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Logic Group
Preprint Series
No. 78
June 1992

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Belief functions and inner measures

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Abstract

In this note we study the relation between belief functions of Dempster-Shafer theory and inner measures induced by probability functions. In [3,4] Joe Halpern and Ron Fagin claim that these classes of functions are essentially the same, or, more precisely, that they are *exactly* the same in case the functions are defined on *formulas* rather than sets. We show that when the functions are defined on sets only a proper subclass of the belief functions over a frame *S* corrsponds to the class of inner measures induced by a probability measure on some algebra on *S*. However, belief functions over *S* do correspond to inner measures induced by probability measures defined on algebras on *refinements* of *S*. The fact that in general refinements of S are needed to obtain all belief functions over *S* is shown to be obscured by the particular way formulas are assigned probabilities or weights in [3].

1. Introduction

In [4] Joe Halpern and Ron Fagin claim that belief functions of Dempster-Shafer theory (DS theory) are essentially inner measures induced by a probability function. They refer to [3] for support of their claim. In that paper it is shown that the inner measure induced by a probability function on some algebra on a (finite) sample space *S* is a belief function over *S*. They remark that the converse is not true: not every belief function over *S* is the inner measure of some probability function on some algebra on *S*.

However, they maintain that belief functions and inner measures induced by probability functions are precisely the same if their domains are considered to be formulas rather than sets. This latter claim is made precise in a theorem stating the equivalence of probability structures and DS structures, where probability structures and DS structures are two kinds of weight structures which assign weights to formulas using inner measures and belief functions, respectively.

We will argue that the validity of this theorem depends on the fact that the weight structures considered in [3] do not satisfy some reasonable constraint, namely that indistinguishable states should not be distinguished by the probability functions, inner measures or belief functions. Weight structures satisfying this constraint do not equate belief functions with inner measures.

By not identifying indistinguishable states one in fact allows inner measures to be induced by probability functions on refinements of the sample space or frame induced by the formulas of the language. It can be shown that, even when defined on sets and not on formulas, belief functions over S correspond precisely to restrictions to S of inner measures induced by probability functions on refinements of S.

Therefore, the correspondence between belief functions and inner measures is not reserved to functions defined on formulas, since it applies equally well to functions defined on sets. However, the correspondence is only valid when the inner measures are allowed to be induced by probability functions on refinements of the frame over which the belief functions are defined.

The rest of the paper is organized as follows. In section 2, we summarize some definitions and results from probability theory and DS theory. In section 3 we discuss the claim that belief functions and inner measures induced by probability functions are precisely the same if their domains are formulas rather than sets. In section 4 we show that belief functions over a frame S correspond to inner measures induced by probability functions on refinements of S.

2. Probability theory and DS theory

In this section we recall some relevant basic definitions and results of probability theory and DS theory, closely following the notation of [3,4].

A *finite probability space* is a triple $\langle S,\mathcal{X},P\rangle$, where

- S is a *finite sample space*, i.e., a finite exhaustive set of mutually exclusive possibilities
- $\mathcal{X}$ is an *algebra* on S, i.e., a set of subsets of S containing S and closed under union and under complementation relative to S
- P is a *probability function* on $\mathcal{X}$, i.e., P is a function $\mathcal{X} \to [0,1]$ satisfying

P1 $P(X) \geq 0$, for all $X \in \mathcal{X}$
P2 $P(S) = 1$
P3 $P(X \cup Y) = P(X) + P(Y)$, if $X \cap Y = \emptyset$.

Throughout this paper we will only consider *finite* probability spaces. Here the notions of probability function and probability measure coincide. The *basis* $\mathcal{Y}$ of an algebra $\mathcal{X}$ is the set of minimal elements of $\mathcal{X}$. The elements of $\mathcal{X}$ are precisely the unions of some elements of the basis of $\mathcal{X}$. Thus, by P3, a probability function on $\mathcal{X}$ is completely determined by its values on the basis $\mathcal{Y}$ of $\mathcal{X}$.

The elements of $\mathcal{X}$ are called the *measurable sets* of $\langle S,\mathcal{X},P\rangle$. There are two standard extensions of P to 2^S (i.e., the set of *all* subsets of S), namely the *inner measure* P_* and the *outer measure* P^* defined as follows:

$$P_*(A) = \sup\{P(X) \mid X \subseteq A, X \in \mathcal{X}\}$$
$$P^*(A) = \inf\{P(X) \mid X \supseteq A, X \in \mathcal{X}\}.$$

Since we restrict ourselves to finite probability spaces, $P_*(A)$ is equal to $P(B)$, where B is the largest measurable set contained in A, and $P^*(A)$ is the value of $P(B)$, where B is the smallest measurable set containing A.

The Dempster-Shafer theory of evidence [8] is an alternative for probability theory. The central notion of DS theory is the notion of a belief function over a sample space S. In DS theory a sample space is called a frame of discernment, or simply frame. A *belief function* over S is a function $2^S \to [0,1]$ satisfying

B1 $\quad Bel(\varnothing) = 0$

B2 $\quad Bel(S) = 1$

B3 $\quad Bel(A_1 \cup \ldots \cup A_n) \geq \sum_{\varnothing \neq I \subseteq \{1\ldots n\}} (-1)^{|I|+1}\, Bel(\bigcap_{i \in I} A_i).$

Belief functions are easier understood in terms of mass functions (also called basic probability assignments). A *mass function* m over S is a function $2^S \to [0,1]$ satisfying

M1 $\quad m(\varnothing) = 0$

M2 $\quad \sum_{A \subseteq S} m(A) = 1.$

The belief function *Bel* induced by a mass function m is given by

BM $\quad Bel(A) = \sum_{B \subseteq A} m(B).$

Hence the belief in a set A is the sum of the masses assigned to its subsets. In addition to mass functions and belief functions, DS theory also considers plausibility functions: the *plausibility function Pl* induced by a belief function Bel is given by $Pl(A) = 1 - Bel(\overline{A}))$, where $\overline{A}$ is the complement of A.

Mass functions, belief functions and plausibility functions represent exactly the same information, since each function in one of the three categories induces a unique function in one each of the other two categories. For example, the mass function m induced by a belief fuctions Bel is given by:

$$\text{MB} \qquad m(A) = \sum_{B \subseteq A} (-1)^{|A - B|} Bel(B)$$

If $m(A) > 0$, then A is called a *focal element* of Bel. (Here, as in the rest of the paper, we implicitly assume that Bel is the belief function corresponding to the mass function m.) If all focal elements of a belief function Bel over S are singletons, then $Bel = Pl$ and Bel is a probability function on 2^S. Such belief functions are called *Bayesian belief functions*.

If $\langle S,\mathcal{X},P\rangle$ is a probability space, then the inner measure P_* is a belief function over S (and the outer measure P^* is the corresponding plausibility function). Not every belief function over S is the inner measure induced by a probability function on an algebra on S. This is only true for belief functions which have pairwise disjoint focal elements ($m(A) > 0, m(B) > 0 \Rightarrow A \cap B = \emptyset$). Let us call such belief functions *disjoint*.

Proposition 2.1 (Fagin and Halpern [3])

(1) If $\langle S,\mathcal{X},P\rangle$ is a probability space, then the inner measure P_* is a disjoint belief function over S.

(2) If Bel is a disjoint belief function over S, then Bel is the inner measure of some probability function P on an algebra $\mathcal{X}$ on S.

Proof. (1) Assume that $\langle S,\mathcal{X},P\rangle$ is a probability space. Let $\mathcal{Y}$ be the basis for $\mathcal{X}$. Let Bel be the belief function corresponding to the mass function m given by $m(A) = P(A)$, for all $A \in \mathcal{Y}$. Then Bel is disjoint and $Bel = P_*$.
(2) Assume that Bel is a disjoint belief function over S. Let $\mathcal{X}$ be the algebra on S with as basis the set $\mathcal{Y}$ of focal elements of Bel and let P be the probability function on $\mathcal{X}$ determined by $P(A) = m(A)$, for all $A \in \mathcal{Y}$. Then $Bel = P_*$. ■

In [8] it is admitted that perhaps not all belief fuctions are appropriate for the representation of evidence. That is the reason why several special classes of belief functions are considered: support functions, separable support functions, and simple support functions. (Each class is contained in the preceding one.) A typical example of a simple support function is a belief function corresponding to evidence supporting a single proper subset of the frame to a degree $s < 1$. (For example, the belief function induced by $m(A) = 0.7$, $m(S) = 0.3$.)

Since a typical simple support function has two focal elements which are not disjoint, it is *not* a disjoint belief function. Hence the class of disjoint belief functions does not contain the belief functions which are considered appropriate for representing evidence by Glenn Shafer. Nevertheless, one can argue that the disjoint belief functions form an interesting subclass of belief functions, since they can be understood completely in terms of standard probabilistic notions.

There is another relation between belief functions and standard probabilistic notions. The class of belief functions over S is a proper subset of the class of lower envelopes of closed sets of probability functions on 2^S, where P_{low} is called a *lower envelope* of the closed set $\mathcal{P}$ of probability functions iff $P_{low}(X) = \min\{P(X) \mid P \in \mathcal{P}\}$. (See for example [5].) Such lower envelopes of closed sets of probability functions are sometimes called *lower probability functions*. Hence the class of belief functions can be approximated from above and from below by classes of standard functions of probability theory. See Fig. 1.

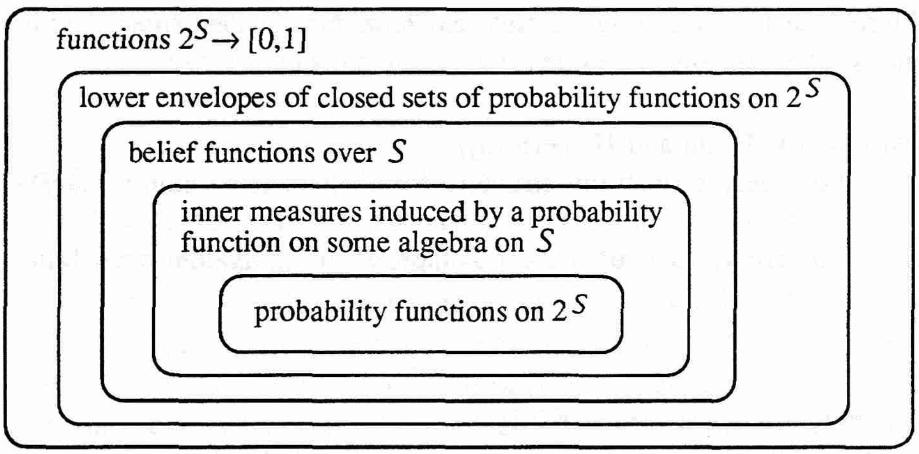

Fig. 1. The class of belief functions in relation to classes of standard probabilistic functions.

In the following section we look at the argument given in [3] for considering belief functions to be essentially inner measures.

3. Probability structures

In [3] probability structures are introduced to give a meaning to the notion of a probability of a *formula* (rather than a set). Let $\mathcal{L}$ be the propositional languange built up from the finite set PL of proposition letters and the logical connectives $\wedge$, $\neg$. A *probability structure* (for $\mathcal{L}$) is a tuple $\langle S,\mathcal{X},P,\pi\rangle$, where $\langle S,\mathcal{X},P\rangle$ is a probability space and π is a truth assignment $S\times\text{PL} \rightarrow \{\textbf{true}, \textbf{false}\}$. If $\pi(s,p) = \textbf{true}$, then p is said to be true at s. $\pi(s,.)$ can be extended to a truth assignment defined on the whole languange in the standard way.

If $M = \langle S,\mathcal{X},P,\pi\rangle$ is a probability structure, then φ^M denotes the set of states of M in which φ is true, i.e., $\varphi^M = \{s \in S \mid \pi(s,\varphi) = \textbf{true}\}$. If φ^M is measurable, then one can take the (probability or) *weigth* $W_M(\varphi)$ of φ to be $P(\varphi^M)$. If φ^M is not measurable, then one cannot talk about the probability of φ, but one can still assign a weight using the inner measure induced by P: $W_M(\varphi) = P_*(\varphi^M)$. A probability structure is called *measurable* iff φ^M is measurable, for every formula φ.

Completely analogous to the notion of probability structure, one can define the notion of *DS structure* as a triple $M = \langle S,Bel,\pi\rangle$, where Bel is a belief function over S and π is a truth assignment as before. The weigth $W_M(\varphi)$ of φ can now simply be taken to be $Bel(\varphi^M)$, since belief functions over S are defined on *all* subsets of S. Both probability structures and DS structures are called *weight structures*. Weight structures M and N are called *equivalent* iff $W_M(\varphi) = W_N(\varphi)$, for every formula φ.

Proposition 3.1 (Fagin and Halpern [3])

(1) For every probability structure for $\mathcal{L}$ there is an equivalent DS structure for $\mathcal{L}$.

(2) For every DS structure for $\mathcal{L}$ there is an equivalent probability structure for $\mathcal{L}$.

Proof.

(1) is an immediate consequence of 2.1(1).

(2) Let $M = \langle S,Bel,\pi\rangle$ be a DS structure. Define $S' = \{\langle A,s\rangle \mid A \subseteq S, s \in A\}$, $A^* = \{\langle A,s\rangle \mid s \in A\}$. Let $\mathcal{X}$ be the algebra on S' with basis $\{A^* \mid A \subseteq S\}$, and let P be the probability function on $\mathcal{X}$ given by $P(A^*) = m(A)$. Finally, let for all $p \in$ PL $\pi'(\langle A,s\rangle,p) = \pi(s,p)$. Then $N = \langle S',\mathcal{X},P,\pi'\rangle$ is a probability structure which is equivalent to M. ■

The second part of proposition 3.1 is not true for *measurable* probability structures. According to Fagin and Halpern the proposition shows that belief functions and inner measures induced by probability functions are precisely the same when they are be defined on formulas rather than on sets, and this is why they call belief functions to be essentially inner measures.

However, we will argue that in the second part of 3.1 the inner measure corresponding to the belief fuction is in general an inner measure of a probability function defined on some algebra on a proper *refinement* of the frame over which the belief function is defined. In the following section we give a precise description of this refinement. The fact that in general a *refinement* of the frame is necessary is more or less obscured by the particular way formulas are assigned weights. In order to support this claim we introduce some notions borrowed from modal logic.

Let S be the set of states of a probability structure or a DS structure for $\mathcal{L}$. Two states s and t of S are called *distinguishable* iff there is a formula $\varphi \in \mathcal{L}$ such that the truth value of φ at s is different from the truth value of φ at t. A probability structure or DS structure is called *distinguishable* iff all its states are pairwise distinguishable.

Proposition 3.2

(1) For every measurable probability structure for $\mathcal{L}$ there is an equivalent distinguishable measurable proability structure for $\mathcal{L}$.

(2) For every DS structure for $\mathcal{L}$ there is an equivalent distinguishable DS structure for $\mathcal{L}$.

Proof.

(1) Let $M = \langle S,\mathcal{X},P,\pi\rangle$ be a measurable probability structure for $\mathcal{L}$ and let $s \approx t$ denote that s and t are indistinguishable. Define for $s \in S$ $[s]_\approx = \{t \in S \mid s \approx t\}$ and for $X \subseteq S$ $X_\approx = \{[s]_\approx \mid s \in X\}$. Define $M_\approx = \langle S_\approx,\mathcal{X}_\approx,P_\approx,\pi_\approx\rangle$, where $\mathcal{X}_\approx = \{X_\approx \mid X \in \mathcal{X}\}$, $P_\approx$ is given by $P_\approx(X_\approx) = P(\cup_{s \in X} [s]_\approx)$, and $\pi_\approx([s]_\approx,p) =$ **true** (**false**) iff $\forall t \in [s]_\approx\ \pi(t,p) =$ **true** (**false**). (Notice that $\cup_{s \in X} [s]_\approx$ is measurable, since $\cup_{s \in X} [s]_\approx = \varphi^M$, for some $\varphi \in \mathcal{L}$.) Then $M_\approx$ is a distinguishable measurable probability structure for $\mathcal{L}$ which is equivalent to M.

(2) Let $M = \langle S,Bel,\pi\rangle$ be a measurable probability structure for $\mathcal{L}$. Define $M_\approx = \langle S_\approx,Bel_\approx,\pi_\approx\rangle$, where $Bel_\approx$ is given by $Bel_\approx(X_\approx) = Bel(X)$, for all $X_\approx \subseteq S_\approx$, and $\pi_\approx([s]_\approx,p) =$ **true** (**false**) iff $\forall t \in [s]_\approx\ \pi(t,p) =$ **true** (**false**). (Notice that each subset of $S_\approx$ is equal to $X_\approx$, for some $X \subseteq S$.) Then $M_\approx$ is a distinguishable DS structure for $\mathcal{L}$ which is equivalent to M. ■

Proposition 3.2 does *not* hold for probability structures in general. An argument is given in the following example.

Example 3.3 Suppose some sensor indicates that a valve is closed. Assume that the sensor is reliable in 80% of the cases and that in the remaining 20% it either indicates that the valve is closed or that it is open, independent of the actual state of the valve. Given the evidence, there are three possible states: s in which the valve is closed and the sensor is reliable, t in which the valve is closed and the valve is unreliable, and u in which the valve is open and the sensor is unreliable.

Let $\mathcal{L}$ be built up from PL = $\{c\}$, where c stands for "the valve is closed." Define $M = \langle S,\mathcal{X},P,\pi\rangle$, where $S = \{s,t,u\}$, $\mathcal{X}$ is the algebra with basis $\mathcal{Y} = \{\{s\},\{t,u\}\}$, P is determined by $P(\{s\}) = 0.8$, $P(\{t,u\}) = 0.2$, and π is given by $\pi(s,c) = \pi(t,c) =$ **true** and $\pi(u,c) =$ **false**. Then M is a probability structure for $\mathcal{L}$ which represents the evidence above. However, M is not distinguishable, since s and t are indistinguishable. Notice that in fact the probability function P represents information about some issue–the reliability of the sensor–that cannot be expressed in the languange $\mathcal{L}$.

M is not equivalent to any distinguishable probability structure N for $\mathcal{L}$, since such a structure N has exactly two states: s_c in which c is true and $s_{\neg c}$ in which c is false. If $W_N(c) = 0.8$, then $\{s_c\}$ is measurable. But then N is measurable and $W_N(c \vee \neg c) = W_N(c) + W_N(\neg c)$. However, $W_M(c) + W_M(\neg c) = 0.8 + 0 \neq W_M(c \vee \neg c) = 1$. ■

The equivalence of probability structures and DS structures mentioned in 3.1 no longer holds for distinguishable structures: not every distinguishable DS structure for $\mathcal{L}$ has an equivalent distinguishable probability structure for $\mathcal{L}$. (Otherwise, by 3.1(1) and 3.2(2), every probability structure would be equivalent to a distinguishable one.) The situation is summarized in Fig. 2.

Distinguishable structures are used by several authors (e.g., [6,7]) for assigning weights to formulas. Also, using distinguishable structures is trivially equivalent to assigning weights via (probability or belief) functions defined on the algebra which is most naturally associated with a languange, namely the Lindenbaum algebra. (See e.g. [1] for a definition of Lindenbaum algebra.)

Nevertheless, there may be good resons for *not* requiring weight structures to be distinguishable, but one should be aware of the fact that one then allows the structures to represent probabilistic information about issues that cannot be expressed in the langauge under consideration. (Cf. M in example 3.3.)

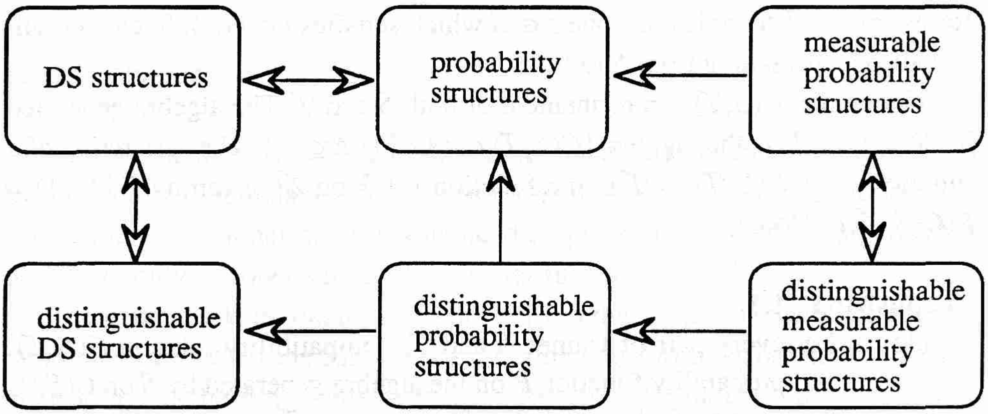

Fig. 2. The relations between classes of weight structures. An arrow from one class to another indicates that for every structure in the first class there is an equivalent one in the second class.

A consequence of proposition 3.2 is that the completeness results for measurable probability structures and DS structures mentioned in [3] remain valid when the structures are required to be distinguishable. In that case, the system for inner measures needs of course to be enriched with an axiom stating essentially that inner measures are *disjoint* belief functions.

4. Refinements, compatibility, and multivalued mappings

In this section we show that a belief function over S is the inner measure induced by some probability function on an algebra on a refinement of S. We use an adaptation of the original definition of belief functions given by Dempster [2] and some notions from [9].

A *multivalued mapping* G from S to $\mathcal{T}$ is a function $S \to 2^{\mathcal{T}}$ which satisfies $\forall s \in S\ G(s) \neq \emptyset$ and $\forall t \in \mathcal{T} \exists s \in S\ t \in G(s)$. The function Bel on $2^{\mathcal{T}}$ induced by a probability function P on 2^S and a multivalued mapping G from S to $\mathcal{T}$ is the function Bel given by $Bel(A) = P(\{s \in S \mid G(s) \subseteq B\})$. The class of *belief functions* over $\mathcal{T}$ is the class of functions on $2^{\mathcal{T}}$ induced by a probability function P on 2^S and a multivalued mapping G from S to $\mathcal{T}$.

A *compatibility relation* $C(S,\mathcal{T})$ between S and $\mathcal{T}$ is a subset of $S \times \mathcal{T}$ which satisfies $\forall s \in S\ \exists t \in \mathcal{T} \langle s,t\rangle \in C(S,\mathcal{T})$ and $\forall t \in \mathcal{T} \exists s \in S\ \langle s,t\rangle \in C(S,\mathcal{T})$. The compatibility relation $C(S,\mathcal{T})$ induced by a multivalued mapping G from S to $\mathcal{T}$ is the set $\{\langle s,t\rangle \in S \times \mathcal{T} \mid t \in G(s)\}$. S is called a *coarsening* of $\mathcal{T}$, and $\mathcal{T}$ a *refinement* of S, if there exist a multivalued mapping G from S to $\mathcal{T}$ such that for every $t \in \mathcal{T}$ there is only one $s \in S$ which satisfies $t \in G(s)$. Such a G will be called a *coarsening* from S to $\mathcal{T}$.

Notice that $C(S,\mathcal{T})$ is a refinement of both S and $\mathcal{T}$. The algebra generated by S on $C(S,\mathcal{T})$ is the algebra $\{C(S,\mathcal{T}) \cap (A \times \mathcal{T}) \mid A \subseteq S\}$. The *restriction* of a function F on $C(S,\mathcal{T})$ to $\mathcal{T}$ is the function $F \mid \mathcal{T}$ on $2^{\mathcal{T}}$ given by $F \mid \mathcal{T}(A) = F(C(S,\mathcal{T}) \cap (S \times A))$.

Proposition 4.1

(1) For every pair of frames S and $\mathcal{T}$, compatibility relation $C(S,\mathcal{T})$, and probability function P on the algebra generated by S on $C(S,\mathcal{T})$ the function $P_* \mid \mathcal{T}$ is a belief function over $\mathcal{T}$.

(2) For every belief function *Bel* over $\mathcal{T}$ there exist a frame S, a compatibility relation $C(S,\mathcal{T})$, and a probability function P on the algebra generated by S on $C(S,\mathcal{T})$ such that $Bel = P_* \mid \mathcal{T}$.

Proof.

(1) By 2.1(1), P_* is a belief fuction over $C(S,\mathcal{T})$, and, by theorem 6.8 of [8], the restriction of a belief function over a frame to a coarsening of that frame is again a belief function.

(2) Assume that *Bel* is a belief function over $\mathcal{T}$. By Dempster's definition, there exist a frame S, a multivalued mapping G from S to $\mathcal{T}$, and a probability function P' on 2^S such that for every $A \subseteq \mathcal{T}$ $Bel(A) = P'(\{s \in S \mid G(s) \subseteq A\})$. Let $C(S,\mathcal{T})$ be the compatibility relation induced by G, and let P be the probability function on the algebra generated by S on $C(S,\mathcal{T})$ defined by $P(C(S,\mathcal{T}) \cap (B \times \mathcal{T})) = P'(B)$, for all $B \subseteq S$. Then for every $A \subseteq \mathcal{T}$ we have that $Bel(A) = P'(\{s \in S \mid G(s) \subseteq A\}) = P(\{\langle s,t\rangle \in C(S,\mathcal{T}) \mid G(s) \subseteq A\}) = P_*(C(S,\mathcal{T}) \cap (S \times A))$. Hence $Bel = P_* \mid \mathcal{T}$. ■

Proposition 4.1 generalizes 2.1, since belief functions induced by a probability function P on 2^S and a coarsening G from S to $\mathcal{T}$ are disjoint belief functions, and $C(S,\mathcal{T})$ induced by such a coarsening G is isomorphic to $\mathcal{T}$. The following is essentially a reformulation of 4.1 in slightly more general terms.

Corollary 4.2

(1) For every probability function P on an algebra on a refinement S of a frame $\mathcal{T}$ the function $P_* \mid \mathcal{T}$ is a belief function over $\mathcal{T}$.

(2) For every belief function *Bel* over $\mathcal{T}$ there is a probability function P on an algebra on some refinement S of $\mathcal{T}$ such that $Bel = P_* \mid \mathcal{T}$.

5. Conclusion

Belief functions over S are exactly inner measures (restricted to S) induced by probability functions on algebras on refinements of S. Allowing refinements is essential, since inner measures induced by probability functions on algebras on S form only a proper subset of the belief functions over S.

The claim made by Fagin and Halpern [3,4] that belief functions are exactly inner measures when the functions are defined on formulas rather than sets is misleading. If the inner measures are allowed to be induced by probability functions on refinements of the frame over which the belief functions are defined, then the correspondence not only holds for functions defined on formulas, but also for functions defined on sets. If refinements are not allowed, then it can be argued that one should not assign weights to formulas in the way it is done in [3], but one should use distinguishable weight structures, which do not equate belief functions and inner measures.

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