wedge+ \wedge} \circ G^{\wedge+ \wedge})y. \end{aligned}$$

Moreover:

$$\begin{aligned} \langle x,y \rangle \in (a \bullet b) \mathfrak{f} &\Leftrightarrow x(F \circ G)^{+}y \text{ or } x(F \circ G)^{\wedge+ \wedge}y \\ &\Leftrightarrow x(F^{+} \circ G^{+})y \text{ or } x(F^{\wedge+ \wedge} \circ G^{\wedge+ \wedge})y \end{aligned}$$

We already know the left to right inclusion (by the well-definedness of the  $in_i$ ), so we are done. (Alternatively suppose e.g.  $x(F^{\wedge+ \wedge} \circ G^{+})y$  and not  $x(F^{+} \circ G^{+})y$ , then  $x=0$ ,  $y \neq 0$  and for some  $u \neq 0$ :  $x(F^{\wedge+ \wedge})uG^{+}y$ . But then  $uGy$  and hence:  $u(G^{\wedge+ \wedge})y$ , so  $x(F^{\wedge+ \wedge} \circ G^{\wedge+ \wedge})y$ .) □

$\mathfrak{D}$  has some pleasant properties introduced in Visser[92b] w.r.t. the interaction of the categorical structure and the  $\bullet$ . These properties are:

C1 if  $in_i(a,b) \circ \phi = in_i(a,b) \circ \psi$ , for  $i=1,2$  then  $\phi=\psi$

C2 There is a (unique) isomorphism  $\alpha$  satisfying:

- (i)  $in_1(x,y) \circ in_1(x \bullet y, z) \circ \alpha(x,y,z) = in_1(x,y \bullet z)$ ,
- (ii)  $in_2(x,y) \circ in_1(x \bullet y, z) \circ \alpha(x,y,z) = in_1(y,z) \circ in_2(x,y \bullet z)$ ,
- (iii)  $in_2(x \bullet y, z) \circ \alpha(x,y,z) = in_2(y,z) \circ in_2(x,y \bullet z)$ .

C3  $in_1(x,1)$  and  $in_2(1,y)$  are isomorphisms,  $in_1(1,1)=in_2(1,1)=id(1)$ .

C4 If  $\phi_i: x_i \rightarrow x'_i$  for  $i=1,2$  are isomorphisms in  $\mathfrak{D}$ , then there is an isomorphism  $\phi_1 \bullet \phi_2: x_1 \bullet x_2 \rightarrow x'_1 \bullet x'_2$  such that for  $i=1,2$ :  $\phi_i \circ \text{in}_i(x'_1, x'_2) = \text{in}_i(x_1, x_2) \circ (\phi_1 \bullet \phi_2)$ .

We call a category satisfying C1-C4 an *m-category*.

**8.5 Theorem:**  $\mathfrak{D}$  is an m-category.

**Proof:** We check C1. Consider  $\langle c, c' \rangle$  and  $\langle d, d' \rangle$  such that  $\text{in}_i(a, b) \circ \langle c, c' \rangle = \text{in}_i(a, b) \circ \langle d, d' \rangle$  for  $i=1,2$ . We have:

$$\langle c, b \circ c' \rangle = \langle 1, b \rangle \circ \langle c, c' \rangle = \langle 1, b \rangle \circ \langle d, d' \rangle = \langle d, b \circ d' \rangle, \text{ so } c=d;$$

$$\langle a \circ c, c' \rangle = \langle a, 1 \rangle \circ \langle c, c' \rangle = \langle a, 1 \rangle \circ \langle d, d' \rangle = \langle a \circ d, d' \rangle, \text{ so } c'=d'.$$

We turn to C2. We take the witnessing morphism simply 1. We find:

$$\text{in}_1(a, b) \circ \text{in}_1(a \bullet b, c) = \langle 1, b \rangle \circ \langle 1, c \rangle = \langle 1, b \bullet c \rangle = \text{in}_1(a, b \bullet c),$$

$$\text{in}_2(a, b) \circ \text{in}_1(a \bullet b, c) = \langle a^\wedge, 1 \rangle \circ \langle 1, c \rangle = \langle a^\wedge, c \rangle = \langle 1, c \rangle \circ \langle a^\wedge, 1 \rangle = \text{in}_1(b, c) \circ \text{in}_2(a, b \bullet c),$$

$$\text{in}_2(a \bullet b, c) = \langle (a \bullet b)^\wedge, 1 \rangle = \langle b^\wedge, 1 \rangle \circ \langle a^\wedge, 1 \rangle = \text{in}_2(b, c) \circ \text{in}_2(a, b \bullet c).$$

C3 is evident since  $\text{in}_1(a, 1) = \text{in}_2(1, b) = \langle 1, 1 \rangle$ . C4 is also evident since the  $\text{id}(a)$  are the only isomorphisms.  $\square$

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